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Random Airy type differential equations: Mean square exact and numerical solutions

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ABSTRACT

This paper deals with the construction of power series solutions of random Airy type differential equations containing uncertainty through the coefficients as well as the initial conditions. Under appropriate hypotheses on the data, we establish that the constructed random power series solution is mean square convergent over the whole real line. In addition, the main statistical functions, such as the mean and the variance, of the approximate solution stochastic process generated by truncation of the exact power series solution are given. Finally, we apply the proposed technique to several illustrative examples which show substantial speed-up and improvement in accuracy compared to other approaches such as Monte Carlo simulations.

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1. Introduction

Solutions of differential equations with parameters are among the most important special functions, because of the very diverse applications in physics, engineering and mathematical analysis itself. These parameters aggregate complexity, uncertainty and ignorance, making the mathematical models available for approximation very complex [1]. Parameters may also represent rates of contagion in epidemiological models, which are very difficult to estimate [2]. These facts motivate the study of random differential equations with parameters. Some first-order random differential equations have been recently treated in [3,4]. Random vector differential equations where randomness is involved in the forcing term and in the initial condition have been treated in [5]. Random numerical methods for both ordinary and partial random differential equations are treated in [6,7]. As regards applications using explicit analytic solutions or numerical methods, a few results may be found in [8–11].

This paper deals with the random Airy type differential equation

$$\ddot{X}(t) + AtX(t) = 0, \quad -\infty < t < +\infty, \qquad X(0) = Y_0, \qquad \dot{X}(0) = Y_1, \tag{1}$$

where A, Y_0 and Y_1 are random variables satisfying certain conditions to be specified later. The goal of this article is twofold: firstly, constructing a mean square convergent random power series solution of (1) and, secondly, proving the accuracy of the approximations of the statistical average and variance for the solution stochastic process.

The paper is organized as follows. Section 2 deals with some preliminaries regarding the mean square calculus that will be required throughout the paper. Section 3 deals with the construction of a mean square convergent power series solution of (1) in the case where A is a random variable satisfying certain conditions related to the exponential growth of its absolute

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moments with respect to the origin. Statistical functions (mean and variance) of the truncated random power series solution which converge to the corresponding exact ones are studied in Section 4. Finally, in Section 5 some illustrative examples are presented showing substantial the speed-up and improvement in accuracy as compared against Monte Carlo simulations and the so-called dishonest method.

2. Preliminaries

For the sake of clarity in the presentation, we begin this section by introducing some concepts, notation and results that may be found in [8, chap. 4], [12, part IV], [13, chaps 1–3]. Let $(\Omega, \mathcal{F}, P)$ be a probability space. In this paper we will work in the set L_2 whose elements are second-order real random variables (2-r.v.'s), i.e., $X : \Omega \to \mathbb{R}$ such that $\mathrm{E}[X^2] < +\infty$, where $\mathrm{E}[\cdot]$ denotes the expectation operator. One can demonstrate that L_2 endowed with the so-called 2-norm

$$\|X\|_2 = \left(\mathrm{E}\left[X^2\right]\right)^{1/2}, \tag{2}$$

has a Banach space structure.

We say that $\{X(t) : t \in \mathcal{T}\}$ is a second-order stochastic process (2-s.p.) if the r.v. $X(t) \in L_2$ for each $t \in \mathcal{T}$, where $\mathcal{T}$ is the so-called space of times. Throughout this paper we will assume that $\mathcal{T}$ is always a real interval. The expectation function of $X(t)$ provides a statistical measure of its mean statistical behavior and it will be denoted by $\mathrm{E}[X(t)]$ or $\mu_X(t)$, while its covariance function will be denoted by $\mathrm{Cov}\,[X(t), X(s)]$ and is defined as follows:

$$\begin{aligned}\mathrm{Cov}\,[X(t), X(s)] &= \mathrm{E}\,[(X(t) - \mu_X(t))\,(X(s) - \mu_X(s))] \\ &= \mathrm{E}\,[X(t)X(s)] - \mu_X(t)\mu_X(s), \quad t, s \in \mathcal{T}.\end{aligned} \tag{3}$$

When $s = t$, one obtains the variance function

$$\mathrm{Var}[X(t)] = \mathrm{Cov}\,[X(t), X(t)] = \mathrm{E}\left[(X(t))^2\right] - (\mu_X(t))^2. \tag{4}$$

The term $\Gamma_X(t, s) = \mathrm{E}\,[X(t)X(s)]$ appearing in (3) is called the correlation function and it plays an important role in the m.s. calculus because many important stochastic results can be characterized through this two-variable deterministic function (see [8, chap. 4]).

