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Concept lattices and order in fuzzy logic

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Abstract

The theory of concept lattices (i.e. hierarchical structures of concepts in the sense of Port-Royal school) is approached from the point of view of fuzzy logic. The notions of partial order, lattice order, and formal concept are generalized for fuzzy setting. Presented is a theorem characterizing the hierarchical structure of formal fuzzy concepts arising in a given formal fuzzy context. Also, as an application of the present approach, Dedekind–MacNeille completion of a partial fuzzy order is described. The approach and results provide foundations for formal concept analysis of vague data—the propositions "object x has attribute y", which form the input data to formal concept analysis, are now allowed to have also intermediate truth values, meeting reality better.

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1. Introduction

The notion of partial and lattice order goes back to 19th century investigations in logic [15]. The origins are in the study of hierarchy of concepts, i.e. the relation of being a subconcept of a superconcept. This view on order has been pursued lately by Wille et al. in the study of concepts in the sense of Port–Royal [1] (so-called formal concepts) and the corresponding hierarchical structures (so-called concept lattices) [7] as a part of a program of "restructuring lattice theory" [16] (restructuring means shifting lattice theory closer to its original motivations). Note also that concept lattices have

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found several real-world applications in data analysis (so-called formal concept analysis, see [7]).

Recent years brought thorough investigations in fuzzy logic in the so-called narrow sense (see e.g. [9,10]). Recall that the main distinguishing feature of fuzzy logic is that it allows propositions to have also intermediate truth values, not just full truth (1) or full falsity (0), i.e. fuzzy logic denies the principle of bivalence. Thus, for instance, the truth value of "Ivan is young" can be 0.9. Fuzzy logic seems to be an appropriate tool for reasoning in the presence of vagueness.

The aim of the present paper is to investigate Port–Royal concepts, their order, and partial order in general from the point of view of fuzzy logic. Our main motivation is that, from the point of view of fuzzy approach, the assumption of bivalence of concepts and their hierarchy is, especially in the context of empirical concepts, unrealistic. Taking the above-mentioned concept "young" as an example, there are surely individuals that are not fully young nor fully old (not young), Ivan being one of them. In this sense, the concept "young" is a typical example of a fuzzy concept. Also, the hope is that taking into account the vagueness phenomenon and modeling vagueness adequately should improve the application capabilities of formal concept analysis.

The paper is organized as follows. Section 2 surveys preliminaries. In Section 3, the notions of a formal fuzzy concept and fuzzy order are introduced, and some properties of fuzzy order are investigated. Section 4 presents the main result, the generalization of the so-called main theorem of concept lattices characterizing the hierarchical structure of formal fuzzy concepts. In the classical (i.e. bivalent) case, the well-known Dedekind–MacNeille completion of a partially ordered set is a particular concept lattice (that one induced by the partial order). As an application, Section 5 describes the Dedekind–MacNeille completion in fuzzy setting.

2. Preliminaries

First, we recall some basic facts about concept lattices. Let I be a binary relation between the sets X and Y. For $A \subseteq X$ and $B \subseteq Y$ put $A^{\uparrow} = \{y \in Y \mid \langle x, y\rangle \in I$ for each $x \in A\}$ and $B^{\downarrow} = \{x \in X \mid \langle x, y\rangle \in I$ for each $y \in B\}$. The pair $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ of thus defined mappings ${}^{\uparrow}: 2^X \to 2^Y$ and ${}^{\downarrow}: 2^Y \to 2^X$ is called a polarity induced by I. Each polarity satisfies the axioms of a Galois connection between X and Y and, conversely, each Galois connection between X and Y is a polarity [12]. The class $\mathscr{B}(X, Y, I) = \{\langle A, B\rangle \in 2^X \times 2^Y \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ of all fixed points of $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ equipped with binary relation $\leqslant$ defined by $\langle A_1, B_1\rangle \leqslant \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (or, equivalently, $B_2 \subseteq B_1$) forms thus a complete lattice. The following interpretation is crucial for our purpose: Let X and Y denote a set of objects and a set of (object) attributes, respectively, let $\langle x, y\rangle \in I$ mean that object x has the attribute y. Then $\langle A, B\rangle \in \mathscr{B}(X, Y, I)$ means that B is the set of all attributes common to all objects from A and A is the set of all objects sharing all the attributes from B. The triple $\langle X, Y, I\rangle$ is called a formal context, each $\langle A, B\rangle \in \mathscr{B}(X, Y, I)$ is called a formal concept (in the respective context), and $\mathscr{B}(X, Y, I)$ is called a concept lattice [16]. Note that the above interpretation takes its inspiration in the Port–Royal logic (see [1]), A and B play the role of the extent (i.e. the set of covered objects) and of the intent (i.e. the set of covered attributes) of the concept $\langle A,B\rangle$.

Except for the general case, there are well-known examples of concept lattices. We will need the following one: If $X=Y$ and I is a partial order, then $\langle A,B\rangle\in\mathscr{B}(X,Y,I)$ iff $\langle A,B\rangle$ is a cut; and $\mathscr{B}(X,Y,I)$ is the Dedekind–MacNeille completion of I [11].

Next, we recall some basic notions of fuzzy logic. The crucial point is to choose an appropriate structure of truth values. As it follows from the investigations in fuzzy logic [8–10], a general one is that of a complete residuated lattice.

Definition 1. A *residuated lattice* is an algebra $\mathbf{L}=\langle L,\wedge,\vee,\otimes,\rightarrow,0,1\rangle$ such that

(1) $\langle L,\wedge,\vee,0,1\rangle$ is a lattice with the least element 0 and the greatest element 1,
(2) $\langle L,\otimes,1\rangle$ is a commutative monoid,
(3) $\otimes,\rightarrow$ form an adjoint pair, i.e.

$$x\otimes y\leqslant z \quad\text{iff}\quad x\leqslant y\rightarrow z \tag{1}$$

holds for all $x,y,z\in L$.

Residuated lattice $\mathbf{L}$ is called *complete* if $\langle L,\wedge,\vee\rangle$ is a complete lattice.

$\otimes$ and $\rightarrow$ are called *multiplication* and *residuum*, respectively. Multiplication is isotone, residuum is isotone in the first and antitone in the second argument (w.r.t. lattice order $\leqslant$). For further properties of residuated lattices we refer to [8].

Several important algebras are special residuated lattices: Boolean algebras (algebraic counterpart of classical logic), Heyting algebras (intuitionistic logic), BL-algebras (logic of continuous t-norms), MV-algebras (Łukasiewicz logic), Girard monoids (linear logic) and others (see e.g. [9,10] for further information and references).

The most studied and applied set of truth values is the real interval [0,1] with $a\wedge b=\min(a,b)$, $a\vee b=\max(a,b)$, and with three important pairs of adjoint operations: the Łukasiewicz one ($a\otimes_{\mathrm{L}} b=\max(a+b-1,0)$, $a\rightarrow_{\mathrm{L}} b=\min(1-a+b,1)$), Gödel one ($a\otimes_{\mathrm{G}} b=\min(a,b)$, $a\rightarrow_{\mathrm{G}} b=1$ if $a\leqslant b$ and $=b$ else), and product one ($a\otimes_{\mathrm{P}} b=a\cdot b, a\rightarrow_{\mathrm{P}} b=1$ if $a\leqslant b$ and $=b/a$ else). More generally, if $\otimes$ is a continuous t-norm (i.e. a continuous operation making $\langle[0,1],\otimes,1,\leqslant\rangle$ an ordered monoid, see [9]) then putting $x\rightarrow=y=\sup\{z\,|\,x\otimes z\leqslant y\}$, $\langle[0,1],\min,\max,\otimes,\rightarrow,0,1\rangle$ is a complete residuated lattice—so-called t-norm algebra determined by $\otimes$. Each continuous t-norm is an ordered sum of $\otimes_{\mathrm{L}},\otimes_{\mathrm{G}}$, and $\otimes_{\mathrm{P}}$, see e.g. [9]. Another important set of truth values is the set $\{a_0=0,a_1,\ldots,a_n=1\}$ ($a_0<\cdots<a_n$) with $\otimes$ given by $a_k\otimes a_l=a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k\rightarrow a_l=a_{\min(n-k+l,n)}$. A special case of the latter algebras is the Boolean algebra $\mathbf{2}$ of classical logic with the support $2=\{0,1\}$. It may be easily verified that the only residuated lattice on $\{0,1\}$ is given by the classical conjunction operation $\wedge$, i.e. $a\wedge b=1$ iff $a=1$ and $b=1$; and by the classical implication operation $\rightarrow$, i.e. $a\rightarrow b=0$ iff $a=1$ and $b=0$. Note that each of the preceding residuated lattices is complete.

In what follows, we assume that all residuated lattices under consideration are complete. Elements of residuated lattices are interpreted as truth degrees, 0 and 1 representing (full) falsity and (full) truth. Multiplication $\otimes$ and residuum $\to$ are intended for modeling of the conjunction and implication, respectively. Supremum ($\bigvee$) and infimum ($\bigwedge$) are intended for modeling of general and existential quantifier, respectively. A syntactico-semantically complete first-order logic with semantics defined over complete residuated lattices can be found in [10], for logics complete w.r.t. semantics defined over various special residuated lattices see [9].

Analogously to the bivalent case, one can start developing a naive set theory with truth values in an (appropriately chosen) complete residuated lattice $\mathbf{L}$ (the classical bivalent case being a special case for $\mathbf{L}=\mathbf{2}$). We recall the basic notions. An $\mathbf{L}$-*set* (or fuzzy set, if $\mathbf{L}$ is obvious or not important) [17,8] A in a universe set X is any map $A:X\to L, A(x)$ being interpreted as the truth degree of the fact "x belongs to A". By L^X we denote the set of all $\mathbf{L}$-sets in X. The concept of an $\mathbf{L}$-relation is defined obviously; we will use both prefix and infix notation (thus, the truth degrees to which elements x and y are related by an $\mathbf{L}$-relation R are denoted by $R(x,y)$ or (xRy)). Operations on L extend pointwise to L^X, e.g. $(A\vee B)(x)=A(x)\vee B(x)$ for $A,B\in L^X$. Following common usage, we write $A\cup B$ instead of $A\vee B$, etc. Given $A,B\in L^X$, the subsethood degree [8] $\mathrm{S}(A,B)$ of A in B is defined by $\mathrm{S}(A,B)=\bigwedge_{x\in X} A(x)\to B(x)$. We write $A\subseteq B$ if $\mathrm{S}(A,B)=1$. Analogously, the equality degree $(A\approx B)$ of A and B is defined by $(A\approx B)=\bigwedge_{x\in X}(A(x)\leftrightarrow B(x))$. It is immediate that $(A\approx B)=\mathrm{S}(A,B)\wedge\mathrm{S}(B,A)$. For $A\in L^X$ and $a\in L$, the set ${}^aA=\{x\in X \mid A(x)\geqslant a\}$ is called the a-cut of A. For $x\in X$ and $a\in L$, $\{a/x\}$ is the $\mathbf{L}$-set in X defined by $\{a/x\}(x)=a$ and $\{a/x\}(y)=0$ for $y\neq x$.

3. Formal fuzzy concepts and fuzzy order

We are going to define the notions of a formal concept and order from the point of view of fuzzy logic so that the classical bivalent (notions) become a special cases for $\mathbf{L}=\mathbf{2}$. Our aim is to prepare necessary notions and facts to obtain the fuzzy version of the main theorem of concept lattices (which will be the subject of the next section). In the bivalent case, a set on which an order is defined is equipped by equality relation. The equality relation is explicitly used in the axiom of antisymmetry (if $x\leqslant y$ and $y\leqslant x$ then $x=y$). An appropriate generalization to the underlying logic with truth values in a complete residuated lattice is to define an $\mathbf{L}$-order on a set equipped with an $\mathbf{L}$-valued equality.

A binary $\mathbf{L}$-relation $\approx$ on X is called an $\mathbf{L}$-*equality* if it satisfies $(x\approx x)=1$ (reflexivity), $(x\approx y)=(y\approx x)$ (symmetry), $(x\approx y)\otimes(y\approx z)\leqslant(x\approx z)$ (transitivity), and $(x\approx y)=1$ implies $x=y$. Binary $\mathbf{L}$-relations satisfying reflexivity, symmetry, and transitivity are called $\mathbf{L}$-equivalences or $\mathbf{L}$-similarities. Note that $\mathbf{2}$-equality on X is precisely the usual equality (identity) id_X (i.e. $\mathrm{id}_X(x,y)=1$ for $x=y$ and $\mathrm{id}_X(x,y)=0$ for $x\neq y$). Therefore, the notion of $\mathbf{L}$-equality is a natural generalization of the classical (bivalent) notion. For an $\mathbf{L}$-set A in $\mathbf{X}$ and an $\mathbf{L}$-equality $\approx$ on X we define the $\mathbf{L}$-set $\mathrm{C}_\approx(A)$ by $C_\approx(A)(x)=\bigvee_{x'\in X} A(x')\otimes(x'\approx x)$. It is easy to see that $\mathrm{C}_\approx(A)$ is the smallest (w.r.t. $\subseteq$) $\mathbf{L}$-set in X that is compatible with $\approx$ and contains A.

Example 2. The equality degree $\approx$ is an **L**-equality on L^X, for any X.

