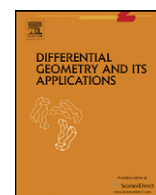


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On a new class of Finsler metrics

Changtao Yu ^{a,*}, Hongmei Zhu ^b^a School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China^b School of Mathematical Sciences, Peking University, Beijing 100871, China

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ABSTRACT

In this paper, the geometric meaning of (α, β) -norms is made clear. On this basis, a new class of Finsler metrics called general (α, β) -metrics are introduced, which are defined by a Riemannian metric and a 1-form. These metrics not only generalize (α, β) -metrics naturally, but also include some metrics structured by R. Bryant. The spray coefficients formula of some kinds of general (α, β) -metrics is given and the projective flatness is also discussed.

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1. Introduction

(α, β) -metrics form a special class of Finsler metrics partly because they are “computable” [1]. The researches on (α, β) -metrics enrich Finsler geometry and the approaches offer references for further study.

Randers metrics arising from physical applications [11] are the simplest (α, β) -metrics. They are expressed in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_\alpha < 1$. The following Randers metric

$$F = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle^2}}{1 - |x|^2} + \frac{\langle x, y \rangle}{1 - |x|^2} \quad (1)$$

is called *Funk metric* [8]. It is a projectively flat Finsler metric on $\mathbb{B}^n(1)$ with flag curvature $K = -\frac{1}{4}$. Recall that a Finsler metric F on an open domain $U \subset \mathbb{R}^n$ is said to be *projectively flat*, if all the geodesics of F are straight lines [7].

Another important example of (α, β) -metric was given by L. Berwald [3],

$$F = \frac{(\sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle^2} + \langle x, y \rangle)^2}{(1 - |x|^2)^2 \sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle^2}}. \quad (2)$$

It is of a special kind of (α, β) -metrics in the form $F = \frac{(\alpha + \beta)^2}{\alpha}$ with $\|\beta\|_\alpha < 1$. Berwald's metric is also a projectively flat Finsler metric on $\mathbb{B}^n(1)$ with flag curvature $K = 0$.

The concept of (α, β) -metrics was firstly proposed by M. Matsumoto in 1972 as a direct generalization of Randers metrics [9]. But some basic concepts of (α, β) -metrics were omitted. In Section 2, we make clear the geometric property about the

* Corresponding author.

E-mail addresses: aizhenli@gmail.com (C. Yu), zhuhongmei1981@pku.edu.cn (H. Zhu).¹ Supported by the National Natural Science Foundation of China (No. 11026097).

indicatrices of (α, β) -metrics. Roughly speaking, a Minkowski norm F is an (α, β) -norm if and only if the indicatrix of F is a rotation hypersurface with the rotation axis passing the origin.

The aim of this paper is to study a new class of Finsler metrics given by

$$F = \alpha \phi \left(b^2, \frac{\beta}{\alpha} \right), \tag{3}$$

where $\phi = \phi(b^2, s)$ is a C^∞ positive function and $b^2 := \|\beta\|_\alpha^2$. This kind of Finsler metrics generalize (α, β) -metrics in a natural way. They are a special class of general (α, β) -metrics which are defined in Section 3. But the most important reason that we are interested in them is that they include some Finsler metrics constructed by R. Bryant.

Bryant's metrics [4–6] are rectilinear Finsler metrics on S^n with flag curvature $K = 1$ and given in the following form with $X \in S^n, Y \in T_X S^n$,

$$F(X, Y) = \Re \left\{ \frac{\sqrt{Q(X, X)Q(Y, Y) - Q(X, Y)^2}}{Q(X, X)} - i \frac{Q(X, Y)}{Q(X, X)} \right\}, \tag{4}$$

where

$$Q(X, Y) = x_0 y_0 + e^{ip_1} x_1 y_1 + e^{ip_2} x_2 y_2 + \dots + e^{ip_n} x_n y_n$$

are complex quadratic forms on \mathbb{R}^{n+1} for $n \geq 2$ with the parameters satisfying

$$0 \leq p_1 \leq p_2 \leq \dots \leq p_n < \pi.$$

Note that the branch of the complex square root being used is the one satisfying $\sqrt{1} = 1$ and having the negative real axis as its branch locus (cf. [5]).

The following result is related to Bryant's metrics, where the constant r_μ is given by $r_\mu = \frac{1}{\sqrt{-\mu}}$ if $\mu < 0$ and $r_\mu = +\infty$ if $\mu \geq 0$.

Theorem 1.1. *The following general (α, β) -metrics are projectively flat on $\mathbb{B}^n(r_\mu)$ with $n \geq 2$,*

$$F = \Re \frac{\sqrt{(e^{ip} + b^2)\alpha^2 - \beta^2} - i\beta}{e^{ip} + b^2} \quad \left(-\frac{\pi}{2} \leq p \leq \frac{\pi}{2} \right), \tag{5}$$

where α and β are given by

$$\alpha = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y \rangle^2}}{1 + \mu|x|^2}, \tag{6}$$

$$\beta = \frac{\lambda\langle x, y \rangle + (1 + \mu|x|^2)\langle a, y \rangle - \mu\langle a, x \rangle\langle x, y \rangle}{(1 + \mu|x|^2)^{\frac{3}{2}}}, \tag{7}$$

in which μ is the sectional curvature of α , λ is a constant and $a \in \mathbb{R}^n$ is a constant vector.

Remark 1. When $\mu = 0, \lambda = 1, a = 0$, the general (α, β) -metrics (5) belong to Bryant's metrics in some appropriate coordinate. One can see Section 4 for details. At the same time, we will point out that the previous metrics (4) are not always regular on the whole sphere. Recall that a Finsler metric is said to be *regular*, if its fundamental tensor is positive definite everywhere.

Moreover, we provide a sufficient condition for the general (α, β) -metrics (3) to be projectively flat. In this paper, a 1-form is called *conformal* with respect to a Riemannian metric if its dual vector field with respect to the Riemannian metric is conformal.