We say that a sequence of 2-r.v.'s $\{X_n : n \geq 0\}$ is mean square (m.s.) convergent to $X \in L_2$ if

$$\lim_{n\to\infty} \|X_n - X\|_2 = \lim_{n\to\infty} \left(\mathrm{E}\left[(X_n - X)^2\right]\right)^{1/2} = 0.$$

Later we will present a method for providing the approximate s.p. of the solution $X(t)$ of the random differential equation (1). The following properties will play a fundamental role when we are interested in computing the mean and the variance of such approximations as well as ensuring they are close to the corresponding exact values.

Lemma 1 *([8, p. 88]). Let $\{X_n : n \geq 0\}$ be a sequence of 2-r.v.'s m.s. convergent to X; then*

$$\mathrm{E}\,[X_n] \xrightarrow[n\to\infty]{} \mathrm{E}\,[X], \qquad \mathrm{Var}\,[X_n] \xrightarrow[n\to\infty]{} \mathrm{Var}\,[X]. \tag{5}$$

We say that a 2-s.p. $\{X(t) : t \in \mathcal{T}\}$ is m.s. continuous in $\mathcal{T}$ if

$$\lim_{\tau\to 0} \|X(t+\tau) - X(t)\|_2 = 0,$$

for each $t \in \mathcal{T}$ such that $t + \tau \in \mathcal{T}$.

Example 2. Let $\{X_n : n \geq 1\}$ be a sequence of r.v.'s in L_2 and suppose $t \in \mathcal{T}$ with $\mathcal{T}$ a real interval; then for each positive integer n_0, the 2-s.p. $\left\{n_0 X_{n_0} t^{n_0-1} : t \in \mathcal{T}\right\}$ is m.s. continuous for all $t \in \mathcal{T}$. In fact,

$$\left\|n_0 X_{n_0}(t+\tau)^{n_0-1} - n_0 X_{n_0} t^{n_0-1}\right\|_2 = n_0 \left|(t+\tau)^{n_0-1} - t^{n_0-1}\right| \left\|X_{n_0}\right\|_2 \xrightarrow[\tau\to 0]{} 0,$$

because $\left\|X_{n_0}\right\|_2 < +\infty$ as $X_{n_0} \in L_2$ for each n_0 and by the continuity of the deterministic function $f(t) = t^{n_0-1}$ with respect to t.

A 2-s.p. $\{X(t) : t \in \mathcal{T}\}$ is said to be m.s. differentiable at $t \in \mathcal{T}$ with $\dot{X}(t)$ denoting its m.s. derivative if

$$\lim_{\tau\to 0} \left\|\frac{X(t+\tau) - X(t)}{\tau} - \dot{X}(t)\right\|_2 = 0,$$

for all $t \in \mathcal{T}$ such that $t + \tau \in \mathcal{T}$.

Example 3. With the notation of Example 2, the process $\left\{X_{n_0}t^{n_0} : t \in \mathcal{T}\right\}$ is m.s. differentiable in $\mathcal{T}$ and its m.s. derivative is $\left\{n_0X_{n_0}t^{n_0-1} : t \in \mathcal{T}\right\}$. Note that

$$\left\|\frac{X_{n_0}(t+\tau)^{n_0} - X_{n_0}t^{n_0}}{\tau} - n_0X_{n_0}t^{n_0-1}\right\|_2 = \left|\frac{(t+\tau)^{n_0} - t^{n_0}}{\tau} - n_0t^{n_0-1}\right| \left\|X_{n_0}\right\|_2 \xrightarrow[\tau\to 0]{} 0.$$

Later, due to the random differential equation (1) involving m.s. derivatives up to second order, one will require the following result. For $\{X(t) : t \in \mathcal{T}\}$ a 2-s.p. twice m.s. differentiable on $\mathcal{T}$, one can demonstrate that the means of its two first m.s. derivatives exist and they are given by

$$\mathrm{E}\left[\dot{X}(t)\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{E}[X(t)]\right), \qquad \mathrm{E}\left[\ddot{X}(t)\right] = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mathrm{E}[X(t)]\right), \quad \forall t \in \mathcal{T}, \tag{6}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ denote the first and second deterministic derivatives, respectively. This result can be extended to the n-th m.s. derivative whenever $\{X(t) : t \in \mathcal{T}\}$ is n-th times m.s. differentiable on $\mathcal{T}$ (see [8, p. 97]).

