Unicity and Strong Unicity in Approximation Theory

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1. Introduction

It is the object of this paper to discuss the following question: Is it possible to characterize unicity and strong unicity of elements of best approximation by modified Kolmogorov-criteria? Furthermore, we examine the relationship between these two properties.

Let $G$ be a nonempty set in a normed linear space $E$, and let $f$ be an element of $E$. Consider $P(f) := P_0(f) := \{g \in G : \|f - g\| \leq \|f - g', g \in G\}$, i.e., the set of elements of best approximation of $f$ in $G$. The set-valued map $P : E \to 2^G$ defined in this way is called the metric projection. The set $G$ is called proximinal (respectively, semi-Chebyshev) if $P(f)$ is nonempty (respectively, contains at most one element) for each $f \in E$. If $G$ is both proximinal and semi-Chebyshev, then it is called Chebyshev. For each $f \in E$ let $S_f := \{L \in E' : \|L\| = 1, L(f) = \|f\|\}$ and let $E_f$ be the set of extreme points of $S_f$ in the $\alpha(F', E)$-topology. We say the pair $(g_0, f)$, with $g_0 \in G$ and $f \in E \setminus \overline{G}$, satisfies the Kolmogorov-criterion if, for each $g \in G$,

$$\min \{\text{Re} \, L(g - g_0) : L \in E_{f-g_0}\} \leq 0;$$

the strict Kolmogorov-criterion if, for each $g \in G$, $g \neq g_0$,

$$\min \{\text{Re} \, L(g - g_0) : L \in E_{f-g_0}\} < 0;$$

and the strong Kolmogorov-criterion if there exists a constant $K > 0$ such that, for each $g \in G$,

$$\min \{\text{Re} \, L(g - g_0) : L \in E_{f-g_0}\} \leq -K \|g - g_0\|.$$ 

Brosowski [3] proved that a set $G$ in a normed linear space $E$ is a sun if and only if for each $f \in E \setminus \overline{G}$ the element $g_0 \in G$ is in $P(f)$ if and only if the pair $(g_0, f)$ satisfies the Kolmogorov-criterion. This result has led us to ask whether
it is possible to characterize certain semi-Chebyshev sets $G$ in an arbitrary normed linear space $E$ by the strict Kolmogorov-criterion. In Section 3 we will see that this is in general not possible, not even if $G$ is a finite-dimensional subspace of $E$. But we show in Section 2 that it can be done for finite-dimensional convex sets $G$ in $E = I_a$, which includes the cases $E = I_\omega = C(X)$ and $E = I_\omega = C_0(T)$, and for suns $G$ in $E = L_q(T, m)$ using characterizations of best approximations given by Brosowski [3], Deutsch [5], Deutsch-Maserick [6] and Havinson [9].

Furthermore in general it is not possible to characterize those elements $f$ in $E$ for which $P(f)$ is a singleton by the strict Kolmogorov-criterion, not even for the finite-dimensional subspace of the splinefunctions in $C[a, b]$. But we are able to verify in this case that under certain alternation properties on $f - g_0$ the pair $(g_0, f)$ satisfies the strict Kolmogorov-criterion.

In Section 3 we show by using results of Bartelt, McLaughlin [2] and Wulbert [17] that strongly unique elements of best approximation (see Definition 3.1) can be characterized by the strong Kolmogorov-criterion in the case when $G$ is a linear subspace in an arbitrary normed linear space $E$.

Finally we apply our theorems of Section 2 to obtain statements concerning strong unicity and pointwise Lipschitzian metric projections (see Definition 3.9), which include results of Ault, Deutsch, Morris, Olson [1], Freud [8], Newman, Shapiro [12], Schumaker [14] and Wulbert [17].

**Notation.** For a normed linear space $E$ we denote by $E'$ the dual space of $E$ and by $S_E := \{ f \in E : \| f \| \leq 1 \}$ its unit ball. For a set $A$ in $E$ and a function $f$ defined on $E$ we denote the restriction of $f$ to $A$ by $f|_A$ and the extreme points of $A$ by $Ep(A)$. We say that a set $G$ in $E$ is a convex cone, if $G$ is closed, convex and $ag \in G$ for each $g \in G$ and $a \geq 0$. For a function $f$ defined on a set $T$ we denote $Z(f) := \{ t \in T : f(t) = 0 \}$.

## 2. Unicity of Best Approximations

The condition that the pair $(g_0, f)$ satisfies the strict Kolmogorov-criterion is sufficient that $g_0$ is the only best approximation of $f$:

**2.1. Lemma.** Let $E$ be a normed linear space, $G$ a nonempty subset of $E$, $f \in E' \setminus G$ and $g_0 \in G$. If $(g_0, f)$ satisfies the strict Kolmogorov-criterion then $\{g_0\} = P(f)$.

**Proof.** According to the assumption for each $g \in G, g \neq g_0$, there exists a functional $L_g \in E_{f-g_0}$ with $\text{Re} \, L_g(g - g_0) < 0$. Then for each $g \in G, g \neq g_0$, $\| f - g \| \geq \| L_g(f - g) \| > \text{Re} \, L_g(f - g) - \text{Re} \, L_0(f - g_0) - \| f - g_0 \|$. Clearly this implies $P(f) = \{g_0\}$. 
In view of Lemma 2.1 it is natural to ask whether semi-Chebyshev sets $G$ in a normed linear space can be characterized by the strict Kolmogorov-criterion. As we will see in Section 3 this is not possible even for finite-dimensional subspaces $G$ in an arbitrary normed linear space $E$. But we are able to prove theorems of this type for certain normed linear spaces.