Theorem 1.2. *Let $F = \alpha \phi(b^2, \frac{\beta}{\alpha})$ be a general (α, β) -metric on a manifold M with dimension $n \geq 2$. Then F is locally projectively flat if the following conditions hold:*

1. The function $\phi(b^2, s)$ satisfies the following partial differential equation

$$\phi_{22} = 2(\phi_1 - s\phi_{12}). \tag{8}$$

2. α is locally projectively flat, β is closed and conformal with respect to α .

Remark 2. Note that ϕ_1 means the derivation of ϕ with respect to the first variable b^2 . On the other hand, a Riemannian metric α is locally projectively flat if and only if it is of constant sectional curvature by Beltrami's theorem [7].

The projective flatness is connected with the Hilbert’s Fourth Problem. Recently, Z. Shen has characterized all the projectively flat (α, β) -metrics for dimension $n \geq 3$ [12]. The first author rewrote the (α, β) -metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ as $F = \frac{(\sqrt{1+b^2}\bar{\alpha}+\bar{\beta})^2}{\bar{\alpha}}$ in his doctoral dissertation, where $\bar{\alpha} = (1 - b^2)\alpha$, $\bar{\beta} = \sqrt{1 - b^2}\beta$, and proved that this kind of Finsler metrics are locally projectively flat if and only if $\bar{\alpha}$ is locally projectively flat while $\bar{\beta}$ is closed and conformal with respect to $\bar{\alpha}$.

Moreover, the first author has classified all the locally projectively flat (α, β) -metrics for dimension $n \geq 3$ in his doctoral dissertation. The results show that the projective flatness of an (α, β) -metric always arises from that of some Riemannian metric by doing some special deformations. Therefore, we claim that the conditions in Theorem 1.2 are, in a sense, also a necessary condition for a non-Randers general (α, β) -metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ to be locally projectively flat for $n \geq 3$.

To be specific, if F is a non-Randers locally projectively flat general (α, β) -metric, then F can be represented as $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ such that $\phi(b^2, s), \alpha$ and β satisfy the conditions in Theorem 1.2. For instance, suppose that $F = \frac{(\alpha+\beta)^2}{\alpha}$ is a locally projectively flat (α, β) -metric. In this case, the corresponding function $\phi(s) = (1 + s)^2$ does not satisfy Eq. (8). Also α is not locally projectively flat and β is not conformal with respect to α in general [12]. But if we rewrite F as $F = \frac{(\sqrt{1+b^2}\bar{\alpha}+\bar{\beta})^2}{\bar{\alpha}}$, then the function $\phi(\bar{b}^2, \bar{s}) = (\sqrt{1 + \bar{b}^2} + \bar{s})^2$ satisfies Eq. (8) now. Although $F = \frac{(\alpha+\beta)^2}{\alpha}$ is simple in this form, the properties of α and β are not so simple. This phenomenon is similar to that of Randers metrics of constant flag curvature [2].

2. The geometric meaning of (α, β) -norms

Let V be an n -dimensional vector space. By definition, an (α, β) -norm on V is a Minkowski norm expressed in the following form,

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where $\alpha = \sqrt{a_{ij}y^i y^j}$ is an Euclidean norm and $\beta = b_i y^i \in V^*$ is a linear functional on V . The function $\phi = \phi(s)$ is a C^∞ positive function on some open interval $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad \forall |s| \leq b < b_0,$$

where $b =: \|\beta\|_\alpha$ [7].

Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of α . Then

$$\alpha(y) = \sqrt{(y^1)^2 + (y^2)^2 + \dots + (y^n)^2}, \quad y = y^i e_i \in V \cong \mathbb{R}^n.$$

It is obvious that the orthogonal group $O(n)$ acting on V preserves α . Conversely, a Minkowski norm on V preserved under the action of $O(n)$ must be Euclidean. In other words, Euclidean norms are the most symmetric Minkowski norms.

By considering the symmetry of (α, β) -norms, Theorem 2.2 shows that the symmetry of (α, β) -norms is just next to that of Euclidean norms. Firstly, we give a description of the symmetry of a Minkowski norm.

Definition 2.1. Let F be a Minkowski norm on an n -dimensional vector space V and G be a subgroup of $GL(n, \mathbb{R})$. Then F is called G -invariant if the following condition holds for some affine coordinate (y^1, y^2, \dots, y^n) of V ,

$$F(y^1, y^2, \dots, y^n) = F((y^1, y^2, \dots, y^n)g), \quad \forall y \in V, \forall g \in G. \tag{9}$$

The symmetry of Minkowski norms should be paid more attentions since it restricts the global symmetry of Finsler manifolds.

Theorem 2.2. Let F be a Minkowski norm on a vector space V of dimension $n \geq 2$. Then F is an (α, β) -norm if and only if F is G -invariant, where

$$G = \left\{ g \in GL(n, \mathbb{R}) \mid g = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, A \in O(n-1) \right\}.$$

Remark 3. The above theorem is trivial when $n = 1$ because every Finsler curve is of Randers type by the navigation problem.

Proof. Let $F = \alpha\phi(\frac{\beta}{\alpha})$ be an (α, β) -norm. Take an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ with respect to α , such that $\ker \beta = \text{span}\{e_1, e_2, \dots, e_{n-1}\}$. Then

$$F(y) = \sqrt{(y^1)^2 + (y^2)^2 + \dots + (y^n)^2} \phi\left(\frac{by^n}{\sqrt{(y^1)^2 + (y^2)^2 + \dots + (y^n)^2}}\right),$$

where $y = y^i e_i$ and $b = \|\beta\|_\alpha$. Obviously, F is G -invariant.

Conversely, assume that (9) holds for the affine coordinate (y^1, y^2, \dots, y^n) .

Case 1. $n \geq 3$.

By restricting F on the linear subspace given by $y^n = 0$, one can obtain an $O(n - 1)$ -invariant Minkowski norm, which must be Euclidean by the previous discussions. So we can choose a positive number a , such that the Euclidean norm $\alpha = a\sqrt{(y^1)^2 + (y^2)^2 + \dots + (y^{n-1})^2}$ on V satisfies $\alpha|_{y^n=0} = F|_{y^n=0}$.