If X and Y are 2-r.v.'s, the Schwarz inequality establishes that

$$\mathrm{E}\left[|XY|\right] \le \left(\mathrm{E}\left[X^2\right]\right)^{1/2}\left(\mathrm{E}\left[Y^2\right]\right)^{1/2}. \tag{7}$$

Later we will require the use of the following basic property:

$$AX_n \xrightarrow[n\to\infty]{\text{m.s.}} AX, \tag{8}$$

which holds true if $A \in L_2$, $\{X_n : n \ge 0\}$ is a sequence of 2-r.v.'s such that $X_n \xrightarrow[n\to\infty]{\text{m.s.}} X$ and A, X_n are independent r.v.'s for each n. However, the independence hypothesis cannot be assumed to hold in many practical cases like those that we will consider below. This motivates the introduction of r.v.'s X such that $\mathrm{E}\left[X^4\right] < \infty$, which will be denoted as 4-r.v.'s. Note that a 4-r.v. is a 2-r.v. The set L_4 of all the 4-r.v.'s endowed with the norm

$$\|X\|_4 = \sqrt[4]{\mathrm{E}[X^4]},$$

is a Banach space (see [14, p. 9]). A stochastic process $\{X(t) : t \in T\}$, where $\mathrm{E}\left[(X(t))^4\right] < \infty$ for all $t \in T$, will be called a 4-s.p.

Definition 4. A sequence of 4-r.v.'s $\{X_n : n \ge 0\}$ is said to be mean fourth-power (m.f.) convergent to a 4-r.v. X if

$$\lim_{n\to\infty} \|X_n - X\|_4 = 0.$$

This type of convergence will be represented by $X_n \xrightarrow[n\to\infty]{\text{m.f.}} X$.

The following lemma establishes the link between the two types of convergence introduced previously.

Lemma 5. *Let $\{X_n : n \ge 0\}$ be a sequence of 4-r.v.'s and suppose that $X_n \xrightarrow[n\to\infty]{\text{m.f.}} X$. Then $X_n \xrightarrow[n\to\infty]{\text{m.s.}} X$.*

Proof. Using the Schwarz inequality (7), one gets

$$(\|X_n - X\|_2)^2 = \mathrm{E}\left[1 \times (X_n - X)^2\right] \le 1 \times \left(\mathrm{E}\left[(X_n - X)^4\right]\right)^{1/2} = (\|X_n - X\|_4)^2.$$

Since $\|X_n - X\|_4 \xrightarrow[n\to+\infty]{} 0$ (because X_n is m.f. convergent to X), it immediately follows that $\|X_n - X\|_2 \xrightarrow[n\to\infty]{} 0$ and therefore $X_n \xrightarrow[n\to\infty]{\text{m.s.}} X$. □

Now, we can give sufficient conditions for property (8) holding true without assuming hypotheses based on independence.

Lemma 6. *Let A be a 2-r.v. and $\{X_n : n \ge 0\}$ a sequence of 4-r.v.'s such that $X_n \xrightarrow[n\to\infty]{\text{m.f.}} X$. Then $AX_n \xrightarrow[n\to\infty]{\text{m.s.}} AX$.*

Proof. By the definition of the norm $\|\cdot\|_2$, we have

$$(\|A(X_n - X)\|_2)^2 = \mathrm{E}\left[A^2(X_n - X)^2\right]. \tag{9}$$

On the other hand, the hypothesis $X_n \xrightarrow[n\to\infty]{\text{m.f.}} X$ implies by definition $(X_n - X)^2 \xrightarrow[n\to\infty]{\text{m.s.}} 0$ and, as clearly $A^2 \xrightarrow[n\to\infty]{\text{m.s.}} A^2$, then from (9) and Lemma 1 one obtains $\|A(X_n - X)\|_2 \xrightarrow[n\to\infty]{} 0$ and hence $AX_n \xrightarrow[n\to\infty]{\text{m.s.}} AX$. □

Next we extend from the deterministic framework to the random one the concept of a fundamental set of solutions.