We say that a binary **L**-relation R between X and Y is *compatible* w.r.t. $\approx_X$ and $\approx_Y$ if $R(x_1, y_1) \otimes (x_1 \approx_X x_2) \otimes (x_2 \approx_Y y_2) \leqslant R(y_1, y_2)$ for any $x_i \in X, y_i \in Y$ $(i = 1, 2)$. By $L^{\langle X,\approx_X\rangle \times \langle Y,\approx_Y\rangle}$ we denote the set of all **L**-relations between X and Y compatible w.r.t. $\approx_X$ and $\approx_Y$. Analogously, $A \in L^X$ is compatible w.r.t. $\approx_X$ if $A(x_1) \otimes (x_1 \approx_X x_2) \leqslant A(x_2)$. Note that $L^X = L^{\langle X,\mathrm{id}_X\rangle}$. An **L**-set $A \in L^{\langle X,\approx\rangle}$ is called an $\approx$-*singleton* if there is some $x_0 \in X$ such that $A(x) = (x_0 \approx x)$ for any $x \in X$. Clearly, an $\approx$-singleton is the least **L**-set A compatible w.r.t. $\approx$ such that $A(x_0) = 1$. For $\mathbf{L} = \mathbf{2}$, singletons coincide with one-elements sets.

Definition 3. An **L**-*order* on a set X with an **L**-equality relation $\approx$ is a binary **L**-relation $\preccurlyeq$ which is compatible w.r.t. $\approx$ and satisfies

$$
\begin{aligned}
x \preccurlyeq x &= 1 && \text{(reflexivity)},\\
(x \preccurlyeq y) \wedge (y \preccurlyeq x) &\leqslant x \approx y && \text{(antisymmetry)},\\
(x \preccurlyeq y) \otimes (y \preccurlyeq z) &\leqslant x \preccurlyeq z && \text{(transitivity)}.
\end{aligned}
$$

If $\preccurlyeq$ is an **L**-order on a set X with an **L**-equality $\approx$, we call the pair $\mathbf{X} = \langle\langle X,\approx\rangle,\preccurlyeq\rangle$ an **L**-ordered set.

Remark. (1) Clearly, if $\mathbf{L} = \mathbf{2}$, the notion of **L**-order coincides with the usual notion of (partial) order.

(2) For a similar approach to fuzzy order (however, with a different formulation of antisymmetry) see [5].

We say that **L**-ordered sets $\langle\langle X,\approx_X\rangle,\preccurlyeq_X\rangle$ and $\langle\langle Y,\approx_Y\rangle,\preccurlyeq_Y\rangle$ are isomorphic if there is a bijective mapping $h : X \to Y$ such that $(x \approx_X x') = (h(x) \approx_Y h(x'))$ and $(x \preccurlyeq_X x') = (h(x) \preccurlyeq_Y h(x'))$ holds for all $x, x' \in X$.

Lemma 4. *In an* **L**-*ordered set* $\langle\langle X,\approx\rangle,\preccurlyeq\rangle$ *we have* $(x \preccurlyeq y) \wedge (y \preccurlyeq x) = (x \approx y)$.

Proof. The "$\leqslant$" part of the equality is the antisymmetry condition. The "$\geqslant$" part follows from compatibility of $\preccurlyeq$: $(x \approx y) = (x \preccurlyeq x) \otimes (x \approx y) \leqslant (x \preccurlyeq y)$, and similarly $(x \approx y) \leqslant (y \preccurlyeq x)$, whence the conclusion follows. □

Lemma 5. *If* $\mathbf{X} = \langle\langle X,\approx_X\rangle,\preccurlyeq_X\rangle$ *and* $\mathbf{Y} = \langle\langle Y,\approx_Y\rangle,\preccurlyeq_Y\rangle$ *are* **L**-*ordered sets and* $h : X \to Y$ *is a mapping satisfying* $(x \preccurlyeq_X x') = (h(x) \preccurlyeq_Y h(x'))$ *then* **X** *and* **Y** *are isomorphic.*

Proof. By Lemma 4, $(h(x) \approx_Y h(x')) = ((h(x) \preccurlyeq_Y h(x')) \wedge (h(x') \preccurlyeq_Y h(x))) = (x \preccurlyeq_X x') \wedge (x' \preccurlyeq_X x) = (x \approx_X x')$, verifying the remaining condition of the definition of isomorphic **L**-ordered sets. □

Example 6. (1) For any set $X \neq \emptyset$ and any subset $\emptyset \neq M \subseteq L^X$, $\langle\langle M, \approx\rangle, \mathrm{S}\rangle$ is an **L**-ordered set. Indeed, reflexivity and antisymmetry is trivial. Transitivity: $\mathrm{S}(A,B) \otimes \mathrm{S}(B,C) \leqslant \mathrm{S}(A,C)$ holds iff $\mathrm{S}(A,B) \otimes \mathrm{S}(B,C) \leqslant A(x) \to C(x)$ is true for each $x \in X$ which is equivalent to $A(x) \otimes \mathrm{S}(A,B) \otimes \mathrm{S}(B,C) \leqslant C(x)$ which is true since $A(x) \otimes \mathrm{S}(A,B) \otimes \mathrm{S}(B,C) \leqslant A(x) \otimes (A(x) \to B(x)) \otimes (B(x) \to C(x)) \leqslant C(x)$. Compatibility i.e. $\mathrm{S}(A,B) \otimes (A \approx A') \otimes (B \approx B') \leqslant \mathrm{S}(A', B')$ can be verified analogously.

(2) For a residuated lattice **L** define $\approx$ and $\preccurlyeq$ by $(x \approx y) := (x \to y) \wedge (y \to x)$ and $(x \preccurlyeq y) := x \to y$. Then $\langle\langle L, \approx\rangle, \preccurlyeq\rangle$ is an **L**-ordered set. Note that $(x \approx y) = 1$ implies $x = y$ since $x \to y = 1$ iff $x \leqslant y$.

We are going to introduce the notion of polarity in many-valued setting. Let X and Y be sets with **L**-equalities $\approx_X$ and $\approx_Y$, respectively; I be an **L**-relation between X and Y which is compatible w.r.t. $\approx_X$ and $\approx_Y$. For $A \in L^X$ and $B \in L^Y$ let $A^\uparrow \in L^Y$ and $B^\downarrow \in L^Y$ be defined by

$$A^\uparrow(y) = \bigwedge_{x \in X} A(x) \to I(x,y) \tag{2}$$

and

$$B^\downarrow(x) = \bigwedge_{y \in Y} B(y) \to I(x,y). \tag{3}$$

Clearly, $A^\uparrow(y)$ is the truth degree to which "for each x from A, x and y are in I", and similarly for $B^\downarrow(x)$. Thus, (2) and (3) are natural generalizations of the classical case. We call the thus defined pair $\langle {}^\uparrow, {}^\downarrow \rangle$ of mappings ${}^\uparrow : L^X \to L^Y$ and ${}^\downarrow : L^Y \to L^X$ an **L**-*polarity* induced by I (and denote it also by $\langle {}^{\uparrow_I}, {}^{\downarrow_I} \rangle$). The one-to-one relationship between polarities and Galois connections [12] generalizes as follows: Let $\approx_X$ and $\approx_Y$ be **L**-equalities on X and Y, respectively. An **L**-Galois connection between $\langle X, \approx_X\rangle$ and $\langle Y, \approx_Y\rangle$ is a pair $\langle {}^\uparrow, {}^\downarrow \rangle$ of mappings ${}^\uparrow : L^{\langle X, \approx_X\rangle} \to L^{\langle Y, \approx_Y\rangle}$, ${}^\downarrow : L^{\langle Y, \approx_Y\rangle} \to L^{\langle X, \approx_X\rangle}$ satisfying

$$\mathrm{S}(A_1, A_2) \leqslant \mathrm{S}(A_2^\uparrow, A_1^\uparrow), \tag{4}$$

$$\mathrm{S}(B_1, B_2) \leqslant \mathrm{S}(B_2^\uparrow, B_1^\uparrow), \tag{5}$$

$$A \subseteq A^{\uparrow\downarrow}, \tag{6}$$

$$B \subseteq B^{\downarrow\uparrow} \tag{7}$$

for any $A, A_1, A_2 \in L^X, B, B_1, B_2 \in L^Y$. For the following proposition see [3]:

Proposition 7. *Let $I \in L^{\langle X, \approx_X\rangle \times \langle Y, \approx_Y\rangle}$, $\langle {}^\uparrow, {}^\downarrow \rangle$ be an* **L**-*Galois connection between $\langle X, \approx_X\rangle$ and $\langle Y, \approx_Y\rangle$. Denote by $I_{\langle\uparrow,\downarrow\rangle}$ the binary* **L**-*relation $I_{\langle\uparrow,\downarrow\rangle} \in L^{X \times Y}$ defined for $x \in X$, $y \in Y$ by $I_{\langle\uparrow,\downarrow\rangle}(x,y) = \{1/x\}^\uparrow(y)$ (or, equivalently, $= \{1/y\}^\downarrow(x)$). Then $\langle {}^{\uparrow_I}, {}^{\downarrow_I} \rangle$ is an* **L**-*Galois connection between $\langle X, \approx_X\rangle$ and $\langle Y, \approx_Y\rangle$; $I_{\langle\uparrow,\downarrow\rangle}$ is compatible w.r.t. $\approx_X$ and $\approx_Y$; and we have*

$$\langle {}^\uparrow, {}^\downarrow \rangle = \langle {}^{\uparrow_{I_{\langle\uparrow,\downarrow\rangle}}}, {}^{\downarrow_{I_{\langle\uparrow,\downarrow\rangle}}} \rangle \quad \text{and} \quad I = I_{\langle \uparrow_I, \downarrow_I \rangle}.$$

Note that, in fact, Proposition 7 is proved for $\approx_X = \mathrm{id}_X$ and $\approx_Y = \mathrm{id}_Y$ in [3]. The extension to the case of arbitrary **L**-equalities is an easy exercise.

We are now able to present the basic notions of concept lattices in fuzzy setting. A formal **L**-*context* is a triple $\langle X, Y, I\rangle$ where I is an **L**-relation between the set X and Y (elements of X and Y are called objects and attributes, respectively). For the **L**-polarity $^\uparrow, ^\downarrow$ induced by I, denote $\mathscr{B}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y\}$ the set of all fixed points of $\langle ^\uparrow, ^\downarrow\rangle$; and call $\mathscr{B}(X, Y, I)$ the corresponding **L**-*concept lattice*. Note that the thus defined notions are direct interpretations of the Port–Royal definition of concept in fuzzy setting. Doing so, the extent A of an **L**-concept $\langle A, B\rangle$ is a fuzzy set and may thus contain objects to different truth degrees, meeting the intuition about fuzziness (vagueness) of concepts. Our aim is to investigate the (hierarchical) structure of $\mathscr{B}(X, Y, I)$. Let $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathscr{B}(X, Y, I)$. By (4) we get $\mathrm{S}(A_1, A_2) \leqslant \mathrm{S}(A_2^\uparrow, A_1^\uparrow)$ and $\mathrm{S}(A_2, A_1) \leqslant \mathrm{S}(A_1^\uparrow, A_2^\uparrow)$. Therefore, since $A_i^\uparrow = B_i$ $(i = 1, 2)$, we have $\mathrm{S}(A_1, A_2) \leqslant \mathrm{S}(B_2, B_1)$ and $\mathrm{S}(A_2, A_1) \leqslant \mathrm{S}(B_1, B_2)$, whence $(A_1 \approx A_2) \leqslant (B_1 \approx B_2)$. Analogously, $\mathrm{S}(B_1, B_2) \leqslant \mathrm{S}(A_2, A_1)$, $\mathrm{S}(B_2, B_1) \leqslant \mathrm{S}(A_1, A_2)$, and $(B_1 \approx B_2) \leqslant (A_1 \approx A_2)$. We thus conclude $\mathrm{S}(A_1, A_2) = \mathrm{S}(B_2, B_1)$, and $\mathrm{S}(A_2, A_1) = \mathrm{S}(B_1, B_2)$, and $(A_1 \approx A_2) = (B_1 \approx B_2)$. Therefore, putting $(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle) = (A_1 \approx A_2)$ (or, equivalently, $= (B_1 \approx B_2)$) and $(\langle A_1, B_1\rangle \preccurlyeq \langle A_2, B_2\rangle) = \mathrm{S}(A_1, A_2)$ (or, equivalently, $= \mathrm{S}(B_2, B_1)$), Example 6 (1) implies that $\langle\langle \mathscr{B}(X, Y, I), \approx\rangle, \preccurlyeq\rangle$ is an **L**-ordered set.

An **L**-order $\preccurlyeq$ on $\langle X, \approx\rangle$ is a binary **L**-relation between $\langle X, \approx\rangle$ and $\langle X, \approx\rangle$. Therefore, $\preccurlyeq$ induces an **L**-Galois connection $\langle ^{\uparrow_\preccurlyeq}, ^{\downarrow_\preccurlyeq}\rangle$ between $\langle X, \approx\rangle$ and $\langle X, \approx\rangle$. Clearly, for an **L**-set A in X, $A^{\uparrow_\preccurlyeq}$ ($A^{\downarrow_\preccurlyeq}$) can be verbally described as the **L**-set of elements which are greater (smaller) than all elements of A. Therefore, we call $A^{\uparrow_\preccurlyeq}$ and $A^{\downarrow_\preccurlyeq}$ the *upper cone* and the *lower cone* of A, respectively. For $\mathbf{L} = \mathbf{2}$, we get the usual notions of upper and lower cone. Thus, following the common usage in the theory of ordered sets, we denote $A^{\uparrow_\preccurlyeq}$ by $U(A)$ and $A^{\downarrow_\preccurlyeq}$ by $L(A)$, and write $UL(A)$ instead of $U(L(A))$ etc. We now introduce the notion of an infimum and supremum in an **L**-ordered set, and the notion of an completely lattice **L**-ordered set.