First we consider the case $E = I_A$: For a compact space $X$ we denote by $C(X)$ the space of all real-valued, continuous functions on $X$ endowed with the usual vector operations and with the norm $\|f\| = \max\{|f(x)| : x \in E\}$ for each $f$ in $C(X)$.

For a locally compact space $T$ we denote by $C_0(T)$ the space of all continuous functions on $T$, vanishing at infinity, endowed with the usual vector operations and with the norm

$$\|f\| = \sup\{|f(t)| : t \in T\}$$

for each $f$ in $C_0(T)$.

If $A$ is a closed set in $X$ we denote by $I_A$ the linear subspace

$$I_A = \{f \in C(X) : f(x) = 0 (x \in A)\}$$

of $C(X)$. In particular $I_A = C(X)$. Compactifying $T$ by adding a point $\infty$ we obtain a compact Hausdorff space $X_0 = T \cup \{\infty\}$ and $C_0(T)$ may be considered as $I_{(\infty)} \subset C(X_0)$.

We need the following characterization for the elements of best approximation in some finite-dimensional convex set of a normed linear spaces given by Deutsch, Maserick [6] and, independently, by Havinson [9]:

2.2. THEOREM. Let $G$ be a convex set in a real normed linear space $E$, let $f \in E \setminus G$, $g_0 \in G$ and suppose the span $G$ is $n$-dimensional. Then $g_0 \in P(f)$ if and only if there exist $m$ linear independent functionals $L_1, \ldots, L_m \in E - g_0$ and $m$ numbers $a_1, \ldots, a_m > 0$ with $\sum_{i=1}^m a_i = 1$, where $1 \leq m \leq n + 1$, such that

$$\sum_{i=1}^m a_i L_i(g - g_0) \leq 0$$

for each $g \in G$.

It is well known that for the case $E = I_A$ each $L \in E - g_0$ is of the form

$$L(h) = \frac{(f - g_0)(x)}{\|f - g_0\|} h(x)$$

for each $h \in I_A$

where $x \in M_{f - g_0} \setminus A$ and $M_{f - g_0} = \{x \in X : |(f - g_0)(x)| = \|f - g_0\|\}$. The converse is true for $I_0 = C(X)$ (see Dunford-Schwartz [7]). Using this fact and Theorem 2.2 we have the following corollary:
2.3. **Corollary.** Let $G$ be a convex set of $I_A$, $f \in E \setminus \overline{G}$, $g_0 \in G$ and suppose the span $G$ is $n$-dimensional. Then $g_0 \in P(f)$ if and only if there exist $m$ distinct points $x_1, \ldots, x_m \in M_{f-g_0} \setminus A$ and $m$ numbers $a_1, \ldots, a_m > 0$ with $\sum_{i=1}^{m} a_i = 1$ where $1 \leq m \leq n + 1$, such that
\[
\sum_{i=1}^{m} a_i (f - g_0)(g - g_0)(x_i) \leq 0 \text{ for each } g \in G.
\]

Using Corollary 2.3 we give a characterization of finite-dimensional convex semi-Chebyshev sets in $I_A$:

2.4. **Theorem.** Let $G$ be a finite-dimensional convex set in $E = I_A$. Then the following statements are equivalent:

1. $G$ is semi-Chebyshev
2. For each $f \in E \setminus \overline{G}$, $g_0 \in P(f)$, $g \in G$, $g \neq g_0$,

\[\min \{(f - g_0)(g - g_0)(x) : x \in M_{f-g_0}\} < 0\]

If $E = I_\phi = C(X)$ the conditions (1) and (2) are equivalent to the following statement:

3. For each $f \in E \setminus \overline{G}$, $g_0 \in P(f)$ the pair $(g_0, f)$ satisfies the strict Kolmogorov-criterion.

**Proof.** Assume we have (2), then for $f \in E \setminus \overline{G}$, $g_0 \in P(f)$ and $g \in G$, $g \neq g_0$, there exists a point $x \in M_{f-g_0} \setminus A$ with $(f - g_0)(g - g_0)(x) < 0$. Therefore $\|f - g_0\| = \|(f - g_0)(x)\| < \|(f - g_0)(x) - (g - g_0)(x)\| = \|(f - g)(x)\| \leq \|f - g\|$. Thus $\{g_0\} = P(f)$. Proving (1).