For $y \neq 0$, define

$$\tilde{\phi}(y^1, y^2, \dots, y^n) = \frac{F(y^1, y^2, \dots, y^n)}{\alpha(y^1, y^2, \dots, y^n)}, \tag{10}$$

then $\tilde{\phi}$ is G -invariant, i.e.

$$\tilde{\phi}(y^1, y^2, \dots, y^n) = \tilde{\phi}((y^1, y^2, \dots, y^n)g), \quad \forall y \neq 0, \forall g \in G.$$

In particular,

$$\tilde{\phi}(\cos ty^1 + \sin ty^2, -\sin ty^1 + \cos ty^2, y^3, \dots, y^n) = \tilde{\phi}(y^1, y^2, \dots, y^n).$$

Differentiating the above equality with respect to t and setting $t = 0$, one obtains $\frac{\partial \tilde{\phi}}{\partial y^1} y^2 - \frac{\partial \tilde{\phi}}{\partial y^2} y^1 = 0$. The same argument yields

$$\frac{\partial \tilde{\phi}}{\partial y^i} y^j - \frac{\partial \tilde{\phi}}{\partial y^j} y^i = 0, \quad 1 \leq i < j \leq n - 1. \tag{11}$$

Moreover, since F and α are both positively homogeneous with degree one, $\tilde{\phi}$ is positively homogeneous with degree zero, i.e., $\tilde{\phi}(\lambda y) = \tilde{\phi}(y)$, $\forall \lambda > 0$. Differentiating this equality with respect to λ and setting $\lambda = 1$, one obtains

$$\frac{\partial \tilde{\phi}}{\partial y^i} y^i = 0. \tag{12}$$

Taking the spherical coordinate transformation

$$\begin{cases} y^1 = r \cos \theta^1 \cos \theta^2 \dots \cos \theta^{n-2} \cos \theta^{n-1}, \\ y^2 = r \cos \theta^1 \cos \theta^2 \dots \cos \theta^{n-2} \sin \theta^{n-1}, \\ \dots \\ y^{n-1} = r \cos \theta^1 \sin \theta^2, \\ y^n = r \sin \theta^1, \end{cases}$$

where $r > 0$, $-\frac{\pi}{2} \leq \theta^\gamma \leq \frac{\pi}{2}$ ($\gamma = 1, \dots, n - 2$), $0 \leq \theta^{n-1} < 2\pi$, and using (11), (12), we have

$$\begin{aligned} \frac{\partial \tilde{\phi}}{\partial r} &= \frac{\partial \tilde{\phi}}{\partial y^i} \frac{\partial y^i}{\partial r} = \frac{\partial \tilde{\phi}}{\partial y^i} \frac{y^i}{r} = 0, \\ \frac{\partial \tilde{\phi}}{\partial \theta^\gamma} &= -\frac{\partial \tilde{\phi}}{\partial y^1} y^{n-\gamma+1} \cos \theta^{\gamma+1} \dots \cos \theta^{n-2} \cos \theta^{n-1} - \frac{\partial \tilde{\phi}}{\partial y^2} y^{n-\gamma+1} \cos \theta^{\gamma+1} \dots \cos \theta^{n-2} \sin \theta^{n-1} \\ &\quad - \dots - \frac{\partial \tilde{\phi}}{\partial y^{n-\gamma}} y^{n-\gamma+1} \sin \theta^{\gamma+1} + \frac{\partial \tilde{\phi}}{\partial y^{n-\gamma+1}} r \cos \theta^1 \dots \cos \theta^\gamma \\ &= -\frac{\partial \tilde{\phi}}{\partial y^{n-\gamma+1}} y^1 \cos \theta^{\gamma+1} \dots \cos \theta^{n-2} \cos \theta^{n-1} - \frac{\partial \tilde{\phi}}{\partial y^{n-\gamma+1}} y^2 \cos \theta^{\gamma+1} \dots \cos \theta^{n-2} \sin \theta^{n-1} \\ &\quad - \dots - \frac{\partial \tilde{\phi}}{\partial y^{n-\gamma+1}} y^{n-\gamma} \sin \theta^{\gamma+1} + \frac{\partial \tilde{\phi}}{\partial y^{n-\gamma+1}} r \cos \theta^1 \dots \cos \theta^\gamma \\ &= 0, \quad \gamma = 2, \dots, n - 2, \\ \frac{\partial \tilde{\phi}}{\partial \theta^{n-1}} &= -\frac{\partial \tilde{\phi}}{\partial y^1} y^2 + \frac{\partial \tilde{\phi}}{\partial y^2} y^1 = 0. \end{aligned}$$

So $\tilde{\phi} = \tilde{\phi}(\theta^1) = \phi(\frac{y^n}{\alpha})$ where the function $\phi(s) = \tilde{\phi}(\arcsin as)$, which means $F = \alpha \phi(\frac{y^n}{\alpha})$ is an (α, β) -norm.

Case 2. $n = 2$.

In this case, (9) is equivalent to $F(y^1, y^2) = F(-y^1, y^2), \forall y \in V$. This equation implies that the indicatrix of F is reflection symmetric with respect to y^2 -axis. It is easy to see that it means that the function defined by (10) has the form $\tilde{\phi} = \phi(\frac{y^2}{\alpha})$ for some function ϕ . \square

Remark 4. (10) shows that the function $\phi(s)$ contains the informations about the shape of the indicatrix.

By Zermelo's viewpoint [2], we can obtain new Minkowski norms by shifting the indicatrix of an (α, β) -norm. We call them navigation (α, β) -norms. The indicatrix of a navigation (α, β) -norm is still a rotation hypersurface, but the rotation axis does not pass the origin in general.

There will not be more discussions about this kind of Minkowski norms in this paper. It shouldn't be omitted if one study the properties of (α, β) -metrics besides Randers metrics [10,13], although it may be very complicated in algebraic form.