Definition 7. Let A_1 and A_2 be r.v.'s, and let $X_1(t)$ and $X_2(t)$ be two solutions of the second-order random differential equation

$$\ddot{X}(t) + A_1\dot{X}(t) + A_2X(t) = 0, \quad -\infty < t < \infty. \tag{10}$$

We say that $\{X_1(t), X_2(t)\}$ is a fundamental set of solution processes of (10) in $-\infty < t < +\infty$ if any solution $X(t)$ of (10) admits a unique representation of the form

$$X(t) = C_1X_1(t) + C_2X_2(t), \quad t \in (-\infty, \infty), \tag{11}$$

where C_1 and C_2 are r.v.'s uniquely determined by $X(t)$.

Definition 8. Let $\mathcal{S} = \{X_1(t), X_2(t)\}$ be two solutions of (10). The s.p. given by

$$W_{\mathcal{S}}(t) = X_1(t)\dot{X}_2(t) - X_2(t)\dot{X}_1(t),$$

is called the wronskian determinant of $\mathcal{S}$.

The following result provides sufficient conditions for a pair of solutions of (10) defining a fundamental set. We omit the proof because it follows in broad outline the same steps as in the deterministic case.

Proposition 9. *If $\mathcal{S} = \{X_1(t), X_2(t)\}$ is a pair of solution processes of the random differential equation (10) in $-\infty < t < \infty$ and there exists $t_0 \in (-\infty, \infty)$ such that $W_{\mathcal{S}}(t_0) \neq 0$, then $\mathcal{S} = \{X_1(t), X_2(t)\}$ is a fundamental set of solution processes of (10).*

We close this section by recalling the following differentiation theorem for m.s. random convergent series that will be required later.

Theorem 10 ([7]). *Let us assume that for each integer $k \geq 0$, the s.p. $\{U_k(t) : t \in \mathcal{T}\}$ is m.s. differentiable for each $t \in \mathcal{T}$, $\left\{\dot{U}_k(t) : t \in \mathcal{T}\right\}$ is m.s. continuous, $U(t) = \sum_{k\geq 0} U_k(t)$ is m.s. convergent and the series $\sum_{k\geq 0} \dot{U}_k(t)$ is m.s. uniformly convergent. Then the s.p. $\{U(t) : t \in \mathcal{T}\}$ is m.s. differentiable and*

$$\dot{U}(t) = \sum_{k\geq 0} \dot{U}_k(t).$$

3. Solving random Airy type differential equations

This section is addressed to obtaining the solution of the random Airy type differential equation (1) by means of a random power series as well as establishing its m.s. convergence. In the following we will assume that r.v. A is independent of initial conditions Y_0 and Y_1 and that the absolute moments with respect to the origin of r.v. A increase at the most exponentially, i.e., there exist a nonnegative integer n_0 and positive constants H and M such that

$$\mathrm{E}\left[|A|^n\right] \leq HM^n < +\infty, \quad \forall n \geq n_0. \tag{12}$$

Let us seek a formal solution process of problem (1) of the form

$$X(t) = \sum_{n\geq 0} X_n t^n, \tag{13}$$

where the coefficients X_n are 2-r.v.'s to be determined. Assuming that $X(t)$ is termwise m.s. differentiable and applying Example 3, it follows that

$$\ddot{X}(t) = \sum_{n\geq 2} n(n-1)X_n t^{n-2} = 2X_2 + \sum_{n\geq 1}(n+2)(n+1)X_{n+2}t^n. \tag{14}$$

By imposing that (13)–(14) satisfy (1), one gets

$$2X_2 + \sum_{n\geq 1}\left((n+2)(n+1)X_{n+2} + AX_{n-1}\right)t^n = 0. \tag{15}$$

Therefore a candidate m.s. solution process of problem (1) can be obtained by imposing

$$X_2 = 0, \qquad X_{n+2} = -\frac{AX_{n-1}}{(n+2)(n+1)}, \quad n \geq 1. \tag{16}$$

By recursive reasoning, these coefficients X_n can be computed as follows:

$$\left.\begin{aligned} X_{3n} &= \frac{(-1)^n A^n (3n-2)!!!}{(3n)!}X_0, \\ X_{3n+1} &= \frac{(-1)^n A^n (3n-1)!!!}{(3n+1)!}X_1, \end{aligned}\right\} \quad n \geq 1, \tag{17}$$

where $k!!! = k(k-3)(k-6)\cdots$ and $X_2 = 0$, $X_{3n+2} = 0$, $n \geq 0$. Therefore the s.p. (13) can be expressed in terms of the data as follows:

$$X(t) = X_0X_1(t) + X_1X_2(t), \tag{18}$$

where

$$X_1(t) = 1 + \sum_{n\geq 1} \frac{(-1)^n A^n (3n-2)!!!}{(3n)!} t^{3n},$$
$$X_2(t) = t + \sum_{n\geq 1} \frac{(-1)^n A^n (3n-1)!!!}{(3n+1)!} t^{3n+1}. \tag{19}$$