We show that (2) follows from (1): Assume that there exist $f \in E \setminus \overline{G}$, $g_0 \in P(f)$ and $g_1 \in G$, $g_1 \neq g_0$, such that for each $x \in M_{f-g_0}$

\[(f - g_0)(g_1 - g_0)(x) \geq 0.\]  

(a)

We show that there exists a function $f_0$ in $E$ with $g_0, g_1 \in P(f_0)$. Since span $G$ is $n$-dimensional and $g_0 \in P(f)$ by Corollary 3 there exist $m$ distinct points $x_1, \ldots, x_m \in M_{f-g_0} \setminus A$ and $m$ numbers $a_1, \ldots, a_m > 0$ with $\sum_{i=1}^{m} a_i = 1$, where $1 \leq m \leq n + 1$, such that

\[
\sum_{i=1}^{m} a_i (f - g_0)(g - g_0)(x_i) \leq 0 \text{ for each } g \in G.
\]

(b)

Since $a_1, \ldots, a_m > 0$ it follows from (a) and (b)

\[(f - g_0)(g_1 - g_0)(x_i) = 0, \quad 1 \leq i \leq m\]
and since \( x_1, \ldots, x_m \in M_{f-g_0} \setminus A \)
\[
(g_1 - g_0)(x_i) = 0, \quad 1 \leq i \leq m. \tag{c}
\]
We define for each \( x \in X \)
\[
f_0(x) := \frac{(f - g_0)(x)}{\|f - g_0\|} \left( \|g_1 - g_0\| - |(g_1 - g_0)(x)| \right) + g_0(x).
\]
Obviously \( f_0 \) is in \( E \). Then
\[
|f - g_0)(x)| = \frac{|(f - g_0)(x)|}{\|f - g_0\|} \left( \|g_1 - g_0\| - |(g_1 - g_0)(x)| \right) \leq \|g_1 - g_0\|.
\]
and because of (c) and \( x_1, \ldots, x_m \in M_{f-g_0} \setminus A \)
\[
|(f_0 - g_0)(x_i)| = \|g_1 - g_0\|, \quad 1 \leq i \leq m.
\]
Therefore \( \|f_0 - g_0\| = \|g_1 - g_0\| \).
Because of (b) and \( a_1, \ldots, a_m > 0 \) for each \( g \in G \) there exists a point \( x_i \in M_{f-g_0} \setminus A, \ 1 \leq i \leq m \), with
\[
(f - g_0)(g - g_0)(x_i) \leq 0. \tag{d}
\]
Therefore by (c) and (d)
\[
\|f_0 - g\| \geq \left\| \frac{(f - g_0)(x_i)}{\|f - g_0\|} \left( \|g_1 - g_0\| - |(g_1 - g_0)(x_i)| \right) + (g_0 - g)(x_i) \right\|
\]
\[
= \left\| \frac{(f - g_0)(x_i)}{\|f - g_0\|} \left( \|g_1 - g_0\| + (g_0 - g)(x_i) \right) \right\|
\]
\[
= \frac{|(f - g_0)(x_i)|}{\|f - g_0\|} \left( \|g_1 - g_0\| + |(g_0 - g)(x_i)| \right) \geq \|g_1 - g_0\|
\]
\[
= \|f_0 - g_0\|.
\]
Therefore \( g_0 \in P(f_0) \).
Moreover for each \( x \in X \)
\[
|(f_0 - g_1)(x)|
\]
\[
= \left| \frac{(f - g_0)(x_i)}{\|f - g_0\|} \left( \|g_1 - g_0\| - |(g_1 - g_0)(x)| \right) + (g_0 - g_1)(x) \right|
\]
\[
\leq \left| \frac{(f - g_0)(x)}{\|f - g_0\|} \left( \|g_1 - g_0\| - |(g_1 - g_0)(x)| \right) + |(g_0 - g_1)(x)| \right|
\]
\[
\leq \|g_1 - g_0\| - |(g_1 - g_0)(x)| + |(g_0 - g_1)(x)| = \|g_1 - g_0\|
\]
\[
= \|f_0 - g_0\|.
\]
Therefore \( \| f_0 - g_1 \| = \| f_0 - g_0 \| \) and since \( g_0 \in P(f) \) the function \( g_1 \) is in \( P(f) \). The fact that \( g_0, g_1 \in P(f) \) and \( g_1 \neq g_0 \) is a contradiction to \( G \) being semi-Chebyshev. Thus (1) implies (2).

The equivalence of (2) and (3) in the case \( E = I_\delta = C(X) \) follows from the representation of the extreme points of the unit sphere in \( C(X)' \).

Using Theorem 2.4 we can prove the following necessary condition for finite-dimensional convex sets \( G \) in \( I_A \) to be Chebyshev, for \( X \) metric.

2.5. COROLLARY. Let \( G \) be an \( n \)-dimensional convex set in \( E = I_A \), such that \( 0 \) is in \( G \). If \( G \) is Chebyshev then each \( g \in G, \ g \neq 0, \) has at most \( n - 1 \) distinct zeros in \( X \setminus A \).

Proof: Assume that there exists a function \( g_0 \in G, \ g_0 \neq 0, \) with \( n \) distinct zeros \( x_1, \ldots, x_n \in X \setminus A \). Then by a standard argument there exist \( n \) numbers \( a_1, \ldots, a_n \) with \( \sum_{i=1}^{n} \left| a_i \right| > 0 \) such that for each \( g \in G, \ \sum_{i=1}^{n} a_i g(x_i) = 0 \).

By Tietze's Lemma there exist a function \( f \in E \) with \( f(x_i) = \text{sgn} \ a_i \) and \( |f(x)| < 1 \) elsewhere. Then for each \( g \in G, \ \sum_{i=1}^{n} \left| a_i \right| f(x_i) = \sum_{i=1}^{n} \left| a_i \right| \text{sgn} \ a_i g(x_i) = \sum_{i=1}^{n} a_i g(x_i) = 0 \). Replacing, if necessary, each \( \left| a_i \right| \) by \( \left| a_i \right| / \sum_{i=1}^{n} \left| a_i \right| \) we may assume that \( \sum_{i=1}^{n} \left| a_i \right| = 1 \).