3. General (α, β) -metrics

Suppose that F is a Finsler metric on a manifold M such that $F(x, y)$ is an (α, β) -norm on T_xM for any $x \in M$. F is not an (α, β) -metric in general. This is because the shape of the indicatrix for different point may be different. This observation leads to the following definition.

Definition 3.1. Let F be a Finsler metric on a manifold M . F is called a general (α, β) -metric, if F can be expressed as the form $F = \alpha\phi(x, \frac{\beta}{\alpha})$ for some C^∞ function $\phi(x, s)$ where $x \in M$, some Riemannian metric α and some 1-form β . F is called a (special) (α, β) -metric, if F can be expressed as $F = \alpha\phi(\frac{\beta}{\alpha})$ for some C^∞ function $\phi(s)$, some Riemannian metric α and some 1-form β .

The Finsler metrics in the form (3) become the simplest class of general (α, β) -metrics except for special (α, β) -metrics. $\phi(b^2, s)$ is a positive C^∞ function with b^2, s as its variables and $|s| \leq b < b_0$ as its definitional domain for some $0 < b_0 \leq +\infty$. We use b^2 instead of b as the first variable, partly because it is convenient for computations. In the rest part of this paper, we will focus on this special kind of general (α, β) -metrics. Firstly, we can obtain the basic facts of the general (α, β) -metrics immediately from the corresponding ones of (α, β) -metrics given in [7].

Proposition 3.2. For a general (α, β) -metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$, the fundamental tensor is given by

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_{y^j} + b_j \alpha_{y^i}) - s \rho_1 \alpha_{y^i} \alpha_{y^j},$$

where

$$\rho = \phi(\phi - s\phi_2), \quad \rho_0 = \phi\phi_{22} + \phi_2\phi_2, \quad \rho_1 = (\phi - s\phi_2)\phi_2 - s\phi\phi_{22}.$$

Moreover,

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi_2)^{n-2}(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}) \det(a_{ij}),$$

$$g^{ij} = \rho^{-1} \{ a^{ij} + \eta b^i b^j + \eta_0 \alpha^{-1} (b^i y^j + b^j y^i) + \eta_1 \alpha^{-2} y^i y^j \},$$

where $(g^{ij}) = (g_{ij})^{-1}, (a^{ij}) = (a_{ij})^{-1}, b^i = a^{ij} b_j,$

$$\eta = -\frac{\phi_{22}}{(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \quad \eta_0 = -\frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})},$$

$$\eta_1 = \frac{(s\phi + (b^2 - s^2)\phi_2)((\phi - s\phi_2)\phi_2 - s\phi\phi_{22})}{\phi^2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}.$$

Proposition 3.3. Let M be an n -dimensional manifold. $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is a Finsler metric on M for any Riemannian metric α and 1-form β with $\|\beta\|_\alpha < b_0$ if and only if $\phi = \phi(b^2, s)$ is a positive C^∞ function satisfying

$$\phi - s\phi_2 > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \tag{13}$$

when $n \geq 3$ or

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0,$$

when $n = 2$, where s and b are arbitrary numbers with $|s| \leq b < b_0$.

Remark 5. Note that the second inequality in (13) doesn't imply the first one, even though it does for special (α, β) -metrics (cf. [7]).

Let b_{ij} denote the coefficients of the covariant derivative of β with respect to α . Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_{00} = r_{ij}y^i y^j, \quad s^i_0 = a^{ij}s_{jk}y^k,$$

$$r_i = b^j r_{ji}, \quad s_i = b^j s_{ji}, \quad r_0 = r_i y^i, \quad s_0 = s_i y^i, \quad r^i = a^{ij}r_j, \quad s^i = a^{ij}s_j, \quad r = b^i r_i.$$

It is easy to see that β is closed if and only if $s_{ij} = 0$.

Proposition 3.4. For a general (α, β) -metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$, its spray coefficients G^i are related to the spray coefficients G^i_α of α by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \{\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Omega(r_0 + s_0)\} \frac{y^i}{\alpha}$$

$$+ \{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 Rr) + \alpha\Pi(r_0 + s_0)\} b^i - \alpha^2 R(r^i + s^i),$$

where

$$Q = \frac{\phi_2}{\phi - s\phi_2}, \quad R = \frac{\phi_1}{\phi - s\phi_2},$$

$$\Theta = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \quad \Psi = \frac{\phi_{22}}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})},$$

$$\Pi = \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \quad \Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi} \Pi.$$

Proof. Recall that the spray coefficients of a Finsler metric F are given by

$$G^i = \frac{1}{4}g^{il}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\}.$$

For the general (α, β) -metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$, direct computations yield

$$[F^2]_{x^k} = [\alpha^2]_{x^k} \phi^2 + 2\alpha^2 \phi \phi_1 [b^2]_{x^k} + 2\alpha^2 \phi \phi_2 s_{x^k},$$

$$[F^2]_{x^k y^l} = [\alpha^2]_{x^k y^l} \phi^2 + 2[\alpha^2]_{x^k} \phi \phi_2 s_{y^l} + 2[\alpha^2]_{y^l} \phi \phi_1 [b^2]_{x^k} + 2\alpha^2 \phi_1 \phi_2 [b^2]_{x^k} s_{y^l} + 2\alpha^2 \phi \phi_{12} [b^2]_{x^k} s_{y^l}$$

$$+ 2[\alpha^2]_{y^l} \phi \phi_2 s_{x^k} + 2\alpha^2 (\phi_2)^2 s_{x^k} s_{y^l} + 2\alpha^2 \phi \phi_{22} s_{x^k} s_{y^l} + 2\alpha^2 \phi \phi_2 s_{x^k} y^l.$$

Set $G^i = G^i_1 + G^i_2$, where G^i_1 includes ϕ_1 and ϕ_{12} but G^i_2 does not, i.e.,

$$G^i_1 = \frac{1}{2}g^{il}\{[\alpha^2]_{y^l} \phi \phi_1 [b^2]_{x^k} y^k + \alpha^2 \phi_1 \phi_2 [b^2]_{x^k} y^k s_{y^l} + \alpha^2 \phi \phi_{12} [b^2]_{x^k} y^k s_{y^l} - \alpha^2 \phi \phi_1 [b^2]_{x^l}\}. \tag{14}$$