Note that in (15) we have formally commuted r.v. A and a random infinite sum (which we will prove is m.s. convergent); thus property (8) has been implicitly applied. In order to legitimate this commutation, the hypotheses of Lemma 6 must be checked. Therefore, the m.f. convergence of the two random series given by (19) has to be established. Since $(L_4, \|\cdot\|_4)$ is a Banach space, that is equivalent to proving that both series are absolutely convergent in the 4-norm. Let $t \in (-\infty, +\infty)$. By applying that r.v. A satisfies condition (12) one gets

$$\begin{aligned}\left\| \frac{A^n (3n-2)!!!}{(3n)!} \right\|_4 |t|^{3n} &= \left\| A^n \right\|_4 \frac{(3n-2)!!!}{(3n)!} |t|^{3n} \\ &= \left(\mathrm{E}\left[|A|^{4n} \right] \right)^{1/4} \frac{(3n-2)!!!}{(3n)!} |t|^{3n} \\ &\leq H_1 M^n \frac{(3n-2)!!!}{(3n)!} |t|^{3n},\end{aligned} \tag{20}$$

where $H_1 = \sqrt[4]{H} > 0$. As a consequence, we have obtained as a majorant series

$$\sum_{n\geq 0} \alpha_n, \quad \text{where } \alpha_n = H_1 M^n \frac{(3n-2)!!!}{(3n)!} |t|^{3n}, \tag{21}$$

which is convergent for all $t \in \mathbb{R}$ as can be directly checked by using the D'Alembert test:

$$\frac{\alpha_{n+1}}{\alpha_n} = M \frac{(3n+1)}{(3n+3)(3n+2)(3n+1)} |t|^3 \xrightarrow[n\to\infty]{} 0, \quad t \in \mathbb{R}. \tag{22}$$

Therefore, we have proven that the numerical series given by (21) is convergent for all $t \in \mathbb{R}$. Thus the first random series of (19) is also m.f. convergent, and so by Lemma 5, it is also m.s. convergent for all $t \in \mathbb{R}$. Following an analogous procedure, it is straightforward to establish the m.s. convergence of the second series in (19) for all t.

Note that the above reasoning shows that both solution series $X_1(t)$ and $X_2(t)$ given by (19) are m.s. uniformly convergent. Therefore taking into account Example 3 and Theorem 10, the formal differentiation considered in (14) is justified. On the other hand, taking $t_0 = 0$ and considering that $X_1(0) = 1, \dot{X}_1(0) = 0, X_2(0) = 0$ and $\dot{X}_2(0) = 1$, one gets that $W_{\mathcal{S}}(0) = 1 \neq 0$. Then by Proposition 9 and (11), the solution of random differential equation (1) with random initial conditions $X(0) = Y_0$ and $\dot{X}(0) = Y_1$ is given by

$$X(t) = Y_0 X_1(t) + Y_1 X_2(t), \quad t \in (-\infty, +\infty), \tag{23}$$

where $X_1(t)$ and $X_2(t)$ are defined by (19).

Note that in expressions (18)–(19) we have commuted r.v. $X_0 = Y_0$ ($X_1 = Y_1$) and the infinite sum $X_1(t)$ ($X_2(t)$), due to the hypothesis of independence between r.v. A and r.v.'s Y_0 and Y_1.

Summarizing, the following result has been established:

Theorem 11. *The random differential equation (1) with initial conditions $X(0) = Y_0$ and $\dot{X}(0) = Y_1$, where A is a r.v. independent of Y_0 and Y_1 and A satisfies condition (12), admits a random power series solution given by (23) where $X_1(t)$ and $X_2(t)$ are defined by (19). Moreover this solution is m.s. convergent for each $t \in (-\infty, +\infty)$.*

The most important class of r.v.'s satisfying condition (12) are those r.v.'s A having finite domain, i.e., such that $a_1 \leq A(\omega) \leq a_2$, for each $\omega \in \Omega$. Indeed, let us assume that A is a continuous r.v. with density function $f_A(a)$. Then taking $M = \max(|a_1|, |a_2|)$, one gets

$$\mathrm{E}\left[|A|^n \right] = \int_{a_1}^{a_2} |a|^n f_A(a) \mathrm{d}a \leq M^n.$$

Note that, by substituting the integral for a sum, the previous conclusion remains true if A is a discrete r.v.