Definition 8. For an **L**-ordered set $\langle\langle X, \approx\rangle, \preccurlyeq\rangle$ and $A \in L^X$ we define the **L**-sets $\inf(A)$ and $\sup(A)$ in X by

$$(\inf(A))(x) = (L(A))(x) \wedge (UL(A))(x),$$
$$(\sup(A))(x) = (U(A))(x) \wedge (LU(A))(x).$$

$\inf(A)$ and $\sup(A)$ are called the *infimum* and *supremum* of A, respectively.

Remark. The notions of infimum and supremum are generalizations of the classical notions. Indeed, if $\mathbf{L} = \mathbf{2}$, $(\inf(A))(x)$ is the truth value of the fact that x belongs to both the lower cone of A and the upper cone of the lower cone of A, i.e. x is the greatest lower bound of A; similarly for $\sup(A)$.

Lemma 9. *Let* $\langle\langle X, \approx\rangle, \preccurlyeq\rangle$ *be an* **L**-*ordered set*, $A \in L^X$. *If* $(\inf(A))(x) = 1$ *and* $(\inf(A))(y) = 1$ *then* $x = y$ (*and similarly for* $\sup(A)$).

Proof. $(\inf(A))(x)=1$ and $(\inf(A))(y)=1$ implies $(L(A))(x)=1$, $(L(A))(y)=1$, $(UL(A))(x)=1$, $(UL(A))(y)=1$. Since $(UL(A))(y)=\bigwedge_{z\in X}(L(A))(z)\to(z\preccurlyeq y)=1$ we have $(x\preccurlyeq y)=1\to(x\preccurlyeq y)=(L(A))(x)\to(x\preccurlyeq y)=1$. In a similar way, we get $y\preccurlyeq x=1$, therefore, by antisymmetry, $1=(x\preccurlyeq y)\wedge(y\preccurlyeq x)\leqslant x\approx y$. Since $\approx$ is an **L**-equality, we have $x=y$. □

Definition 10. An **L**-ordered set $\langle\langle X,\approx\rangle,\preccurlyeq\rangle$ is said to be completely lattice **L**-ordered if for any $A\in L^X$ both $\sup(A)$ and $\inf(A)$ are $\approx$-singletons.

Remark. Lemma 9 and Definition 10 imply that in a completely lattice **L**-ordered set **X**, supremum $\sup(A)$ of $A\in L^X$ is uniquely determined by the element $x\in X$ such that ${}^1\sup(A)=\{x\}$ (i.e. $(\sup(A))(x)=1$).

Checking that an **L**-ordered set is completely lattice **L**-ordered may be simplified:

Lemma 11. *For an* **L***-ordered set* **X** *and* $A\in L^X$ *we have*: $\inf(A)$ *is a* $\approx$*-singleton iff there is some* $x\in X$ *such that* $(\inf(A))(x)=1$. *The same is true for suprema.*

Proof. Obviously, we have to show that if $(\inf(A))(x)=1$ for some $x\in X$ then $\inf(A)$ is a $\approx$-singleton, i.e. $(\inf(A))(x')=(x\approx x')$ for all $x'\in X$. First, we show $(\inf(A))(x')\geqslant(x\approx x')$: we have to show that $(x\approx x')\leqslant(L(A))(x')$ and $(x\approx x')\leqslant(UL(A))(x')$. We show only the first inequality, the second one is analogous. By the definition of L, $(x\approx x')\leqslant(L(A))(x')$ holds iff $(x\approx x')\leqslant A(y)\to(x'\preccurlyeq y)$ for any $y\in X$ which is true iff $(x\approx x')\otimes A(y)\leqslant(x'\preccurlyeq y)$. Since $(L(A))(x)=1$, we have $(x\approx x')\otimes A(y)=(x\approx x')\otimes A(y)\otimes(L(A))(x)\leqslant(x\approx x')\otimes A(y)\otimes(A(y)\to(x\preccurlyeq y))\leqslant(x\approx x')\otimes(x\preccurlyeq y)\leqslant(x'\preccurlyeq y)$, verifying the required inequality.

Second, we show $(\inf(A))(x')\leqslant(x\approx x')$. As $\langle L,U\rangle$ forms a Galois connection between complete lattices $\langle L^X,\subseteq\rangle$ and $\langle L^X,\subseteq\rangle$, we have $L=LUL$, see [12, 3, Remark]. We thus have

$$
\begin{aligned}
(\inf(A))(x') &= (L(A))(x')\wedge(UL(A))(x')\\
&= (LUL(A))(x')\wedge(UL(A))(x')\\
&= \bigwedge_{y\in X}((UL(A))(y)\to(x'\preccurlyeq y))\wedge\bigwedge_{y\in X}((L(A))(y)\to(y\preccurlyeq x'))\\
&\leqslant ((UL(A))(x)\to(x'\preccurlyeq x))\wedge((L(A))(x)\to(x\preccurlyeq x'))\\
&= (1\to(x'\preccurlyeq x))\wedge(1\to(x\preccurlyeq x'))\\
&= (x'\preccurlyeq x)\wedge(x\preccurlyeq x')=(x\approx x').
\end{aligned}
$$

The case of suprema is dual. □

The following assertion generalizes the well-known fact that “infimum of a larger subset is smaller” and “supremum of a larger subset is bigger”.

Lemma 12. *For an* **L**-*ordered set* **L**, $A, B \in L^X$, *and* $x, y \in X$ *we have*

$$S(A,B) \otimes (\inf(A))(x) \otimes (\inf(B))(y) \leqslant (y \preccurlyeq x),$$
$$S(A,B) \otimes (\sup(A))(x) \otimes (\sup(B))(y) \leqslant (x \preccurlyeq y).$$

Proof. We have

$$\begin{aligned}
&S(A,B) \otimes (\inf(A))(x) \otimes (\inf(B))(y)\\
&\quad \leqslant S(L(B), L(A)) \otimes (\inf(A))(x) \otimes (\inf(B))(y)\\
&\quad \leqslant (L(B)(y) \to L(A)(y)) \otimes (\inf(A))(x) \otimes (L(B))(y)\\
&\quad \leqslant L(A)(y) \otimes (\inf(A))(x) \leqslant (L(A))(y) \otimes (UL(A))(x)\\
&\quad \leqslant (L(A))(y) \otimes \bigwedge_{x' \in X} ((L(A))(x') \to (x' \preccurlyeq x))\\
&\quad \leqslant (L(A)(y)) \otimes (L(A)(y) \to (y \preccurlyeq x)) \leqslant (y \preccurlyeq x)
\end{aligned}$$

proving the first inequality. The second one is dual. □

Note that for an **L**-order $\preccurlyeq$, ${}^1\!\preccurlyeq$ (the one-cut of $\preccurlyeq$, i.e. ${}^1\!\preccurlyeq = \{\langle x, y\rangle \in X \times X \mid (x \preccurlyeq y) = 1\}$) is a binary relation on X. $\langle x, y\rangle \in {}^1\!\preccurlyeq$ means that the fact that x is less or equal to y is "fully true". The basic properties of the "fully true"-part of an **L**-order are the subject of the following theorem.

Theorem 13. *For an* **L**-*ordered set* $\mathbf{X} = \langle\langle X, \approx\rangle, \preccurlyeq\rangle$, *the relation* ${}^1\!\preccurlyeq$ *is an order on* X. *Moreover, if* **X** *is completely lattice* **L**-*ordered then* ${}^1\!\preccurlyeq$ *is a lattice order on* X.

Proof. Denote $\sqsubseteq = {}^1\!\preccurlyeq$. Reflexivity of $\sqsubseteq$ follows from reflexivity of $\preccurlyeq$. Antisymmetry of $\sqsubseteq$: $x \sqsubseteq y$ and $y \sqsubseteq x$ implies $(x \preccurlyeq y) = 1$ and $(y \preccurlyeq x) = 1$. Antisymmetry of $\preccurlyeq$ thus yields $(x \approx y) = 1$. Since $\approx$ is an **L**-equality, we conclude $x = y$. If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $(x \preccurlyeq y) = 1$ and $(y \preccurlyeq z) = 1$, therefore $1 = (x \preccurlyeq y) \otimes (y \preccurlyeq z) \leqslant (x \preccurlyeq z)$, whence $(x \preccurlyeq z) = 1$, i.e. $x \sqsubseteq z$, by transitivity of $\preccurlyeq$.

Let **X** be completely lattice **L**-ordered and let A be a subset of X; denote by A' the **L**-set in X corresponding to A, i.e. $A'(x) = 1$ for $x \in A$ and $A'(x) = 0$ for $x \notin A$. We show that there exists a supremum $\bigwedge A$ of A in $\langle X, \sqsubseteq\rangle$ (the case of infimum is dual). Since **X** is completely lattice **L**-ordered, $\sup(C_\approx(A'))$ is a $\approx$-singleton in $\langle X, \approx\rangle$. Denote by x^* the element of X such that $(\sup(C_\approx(A')))(x^*) = 1$. Since $(\sup(C_\approx(A')))(x^*) = (U(C_\approx(A')))(x^*) \wedge (LU(C_\approx(A')))(x^*)$ we have both $(U(C_\approx(A')))(x^*) = 1$ and $(LU(C_\approx(A')))(x^*) = 1$. From the former we have $\bigwedge_{x \in X} (C_\approx(A'))(x) \to (x \preccurlyeq x^*) = 1$, i.e. $(C_\approx(A'))(x) \leqslant (x \preccurlyeq x^*)$ by adjointness. Since $A'(x) \leqslant (C_\approx(A'))(x)$ for any $x \in X$, we further conclude $(x \preccurlyeq x^*) = 1$ for any $x \in X$ such that $A'(x) = 1$ (i.e. $x \in A$). Therefore, x^* belongs to the upper cone (w.r.t. $\sqsubseteq$) of A. In a similar way, using $U(C_\approx(A)) = U(A)$ (this equality can be easily established), we can show that $(LU(C_\approx(A')))(x^*) = 1$ implies that x^* belongs to the lower cone of the upper cone (cones w.r.t. $\sqsubseteq$) of A. Thus, x^* is the supremum of A w.r.t. $\sqsubseteq$. □

Remark. (1) Theorem 13 has the following consequence: if $\mathbf{X}$ is a completely lattice $\mathbf{L}$-ordered set, we may speak about the infimum (supremum) of a (crisp) subset A of X w.r.t. ${}^1\preccurlyeq$. Unless otherwise specified, we adopt the following conventions to be used in what follows: whenever the context avoids possible confusion with the symbols related to the order on $\mathbf{L}$ (the structure of truth values), ${}^1\preccurlyeq$ will be denoted by $\leqslant$; infimum (supremum) of $A \subseteq X$ will be denoted by $\bigwedge A (\bigvee A)$ or any obvious modification of this notation. Due to the proof of Theorem 13, we have that for $A \subseteq X$ it holds ${}^1\inf(\mathrm{C}_\approx(A')) = \{\bigwedge A\}$ and ${}^1\sup(\mathrm{C}_\approx(A')) = \{\bigvee A\}$ where A' is the $\mathbf{L}$-set in X corresponding to A (i.e. $A'(x) = 1$ for $x \in A$ and $A'(x) = 0$ for $x \notin A$). Therefore, in a sense, the infima of crisp subsets of X w.r.t. to $\preccurlyeq$ and w.r.t. to $\leqslant$ (i.e. ${}^1\preccurlyeq$) are consistent.

(2) Note, however, that a completely lattice $\mathbf{L}$-ordered set $\mathbf{X} = \langle\langle X, \approx\rangle, \preccurlyeq\rangle$ is in general not determined by ${}^1\preccurlyeq$. Indeed, consider the following example: Let $\mathbf{L}$ be the Gödel algebra on $[0,1]$ (i.e. $a \otimes b = \min(a,b)$), let $X = \{x, y\}$. Consider the bivalent order $\leqslant = \{\langle x,x\rangle, \langle x,y\rangle, \langle y,y\rangle\}$ on X. Then $\mathbf{X}_1 = \langle\langle X_1, \approx_1\rangle, \preccurlyeq_1\rangle$ and $\mathbf{X}_2 = \langle\langle X_2, \approx_2\rangle, \preccurlyeq_2\rangle$ defined by $(x \preccurlyeq_1 x) = 1$, $(x \preccurlyeq_1 y) = 1$, $(y \preccurlyeq_1 x) = 0.6$, $(y \preccurlyeq_1 y) = 1$; $(x \preccurlyeq_2 x) = 1$, $(x \preccurlyeq_2 y) = 1$, $(y \preccurlyeq_2 x) = 0.8$, $(y \preccurlyeq_2 y) = 1$; and $(u \approx_i v) = \min((u \preccurlyeq_i v), (v \preccurlyeq_i u))$ (for $u, v \in X$, $i = 1, 2$), are two different completely lattice $\mathbf{L}$-ordered sets such that $\leqslant$ equals both ${}^1\preccurlyeq_1$ and ${}^1\preccurlyeq_2$.