It is easy to see that G^i_2 can be obtained immediately by exchanging ϕ' for ϕ_2 and ϕ'' for ϕ_{22} in the spray coefficients of (α, β) -metrics which can be found in [7]. So

$$G^i_2 = G^i_\alpha + \alpha Q s^i_0 + \Theta\{-2\alpha Q s_0 + r_{00}\} \frac{y^i}{\alpha} + \Psi\{-2\alpha Q s_0 + r_{00}\} b^i.$$

In order to compute G^i_1 , we need the following simple facts:

$$[\alpha^2]_{y^l} = 2y_l, \quad [b^2]_{x^l} = 2(r_l + s_l), \quad s_{y^l} = \frac{\alpha b_l - s y_l}{\alpha^2}, \tag{15}$$

where $y_l = a_{lt} y^t$.

By (14) and (15), we have

$$G^i_1 = g^{il}\{A y_l + B b_l + C(r_l + s_l)\} := \rho^{-1}\{D y^i + E b^i + F(r^i + s^i)\},$$

where

$$A = (2\phi\phi_1 - s\phi_1\phi_2 - s\phi\phi_{12})(r_0 + s_0), \quad B = \alpha(\phi_1\phi_2 + \phi\phi_{12})(r_0 + s_0), \quad C = -\alpha^2\phi\phi_1,$$

and by Proposition 3.2,

$$\begin{aligned}
 D &= A + (As + \alpha^{-1}Bb^2 + \alpha^{-1}Cr)\eta_0 + \{A + \alpha^{-1}Bs + \alpha^{-2}C(r_0 + s_0)\}\eta_1, \\
 E &= B + (\alpha As + Bb^2 + Cr)\eta + \{\alpha A + Bs + \alpha^{-1}C(r_0 + s_0)\}\eta_0, \\
 F &= C.
 \end{aligned}$$

Plugging $\eta, \eta_0, \eta_1, A, B, C$ into the above equalities yields

$$\begin{aligned}
 D &= \left\{ \left[2(\phi - s\phi_2) + \frac{s\phi_{22}(s\phi + (b^2 - s^2)\phi_2)}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}} \right] \phi_1 - \frac{(\phi - s\phi_2)(s\phi + (b^2 - s^2)\phi_2)}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}} \phi_{12} \right\} (r_0 + s_0) \\
 &\quad + \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}} \phi_1 \alpha r, \\
 E &= \left\{ \frac{\phi(\phi - s\phi_2)}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}} \phi_{12} - \frac{s\phi\phi_{22}}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}} \phi_1 \right\} \alpha (r_0 + s_0) + \frac{\phi\phi_{22}}{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}} \phi_1 \alpha^2 r.
 \end{aligned}$$

One can obtain the spray coefficients G^i by the above equalities. \square

4. Some constructions of projectively flat general (α, β) -metrics

Bryant’s metrics (4) contain some general (α, β) -metrics. In order to see that, let us take $p_1 = p_2 = \dots = p_{n-1} = 0, p_n = p$. Then (4) is given in the following form in some appropriate coordinate by stereographic projection,

$$F = \mathfrak{N} \frac{\sqrt{(e^{ip} + |x|^2)|y|^2 - \langle x, y \rangle^2} - i \langle x, y \rangle}{e^{ip} + |x|^2}.$$

If we take $p_1 = p_2 = \dots = p_n = p$, then (4) is given by

$$F = \mathfrak{N} \frac{\sqrt{(e^{-ip} + |x|^2)|y|^2 - \langle x, y \rangle^2} - i \langle x, y \rangle}{e^{-ip} + |x|^2}.$$

So it is natural to consider the general (α, β) -metrics in the form (5).

Lemma 4.1. $F = \mathfrak{N} \frac{\sqrt{(e^{ip} + b^2)\alpha^2 - \beta^2} - i\beta}{e^{ip} + b^2}$ is a Finsler metric if and only if $b < b_0$, where

$$b_0 = \begin{cases} +\infty & \text{if } |p| \leq \frac{\pi}{2}, \\ \sqrt{\frac{1}{2} \sec(\frac{2\pi}{3} - \frac{|p|}{3})} & \text{if } \frac{\pi}{2} < |p| < \pi. \end{cases}$$

Proof. There is no need to be discussed when $p = 0$, because in this case $F = \frac{\sqrt{(1+b^2)\alpha^2 - \beta^2}}{1+b^2}$ is just a Riemannian metric.

Define a complex-valued function $\Phi(b^2, s)$ by

$$\Phi(b^2, s) = \frac{\sqrt{e^{ip} + b^2 - s^2} - is}{e^{ip} + b^2} = \frac{1}{\sqrt{e^{ip} + b^2 - s^2} + is}, \tag{16}$$

then $\phi(b^2, s)$ is the real part of Φ . Direct computations yield

$$\Phi - s\phi_2 = \frac{1}{(e^{ip} + b^2 - s^2)^{\frac{1}{2}}}, \tag{17}$$

$$\Phi - s\phi_2 + (b^2 - s^2)\phi_{22} = \frac{e^{ip}}{(e^{ip} + b^2 - s^2)^{\frac{3}{2}}}. \tag{18}$$

When $0 < p < \pi$, it is easy to see that the argument of $e^{ip} + b^2 - s^2$, denoted by θ , satisfies $0 < \theta \leq p$ since $b^2 - s^2 \geq 0$. We conclude ϕ and $\phi - s\phi_2$ are positive because the arguments of Φ and $\Phi - s\phi_2$ belong to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

On the other hand,

$$\arg(\Phi - s\phi_2 + (b^2 - s^2)\phi_{22}) = p - \frac{3}{2}\theta,$$

so $\phi - s\phi_2 + (b^2 - s^2)\phi_{22}$ is positive when $p \leq \frac{\pi}{2}$. In other words, $b_0 = +\infty$ when $0 < p \leq \frac{\pi}{2}$.