Therefore by Corollary 2.3 the function \( 0 \) is in \( P(f) \). Moreover \( M_f = \{ x_1, \ldots, x_n \} \) and for each \( x \in M_f, \ f g_0(x) = 0 \). By Theorem 2.4 it follows that \( G \) is not Chebyshev.

Corollary 2.5 has been proved by Phelps [13] for \( n \)-dimensional subspaces of \( I_A \). The converse of Corollary 2.5 does not hold, as can be seen by easily constructed examples in \( C(\{1, 2\}) \).

Now we consider the case \( E = L_1(T, \mu) \): For a positive measure space \((T, \mu)\) we denote by \( L_1(T, \mu) \) (respectively by \( L_\infty(T, \mu) \)) the space of all equivalence classes of \( \mu \)-integrable (respectively \( \mu \)-measurable and \( \mu \)-essentially bounded) real-valued functions on \( T \), endowed with the usual vector operations and with the norm \( \| f \| = \int_T |f| \ d\mu \) (respectively, \( \| f \| = \text{ess sup}_{t \in T} \{ |f(t)| : t \in T \} \)).

A set \( G \) in a normed linear space \( E \) is called a sun if, for each \( f \in E \) and \( g_0 \in P(f), \) we have \( g_0 \in P(af + (1 - a) g_0) \) for each \( a > 1 \).

Brosowski [3] proved the following characterization:

2.6. THEOREM. A set \( G \) in a normed linear space is a sun if and only if for each \( f \in E \setminus \mathcal{C}, \ g_0 \in G \) the following statements are equivalent:

1. \( g_0 \in P(f) \)
2. \( (g_0, f) \) satisfies the Kolmogorov-criterion.

By Singer [15, Lemma 1.13, p. 83] for \( E = L_1(T, \mu) \) with the property \( L_1(T, \mu)' = L_\infty(T, \mu) \) a functional \( L \) is in \( Ep(S_E) \) if and only if there exists a function \( \beta \in L_\infty(T, \mu) \) such that \( |\beta| = 1 \) a.e. on \( T \) and \( L(f) = \int_T f \beta \ d\mu \) for
each $f \in E$. Thus for $E = L_1(T, m)$ the pair $(g_0, f)$ satisfies the Kolmogorov-criterion (respectively the strict Kolmogorov-criterion) if and only if for each $g \in G$ (respectively $g \in G$, $g \neq g_0$) there exists a $\beta \in L_\infty(T, m)$ such that $|\beta| = 1$ a.e. on $T$, $\int_T (f - g_0)\beta \, dm = \int_T |f - g_0| \, dm$ and $\int_T (g - g_0)\beta \, dm \leq 0$ (respectively $\int_T (g - g_0)\beta \, dm < 0$). Because for a given $g \in G$ we replace $T$ by the union of the supports of $f, g_0$ and $g$, which is $\sigma$-finite, and therefore we may assume $L_1(T, m)' = L_\infty(T, m)$.

Under application of Theorem 2.6 Deutsch [5] has given the following

2.7. COROLLARY. Let $G$ be a sun in $L_1(T, m)$, $f \in E \setminus G$ and $g_0 \in G$. Then $g_0 \in P(f)$ if and only if for each $g \in G$

$$\int_{T \setminus Z(f-g_0)} (g - g_0) \text{sgn}(f - g_0) \, dm \lesssim \int_{Z(f-g_0)} |g - g_0| \, dm.$$ 

Using Theorem 2.6 and Corollary 2.7 we can prove the following characterization of semi-Chebyshev suns in $L_1(T, m)$:

2.8. THEOREM. Let $G$ be a set in $E = L_1(T, m)$. Then the following statements are equivalent:

1. $G$ is a semi-Chebyshev sun
2. For each $f \in E \setminus G$ and $g_0 \in P(f)$ the pair $(g_0, f)$ satisfies the strict Kolmogorov-criterion
3. For each $f \in E \setminus G$, $g_0 \in P(f)$, $g \in G$, $g \neq g_0$, there exists a function $\beta \in L_\infty(T, m)$ such that $|\beta| = 1$ a.e. on $T$, $\int_T (f - g_0)\beta \, dm = \int_T |f - g_0| \, dm$ and $\int_T (g - g_0) \, dm < 0$
4. For each $f \in E \setminus G$, $g_0 \in P(f)$, $g \in G$, $g \neq g_0$,

$$\int_{T \setminus Z(f-g_0)} (g - g_0) \text{sgn}(f - g_0) \, dm < \int_{Z(f-g_0)} |g - g_0| \, dm$$

Proof. The equivalence of (2) and (3) follows from the remark after Theorem 2.6. We show that (4) follows from (1): Assume (4) is not true, then there exist functions $f \in E \setminus G$, $g_0 \in P(f)$ and $g_1 \in G$, $g_1 \neq g_0$, such that

$$\int_{T \setminus Z(f-g_0)} (g - g_0) \text{sgn}(f - g_0) \, dm \geq \int_{Z(f-g_0)} |g - g_0| \, dm.$$

We show that there exists a function $f_0 \in L_1(T, m)$ with $g_0 \in P(f_0)$ and $g_1 \neq g_0$. 