4. The structure of fuzzy concept lattices

Recall that for an $\mathbf{L}$-set A in U and $a \in L$, $a \otimes A$ and $a \to A$ denote the $\mathbf{L}$-sets such that $(a \otimes A)(u) = a \otimes A(u)$ and $(a \to A)(u) = a \to A(u)$, respectively.

If $\mathscr{M}$ is an $\mathbf{L}$-set in Y and each $y \in Y$ is an $\mathbf{L}$-set in X, we define the $\mathbf{L}$-sets $\bigcap \mathscr{M}$ and $\bigcup \mathscr{M}$ in X by

$$\left(\bigcap \mathscr{M}\right)(x) = \bigwedge_{A \in Y} \mathscr{M}(A) \to A(x),$$

$$\left(\bigcup \mathscr{M}\right)(x) = \bigvee_{A \in Y} \mathscr{M}(A) \otimes A(x).$$

Clearly, $\bigcap \mathscr{M}$ and $\bigcup \mathscr{M}$ are generalizations of an intersection and a union of a system of sets, respectively. For an $\mathbf{L}$-set $\mathscr{M}$ in $\mathscr{B}(X, Y, I)$, we put $\bigcap_X \mathscr{M} = \bigcap \mathrm{pr}_X(\mathscr{M})$, $\bigcup_X \mathscr{M} = \bigcup \mathrm{pr}_X(\mathscr{M})$, $\bigcap_Y \mathscr{M} = \bigcap \mathrm{pr}_Y(\mathscr{M})$, $\bigcup_Y \mathscr{M} = \bigcup \mathrm{pr}_Y(\mathscr{M})$, where $\mathrm{pr}_X(\mathscr{M})$ is an $\mathbf{L}$-set in the set $\{A \in L^X \mid A = A^{\uparrow\downarrow}\}$ of all extents of $\mathscr{B}(X, Y, I)$ defined by $(\mathrm{pr}_X \mathscr{M})(A) = \mathscr{M}(A, A^\uparrow)$ and, similarly, $\mathrm{pr}_Y(\mathscr{M})$ is an $\mathbf{L}$-set in the set $\{B \in L^Y \mid B = B^{\uparrow\downarrow}\}$ of all intents of $\mathscr{B}(X, Y, I)$ defined by $(\mathrm{pr}_Y \mathscr{M})(B) = \mathscr{M}(B^\downarrow, B)$. Thus, $\bigcap_X \mathscr{M}$ is the "intersection of all extents from $\mathscr{M}$" etc.

Let $\mathbf{X}$ be a completely lattice $\mathbf{L}$-ordered set, $L' \subseteq L$. A subset $K \subseteq X$ is called L'-infimally dense in $\mathbf{X}$ (L'-supremally dense in $\mathbf{X}$) if for each $x \in X$ there is some $A \in L'^X$ such that $A(y) = 0$ for all $y \notin K$ and $(\inf(A))(x) = 1$ $((\sup(A))(x) = 1)$.

Remark. (1) Note that by Remark 3, K is $\{0,1\}$-infimally dense ($\{0,1\}$-supremally dense) in $\mathbf{X}$ if for each $x \in X$ there is some $K' \subseteq K$ such that $x = \bigwedge K'$ ($x = \bigvee K'$). Here, $\bigwedge$ ($\bigvee$) refers to the infimum (supremum) w.r.t. $\leqslant$, i.e. w.r.t. the one-cut of $\preccurlyeq$.

(2) For $\mathbf{L} = \mathbf{2}$ (the classical (bivalent) case), the above notions coincide with the usual notions of infimal and supremal density.

We are ready to present the main result characterizing the hierarchical structure of $\mathscr{B}(X,Y,I)$.

Theorem 14. *Let $\langle X,Y,I\rangle$ be an $\mathbf{L}$-context.* (1) *$\langle\langle\mathscr{B}(X,Y,I),\approx\rangle,\preccurlyeq\rangle$ is completely lattice $\mathbf{L}$-ordered set in which infima and suprema can be described as follows: for an $\mathbf{L}$-set $\mathscr{M}$ in $\mathscr{B}(X,Y,I)$ we have*

$$
\begin{aligned}
{}^1\inf(\mathscr{M}) &= \left\{\left\langle \bigcap_X \mathscr{M}, \left(\bigcap_X \mathscr{M}\right)^{\uparrow} \right\rangle\right\} \\
&= \left\{\left\langle \left(\bigcup_Y \mathscr{M}\right)^{\downarrow}, \left(\bigcup_Y \mathscr{M}\right)^{\downarrow\uparrow} \right\rangle\right\},
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
{}^1\sup(\mathscr{M}) &= \left\{\left\langle \left(\bigcap_Y \mathscr{M}\right)^{\downarrow}, \bigcap_Y \mathscr{M} \right\rangle\right\} \\
&= \left\{\left\langle \left(\bigcup_X \mathscr{M}\right)^{\uparrow\downarrow}, \left(\bigcup_X \mathscr{M}\right)^{\uparrow} \right\rangle\right\}.
\end{aligned} \tag{9}
$$

(2) *Moreover, a completely lattice $\mathbf{L}$-ordered set $\mathbf{V} = \langle\langle V,\approx\rangle,\preccurlyeq\rangle$ is isomorphic to $\langle\langle\mathscr{B}(X,Y,I),\approx\rangle,\preccurlyeq\rangle$ iff there are mappings $\gamma : X \times L \to V$, $\mu : Y \times L \to V$, such that $\gamma(X \times L)$ is $\{0,1\}$-supremally dense in $\mathbf{V}$, $\mu(Y \times L)$ is $\{0,1\}$-infimally dense in $\mathbf{V}$, and $((a \otimes b) \to I(x,y)) = (\gamma(x,a) \preccurlyeq \mu(y,b))$ for all $x \in X$, $y \in Y$, $a,b \in L$. In particular, $\mathbf{V}$ is isomorphic to $\mathscr{B}(V,V,\preccurlyeq)$.*

Proof. For brevity, we write also $\mathscr{B}$ instead of $\mathscr{B}(X,Y,I)$.

Part 1: From Example 6, we know that $\langle\langle\mathscr{B}(X,Y,I),\approx\rangle,\preccurlyeq\rangle$ is an $\mathbf{L}$-ordered set. We therefore have to show that $\inf(\mathscr{M})$ and $\sup(\mathscr{M})$ are $\approx$-singletons and that (8) and (9) hold for any $\mathbf{L}$-set $\mathscr{M}$ in $\mathscr{B}(X,Y,I)$. We proceed only for suprema, the case of infima is symmetric. First, we show (9). To this end, denote $\langle A^*,B^*\rangle = \langle(\bigcup_X \mathscr{M})^{\uparrow\downarrow}, (\bigcup_X \mathscr{M})^{\uparrow}\rangle$. We start by proving $B^* = \bigcap_Y \mathscr{M}$, i.e. we have to prove

$$
\begin{aligned}
&\bigwedge_{x\in X}\left(\left(\bigvee_{\langle A,B\rangle\in\mathscr{B}} \mathscr{M}(A,B)\otimes A(x)\right) \to I(x,y)\right) \\
&\quad = \bigwedge_{\langle A,B\rangle\in\mathscr{B}} (\mathscr{M}(A,B) \to B(y))
\end{aligned} \tag{10}
$$

for any $y\in Y$. The "$\leqslant$" part of (10) holds iff $\bigwedge_{x\in X}((\bigvee_{\langle A,B\rangle\in\mathscr{B}}\mathscr{M}(A,B)\otimes A(x))\to I(x,\ y))\leqslant\mathscr{M}(A,B)\to B(y)$ is true for any $\langle A,B\rangle\in\mathscr{B}$ which holds iff $\mathscr{M}(A,B)\otimes\bigwedge_{x\in X}((\bigvee_{\langle A,B\rangle\in\mathscr{B}}\mathscr{M}(A,B)\otimes A(x))\to I(x,y))\leqslant B(y)$. The last inequality is true since

$$\begin{aligned}\mathscr{M}(A,B)\otimes\bigwedge_{x\in X}&\left(\left(\bigvee_{\langle A,B\rangle\in\mathscr{B}}\mathscr{M}(A,B)\otimes A(x)\right)\to I(x,y)\right)\\ &\leqslant\mathscr{M}(A,B)\otimes\bigwedge_{x\in X}((\mathscr{M}(A,B)\otimes A(x))\to I(x,y))\\ &\leqslant\bigwedge_{x\in X}(\mathscr{M}(A,B)\otimes(\mathscr{M}(A,B)\to(A(x)\to I(x,y)))\\ &\leqslant\bigwedge_{x\in X}A(x)\to I(x,y)=B(y).\end{aligned}$$

The "$\geqslant$" part of (10) holds iff $\bigwedge_{\langle A,B\rangle\in\mathscr{B}}(\mathscr{M}(A,B)\to B(y))\leqslant(\bigvee_{\langle A,B\rangle\in\mathscr{B}}\mathscr{M}(A,B)\otimes A(x))\to I(x,y)$ is valid for each $x\in X$ which holds iff $(\bigvee_{\langle A,B\rangle\in\mathscr{B}}\mathscr{M}(A,B)\otimes A(x))\otimes\bigwedge_{\langle A,B\rangle\in\mathscr{B}}(\mathscr{M}(A,B)\to B(y))\leqslant I(x,y)$ which is true. Indeed,

$$\begin{aligned}&\left(\bigvee_{\langle A,B\rangle\in\mathscr{B}}\mathscr{M}(A,B)\otimes A(x)\right)\otimes\bigwedge_{\langle A,B\rangle\in\mathscr{B}}(\mathscr{M}(A,B)\to B(y))\\ &=\bigvee_{\langle A,B\rangle\in\mathscr{B}}(\mathscr{M}(A,B)\otimes A(x)\otimes\bigwedge_{\langle A,B\rangle\in\mathscr{B}}(\mathscr{M}(A,B)\to B(y)))\\ &\leqslant\bigvee_{\langle A,B\rangle\in\mathscr{B}}(A(x)\otimes\mathscr{M}(A,B)\otimes(\mathscr{M}(A,B)\to B(y)))\\ &\leqslant\bigvee_{\langle A,B\rangle\in\mathscr{B}}(A(x)\otimes B(y))\leqslant I(x,y),\end{aligned}$$

the last inequality being true by adjunction and the fact that $\langle A,B\rangle\in\mathscr{B}(X,Y,I)$. Therefore, (10) is established. Obviously, to prove (9) it is now sufficient to check ${}^{1}\mathrm{sup}(\mathscr{M})=\{\langle A^*,B^*\rangle\}$. That is, we have to show $(U(\mathscr{M}))(A^*,B^*)\wedge(LU(\mathscr{M}))(A^*,B^*)=1$, i.e. $(U(\mathscr{M}))(A^*,B^*)=1$ and $(LU(\mathscr{M}))(A^*,B^*)=1$. We show $(U(\mathscr{M}))(A^*,B^*)=1$. By definition of U, we have to show that for any $\langle A,B\rangle\in\mathscr{B}(X,Y,I)$ we have $1\leqslant\mathscr{M}(A,B)\to(\langle A,B\rangle\preccurlyeq\langle A^*,B^*\rangle)$, i.e. $1\leqslant\mathscr{M}(A,B)\to\mathrm{S}(A,A^*)$ which is equivalent to $\mathscr{M}(A,B)\leqslant\mathrm{S}(A,A^*)$. The last inequality holds iff $\mathscr{M}(A,B)\leqslant A(x)\to A^*(x)$ holds for each $x\in X$, i.e. iff $\mathscr{M}(A,B)\otimes A(x)\leqslant A^*(x)$ which is true by definition of $\langle A^*,B^*\rangle$. We established $(U(\mathscr{M}))(A^*,B^*)=1$. We show $(LU(\mathscr{M}))(A^*,B^*)=1$: By definition of L we have to show $1\leqslant(U(\mathscr{M}))(A,B)\to(\langle A^*,B^*\rangle\preccurlyeq\langle A,B\rangle)$, i.e. $(U(\mathscr{M}))(A,B)\leqslant(\langle A^*,B^*\rangle\preccurlyeq\langle A,B\rangle)$ for any $\langle A,B\rangle\in\mathscr{B}(X,Y,I)$. Since $(\langle A^*,B^*\rangle\preccurlyeq\langle A,B\rangle)=\mathrm{S}(B,B^*)$, we have to prove $(U(\mathscr{M}))(A,B)\leqslant B(y)\to B^*(y)$, i.e. $B(y)\otimes(U(\mathscr{M}))(A,B)\leqslant B^*(y)$ for each $y\in Y$. As $B^*=\bigcap_Y\mathscr{M}$, we have to show $B(y)\otimes(U(\mathscr{M}))(A,B)\leqslant\bigwedge_{\langle A',B'\rangle\in\mathscr{B}(X,Y,I)}\mathscr{M}(A',B')\to B'(y)$, which holds iff for each $\langle A',B'\rangle\in\mathscr{B}(X,Y,I)$ we have $\mathscr{M}(A',B')\otimes B(y)\otimes(U(\mathscr{M}))(A,B)\leqslant\rightarrow B'(y)$ which is true since

$$\begin{aligned}&\mathscr{M}(A',B')\otimes B(y)\otimes(U(\mathscr{M}))(A,B)\\&=\mathscr{M}(A',B')\otimes B(y)\otimes\left(\bigwedge_{\langle A'',B''\rangle\in\mathscr{B}}\mathscr{M}(A'',B'')\rightarrow(\langle A'',B''\rangle\preccurlyeq\langle A,B\rangle)\right)\\&=\mathscr{M}(A',B')\otimes B(y)\otimes\left(\bigwedge_{\langle A'',B''\rangle\in\mathscr{B}}\mathscr{M}(A'',B'')\rightarrow \mathrm{S}(B,B'')\right)\\&\leqslant\mathscr{M}(A',B')\otimes B(y)\otimes(\mathscr{M}(A',B')\rightarrow\mathrm{S}(B,B'))\\&\leqslant B(y)\otimes\mathrm{S}(B,B')\leqslant B'(y).\end{aligned}$$

We proved $(LU(\mathscr{M}))(A^*,B^*)=1$.