In the case $p > \frac{\pi}{2}$, $\phi - s\phi_2 + (b^2 - s^2)\phi_{22}$ is not always positive because θ may be very small. Let b_0 be the largest number such that for all $|s| \leq b < b_0, \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0$. Then b_0 must be the solution, which is given in the lemma, of the following equation,

$$\arg \frac{e^{ip}}{(e^{ip} + b_0^2)^{\frac{3}{2}}} = \frac{\pi}{2}.$$

We can finish the proof by the similar argument for the case $-\pi < p < 0$. \square

Remark 6. By the above lemma, Bryant’s metrics (4) do not always define on the whole sphere. This conclusion have been confirmed by R. Bryant. That is to say, in order to ensure the regularity of (4) on the whole sphere, some more conditions on the parameters p_i ($1 \leq i \leq n$) should be provided.

Proof of Theorem 1.2. Since α is locally projectively flat, we can assume that $G_\alpha^i = \theta y^i$ in some local coordinate system $(\mathcal{U}; x^i)$, where $\theta = \theta_i(x) y^i$ is a 1-form on \mathcal{U} . On the other hand, $b_{ij} = c(x) a_{ij}$ for some function $c(x)$ because β is closed and conformal with respect to α . It is obvious that

$$r_{00} = c\alpha^2, \quad r_0 = c\beta, \quad r = cb^2, \quad r^i = cb^i, \quad s^i_0 = 0, \quad s_0 = 0, \quad s^i = 0. \tag{19}$$

Substituting (19) into the spray coefficients in Proposition 3.4 yields

$$\begin{aligned} G^i &= \left\{ \theta + c\alpha \left[\Theta(1 + 2Rb^2) + s\Omega \right] \right\} y^i + c\alpha^2 \left\{ \Psi(1 + 2Rb^2) + s\Pi - R \right\} b^i \\ &= \left\{ \theta + c\alpha \left[\frac{\phi_2 + 2s\phi_1}{2\phi} - \frac{(\phi_{22} - 2(\phi_1 - s\phi_{12}))(s\phi + (b^2 - s^2)\phi_2)}{2\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})} \right] \right\} y^i + c\alpha^2 \left\{ \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})} \right\} b^i. \end{aligned}$$

So the spray coefficients are given by

$$G^i = \left\{ \theta + c\alpha \frac{\phi_2 + 2s\phi_1}{2\phi} \right\} y^i \tag{20}$$

if ϕ satisfies the first condition of Theorem 1.2. Recall that a Finsler metric is projectively flat if and only if its spray coefficients are in the form $G^i = P y^i$ [7]. Therefore F is projectively flat on \mathcal{U} . \square

Proof of Theorem 1.1. The function $\Phi(b^2, s)$ is defined by (16). Differentiating (17) with respect to b^2 yields

$$\Phi_1 - s\Phi_{12} = -\frac{1}{2(e^{ip} + b^2 - s^2)^{\frac{3}{2}}}.$$

So by the above equality and (18), Φ satisfies the following equality,

$$\Phi_{22} = 2(\Phi_1 - s\Phi_{12}).$$

The same relation is true for ϕ by taking the real parts of the above equality.

On the other hand, set $\varrho = \sqrt{1 + \mu|x|^2}$, then the Christoffel symbols of (6) are given by $\Gamma^{k}_{ij} = -\varrho^{-2}\mu(x^i\delta^k_j + x^j\delta^k_i)$, and

$$\begin{aligned} b_i &= \varrho^{-3}\lambda x^i + \varrho^{-1}a^i - \varrho^{-3}\mu\langle a, x \rangle x^i, \\ \frac{\partial b_i}{\partial x^j} &= \varrho^{-3}\lambda\delta_{ij} - 3\varrho^{-5}\mu\lambda x^i x^j - \varrho^{-3}\mu a^i x^j - \varrho^{-3}\mu\langle a, x \rangle\delta_{ij} - \varrho^{-3}\mu a^j x^i + 3\varrho^{-5}\mu^2\langle a, x \rangle x^i x^j, \\ b_{ij} &= \frac{\partial b_i}{\partial x^j} - b_k \Gamma^{k}_{ij} \\ &= \varrho^{-3}(\lambda - \mu\langle a, x \rangle)\delta_{ij} - \varrho^{-5}(\lambda - \mu\langle a, x \rangle)\mu x^i x^j. \end{aligned}$$

The last equality implies $s_{ij} = 0$ and $r_{ij} = \varrho^{-1}(\lambda - \mu\langle a, x \rangle)a_{ij}$. So β is closed and conformal with respect to α with conformal factor $c(x) = \varrho^{-1}(\lambda - \mu\langle a, x \rangle)$.

Moreover, the spray coefficients of F are given by

$$G^i = \left\{ -\frac{\mu\langle x, y \rangle}{1 + \mu|x|^2} + \frac{(\lambda - \mu\langle a, x \rangle)}{\sqrt{1 + \mu|x|^2}} \Im \frac{\sqrt{(e^{ip} + b^2)\alpha^2 - \beta^2 - i\beta}}{e^{ip} + b^2} \right\} y^i,$$

which are obtained by the simple equality $\Phi_2 + 2s\Phi_1 = -i\Phi^2$ and (20). \square

Example 4.2. Take $\lambda = 1, a = 0$ in Theorem 1.1, then the following general (α, β) -metrics are projectively flat for $-\frac{\pi}{2} \leq p \leq \frac{\pi}{2}$:

$$F = \Re \frac{\sqrt{(e^{ip} + |x|^2 + \mu e^{ip}|x|^2)|y|^2 - (1 + \mu e^{ip})\langle x, y \rangle^2} - \frac{i\langle x, y \rangle}{\sqrt{1 + \mu|x|^2}}}{e^{ip} + |x|^2 + \mu e^{ip}|x|^2}.$$

Example 4.3. It is easy to verify that the function $\phi(b^2, s) = (\sqrt{1+b^2} + s)^2$ satisfies the first condition of [Theorem 1.2](#). Take $\lambda = 1, a = 0$, then the following general (α, β) -metrics are projectively flat:

$$F = \frac{(\sqrt{1 + (1 + \mu)|x|^2} \sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y \rangle^2} + \langle x, y \rangle)^2}{(1 + \mu|x|^2)^2 \sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y \rangle^2}}.$$

In particular, F is the Berwald's metric when $\mu = -1$.