Since $g_0 \in P(f)$ by Corollary 2.7 for each $g \in G$

$$\int_{T \setminus Z(f-g_0)} (g - g_0) \, \text{sgn}(f - g_0) \, dm \leq \int_{Z(f-g_0)} |g - g_0| \, dm. \tag{b}$$

Combining (a) and (b) it follows

$$\int_{T \setminus Z(f-g_0)} (g_1 - g_0) \, \text{sgn}(f - g_0) \, dm = \int_{Z(f-g_0)} |g_1 - g_0| \, dm. \tag{c}$$

We define:

$$f_0(t) := \begin{cases} (g_1 - g_0)(t) \, \text{sgn}(f - g_0)(t) + g_0(t) & \text{if } t \in T \setminus Z(f - g_0), \\ g_0(t) & \text{if } t \in Z(f - g_0), \end{cases}$$

Then it holds:

$$\|f_0 - g_0\| = \int_T |f - g_0| \, dm$$

$$= \int_{T \setminus Z(f-g_0)} \|g_1 - g_0\| \, \text{sgn}(f - g_0) \, dm$$

$$= \int_{T \setminus Z(f-g_0)} |g_1 - g_0| \, dm.$$

From (c) it follows:

$$\|f_0 - g_1\| = \int_T |f_0 - g_1| \, dm$$

$$= \int_{T \setminus Z(f-g_0)} \|g_1 - g_0\| \, \text{sgn}(f - g_0) \, dm$$

$$+ \int_{Z(f-g_0)} |g_0 - g_1| \, dm$$

$$= \int_{T \setminus Z(f-g_0)} (|g_1 - g_0| \, \text{sgn}(f - g_0)$$

$$\quad + (g_0 - g_1) \, \text{sgn}(f - g_0)) \, dm$$

$$+ \int_{Z(f-g_0)} |g_0 - g_1| \, dm.$$
Moreover from (b) it follows that for each $g \in G$

\[
\|f_0 - g\| = \int_T |f_0 - g| \, dm
\]

\[
= \int_{T \setminus Z(f - g_0)} \|g_1 - g_0\| \, dm - \int_{T \setminus Z(f - g_0)} (g_1 - g_0) \, \text{sgn}(f - g_0) \, dm
\]

\[
+ \int_{Z(f - g_0)} |g_0 - g| \, dm
\]

\[
\geq \int_{T \setminus Z(f - g_0)} (|g_1 - g_0| \, \text{sgn}(f - g_0) + g_0 - g) \, \text{sgn}(f - g_0) \, dm
\]

\[
+ \int_{Z(f - g_0)} |g_0 - g| \, dm
\]

\[
= \int_{T \setminus Z(f - g_0)} |g_1 - g_0| \, dm
\]

\[
- \int_{T \setminus Z(f - g_0)} (g - g_0) \, \text{sgn}(f - g_0) \, dm
\]

\[
+ \int_{Z(f - g_0)} |g_0 - g| \, dm
\]

\[
\geq \int_{T \setminus Z(f - g_0)} |g_1 - g_0| \, dm - \|f_0 - g_0\|.
\]

Therefore $g_0, g \in P(f), g_1 \neq g_0$. This is a contradiction to $G$ being semi-Chebyshev. Thus (1) implies (4).

We show that (3) follows from (4). If we have (4) then for $g \in G$ we define

\[
\beta(t) := \begin{cases} 
\text{sgn}(f - g_0)(t) & \text{if } t \in T \setminus Z(f - g_0), \\
\text{sgn}(g_0 - g)(t) & \text{if } t \in Z(f - g_0) \cap Z(g - g_0), \\
1 & \text{if } t \in Z(f - g_0) \cap Z(g - g_0).
\end{cases}
\]
Then $|\beta| = 1$ on $T$ and
\[
\int_T (f - g_0) \beta \, dm = \int_{T \setminus Z(f - g_0)} (f - g_0) \beta \, dm \\
= \int_{T \setminus Z(f - g_0)} \text{sgn}(f - g_0) \, dm \\
= \int_{T \setminus Z(f - g_0)} |f - g_0| \, dm = \int_T |f - g_0| \, dm.
\]

\[
\int_T (g - g_0) \beta \, dm = \int_{T \setminus Z(f - g_0)} (g - g_0) \text{sgn}(f - g_0) \, dm \\
+ \int_{Z(f - g_0) \setminus Z(g - g_0)} (g - g_0) \text{sgn}(g - g_0) \, dm \\
+ \int_{Z(f - g_0) \cap Z(g - g_0)} (g - g_0) \, dm \\
= \int_{T \setminus Z(f - g_0)} (g - g_0) \text{sgn}(f - g_0) \, dm \\
- \int_{Z(f - g_0)} g - g_0 \, dm \\
+ \int_{Z(f - g_0) \cap Z(g - g_0)} (g - g_0) \, dm < 0
\]

Thus (4) implies (3).

If we have (2), the fact that $G$ is semi-Chebyshev follows from Lemma 2.1 and that $G$ is a sun follows from Theorem 2.6. Thus (2) implies (1).

Now we will give some examples of semi-Chebyshev sets in $L_1(T, m)$.