It remains to show that $\sup(\mathscr{M})$ is a $\approx$-singleton in $\mathscr{B}(X,Y,I)$. This fact, however, follows by Lemma 11.

Part 2: Let $\mathscr{B}(X,Y,I)$ and $\mathbf{V}$ be isomorphic. We show the existence of γ, μ with the desired properties. It suffices to show the existence for $\mathbf{V}=\mathscr{B}(X,Y,I)$ because for the general case $\mathbf{V}\cong\mathscr{B}(X,Y,I)$ one can take $\gamma\circ\varphi:X\times L\rightarrow V$, $\mu\circ\varphi:Y\times L\rightarrow V$, where φ is the isomorphism of $\mathscr{B}(X,Y,I)$ onto $\mathbf{V}$. Let then $\gamma:X\times L\rightarrow\mathscr{B}(X,Y,I)$, $\mu:Y\times L\rightarrow\mathscr{B}(X,Y,I)$ be defined by

$$\begin{aligned}&\gamma(x,a)=\langle\{a/x\}^{\uparrow\downarrow},\{a/x\}^{\uparrow}\rangle,\\&\mu(y,b)=\langle\{b/y\}^{\downarrow},\{b/y\}^{\downarrow\uparrow}\rangle\end{aligned}$$

for every $x\in X$, $y\in Y$, $a,b\in L$. Since for each $\langle A,B\rangle\in\mathscr{B}(X,Y,I)$ it holds $A=\bigcup_{x\in X}\{A(x)/x\}$, and $B=\bigcup_{y\in Y}\{B(y)/y\}$, it follows from (8) and (9) that $\gamma(X\times L)$ and $\mu(Y\times L)$ are $\{0,1\}$-supremally dense and $\{0,1\}$-infimally dense in $\mathscr{B}(X,Y,I)$, respectively. We show that $((a\otimes b)\rightarrow I(x,y))=(\gamma(x,a)\preccurlyeq\mu(y,b))$ is true for any $a,b\in L$, $x\in X$, and $y\in Y$: the equality easily follows by observing that $(\gamma(x,a)\preccurlyeq\mu(y,b))=\mathrm{S}(\{a/x\}^{\uparrow\downarrow},\{b/y\}^{\downarrow})=\mathrm{S}(\{b/y\},\{a/x\}^{\uparrow})=b\rightarrow(a\rightarrow I(x,y))=(a\otimes b)\rightarrow I(x,y)$.

Conversely, let γ and μ with the above properties exist. We prove the assertion by showing that there are mappings $\varphi:\mathscr{B}(X,Y,I)\rightarrow V$, $\psi:V\rightarrow\mathscr{B}(X,Y,I)$, such that $\varphi\circ\psi=\mathrm{id}_{\mathscr{B}(X,Y,I)}$, $\psi\circ\varphi=\mathrm{id}_V$, and $(\langle A_1,B_1\rangle\preccurlyeq\langle A_2,B_2\rangle)=(\varphi(A_1,B_1)\preccurlyeq\varphi(A_2,B_2))$. Then φ is a bijection and, by Lemma 5, $\mathscr{B}(X,Y,I)$ and $\mathbf{V}$ are isomorphic. We will need the following claims.

Claim A. $\gamma(x,\bigvee_{j\in J}a_j)=\bigvee_{j\in J}\gamma(x,a_j)$, $\mu(y,\bigvee_{j\in J}a_j)=\bigwedge_{j\in J}\mu(y,a_j)$ *for each* $x\in X$, $y\in Y$, $\{a_j\mid j\in J\}\subseteq L$, *i.e.* $\gamma(x,\_):L\rightarrow V$ *are complete lattice* $\bigvee$*-morphisms and* $\mu(y,\_):L\rightarrow V$ *are dual complete lattice* $\bigwedge$*-morphisms.*

Proof. The $\{0,1\}$-infimal density of $\mu(Y\times L)$ implies that $\gamma(x,\bigvee_{j\in J}a_j)=\bigwedge_{\langle y,b\rangle\in M}\mu(y,b)$ for some $M\subseteq Y\times L$. Hence, $\gamma(x,\bigvee_{j\in J}a_j)\leqslant\mu(y,b)$ which implies $1=(\gamma(x,\bigvee_{j\in J}a_j)\preccurlyeq\mu(y,b))=((\bigvee_{j\in J}a_j)\otimes b)\rightarrow I(x,y)$, whence $(\bigvee_{j\in J}a_j)\otimes b\leqslant$

$I(x,y)$, for each $\langle y,b\rangle \in M$. From $a_j \otimes b \leqslant (\bigvee_{j\in J} a_j)\otimes b$ we have $a_j \otimes b \leqslant I(x,y)$, i.e. $\gamma(x,a_j) \leqslant \mu(y,b)$ for every $j \in J$. This implies $\bigvee_{j\in J} \gamma(x,a_j) \leqslant \bigwedge_{\langle y,b\rangle \in M} \mu(y,b) = \gamma(x, \bigvee_{j\in J} a_j)$.

Conversely, the $\{0,1\}$-infimal density of $\mu(Y\times L)$ again implies the existence of some $M \subseteq Y\times L$ such that $\bigvee_{j\in J} \gamma(x,a_j) = \bigwedge_{\langle y,b\rangle\in M} \mu(y,b)$. That means that for each $j\in J$, $\langle y,b\rangle \in M$ we have $\gamma(x,a_j)\leqslant \mu(y,b)$, i.e. $a_j\otimes b \leqslant I(x,y)$. This implies $\bigvee_{j\in J}(a_j\otimes b)\leqslant I(x,y)$ and, by $\bigvee_{j\in J}(a_j\otimes b) = (\bigvee_{j\in J} a_j)\otimes b$, furthermore $(\bigvee_{j\in J} a_j) \otimes b \leqslant I(x,y)$, i.e. $\gamma(x,\bigvee_{j\in J} a_j)\leqslant \mu(y,b)$ for each $\langle y,b\rangle\in M$, thus $\gamma(x,\bigvee_{j\in J} a_j)\leqslant \bigwedge_{\langle y,b\rangle\in M}\mu(y,b) = \bigvee_{j\in J}\gamma(x,a_j)$, proving $\gamma(x,\bigvee_{j\in J} a_j) = \bigvee_{j\in J}\gamma(x,a_j)$.

$\mu(y,\bigvee_{j\in J} a_j) = \bigwedge_{j\in J}\mu(y,a_j)$ may be proved analogously using the $\{0,1\}$-supremal density of $\gamma(X\times L)$. □

Claim B. *For any $A\subseteq V$ and $u\in V$ we have $\bigwedge_{v\in A}(u\preccurlyeq v) = (u\preccurlyeq(\bigwedge_{v\in A} v))$ and $\bigwedge_{v\in A}(v\preccurlyeq u) = ((\bigvee_{v\in A} v)\preccurlyeq u)$.*

Proof. Consider the crisp **L**-set A' in V corresponding to A (i.e. $A'(v)=1$ if $v\in A$ and $A'(v)=0$ otherwise). We show $\bigwedge_{v\in A}(u\preccurlyeq v) = (u\preccurlyeq \bigwedge_{v\in A} v)$. Denote $v^* = \bigwedge_{v\in A} v$. Since $\bigwedge$ in the previous formula is derived from inf in **V**, we have $(\inf(A'))(v^*) = ((L(A'))(v^*)) \wedge ((UL(A'))(v^*)) = 1$, therefore also $(UL(A'))(v^*) = 1$. We have $(L(A'))(u) = \bigwedge_{u'\in V} A'(u')\to(u\preccurlyeq u') = \bigwedge_{v\in A}(u\preccurlyeq v)$. On the other hand, $(LUL(A'))(u) = \bigwedge_{u'\in V}(UL(A'))(u')\to(u\preccurlyeq u')\leqslant (UL(A'))(v^*)\to(u\preccurlyeq v^*) = (u\preccurlyeq v^*) = (u\preccurlyeq \bigwedge_{v\in A} v)$. Applying $L(A') = LUL(A')$ we get $\bigwedge_{v\in A}(u\preccurlyeq v)\leqslant (u\preccurlyeq \bigwedge_{v\in A} v)$. Conversely, $(u\preccurlyeq \bigwedge_{v\in A} v)\leqslant \bigwedge_{v\in A}(u\preccurlyeq v)$ holds iff $(u\preccurlyeq\bigwedge_{v\in A} v)\leqslant(u\preccurlyeq v)$ for any $v\in A$ which is true since $(u\preccurlyeq\bigwedge_{v\in A} v) = (u\preccurlyeq \bigwedge_{v\in A} v)\otimes(\bigwedge_{v\in A} v\preccurlyeq v)\leqslant(u\preccurlyeq v)$, by transitivity of $\preccurlyeq$.

The second equality, i.e. $\bigwedge_{v\in A}(v\preccurlyeq u) = ((\bigvee_{v\in A} v)\preccurlyeq u)$ can be proved analogously: put $v^* = \bigvee_{v\in A} v$; one has $(\sup(A'))(v^*) = 1$, and so also $(LU(A'))(v^*) = 1$; one can verify $(U(A'))(u) = \bigwedge_{v\in A}(v\preccurlyeq u)$ and $(ULU(A'))(u)\leqslant(\bigvee_{v\in A} v\preccurlyeq u)$; using $U(A') = ULU(A')$ we get $\bigwedge_{v\in A}(v\preccurlyeq u)\leqslant(\bigvee_{v\in A} v\preccurlyeq u)$; conversely, we have $(\bigvee_{v\in A} v \preccurlyeq u) = (v\preccurlyeq\bigvee_{v\in A} v)\otimes(\bigvee_{v\in A} v\preccurlyeq u)\leqslant(v\preccurlyeq u)$ for any $v\in A$ yielding $(\bigvee_{v\in A} v\preccurlyeq u)\leqslant \bigwedge_{v\in A}(v\preccurlyeq u)$. □

Claim C. $(a\to b)\leqslant(\gamma(x,a)\preccurlyeq\gamma(x,b))$ *for every $x\in X$ and $a,b\in L$.*

Proof. From the $\{0,1\}$-infimal density of $\mu(Y\times L)$ it follows that $\gamma(x,b) = \bigwedge_{\gamma(x,b)\leqslant\mu(y,b')}\mu(y,b')$. Therefore, we have to show $(a\to b)\leqslant(\gamma(x,a)\preccurlyeq(\bigwedge_{\gamma(x,b)\leqslant\mu(y,b')}\mu(y,b')))$. Putting $A = \{\mu(y,b')\mid\gamma(x,b)\leqslant\mu(y,b')\}$. Claim B yields $(\gamma(x,a)\preccurlyeq(\bigwedge_{\gamma(x,b)\leqslant\mu(y,b')}\mu(y,b'))) = \bigwedge_{\gamma(x,b)\leqslant\mu(y,b')}(\gamma(x,a)\preccurlyeq(\mu(y,b')))$.

To prove $(a\to b)\leqslant(\gamma(x,a)\preccurlyeq\gamma(x,b))$ we therefore have to show $(a\to b)\leqslant \bigwedge_{\gamma(x,b)\leqslant\mu(y,b')}(\gamma(x,a)\preccurlyeq\mu(y,b'))$, i.e. we have to show that $(a\to b)\leqslant(\gamma(x,a)\preccurlyeq\mu(y,b'))$ is true for any y and b' such that $\gamma(x,b)\leqslant\mu(y,b')$. By assumption, $(\gamma(x,a)\preccurlyeq\mu(y,b')) = (a\otimes b')\to I(x,y)$. If $\gamma(x,b)\leqslant\mu(y,b')$ then using the assumption we get $1 = \gamma(x,b)\to\mu(y,b') = (b\otimes b')\to I(x,y)$, i.e. $b\otimes b'\leqslant I(x,y)$. We therefore have $a\to b\leqslant(\gamma(x,a)\preccurlyeq\mu(y,b'))$ iff $a\to b\leqslant(a\otimes b')\to I(x,y)$ iff (we now use $(a\otimes b')\to I(x,y)$

$= a \to (b' \to I(x,y))$ and adjointness) $a \otimes (a \to b) \otimes b' \leqslant I(x,y)$ which is true since $a \otimes (a \to b) \otimes b' \leqslant b \otimes b'$ and $b \otimes b' \leqslant I(x,y)$ (by the above observation). □

Claim D. $I(x,y) = \bigvee_{\gamma(x,a) \leqslant \mu(y,b)} a \otimes b$.