5. Some discussions about the PDE

In this section, we will discuss some interesting properties about the partial differential equation

$$\phi_{22} = 2(\phi_1 - s\phi_{12}). \quad (21)$$

We will always assume $\lambda = 1$ and $a = 0$ in [Theorem 1.1](#) in this section. In this case, α and β are given by

$$\alpha_\mu = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y \rangle^2}}{1 + \mu|x|^2}, \quad \beta_\mu = \frac{\langle x, y \rangle}{(1 + \mu|x|^2)^{\frac{3}{2}}}.$$

It is easy to verify that $b_\mu^2 := \|\beta_\mu\|_{\alpha_\mu}^2 = \frac{|x|^2}{1 + \mu|x|^2}$.

For any solution ϕ of (21) satisfying [Proposition 3.3](#), $F = \alpha_\mu \phi(b_\mu^2, \frac{\beta_\mu}{\alpha_\mu})$ is a projectively flat general (α, β) -metric for any constant μ by [Theorem 1.2](#). It is easy to see that such a metric can always be rewrote as the form

$$F = |y| \phi_\mu \left(|x|^2, \frac{\langle x, y \rangle}{|y|} \right), \quad (22)$$

where the function ϕ_μ is given by

$$\phi_\mu(b^2, s) = \frac{\sqrt{1 + \mu(b^2 - s^2)}}{1 + \mu b^2} \phi \left(\frac{b^2}{1 + \mu b^2}, \frac{s}{\sqrt{1 + \mu b^2} \sqrt{1 + \mu(b^2 - s^2)}} \right). \quad (23)$$

In particular, $\phi_0 = \phi$.

(23) defines a family of transformations $\{\mathcal{T}_\mu\}$ by $\phi_\mu = \mathcal{T}_\mu(\phi)$. Such a family of transformations become a transformation group of the solution space of (21) by the following proposition.

Proposition 5.1. For any solution $\phi(b^2, s)$ of (21), the following facts hold:

1. $\phi_\mu = \mathcal{T}_\mu(\phi)$ is also a solution of (21) for any constant μ ;
2. $\mathcal{T}_0(\phi) = \phi$;
3. $\mathcal{T}_\mu \circ \mathcal{T}_\nu(\phi) = \mathcal{T}_{\mu+\nu}(\phi)$.

Proof. Denote ϕ_μ by $\tilde{\phi}$ and set $\tilde{\phi} = A\phi(B, S)$ where

$$A(b^2, s) = \frac{\sqrt{1 + \mu(b^2 - s^2)}}{1 + \mu b^2}, \quad B(b^2, s) = \frac{b^2}{1 + \mu b^2}, \quad S(b^2, s) = \frac{s}{\sqrt{1 + \mu b^2} \sqrt{1 + \mu(b^2 - s^2)}}.$$

Then

$$\tilde{\phi}_2 = A_2\phi(B, S) + AS_2\phi_S(B, S) = -\frac{\mu S\phi(B, S)}{(1 + \mu b^2)\sqrt{1 + \mu(b^2 - s^2)}} + \frac{\phi_S(B, S)}{\sqrt{1 + \mu b^2}(1 + \mu(b^2 - s^2))},$$

$$\tilde{\phi} - s\tilde{\phi}_2 = \frac{1}{\sqrt{1 + \mu(b^2 - s^2)}}(\phi(B, S) - S\phi_S(B, S)).$$

Set $E = \frac{1}{\sqrt{1 + \mu(b^2 - s^2)}}$, then

$$(\tilde{\phi} - s\tilde{\phi}_2)_1 = E(\phi(B, S) - S\phi_S(B, S))_B B_1 + E(\phi(B, S) - S\phi_S(B, S))_S S_1 + E_1(\phi(B, S) - S\phi_S(B, S)), \quad (24)$$

$$(\tilde{\phi} - s\tilde{\phi}_2)_2 = E(\phi(B, S) - S\phi_S(B, S))_S S_2 + E_2(\phi(B, S) - S\phi_S(B, S)). \quad (25)$$

The fact that ϕ is a solution of (21) yields

$$(\phi(B, S) - S\phi_S(B, S))_S = -2S(\phi(B, S) - S\phi_S(B, S))_B. \quad (26)$$

Then by (24), (25) and (26) we have

$$\begin{aligned} 2(\tilde{\phi}_1 - s\tilde{\phi}_{12}) - \tilde{\phi}_{22} &= 2(\tilde{\phi} - s\tilde{\phi}_2)_1 + s^{-1}(\tilde{\phi} - s\tilde{\phi}_2)_2 \\ &= (2ES_1 + s^{-1}ES_2)(\phi(B, S) - S\phi_S(B, S))_S + 2EB_1(\phi(B, S) - S\phi_S(B, S))_B \\ &\quad + (2E_1 + s^{-1}E_2)(\phi(B, S) - S\phi_S(B, S)) \\ &= 2E(B_1 - 2SS_1 - s^{-1}SS_2)(\phi(B, S) - S\phi_S(B, S))_B + (2E_1 + s^{-1}E_2)(\phi(B, S) - S\phi_S(B, S)) \\ &= 0. \end{aligned}$$

The last equality holds because the items $B_1 - 2SS_1 - s^{-1}SS_2$ and $2E_1 + s^{-1}E_2$ are both equal to 0 by direct computations. So (1) holds.

(2) holds since $\phi_0 = \phi$.