First we recall that every convex set in a normed linear space is a sun. An atom of a positive measure space $(T, m)$ is a measurable set $A$ in $T$ such that $m(A) > 0$ and for each measurable set $B$ of $A$ either $m(B) = 0$ or $m(A \setminus B) = 0$.

2.9. EXAMPLES. 1. The space $\mathbb{R}^2$ endowed with the norm $\|(x, y)\| = |x| + |y|$ for each $(x, y) \in \mathbb{R}^2$ is a space of type $L_1(T, m)$. It is easy to verify that the set $G = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 \leq 1, x < 0, y < 0\}$ is a non-convex semi-Chebyshev sun in this space.

2. A. L. Garkavi has shown that in $L_1(T, m)$ such that $L_1(T, m)' = L_\infty(T, m)$ there exists a Chebyshev subspace in $L_1(T, m)$ of dimension $n$ (respectively, of codimension $n$) if and only if $(T, m)$ has at least $n$ atoms (see Singer [10, pp. 233, 331]).

3. Phelps [13] has given an example of a Chebyshev subspace in $L_1(T, m)$.
which has neither finite dimension nor finite codimension (see Singer [10, p. 332]). Here it is not necessary that \((T, m)\) contains an atom.

4. Let \(A\) be an atom in a positive measure space \((T, m)\) with \(m(T\setminus A) > 0\) and \(G = \{f \in L_2(T, m): f = 0 \text{ on } T\setminus A, |f(t)| \leq 1 \text{ on } A\}\). Then \(G\) is a convex Chebyshev set in \(L_2(T, m)\): Let \(f\) be a function in \(L_2(T, m)\). If \(|f(t)| = 1\) on \(A\), then for \(g_f \in G\), defined by \(g_f = f\) on \(A\) and \(g_f = 0\) on \(T\setminus A\), it holds:

\[
\|f - g_f\| = \int_T |f - g_f| \, dm = \int_{T\setminus A} |f| \, dm < \int_{T\setminus A} |f - g| \, dm + \int_A |g_f - g| \, dm = \int_{T\setminus A} |f - g| \, dm
\]

If \(|f(t)| \leq 1\) on \(A\), then for \(g_f \in G\), defined by \(g_f = 1\) on \(A\) and \(g_f = 0\) on \(T\setminus A\), it holds:

\[
\|f - g_f\| = \int_T |f - g_f| \, dm = \int_A |f - 1| \, dm + \int_{T\setminus A} |f| \, dm < \int_A |f - g| \, dm + \int_{T\setminus A} |f - g| \, dm = \|f - g\| \text{ for each } g \in G, g \neq g_f.
\]

The case \(f(t) < -1\) on \(A\) can be proved similarly.

5. Let \(A\) be an atom as in 4., then \(G = \{f \in L_2(T, m): f = 0 \text{ on } T\setminus A, f(t) \geq 0 \text{ on } A\}\) is a one-dimensional convex Chebyshev cone in \(L_2(T, m)\). This can be shown similarly as in 4.

Theorem 2.4 and Theorem 2.8 show that it is actually possible to characterize certain semi-Chebyshev sets in \(C(X)\) and in a certain sense also in \(I_2\), respectively in \(L_2(T, m)\), by the strict Kolmogorov-criterion.

Considering this fact there is the question if it is possible to characterize those elements, which have exactly one best approximation in a non-semi-Chebyshev set, by the strict Kolmogorov-criterion. Examples can easily be constructed to show that this cannot be done, not even for finite-dimensional subspaces in \(C(X)\) that are very close to being Chebyshev as e.g. subspaces of spline functions in \(C[a, b]\). But in this case we can show that under certain alternation conditions the strict Kolmogorov-criterion is valid.

First some definitions: Let \(a = x_0 < x_1 < \cdots < x_k < x_{k+1} = b\) be \(k\)
fixed knots in $[a, b]$. The class of the usual polynomial splines of degree $n$ with $k$ fixed knots is defined by

$$S_{n,k} := S_{n,k}(x_1, \ldots, x_k) = \text{span}\{1, x, \ldots, x^n, (x - x_1)^n, \ldots, (x - x_k)^n\}$$

where

$$(x - \xi)^n := \begin{cases} 0 & \text{for } x \leq \xi \\ (x - \xi)^n & \text{for } x > \xi \end{cases}$$

They form an $n + k + 1$-dimensional subspace of $C[a, b]$. Each function $s \in S_{n,k}$ is in $C^{n-1}[a, b]$ and the restriction of $s$ to the interval $[x_i, x_{i+1}]$, $i = 0, \ldots, k$, represents a polynomial of degree $n$.

It is well known that a function in $C[a, b]$ in general has more than one element of best approximation in $S_{n,k}$.

We need the following restricted interpolation property for spline functions (see Schumaker [14], Karlin [10]).

2.10. THEOREM. The determinant

$$\delta \left( 0, \ldots, 0, x_1, \ldots, x_k \right) = \begin{vmatrix} 1 & t_1 & \cdots & t_1^n & (t_1 - x_1)^n & \cdots & (t_1 - x_k)^n \\ 1 & t_2 & \cdots & t_2^n & (t_2 - x_1)^n & \cdots & (t_2 - x_k)^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{n+k+1} & \cdots & t_{n+k+1}^n & (t_{n+k+1} - x_1)^n & \cdots & (t_{n+k+1} - x_k)^n \end{vmatrix}$$

is nonnegative for all $a < t_1 < t_2 < \cdots < t_{n+k+2} < b$ and strictly positive if and only if

$$t_i < x_i < t_{n+i+1}, \quad i = 1, \ldots, k \quad (a < t_1).$$

Using Theorem 2.10 we can prove the following theorem, which is also true for the more general class of Chebyshevian spline functions (for definition see Schumaker [9]).