We emphasize that important r.v.'s such as Binomial, Hypergeometric, Uniform and Beta have finite support. Otherwise, we can use the truncation method (see [12]) for dealing with unbounded r.v.'s such as Poisson, Gaussian and exponential ones (see Example 13 below).

4. Statistical functions of the mean square random power series solution

This section deals with the computation of the main statistical functions of the m.s. solution of (1) given by (19) and (23), such as the mean and variance, in terms of the data $\mathrm{E}[Y_0]$, $\mathrm{E}[Y_1]$, $\mathrm{E}[Y_0Y_1]$, $\mathrm{E}\left[(Y_0)^2\right]$, $\mathrm{E}\left[(Y_1)^2\right]$ and $\mathrm{E}[A^n]$.

In order to compute the average of the solution s.p. $X(t)$, let us take the expectation operator in the random differential equation (1). Then by applying property (6) one gets

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\mu_X(t)\right) + t\mathrm{E}\left[AX(t)\right] = 0. \tag{24}$$

Note that (24) is not a suitable equation for computing $\mu_X(t)$ because the term $\mathrm{E}[AX(t)]$ cannot be factorized as $A\,\mu_X(t)$, in general. Nevertheless some methods, such as the so-called dishonest method (see [15,16], [17, p. 148]), accept the above factorization as an alternative for handling the problem of computing the mean of the solution process. We shall see right away through examples that our approach avoids the above approximations and allows us to provide reliable values for the mean and the variance for quite general situations.

In practice, as it occurs in the deterministic framework, it will be unfeasible for the computation of the mean to proceed via the infinite series given by (23) and (19). Then, we will consider the truncation of order N

$$X_N(t) = Y_0\left(1 + \sum_{n=1}^{N} \frac{(-1)^n A^n (3n-2)!!!}{(3n)!} t^{3n}\right) + Y_1\left(t + \sum_{n=1}^{N} \frac{(-1)^n A^n (3n-1)!!!}{(3n+1)!} t^{3n+1}\right). \tag{25}$$

Henceforth, we will assume that r.v. A is independent of initial conditions $X(0) = Y_0$ and $\dot{X}(0) = Y_1$ (note that from an applied point of view, this is a realistic assumption). Then taking the expectation operator in (25) one gets

$$\mu_{X_N}(t) = \mathrm{E}[Y_0]\left(1 + \sum_{n=1}^{N} \frac{(-1)^n \mathrm{E}[A^n]\,(3n-2)!!!}{(3n)!} t^{3n}\right) + \mathrm{E}[Y_1]\left(t + \sum_{n=1}^{N} \frac{(-1)^n \mathrm{E}[A^n]\,(3n-1)!!!}{(3n+1)!} t^{3n+1}\right). \tag{26}$$

Now taking into account the expression (4) for computing the variance of the truncated solution process, we only need to calculate $\mathrm{E}\left[(X_N(t))^2\right]$. First we compute $(X_N(t))^2$ from (25) and, after that, we take the expectation operator

$$\begin{aligned}
\mathrm{E}\left[(X_N(t))^2\right] = {} & \mathrm{E}\left[(Y_0)^2\right]\left\{1 + 2\sum_{n=1}^{N} \frac{(-1)^n \mathrm{E}[A^n]\,(3n-2)!!!}{(3n)!} t^{3n} + \sum_{n=1}^{N} \mathrm{E}\left[A^{2n}\right]\left(\frac{(3n-2)!!!}{(3n)!}\right)^2 t^{6n}\right. \\
& \left. + 2\sum_{n=2}^{N}\sum_{i=1}^{n-1} \frac{(-1)^{n+i} \mathrm{E}\left[A^{n+i}\right](3n-2)!!!(3i-2)!!!}{(3n)!(3i)!} t^{3(n+i)}\right\} \\
& + \mathrm{E}\left[(Y_1)^2\right]\left\{t^2 + 2\sum_{n=1}^{N} \frac{(-1)^n \mathrm{E}[A^n]\,(3n-1)!!!}{(3n+1)!} t^{3n+2} + \sum_{n=1}^{N} \mathrm{E}\left[A^{2n}\right]\left(\frac{(3n-1)!!!}{(3n+1)!}\right)^2 t^{6n+2}\right. \\
& \left. + 2\sum_{n=2}^{N}\sum_{i=1}^{n-1} \frac{(-1)^{n+i} \mathrm{E}\left[A^{n+i}\right](3n-1)!!!(3i-1)!!!}{(3n+1)!(3i+1)!} t^{3(n+i)+2}\right\} \\
& + 2\mathrm{E}\left[(Y_0Y_1)^2\right]\left\{t + \sum_{n=1}^{N} \frac{(-1)^n \mathrm{E}[A^n]\,(3n-1)!!!}{(3n+1)!} t^{3n+1} + \sum_{n=1}^{N} \frac{(-1)^n \mathrm{E}[A^n]\,(3n-2)!!!}{(3n)!} t^{3n+1}\right. \\
& \left. + \sum_{n=1}^{N}\sum_{m=1}^{N} \frac{(-1)^{n+m} \mathrm{E}\left[A^{n+m}\right](3n-2)!!!(3m-1)!!!}{(3n)!(3m+1)!} t^{3(n+m)+1}\right\}.
\end{aligned} \tag{27}$$