Proof. The inequality $I(x,y) \geqslant \bigvee_{\gamma(x,a) \leqslant \mu(y,b)} a \otimes b$ follows immediately. For $a = I(x,y)$, $b = 1$ we have $a \otimes b = I(x,y) \otimes 1 \leqslant I(x,y)$, hence $\gamma(x, I(x,y)) \leqslant \mu(y,1)$, thus the equality holds. □

Define the mapping $\varphi : \mathscr{B}(X,Y,I) \to V$ by

$$\varphi(A,B) = \bigvee_{x \in X} \gamma(x, A(x)) \tag{11}$$

for each $\langle A,B \rangle \in \mathscr{B}(X,Y,I)$.

We prove the existence of an inverse mapping ψ of φ. Define $\psi : V \to \mathscr{B}(X,Y,I)$ by

$$\psi(v) = \langle A,B \rangle, \quad \text{where } A(x) = \bigvee_{\gamma(x,a) \leqslant v} a, \; B(y) = \bigvee_{\mu(y,b) \geqslant v} b \tag{12}$$

for each $v \in V$, and every $x \in X$, $y \in Y$. First, we show that for each $v \in V$, $\psi(v)$ is a fixed point of $\mathscr{B}(X,Y,I)$, i.e. $A^{\uparrow} = B$ and $B^{\downarrow} = A$. We show only $B^{\downarrow} = A$, the second case may be proved symmetrically. By Claim D we have

$$B^{\downarrow}(x) = \bigwedge_{y \in Y} B(y) \to I(x,y) = \bigwedge_{y \in Y} \left(\bigwedge_{\mu(y,b) \geqslant v} b \to \bigvee_{\gamma(x,a) \leqslant \mu(y,b)} a \otimes b \right).$$

We show $A(x) \leqslant B^{\downarrow}(x)$. We have

$$\bigvee_{\gamma(x,a) \leqslant v} a \otimes \bigwedge_{\mu(y,b) \geqslant v} b \leqslant \bigvee_{\gamma(x,a) \leqslant v \leqslant \mu(y,b)} a \otimes b \leqslant \bigvee_{\gamma(x,a) \leqslant \mu(y,b)} a \otimes b,$$

i.e.

$$A(x) = \bigvee_{\gamma(x,a) \leqslant v} a \leqslant \bigwedge_{\mu(y,b) \geqslant v} b \to \bigvee_{\gamma(x,a) \leqslant \mu(y,b)} a \otimes b$$

holds for each $y \in Y$, hence also

$$A(x) = \bigvee_{\gamma(x,a) \leqslant v} a \leqslant \bigwedge_{y \in Y} \left(\bigwedge_{\mu(y,b) \geqslant v} b \to \bigvee_{\gamma(x,a) \leqslant \mu(y,b)} a \otimes b \right) = B^{\downarrow}(x),$$

holds. We now show the equality $A(x) = B^{\downarrow}(x)$ as follows. Suppose there is an $a \in L$ such that for each $y \in Y$ it holds

$$a \leqslant \bigvee_{\mu(y,b) \geqslant v} b \to I(x,y) \tag{13}$$

(i.e. a is a lower bound) and show that $a \leqslant \bigvee_{\gamma(x,a') \leqslant v} a' = A(x)$ (i.e. $A(x)$ is the infimum, i.e. $B^{\downarrow}(x)$). Eq. (13) holds iff

$$a \otimes \bigvee_{\mu(y,b) \geqslant v} b \leqslant I(x,y),$$

i.e. by $a \otimes \bigvee_{\mu(y,b) \geqslant v} b = \bigvee_{\mu(y,b) \geqslant v} (a \otimes b)$ we get that for each b such that $\mu(y,b) \geqslant v$ it holds $a \otimes b \leqslant I(x,y)$. The last fact implies that for each b such that $\mu(y,b) \geqslant v$ it holds $\gamma(x,a) \leqslant \mu(y,b)$ which holds for each $y \in Y$. From the $\{0,1\}$-infimal density of $\mu(Y \times L)$ it follows that $v = \bigwedge_{v \leqslant \mu(y,b)} \mu(y,b)$, and hence $\gamma(x,a) \leqslant v$ which implies $a \leqslant \bigvee_{\gamma(x,a') \leqslant v} a' = A(x)$. We have proved $A = B^{\downarrow}$.

Next, we show that $\varphi \circ \psi = \mathrm{id}_{\mathscr{B}(X,Y,I)}$ and $\psi \circ \varphi = \mathrm{id}_V$. For each $v \in V$ we have by Claim A and the $\{0,1\}$-supremal density of $\gamma(X \times L)$

$$\begin{aligned}\psi \circ \varphi(v) = \varphi(A,B) &= \bigvee_{x \in X} \gamma(x, A(x)) \\ &= \bigvee_{x \in X} \gamma\left(x, \bigvee_{\gamma(x,a) \leqslant v} a\right) = \bigvee_{x \in X} \bigvee_{\gamma(x,a) \leqslant v} \gamma(x,a) \\ &= \bigvee_{\gamma(x,a) \leqslant u} \gamma(x,a) = v,\end{aligned}$$

i.e. $\psi \circ \varphi(v) = v$. Consider now $\varphi \circ \psi(A,B)$ for $\langle A,B \rangle \in \mathscr{B}(X,Y,I)$. First, we show

$$\bigvee_{x \in X} \gamma(x, A(x)) = \bigwedge_{y \in Y} \mu(y, B(y)). \tag{14}$$

The inequality $\bigvee_{x \in X} \gamma(x, A(x)) \leqslant \bigwedge_{y \in Y} \mu(y, B(y))$ is inferred from the fact that for every $x \in X$, $y \in Y$ we have $\gamma(x, A(x)) \leqslant \mu(y, B(y))$ which follows from Claim D as here: $\gamma(x, A(x)) \leqslant \mu(y, B(y))$ holds iff $A(x) \otimes B(y) \leqslant I(x,y)$ iff $A(x) \leqslant B(y) \to I(x,y)$ which holds because of $A(x) = \bigwedge_{y' \in Y} (B(y') \to I(x,y)) \leqslant B(y) \to I(x,y)$. To get equality (14), denote $v = \varphi(A,B) = \bigvee_{x \in X} \gamma(x, A(x))$. We show that $\bigwedge_{y \in Y} \mu(y, B(y)) = v$. From the $\{0,1\}$-infimal density of $\mu(Y \times L)$ we have clearly $v = \bigwedge_{\mu(y,b) \geqslant v} \mu(y,b)$. We show that for each y,b such that $\mu(y,b) \geqslant v$ it holds $b \leqslant B(y)$. Indeed, if $\mu(y,b) \geqslant v$ then clearly $\mu(y,b) \geqslant \gamma(x, A(x))$ for all $x \in X$. If $b \leqslant B(y)$ is not the case then consider $b \vee B(y)$. For each $x \in X$ we have $\mu(y,b) \geqslant \gamma(x, A(x))$, $\mu(y, B(y)) \geqslant \gamma(x, A(x))$, hence, by Claim A, $\mu(y, b \vee B(y)) = \mu(y,b) \wedge \mu(y, B(y)) \geqslant \gamma(x, A(x))$. This implies $A(x) \otimes B(y) \leqslant A(x) \otimes (b \vee B(y)) \leqslant I(x,y)$, i.e. $b \vee B(y) \leqslant A(x) \to I(x,y)$ for each $x \in X$, i.e. $b \vee B(y) \leqslant \bigwedge_{x \in X} A(x) \to I(x,y) = B(y)$, i.e. $b \leqslant B(y)$, a contradiction. Furthermore, from $b \leqslant B(y)$ it follows by Claim A that $\mu(y, B(y)) \leqslant \mu(y,b)$. Thus, from $v \leqslant \mu(y,b)$ it follows $\mu(y, B(y)) \leqslant \mu(y,b)$. We conclude

$$v = \bigvee_{x \in X} \gamma(x, A(x)) \leqslant \bigwedge_{y \in Y} \mu(y, B(y)) \leqslant \bigwedge_{\mu(y,b) \geqslant v} \mu(y,b) = v,$$

i.e (14) holds. We therefore have

$$\varphi \circ \psi(A,B) = \psi\left(\bigwedge_{y\in Y} \mu(y,B(y))\right)$$
$$= \left\langle \left\{ \left\langle x, \bigvee_{\gamma(x,a)\leqslant \bigwedge_{y\in Y} \mu(y,B(y))} a \right\rangle \middle| x\in X \right\}, \left\{ \left\langle y, \bigvee_{\mu(y,b)\geqslant \bigwedge_{y\in Y} \mu(y,B(y))} b \right\rangle \middle| y\in Y \right\} \right\rangle.$$

As $\varphi \circ \psi(A,B) \in \mathscr{B}(X,Y,I)$, it suffices to show that

$$\bigvee_{\gamma(x,a)\leqslant \bigwedge_{y\in Y} \mu(y,B(y))} a = A(x).$$

From (14) we have $\gamma(x,A(x)) \leqslant \bigwedge_{y\in Y} \mu(y,B(y))$ and therefore

$$\bigvee_{\gamma(x,a)\leqslant \bigwedge_{y\in Y} \mu(y,B(y))} a \geqslant A(x).$$

Conversely, if $\gamma(x,a) \leqslant \bigwedge_{y\in Y} \mu(y,B(y))$, then $\gamma(x,a) \leqslant \mu(y,B(y))$ for each $y\in Y$, i.e. $a\otimes B(y)\leqslant I(x,y)$, which yields $a\leqslant B(y)\to I(x,y)$, for each $y\in Y$, and hence $a\leqslant \bigwedge_{y\in Y} B(y)\to I(x,y) = A(x)$ which implies $\bigvee_{\gamma(x,a)\leqslant \bigwedge_{y\in Y} \mu(y,B(y))} a \leqslant A(x)$. We have proved $\varphi\circ\psi(A,B) = \langle A,B\rangle$.

It now suffices to show that $(\langle A_1,B_1\rangle \preccurlyeq \langle A_2,B_2\rangle) = (\varphi(A_1,B_1) \preccurlyeq \varphi(A_2,B_2))$ for any $\langle A_1,B_1\rangle$, $\langle A_2,B_2\rangle \in \mathscr{B}(X,Y,I)$. We prove this fact by showing (a) $(\langle A_1,B_1\rangle \preccurlyeq \langle A_2,B_2\rangle) \leqslant (\varphi(A_1,B_1)\preccurlyeq \varphi(A_2,B_2))$ and (b) $(u\preccurlyeq v) \leqslant (\psi(u)\preccurlyeq\psi(v))$ (for any $u,v\in V$).

(a) By definitions, we have to prove $\mathrm{S}(A_1,B_1) \leqslant (\bigvee_{x\in X} \gamma(x,A_1(x)) \preccurlyeq \bigvee_{x\in X} \gamma(x, A_2(x)))$. Denote $v^* = \bigvee_{x\in X} \gamma(x,A_2(x))$. The $\{0,1\}$-infimal density of $\mu(Y\times L)$ yields $v^* = \bigwedge_{v^*\leqslant \mu(y,b)} \mu(y,b)$. Therefore, we have to show $\mathrm{S}(A_1,B_1) \leqslant (\bigvee_{x\in X} \gamma(x,A_1(x)) \preccurlyeq \bigwedge_{v^*\leqslant\mu(y,b)} \mu(y,b))$. Since, by Claim B, we have $(\bigvee_{x\in X} \gamma(x,A_1(x)) \preccurlyeq \bigwedge_{v^*\leqslant\mu(y,b)} \mu(y,b)) = \bigwedge_{x\in X} \bigwedge_{v^*\leqslant\mu(y,b)} (\gamma(x,A_1(x))\preccurlyeq \mu(y,b))$, we have to show that $\mathrm{S}(A_1,A_2) \leqslant (\gamma(x,A_1(x))\preccurlyeq\mu(y,b))$ holds for any $x\in X$, $y\in Y$, $b\in L$ such that $v^*\leqslant\mu(y,b)$. Since $(\gamma(x,A_1(x))\preccurlyeq\mu(y,b)) = (A_1(x)\otimes b)\to I(x,y))$, we have to show that $\mathrm{S}(A_1,A_2)\leqslant (A_1(x) \otimes b)\to I(x,y))$, i.e. $(b\otimes A_1(x))\otimes \mathrm{S}(A_1,A_2) \leqslant I(x,y))$. Since $(b\otimes A_1(x))\otimes\mathrm{S}(A_1,A_2) = b\otimes A_1(x)\otimes \bigwedge_{x\in X}(A_1(x)\to A_2(x)) \leqslant b\otimes A_1(x)\otimes(A_1(x)\to A_2(x)) \leqslant b\otimes A_2(x)$ it is now sufficient to prove $b\otimes A_2(x)\leqslant I(x,y))$. However, since $\bigvee_{x\in X} \gamma(x,A_2(x))\leqslant\mu(y,b)$, thus also $\gamma(x,A_2(x))\leqslant\mu(y,b)$, we have $1 = (\gamma(x,A_2(x))\preccurlyeq\mu(y,b)) = b\otimes A_2(x)\to I(x,y)$ from which the required inequality $b\otimes A_2(x)\leqslant I(x,y)$ directly follows.