2.11. THEOREM. Let $E = C[a, b]$, $G = S_{n,k}$, $f \in E \backslash G$ and $g_0 \in P_0(f)$. If there exist $a \leq t_1 < \cdots < t_{n+k+2} \leq b$ such that

1. $t_{i+1} < x_i < t_{n+i+1}$, $i = 1, \ldots, k$
2. $\epsilon(-1)(f - g_0(t_i)) = \|f - g_0\|, \quad i = 1, \ldots, n + k + 2, \quad \epsilon = \pm 1$,

then $(g_0, f)$ satisfies the strict Kolmogorov-criterion.
Proof. Assume that the conditions (1) and (2) hold, but that \((g_0, f)\) does not satisfy the strict Kolmogorov-Criterion, i.e., there exists a function \(g \in G, g \neq 0\), such that, for each \(x \in M_{f-g_0}\), we have \((f - g_0)g(x) > 0\).

Since by (1) there exist points \(a \leq t_1 < \cdots < t_{n+k+2} \leq b\) such that \(\epsilon(-1)^i(f - g_0)(t_i) = \|f - g_0\|, \ i = 1, \ldots, n + k + 2, \ \epsilon = \pm 1\), it follows that \(\epsilon(-1)^i(g(t_i)) > 0, \ i = 1, \ldots, n + k + 2, \ \epsilon = \pm 1\). From Theorem 2.10 and condition (1) it follows that for each \(n + k + 1\) distinct points \(u_1, \ldots, u_{n+k+1}\) from \(\{t_1, \ldots, t_{n+k+2}\}\) we have

\[
\delta \left( 0, \ldots, 0, x_1, \ldots, x_k \right) \neq 0.
\]

Therefore there exists a basis \(\{g_1, \ldots, g_{n+k+1}\}\) of \(G\) such that for each \(i \in \{1, \ldots, n + k + 1\}\) we have \(g_i(t_j) = 0\), where \(1 \leq j \leq n + k + 1\) and \(j \neq i\), and \(\epsilon(-1)^i g_i(t_i) = 1\). Then \(g = a_1 g_1 + \cdots + a_{n+k+1} g_{n+k+1}\) with \(a_1, \ldots, a_{n+k+1} > 0\) and the scalars \(a_i\) are not all zero.

We define

\[
D = \begin{vmatrix}
g_1(t_1) & \ldots & g_1(t_{n+k+1}) \\
\vdots & & \vdots \\
g_{n+k+1}(t_1) & \ldots & g_{n+k+1}(t_{n+k+1})
\end{vmatrix}
\]

and, for each \(i \in \{1, \ldots, n + k + 1\}\),

\[
D_i = \begin{vmatrix}
g_1(t_i) & \ldots & g_1(t_{i-1}) & g_1(t_{i+1}) & \ldots & g_1(t_{n+k+2}) \\
\vdots & & \vdots & & \vdots & \\
g_{n+k+1}(t_i) & \ldots & g_{n+k+1}(t_{i-1}) & g_{n+k+1}(t_{i+1}) & \ldots & g_{n+k+1}(t_{n+k+2})
\end{vmatrix}.
\]

From Theorem 2.10 it follows that, for each \(i \in \{1, \ldots, n + k + 1\}\),

\[
DD_i = \epsilon(-1)^{n+k+2} g_i(t_{n+k+2}) \geq 0, \ \text{i.e.,} \ \epsilon(-1)^{n+k+2} g_i(t_{n+k+2}) \leq 0.
\]

From this it follows that

\[
0 \leq \epsilon(-1)^{n+k+2} g(t_{n+k+2}) = a_1 \epsilon(-1)^{n+k+2} g_1(t_{n+k+2}) + \cdots + a_{n+k+1} \epsilon(-1)^{n+k+2} g_{n+k+1}(t_{n+k+2}) \leq 0.
\]

Then, since \(a_1, \ldots, a_{n+k+1} > 0\), for each \(i \in \{1, \ldots, n + k + 1\}\) with \(a_i \neq 0\), we have \(g_i(t_{n+k+2}) = 0\).
But this shows that, for such an index \( i \), we have \( g_i(t_j) = 0 \), where \( j \in \{1, \ldots, n + k + 2\} \) and \( j \neq i \), i.e., \( D_i = 0 \). Hence

\[
\delta \left( \frac{0, \ldots, 0, x_1, \ldots, x_k}{t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{n+k+2}} \right) = 0
\]

which, using Theorem 2.10, contradicts condition (1).

3. STRONG UNICITY OF BEST APPROXIMATIONS

In this section we apply the unicity results of Section 2 to obtain statements on strong unicity and show that strongly unique elements of best approximation (see Definition 3.1) can be characterized by the strong Kolmogorov-criterion, if the set \( G \) is a finite-dimensional subspace in an arbitrary normed linear space.