Taking into account property (5) as well as the m.s. convergence of the random series given by (23) and (19), the convergences of the mean and the variance of the truncated solution (25) to the corresponding exact values are guaranteed under the hypotheses of Theorem 11.

5. Examples

In this section we provide several illustrative examples. The results obtained for approximating the mean and the variance by means of the series method presented in this paper are compared to the corresponding ones yielded by the dishonest and Monte Carlo approaches.

Example 12. Let us consider the model (1) where A is a Beta r.v. with parameters $\alpha = 2$ and $\beta = 3$, i.e., $A \sim \mathrm{Be}(\alpha = 2; \beta = 3)$ and the initial conditions Y_0 and Y_1 are independent r.v.'s such as $\mathrm{E}[Y_0] = 1$, $\mathrm{E}\left[(Y_0)^2\right] = 2$, $\mathrm{E}[Y_1] = 2$, $\mathrm{E}\left[(Y_1)^2\right] = 5$, and as a consequence of the independence, $\mathrm{E}[Y_0Y_1] = \mathrm{E}[Y_0]\,\mathrm{E}[Y_1] = 2$. Note that r.v. A satisfies the conditions of Theorem 11 since it takes values on a bounded interval. Then the m.s. solution of model (1) with initial conditions Y_0 and Y_1 is given

Table 1
Comparison of the means obtained by using the truncation power series, Monte Carlo and dishonest approaches for Example 12.

t	$\mu_{X_3}(t)$	$\mu_{X_5}(t)$	$\mu_X^d(t)$	$c\widetilde{\mu}_X^m(t)$ $m = 50\,000$	$c\widetilde{\mu}_X^m(t)$ $m = 100\,000$
0.00	1.00000	1.00000	1.00000	0.998784	0.994909
0.25	1.4987	1.4987	1.4987	1.49676	1.49413
0.50	1.98752	1.98752	1.98752	1.98485	1.9835
0.75	2.45108	2.45108	2.45102	2.44767	2.44761
1.00	2.86856	2.86856	2.86818	2.86444	2.8656
1.25	3.21494	3.21494	3.21339	3.21026	3.21242
1.50	3.46309	3.4631	3.45812	3.45812	3.46087
1.75	3.58654	3.5866	3.5734	3.58178	3.58446
2.00	3.56302	3.56335	3.53286	3.55931	3.56103

Table 2
Comparison of the variances obtained by using the truncation power series and Monte Carlo approaches for Example 12.

t	$\mathrm{Var}[X_3(t)]$	$\mathrm{Var}[X_5(t)]$	$c\widetilde{\mathrm{Var}}_X^m(t)$ $m = 50\,000$	$c\widetilde{\mathrm{Var}}_X^m(t)$ $m = 100\,000$
0.00	1.00000	1.00000	1.00127	0.999234
0.25	1.06035	1.06035	1.05947	1.05857
0.50	1.23142	1.23142	1.22958	1.22832
0.75	1.49261	1.49261	1.49134	1.4879
1.00	1.81392	1.81392	1.81554	1.80733
1.25	2.1587	2.1587	2.16679	2.14992
1.50	2.49379	2.49379	2.5136	2.48256
1.75	2.8056	2.8056	2.84393	2.79184
2.00	3.11558	3.1153	3.17958	3.09936

Table 3
Comparison of the means obtained by using the truncation power series, Monte Carlo and dishonest approaches for Example 13.