(b) Denote $\psi(u) = \langle A_u,B_u\rangle$ and $\psi(v) = \langle A_v,B_v\rangle$. We have to show $(u\preccurlyeq v)\leqslant \mathrm{S}(A_u,A_v)$, i.e. $(u\preccurlyeq v)\leqslant \bigwedge_{x\in X} A_u(x)\to A_v(x)$ which holds iff for each $x\in X$ we have $A_u(x)\otimes(u\preccurlyeq v)\leqslant A_v(x)$. By definition of ψ we thus have to show $(\bigvee_{\gamma(x,a)\leqslant u} a)\otimes(u\preccurlyeq v)$

$\leqslant \bigvee_{\gamma(x,b)\leqslant v} b$, i.e. $\bigvee_{\gamma(x,a)\leqslant u}(a\otimes(u\preccurlyeq v))\leqslant \bigvee_{\gamma(x,b)\leqslant v} b$ which holds iff $(a\otimes(u\preccurlyeq v))\leqslant \bigvee_{\gamma(x,b)\leqslant v} b$ for each $a\in L$ such that $\gamma(x,a)\leqslant u$. The last inequality certainly holds provided $\gamma(x,a\otimes(u\preccurlyeq v))\leqslant v$ which we are now going to prove: first, we show $\gamma(x,a\otimes(u\preccurlyeq v))\leqslant\gamma(x,(\gamma(x,a)\preccurlyeq v))$. $\gamma(x,a)\leqslant u$ yields $(u\preccurlyeq v)\leqslant(\gamma(x,a)\preccurlyeq v)$, and so $a\otimes(u\preccurlyeq v)\leqslant a\otimes(\gamma(x,a)\preccurlyeq v)$. By Claim C, we therefore have $1=a\otimes(u\preccurlyeq v)\to a\otimes(\gamma(x,a)\preccurlyeq v)\leqslant(\gamma(x,a\otimes(u\preccurlyeq v))\preccurlyeq\gamma(x,a\otimes(\gamma(x,a)\preccurlyeq v)))$, hence $\gamma(x,a\otimes(u\preccurlyeq v))\leqslant\gamma(x,a\otimes(\gamma(x,a)\preccurlyeq v))$. Now, the $\{0,1\}$-infimal density of $\mu(Y\times L)$ implies that $v=\bigwedge_{v\leqslant\mu(y,b)}\mu(y,b)$. Therefore, to show $\gamma(x,a\otimes(u\preccurlyeq v))\leqslant v$ it is sufficient to show that $\gamma(x,a\otimes(\gamma(x,a)\preccurlyeq v))\leqslant\mu(y,b)$ for any y and b such that $v\leqslant\mu(y,b)$. The required inequality is equivalent to $(\gamma(x,a\otimes(\gamma(x,a)\preccurlyeq v))\preccurlyeq\mu(y,b))=1$ which is equivalent (using $(\gamma(x,a\otimes(\gamma(x,a)\preccurlyeq v))\preccurlyeq\mu(y,b))=(a\otimes b\otimes(\gamma(x,a)\preccurlyeq v))\to I(x,y))$ to $(a\otimes b\otimes(\gamma(x,a)\preccurlyeq v))\to I(x,y)=1$, i.e. to $(\gamma(x,a)\preccurlyeq v)\leqslant(a\otimes b)\to I(x,y)$. However, the last inequality is true: $v\leqslant\mu(y,b)$ implies $(\gamma(x,a)\preccurlyeq v)\leqslant(\gamma(x,a)\preccurlyeq\mu(y,b))=(a\otimes b)\to I(x,y)$.

We proved that $\langle\langle\mathscr{B}(X,Y,I),\approx\rangle,\preccurlyeq\rangle$ and $\mathbf{V}$ are isomorphic.

To complete the proof we show that any completely lattice $\mathbf{L}$-ordered set $\mathbf{V}$ is isomorphic to $\mathscr{B}(V,V,\preccurlyeq)$. By what we just verified, it is enough to show that there are mappings $\gamma: V\times L\to V$ and $\mu: V\times L\to V$ with the required properties. For $a\in L$ and $x\in V$, let $\gamma(x,a)$ be the (unique) element $x^*\in V$ such that $(\sup(\{a/x\}))(x^*)=1$; $\mu(x,a)$ be the (unique) element $x^*\in V$ such that $(\inf(\{a/x\}))(x^*)=1$. Since $\gamma(x,1)=x$ and $\mu(x,1)=x$, we have $\gamma(V,L)=V$ and $\mu(V,L)=V$, therefore $\gamma(V,L)=V$ and $\mu(V,L)=V$ are $\{0,1\}$-supremally dense and $\{0,1\}$-infimally dense in $\mathbf{V}$. We show $((a\otimes b)\to(x\preccurlyeq y))=(\gamma(x,a)\preccurlyeq\mu(y,b))$ by proving both of the inequalities.

"$\leqslant$": Denote $x^*=\gamma(x,a)$, $y^*=\mu(y,b)$. We have to show $((a\otimes b)\to(x\preccurlyeq y))\leqslant(x^*\preccurlyeq y^*)$. By definition, $(\sup(\{a/x\}))(x^*)=1$ and $(\inf(\{b/y\}))(y^*)=1$. One easily verifies that $U(\{a/x\})(x')=a\to(x\preccurlyeq x')$. Therefore, $(LU(\{a/x\}))(x^*)=\bigwedge_{x'\in X}((a\to(x\preccurlyeq x'))\to(x^*\preccurlyeq x'))\leqslant(a\to(x\preccurlyeq y))\to(x^*\preccurlyeq y)$. Since, by definition of sup, $(LU(\{a/x\}))(x^*)=1$, we conclude $(a\to(x\preccurlyeq y))\leqslant(x^*\preccurlyeq y)$. Similarly, $1=(UL(\{b/y\}))(y^*)=\bigwedge_{x'\in X}((b\to(x'\preccurlyeq y))\to(x'\preccurlyeq y^*))\leqslant(b\to(x^*\preccurlyeq y))\to(x^*\preccurlyeq y^*)$, thus $(b\to(x^*\preccurlyeq y))\leqslant(x^*\preccurlyeq y^*)$. We thus have

$$
\begin{aligned}
(a\otimes b)\to(x\preccurlyeq y) &= b\to(a\to(x\preccurlyeq y))\\
&\leqslant b\to(x^*\preccurlyeq y)\leqslant(x^*\preccurlyeq y^*).
\end{aligned}
$$

"$\geqslant$": $(x^*\preccurlyeq y^*)\leqslant(a\otimes b)\to(x\preccurlyeq y)$ iff $a\otimes b\otimes(x^*\preccurlyeq y^*)\leqslant(x\preccurlyeq y)$. Now, $(\inf(\{b/y\}))(y^*)=1$ and $(\sup(\{a/x\}))(x^*)=1$ yield $(L(\{b/y\}))(y^*)=b\to(y^*\preccurlyeq y)=1$ and $(U(\{a/x\}))(x^*)=a\to(x^*\preccurlyeq x)=1$, respectively. Therefore

$$
\begin{aligned}
& a\otimes b\otimes(x^*\preccurlyeq y^*)\\
&\quad= a\otimes(a\to(x\preccurlyeq x^*))\otimes(x^*\preccurlyeq y^*)\otimes b\otimes(b\to(y^*\preccurlyeq y))\\
&\quad\leqslant(x\preccurlyeq x^*)\otimes(x^*\preccurlyeq y^*)\otimes(y^*\preccurlyeq y)\leqslant(x\preccurlyeq y)
\end{aligned}
$$

proving the inequality.

The proof of Theorem 14 is complete. □

Remark (historical development of Theorem 14). It was Birkhoff (see e.g. [6]) who observed that for any (bivalent) relation I between X and Y, $\mathcal{B}(X,Y,I)$ is a complete lattice. The characterization of $\mathcal{B}(X,Y,I)$ for special cases (I is a quasiorder or an order) has been pursued by Banaschewski [2] and Schmidt [14]. The general case when I is an arbitrary binary relation is due to Wille [16]. Theorem 14 is a further improvement of these results. A moment inspection shows that Wille's theorem is a special case of Theorem 14 for $\mathbf{L}=\mathbf{2}$ (i.e. for $\mathbf{L}$ being the two-element Boolean algebra): The description of infima and suprema is the same in Theorem 14 and in [16]. As to part (2) of Theorem 14, if $\mathbf{L}=\mathbf{2}$ then the conditions may be equivalently reformulated to "...iff there are mappings $\gamma: X \to V$, $\mu: Y \to V$, such that $\gamma(X)$ is supremally dense in $\mathbf{V}$, $\mu(Y)$ is infimally dense in $\mathbf{V}$, and $\langle x,y\rangle \in I$ iff $\gamma(x,a) \leqslant \mu(y,b)$..." which are the conditions of [16].

Theorem 13 implies that $\mathcal{B}(X,Y,I)$, equipped with ${}^1\!\preccurlyeq$, is a complete lattice. The lattice structure of $\mathcal{B}(X,Y,I)$ is characterizes by the following theorem.

Theorem 15. (1) *$\langle \mathcal{B}(X,Y,I), {}^1\!\preccurlyeq\rangle$ is complete lattice where infima and suprema for any $\mathcal{M} \subseteq \mathcal{B}(X,Y,I)$ are described by* (8) *and* (9).

(2) *Moreover, a complete lattice $\mathbf{V}=\langle V, \leqslant\rangle$ is isomorphic to $\langle \mathcal{B}(X,Y,I), {}^1\!\preccurlyeq\rangle$ iff there are mappings $\gamma: X\times L \to V$, $\mu: Y \times L \to V$, such that $\gamma(X\times L)$ is supremally dense in $\mathbf{V}$, $\mu(Y\times L)$ is infimally dense in $\mathbf{V}$, and $a\otimes b \leqslant I(x,y)$ iff $\gamma(x,a)\leqslant \mu(y,b)$ for all $x\in X$, $y\in Y$, $a,b\in L$.*

Proof. Part 1 follows immediately from Theorem 13 and from Theorem 14(1).

Part 2: Consider any $\mathcal{B}(X,Y,I)$ and a complete lattice $\mathbf{V}$. If $\mathcal{B}(X,Y,I)$ and $\mathbf{V}$ are isomorphic as lattices then the order $\leqslant$ of $\mathbf{V}$ can clearly be (in an obvious way) extended to an $\mathbf{L}$-order $\preccurlyeq$ on V in such a way that $\mathcal{B}(X,Y,I)$ and $\langle\langle V,\approx\rangle, \preccurlyeq\rangle$ are isomorphic as $\mathbf{L}$-ordered sets (here, $\approx$ on V is uniquely determined by $(u\approx v)=(u\preccurlyeq v)\wedge(v\preccurlyeq u)$, see Lemma 4). Part 2 of Theorem 14 then yields the mappings γ and μ which satisfy the density conditions and $((a\otimes b)\to I(x,y))=(\gamma(x,a)\preccurlyeq \mu(y,b))$. The required condition $a\otimes b\leqslant I(x,y)$ iff $\gamma(x,a)\leqslant\mu(y,b)$ now follows from $((a\otimes b)\to I(x,y)) =(\gamma(x,a)\preccurlyeq \mu(y,b))$. Conversely, if γ and μ satisfying the conditions of part (2) of Theorem 15 exist, then the order preserving bijections $\varphi: \mathcal{B}(X,Y,I)\to V$ and $\psi: V\to \mathcal{B}(X,Y,I)$ can be constructed as in the proof of Theorem 14 (one just needs to go through the respective parts of the proof of Theorem 14). □

Remark. Note that the characterization of the lattice structure of $\mathcal{B}(X,Y,I)$ contained in Theorem 15 has been obtained independently in [13] (the author uses so-called L-fuzzy-algebra for the structure of truth values; however, L-fuzzy-algebras are complete residuated lattices) and [4] (where a more general case of so-called $\mathbf{L}_K$-concept lattices is considered).

5. Dedekind–MacNeille completion

Let $\mathbf{X}=\langle\langle X,\approx_X\rangle, \preccurlyeq_X\rangle$ and $\mathbf{Y}=\langle\langle Y,\approx_Y\rangle,\preccurlyeq_Y\rangle$ be $\mathbf{L}$-ordered sets. A mapping $g: X\to Y$ is called an embedding of $\mathbf{X}$ into $\mathbf{Y}$ if g is injective, $(x\preccurlyeq_X x')=(g(x)\preccurlyeq_Y$

$g(x'))$, and $(x \approx_X x') = (g(x) \approx_Y g(x'))$ for every $x, x' \in X$. Therefore, the image of X under g is a "copy" of $\mathbf{X}$. We say that an embedding $g: X \to Y$ preserves infima (suprema) if for any $M \in L^X$ and $x \in X$ we have $(L(M))(x) = (L(g(M)))(g(x))$ and $(UL(M))(x) = (UL(g(M)))(g(x))$ $((U(M))(x) = (U(g(M)))(g(x))$ and $(LU(M))(x) = (LU(g(M)))(g(x)))$ where $g(M) \in L^Y$ is defined by $(g(M))(y) = M(x)$ if $y = g(x)$ and $(g(M))(y) = 0$ otherwise. Clearly, the preservation of infima (suprema) implies that $(\inf(M))(x) = \inf(g(M))(g(x))$ $((\sup(A))(x) = \sup(g(A))(g(x)))$.

For an $\mathbf{L}$-ordered set $\mathbf{X}$ and $x \in X$ we put $(x] := L(\{1/x\})$ and $[x) := U(\{1/x\})$. Therefore, $((x])(y) = (y \preccurlyeq x)$ and $([x))(y) = (x \preccurlyeq y)$ for each $y \in X$.