In order to see that (3) is true, we only need to compute $\mathcal{T}_\mu(\phi_\nu)$. By (23) and the definition of \mathcal{T}_μ ,

$$\begin{aligned} \mathcal{T}_\mu(\phi_\nu) &= \frac{\sqrt{1 + \nu(b^2 - s^2)}}{1 + \nu b^2} \sqrt{1 + \mu \left(\frac{b^2}{1 + \nu b^2} - \frac{s^2}{(1 + \nu b^2)(1 + \nu(b^2 - s^2))} \right)} \\ &\quad \times \phi \left(\frac{\frac{b^2}{1 + \nu b^2}}{1 + \mu \frac{b^2}{1 + \nu b^2}}, \frac{\frac{s}{\sqrt{1 + \nu b^2} \sqrt{1 + \nu(b^2 - s^2)}}}{\sqrt{1 + \mu \frac{b^2}{1 + \nu b^2} \sqrt{1 + \mu \left(\frac{b^2}{1 + \nu b^2} - \frac{s^2}{(1 + \nu b^2)(1 + \nu(b^2 - s^2))} \right)}}} \right) \\ &= \phi_{\mu + \nu}(b^2, s), \end{aligned}$$

which means $\mathcal{T}_\mu \circ \mathcal{T}_\nu(\phi) = \mathcal{T}_{\mu + \nu}(\phi)$. \square

Proposition 5.1 implies a simple fact. If $\tilde{\phi}$ can be obtained from some solution ϕ of (21) by some transformation \mathcal{T}_μ , then they will offer the same projectively flat Finsler metrics by **Theorem 1.2**. For instance, obviously $\phi = 1$ is a solution of (21), and $\mathcal{T}_\mu(1) = \frac{\sqrt{1 + \mu(b^2 - s^2)}}{1 + \mu b^2}$. In this case,

$$\phi_\mu \left(b_\nu^2, \frac{\beta_\nu}{\alpha_\nu} \right) = \phi_{\mu + \nu} \left(b_0^2, \frac{\beta_0}{\alpha_0} \right) = \alpha_{\mu + \nu}$$

are just the Riemannian metrics of constant sectional curvature.

We still don't know how to solve Eq. (21) completely, but the following lemma is helpful to get its solutions.

Lemma 5.2. For any C^∞ functions f and g , the following function is the solution of (21):

$$\phi(b^2, s) = f(b^2 - s^2) + 2s \int_0^s f'(b^2 - \sigma^2) d\sigma + g(b^2)s. \tag{27}$$

Proof. It is easy to verify that the above function satisfies (21). \square

Suppose that ϕ satisfies (27). Direct computations show that

$$\phi - s\phi_2 = f(t), \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} = f(t) + 2tf'(t),$$

where $t = b^2 - s^2 \geq 0$. Assume that $f(0) > 0$, then the inequalities $\phi > 0$ and (13) always hold for b small enough. So one can construct infinitely many projectively flat general (α, β) -metrics by **Lemma 5.2**. Some simple examples are given in the following:

- $f(t) = \frac{1}{\sqrt{1-t}}$,

$$\phi(b^2, s) = \frac{\sqrt{1 - b^2 + s^2}}{1 - b^2} + g(b^2)s.$$

In this case, F is of Randers type. In particular, it is the navigation representation of Randers metrics when $g(b^2) = -\frac{1}{1-b^2}$ (cf. [7]).

- $f(t) = 1 + t,$

$$\phi(b^2, s) = 1 + b^2 + s^2 + g(b^2)s.$$

In particular, it is given by Example 4.3 when $g(b^2) = 2\sqrt{1+b^2}$.

- $f(t) = \sqrt{1-t},$

$$\phi(b^2, s) = \sqrt{1-b^2+s^2} - s \ln(\sqrt{1-b^2+s^2}+s) + s \ln \sqrt{1-b^2} + g(b^2)s.$$

- $f(t) = \sqrt{1+t},$

$$\phi(b^2, s) = \sqrt{1+b^2-s^2} + s \arcsin \frac{s}{\sqrt{1+b^2}} + g(b^2)s.$$

- $f(t) = \ln(2+t),$

$$\phi(b^2, s) = \ln(2+b^2-s^2) + \frac{s}{\sqrt{2+b^2}} \ln \frac{\sqrt{2+b^2}+s}{\sqrt{2+b^2}-s} + g(b^2)s.$$

- $f(t) = \ln(2-t),$

$$\phi(b^2, s) = \ln(2-b^2+s^2) - \frac{2s}{\sqrt{2-b^2}} \arctan \frac{s}{\sqrt{2-b^2}} + g(b^2)s.$$

- $f(t) = 1 + \arctan t,$

$$\begin{aligned} \phi(b^2, s) &= 1 + \arctan(b^2 - s^2) + \frac{s}{\sqrt{1+b^4}\sqrt{2\sqrt{1+b^4}-2b^2}} \\ &\cdot \left(\frac{1}{2}(\sqrt{1+b^4}-b^2) \ln \frac{\sqrt{1+b^4} + \sqrt{2\sqrt{1+b^4}+2b^2}s + s^2}{\sqrt{1+b^4} - \sqrt{2\sqrt{1+b^4}+2b^2}s + s^2}} \right. \\ &+ \arctan(\sqrt{2\sqrt{1+b^4}+2b^2}s + \sqrt{1+b^4}+b^2) \\ &\left. + \arctan(\sqrt{2\sqrt{1+b^4}+2b^2}s - \sqrt{1+b^4}-b^2) \right) + g(b^2)s. \end{aligned}$$

Obviously, the general (α, β) -metrics include all the (α, β) -metrics. But it seems a little difficult to determine whether a general (α, β) -metric is an (α, β) -metric or not. If $\phi = \phi(s)$ is independent of b^2 , then there is no doubt that $F = \alpha\phi(\frac{\beta}{\alpha})$ is an (α, β) -metric. But if $\phi = \phi(b^2, s)$, we can't conclude that $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ isn't an (α, β) -metric. For instance, as we know in Section 1, the general (α, β) -metric $F = \frac{(\sqrt{1+b^2\alpha+\beta})^2}{\alpha}$ is actually an (α, β) -metric. So the following problem is still open:

Give an approach to distinguish (α, β) -metrics from general (α, β) -metrics.

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