t	$\mu_{X_5}(t)$	$\mu_{X_{10}}(t)$	$\mu_X^d(t)$	$\widetilde{\mu}_X^m(t)$ $m = 20\,000$	$\widetilde{\mu}_X^m(t)$ $m = 50\,000$
0.00	3.00000	3.00000	3.00000	3.01107	2.99797
0.25	3.23374	3.23374	3.23374	3.24746	3.23043
0.50	3.36575	3.36575	3.36568	3.38185	3.36131
0.75	3.28893	3.28893	3.28817	3.30675	3.28373
1.00	2.91012	2.91012	2.90594	2.92857	2.90481
1.25	2.18225	2.18225	2.16713	2.19983	2.17772
1.50	1.13986	1.13987	1.09882	1.15483	1.13714
1.75	−0.0771305	−0.0770383	−0.166186	−0.0663526	−0.0769893
2.00	−1.23055	−1.22953	−1.38892	−1.22429	−1.22616

by (23) and (19). Table 1 shows the expectation of the truncated solution s.p. for different values of the truncation order N (denoted by $\mu_{X_N}(t)$) at different values of the time parameter t as well as the corresponding values obtained by the dishonest ($\mu_X^d(t)$) and Monte Carlo methods ($\widetilde{\mu}_X^m(t)$) by using m simulations. One observes that for values of t near of the origin (where the initial conditions are imposed and the power series solution s.p. is centered), the approximations obtained by the method proposed in this paper coincide from $t = 0$ to $t = 1.25$ even for small truncation orders such as $N = 3$ and $N = 5$. As regards approximations obtained by means of the Monte Carlo method it is worth pointing out that they improve as the number m of simulations increases, and in general, they provide better approximations than those obtained by the dishonest method. Table 2 compares the values of the variance for the truncation method with respect to those from the Monte Carlo approach. Comments similar to those that we have made for the average results apply for the variance. We outline that in both cases, for the approximation of the mean and variance, only a few terms of the corresponding series are needed to obtain very good approximations, whereas the Monte Carlo computational cost is much greater.

Example 13. In this example we take advantage of the so-called truncation method (see [12]) to deal with a r.v. A having unbounded domain that does not satisfy condition (12). Let us consider model (1) where A is a Gaussian r.v., $A \sim \mathrm{N}(\mu = 2; \sigma = 0.5)$, and the initial conditions Y_0 and Y_1 are uncorrelated r.v.'s such that $\mathrm{E}[Y_0] = 3$, $\mathrm{E}\left[(Y_0)^2\right] = 10$, $\mathrm{E}[Y_1] = 1$, $\mathrm{E}\left[(Y_1)^2\right] = 2$. In order to overcome this difficulty, we will consider the truncation of this r.v. on the interval $[\mu - 3\sigma, \mu + 3\sigma] = [0.5, 3.5]$ that will contain all the values of A with probability 0.997. The probability density function associated with the new censured r.v., say, B, is

$$f_B(b) = \frac{\exp\left(-2\,(b-2)^2\right)}{\int_{0.5}^{3.5} \exp\left(-2\,(x-2)^2\right) \mathrm{d}x}, \quad 0.5 \le b \le 3.5.$$

In this way, B satisfies hypotheses of Theorem 11 since it takes values on a bounded interval. Table 3 shows approximations of the expectation of the solution s.p. computed by the truncation series, dishonest and Monte Carlo methods. For the

Table 4
Comparison of the variances obtained by using the truncation power series and Monte Carlo approaches for Example 13.

t	$\mathrm{Var}[X_5(t)]$	$\mathrm{Var}[X_{10}(t)]$	$\widetilde{\mathrm{Var}}_X^m(t)$ $m = 20\,000$	$\widetilde{\mathrm{Var}}_X^m(t)$ $m = 50\,000$
0.00	1.00000	1.00000	1.00872	1.00236
0.25	1.05181	1.05181	1.06283	1.05202
0.50	1.16005	1.16005	1.17435	1.15767
0.75	1.2469	1.2469	1.26478	1.24238
1.00	1.25398	1.25398	1.27478	1.24917
1.25	1.18919	1.18919	1.21187	1.18719
1.50	1.12328	1.12327	1.1479	1.12739
1.75	1.09547	1.0953	1.12482	1.10771
2.00	1.03352	1.03165	1.07142	1.05297

truncation method, Table 3 only shows results for $N = 5, 10$ because for greater values the results obtained do not change. Table 4 shows approximations for the variance obtained by means of the truncation method as well as the Monte Carlo approach. Comments analogous to those that we have made regarding Example 12 apply again.

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