The above introduced notions generalize the well-known notions from the theory of ordered sets. Our aim in the following is a fundamental construction in the theory of ordered sets, so-called Dedekind–MacNeille completion (or completion by cuts). The objective is to describe a most economic completion of an ordered set which preserves infima and suprema, i.e. to describe "the least" completely lattice ordered set to which the original ordered set can be embedded in such a way that the embedding preserves infima and suprema. For the bivalent case, the completion by cuts has been for the first time exploited by Dedekind by the construction of real numbers from rational numbers. The construction has been generalized for arbitrary ordered sets by McNeille [11]. As it is well-known, the completion by cuts of a (classically) ordered set $\langle X, \leqslant \rangle$ is (up to an isomorphism) the concept lattice $\mathscr{B}(X, X, \leqslant)$. The following theorem describes the completion of an $\mathbf{L}$-ordered set (the classical completion being a special case for $\mathbf{L} = \mathbf{2}$).

Theorem 16 (Dedekind–MacNeille completion for L-order). *Let* $\mathbf{X}$ *be an* $\mathbf{L}$*-ordered set. Then* $g: x \mapsto \langle (x], [x) \rangle$ *is an embedding of* $\mathbf{X}$ *into a completely* $\mathbf{L}$*-ordered set* $\mathscr{B}(X, X, \preccurlyeq)$ *which preserves infima and suprema. Moreover, if* f *is an embedding of* $\mathbf{X}$ *into a completely lattice* $\mathbf{L}$*-ordered set* $\mathbf{Y}$ *which preserves infima and suprema then there is an embedding* h *of* $\mathscr{B}(X, X, \preccurlyeq)$ *into* $\mathbf{Y}$ *such that* $f = g \circ h$.

Proof. Note that by Theorem 14, $\mathscr{B}(X, X, \preccurlyeq)$ is a completely lattice $\mathbf{L}$-ordered set. Furthermore, g is correctly defined since $(x]^{\uparrow} = [x)$ and $[x)^{\downarrow} = (x]$, i.e. $\langle (x], [x) \rangle \in \mathscr{B}(X, X, \preccurlyeq)$: We verify only $(x]^{\uparrow} = [x)$, the second equality is symmetric. On the one hand, $(x]^{\uparrow}(y) = \bigwedge_{z \in X}((x](z) \to (z \preccurlyeq y)) \leqslant (x](x) \to (x \preccurlyeq y) = (x \preccurlyeq y) = [x)(y)$. On the other hand, $[x)(y) \leqslant (x]^{\uparrow}(y)$ holds iff $[x)(y) \leqslant (x](z) \to (z \preccurlyeq y)$, i.e. $(x](z) \otimes [x)(y) \leqslant (z \preccurlyeq y)$ for any $z \in X$. However, this is true since $(x](z) \otimes [x)(y) = (z \preccurlyeq x) \otimes (x \preccurlyeq y) \leqslant (z \preccurlyeq y)$, by transitivity of $\preccurlyeq$.

We show that g is an embedding of $\mathbf{X}$ into $\mathscr{B}(X, X, \preccurlyeq)$. To this end it is clearly sufficient to show that $(x \preccurlyeq y) = (g(x) \preccurlyeq g(y))$. We prove both of the required inequalities:

"$\leqslant$": As $(g(x) \preccurlyeq g(y)) = \mathrm{S}((x], (y])$, the inequality holds iff $(x \preccurlyeq y) \leqslant (x](z) \to (y](z)$ which is equivalent to $(x](z) \otimes (x \preccurlyeq y) \leqslant (y](z)$, i.e. $(z \preccurlyeq x) \otimes (x \preccurlyeq y) \leqslant (z \preccurlyeq y)$ which holds by transitivity of $\preccurlyeq$.

"$\geqslant$": The inequality holds iff $\bigwedge_{z \in X}((x](z) \to (y](z)) \leqslant (x \preccurlyeq y)$ which is true since $\bigwedge_{z \in X}((x](z) \to (y](z)) \leqslant (x](x) \to (y](x) = 1 \to (x \preccurlyeq y) = (x \preccurlyeq y)$.

We now have to prove that g preserves infima and suprema. Due to the symmetry of both of the cases we proceed only for infima. We have to show $(L(M))(x) = (L(g(M)))(g(x))$ and $(UL(M))(x) = (UL(g(M)))(g(x))$ for any $M \in L^X$.

$(L(M))(x) = (L(g(M)))(g(x))$: We have $(L(g(M)))(g(x)) = (L(g(M)))((x],[x)) = \bigwedge_{\langle A,B\rangle \in \mathscr{B}(X,X,\preccurlyeq)} (g(M))(A,B) \to (\langle (x],[x)\rangle \preccurlyeq \langle A,B\rangle) = \bigwedge_{y\in X} (g(M))((y],[y)) \to (\langle (x],[x)\rangle \preccurlyeq \langle (y],[y)\rangle) = \bigwedge_{y\in X} M(y) \to \mathrm{S}((x],(y]) = \bigwedge_{y\in X} M(y) \to (x \preccurlyeq y) = (L(M))(x)$.

$(UL(M))(x) = (UL(g(M)))(g(x))$: On the one hand, $(UL(g(M)))(g(x)) = (UL(g(M)))((x],[x)) = \bigwedge_{\langle A,B\rangle \in \mathscr{B}(X,X,\preccurlyeq)} (L(g(M)))(A,B) \to (\langle A,B\rangle \preccurlyeq \langle (x],[x)\rangle) = \bigwedge_{\langle A,B\rangle \in \mathscr{B}(X,X,\preccurlyeq)} (L(g(M)))(A,B) \to \mathrm{S}(A,(x]) \leqslant \bigwedge_{\langle (y],[y)\rangle \in \mathscr{B}(X,X,\preccurlyeq)} (L(g(M)))((y],[y)) \to \mathrm{S}((y],(x]) = \bigwedge_{y\in Y} (L(M))(y) \to (y \preccurlyeq x) = (UL(M))(x)$.

On the other hand, we have $(UL(M))(x) \leqslant (UL(g(M)))(g(x))$ iff for each $\langle A,B\rangle \in \mathscr{B}(X,X,\preccurlyeq)$ we have $(L(g(M)))(A,B) \otimes (UL(M))(x) \leqslant \mathrm{S}(A,(x])$ which holds (since $\mathrm{S}(A,(x]) = \bigwedge_{y\in Y} A(y) \to (y \preccurlyeq x)$) iff for each $y \in Y$ we have $A(y) \otimes (L(g(M)))(A,B) \otimes (UL(M))(x) \leqslant (y \preccurlyeq x)$. Since $A(y) \otimes (L(g(M)))(A,B) \otimes (UL(M))(x) = A(y) \otimes (L(g(M)))(A,B) \otimes \bigwedge_{u\in X} ((L(M))(u) \to (u \preccurlyeq x)) \leqslant A(y) \otimes (L(g(M)))(A,B) \otimes ((L(M))(y) \to (y \preccurlyeq x))$, it suffices to show that $A(y) \otimes (L(g(M)))(A,B) \leqslant (L(M))(y)$: To this end, observe that (A) $A(y) = \mathrm{S}((y],A)$: on the one hand, $\mathrm{S}((y],A) \leqslant (y](y) \to A(y) = A(y)$; on the other hand, $A(y) \leqslant \mathrm{S}((y],A)$ iff for each $z \in X$ we have $(y](z) \otimes A(y) \leqslant A(z)$, i.e. $(z \preccurlyeq y) \otimes A(y) \leqslant A(z)$; using $A = A^{\uparrow\downarrow}$ we obtain that the last inequality holds iff $(z \preccurlyeq y) \otimes A(y) \otimes A^{\uparrow}(x) \leqslant (z \preccurlyeq x)$ for any $x \in X$ which holds since $(z \preccurlyeq y) \otimes A(y) \otimes A^{\uparrow}(x) \leqslant (z \preccurlyeq y) \otimes A(y) \otimes (A(y) \to (y \preccurlyeq x)) \leqslant (z \preccurlyeq y) \otimes (y \preccurlyeq x) \leqslant (z \preccurlyeq x)$. Now, $A(y) \otimes (L(g(M)))(A,B) \leqslant (L(M))(y)$ is true iff $A(y) \otimes M(z) \otimes (L(g(M)))(A,B) \leqslant (y \preccurlyeq z)$ holds for each $z \in Z$. However, the last inequality is true: $A(y) \otimes M(z) \otimes (L(g(M)))(A,B) \leqslant A(y) \otimes M(z) \otimes (M(z) \to \mathrm{S}(A,(z])) \leqslant A(y) \otimes \mathrm{S}(A,(z]) = \mathrm{S}((y],A) \otimes \mathrm{S}(A,(z]) \leqslant \mathrm{S}((y],(z]) = (y \preccurlyeq z)$, using (A) and the fact that $x \mapsto \langle (x],[x)\rangle$ is an embedding. Thus, g preserves infima and suprema.

Let now f be an embedding of $\mathbf{X}$ into $\mathbf{Y}$ which preserves infima and suprema. Define the mapping $h: \mathscr{B}(X,X,\preccurlyeq) \to Y$ as follows: for $\langle A,B\rangle \in \mathscr{B}(X,X,\preccurlyeq)$ let $h(A,B)$ be the (unique) element of Y such that $(\sup(f(A)))(h(A,B)) = 1$. We have to prove that (1) $f = g \circ h$ and (2) h is an embedding.

(1) Take any $x \in X$. Observe that $(\sup((x]))(x) = 1$: we have $(U((x]))(x) = \bigwedge_{y\in X} (x](y) \to (y \preccurlyeq x) = \bigwedge_{y\in X} (y \preccurlyeq x) \to (y \preccurlyeq x) = 1$. Furthermore, $(LU((x]))(x) = 1$ iff $(U((x]))(y) \leqslant (x \preccurlyeq y)$ for any $y \in X$ which is true since $(U((x]))(y) = \bigwedge_{z\in X} (x](z) \to (z \preccurlyeq y) \leqslant (x](x) \to (x \preccurlyeq y) = (x \preccurlyeq y)$. Therefore, $(\sup((x]))(x) = 1$. Since f preserves suprema, we get $1 = (\sup((x]))(x) = (\sup(f((x])))(f(x))$ which means that $(g \circ h)(x) = h((x],[x)) = f(x)$.

(2) Take $\langle A_1,B_1\rangle, \langle A_2,B_2\rangle \in \mathscr{B}(X,X,\preccurlyeq)$, denote $y_1 = h(A_1,B_1)$, $y_2 = h(A_2,B_2)$. We have to show $(\langle A_1,B_1\rangle \preccurlyeq \langle A_2,B_2\rangle) = (y_1 \preccurlyeq y_2)$, i.e. $\mathrm{S}(A_1,A_2) = (y_1 \preccurlyeq y_2)$.

On the one hand, since $\mathrm{S}(A_1,A_2) = \mathrm{S}(f(A_1),f(A_2))$, Lemma 12 implies $\mathrm{S}(A_1,A_2) = \mathrm{S}(f(A_1),f(A_2)) \otimes (\sup(f(A_1)))(y_1) \otimes (\sup(f(A_2)))(y_2) \leqslant (y_1 \preccurlyeq y_2)$.

On the other hand, by $\mathrm{S}(A_1,A_2) = \mathrm{S}(f(A_1),f(A_2))$, we have to show $(y_1 \preccurlyeq y_2) \leqslant \mathrm{S}(f(A_1),f(A_2))$. $(\sup(f(A_1)))(y_1) = 1$ implies $(U(f(A_1)))(y_1) = 1$ from which one easily gets that $(f(A_1))(y) \leqslant (y \preccurlyeq y_1)$ for any $y \in X$, i.e. in particular $(f(A_1))(f(x))$

$\leqslant (f(x) \preccurlyeq y_1)$. Now, $(y_1 \preccurlyeq y_2) \leqslant \mathrm{S}(f(A_1), f(A_2))$ holds iff for each $x \in X$ we have $(f(A_1))(f(x)) \otimes (y_1 \preccurlyeq y_2) \leqslant (f(A_2))(f(x))$. Since $(f(A_1))(f(x)) \otimes (y_1 \preccurlyeq y_2) \leqslant (f(x) \preccurlyeq y_1) \otimes (y_1 \preccurlyeq y_2) \leqslant (f(x) \preccurlyeq y_2)$, it is sufficient to show that $(f(x) \preccurlyeq y_2) \leqslant (f(A_2))(f(x))$ for any $x \in X$. Moreover, since $\langle A_2, B_2 \rangle \in \mathscr{B}(X, Y, I)$ and since f preserves suprema, we have $(f(A_2))(f(x)) = (f(LU(A_2)))(f(x)) = LU(f(A_2))(f(x))$. We thus have to prove $(f(x) \preccurlyeq y_2) \leqslant (LU(f(A_2)))(f(x))$: we have $(f(x) \preccurlyeq y_2) \leqslant (LU(f(A_2)))(f(x))$ iff $(f(x) \preccurlyeq y_2) \otimes (U(f(A_2)))(f(x)) \leqslant (f(x) \preccurlyeq y)$ for any $y \in Y$ which holds since $(f(x) \preccurlyeq y_2) \otimes (U(f(A_2)))(f(x)) = (f(x) \preccurlyeq y_2) \otimes (ULU(f(A_2)))(f(x)) \leqslant (f(x) \preccurlyeq y_2) \otimes ((UL(f(A_2)))(y_2) \to (y_2 \preccurlyeq y)) = (f(x) \preccurlyeq y_2) \otimes (y_2 \preccurlyeq y) \leqslant (f(x) \preccurlyeq y)$. □

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