3.1. Definition. Let \( G \) be a set in a normed linear space.

1. An element \( g_0 \in G \) is said to be a **strongly unique element of best approximation** of an element \( f \in E \) if there exists a number \( K > 0 \) such that for each \( g \in G \)

\[
\|f - g\| \geq \|f - g_0\| + K\|g - g_0\|.
\]

2. \( G \) is said to be a **strongly Chebychev set** if each \( f \in E \) has a strongly unique element of best approximation in \( G \).

It is easy to see that, in this case, \( P_G(f) = \{g_0\} \).

The following lemma proves sufficiency of the strong Kolmogorov-criterion.

3.2. Lemma. Let \( E \) be a normed linear space, \( G \) a nonempty set in \( E \), \( f \in E \setminus G \) and \( g_0 \in G \). If \( (g_0, f) \) satisfies the strong Kolmogorov-criterion then \( g_0 \) is strongly unique element of best approximation of \( f \).

**Proof.** According to our assumption, for each \( g \in G \), there exists a functional \( L_g \in E_{f-g_0} \) with \( \Re L_g(g - g_0) \leq -K\|g - g_0\| \). Then for each \( g \in G \),

\[
\|f - g\| \geq \|L_g(f - g)\| \geq \Re L_g(f - g) + \Re L_g(g - g_0) + K\|g - g_0\| = \Re L_g(f - g_0) + K\|g - g_0\| = \|f - g_0\| + K\|g - g_0\|.
\]

Thus \( g_0 \) is a strongly unique element of best approximation of \( f \).

3.3. Remark. Let \( E \) be a normed linear space, \( G \) a nonempty set in \( E \), \( f \in E \setminus G \) and \( g_0 \in G \). It is easy to verify that if

\[
G(g_0) := \left\{ \frac{g - g_0}{\|g - g_0\|} : g \in G, g \neq g_0 \right\}
\]
is compact and \((g_0, f)\) satisfies the strict Kolmogorov-criterion, then \((g_0, f)\) satisfies the strong Kolmogorov-criterion. Examples can be easily constructed to show that in general \(G(g_0)\) is not compact, even if \(E\) is finite-dimensional and if \(G\) is compact and convex. However, if \(G\) is a finite-dimensional subspace or a set with span \(G\) is one-dimensional, then \(G(g_0)\) is compact for each \(g_0 \in G\), and if \(G\) is a finite-dimensional convex cone, then \(G(0)\) is also compact.

Using Theorem 2.4, Theorem 2.8 and Theorem 2.11 we immediately obtain the following results on strong unicity:

3.4. Corollary. Let \(G\) be a nonempty set in \(E = I_A\) (respectively in \(E = L_1(T, m)\)).

(1) If \(G\) is a finite-dimensional Chebyshev subspace of \(E\) then \(G\) is a strongly Chebyshev subspace.

(2) If \(G\) is a one-dimensional convex Chebyshev set in \(E\) then \(G\) is strongly Chebyshev.

(3) If \(G\) is a finite-dimensional semi-Chebyshev convex cone of \(E\) then for each \(f \in E\) with \(0 \in P(f)\) the element \(0\) is a strongly unique element of best approximation of \(f\).

Statement (1) in Corollary 3.4 has been proved by Newman, Shapiro [12] for \(E = I_A = C(X)\), by Ault, Deutsch, Morris, Olson [1] for \(E = l_{(\infty)} = C_0(T)\) and by Wulbert [17] for \(E = L_1(T, m)\).

3.5. Corollary. In Theorem 2.11 the element \(g_0\) is a strongly unique element of best approximation of \(f\).

Schumaker [14] has shown that in Theorem 2.11 the element \(g_0\) is the unique element of best approximation of \(f\).

Now we will show that strongly unique elements of best approximation can be characterized by the strong Kolmogorov-criterion. For this we need the following characterization of strongly unique elements of best approximation due to Wulbert [17] for real normed linear spaces and due to Bartelt, McLaughlin [2] for complex normed linear spaces:

3.6. Theorem. Let \(G\) be a linear subspace of a normed linear space \(E\). An element \(g_0 \in G\) is a strongly unique element of best approximation of an element \(f \in E \setminus G\) if and only if there exists a number \(K > 0\) such that, for each \(g \in G\),

\[
\min \{ \Re L(g) : L \in S_{f-g_0} \} \leq -K \| g \| .
\]

Using standard arguments (see Köthe [11] and Brosowski [3], Lemma 2) from Theorem 3.6 we obtain the following
3.7. **Corollary.** Let $G$ be a linear subspace of a normed linear space $E$. An element $g_0 \in G$ is a strongly unique element of best approximation of an element $f \in E \setminus G$ if and only if $(g_0, f)$ satisfies the strong Kolmogorov-criterion.

Corollary 3.7 has been proved by Bartelt, McLaughlin [2] for finite-dimensional subspaces of $C(X)$, where the functions in $C(X)$ are complex-valued.

3.8. **Remark.** Now we can see that it is not possible to characterize finite-dimensional Chebyshev subspaces in an arbitrary normed linear space by the strict Kolmogorov-criterion. Because would this be true then from Remark 3.3 and Theorem 3.8 it would follow that each finite-dimensional Chebyshev subspace in an arbitrary normed linear space is strongly Chebyshev. Wulbert [17], however, has shown that in a smooth normed linear space no Chebyshev subspace is strongly Chebyshev.

3.9. **Definition.** For a nonempty set $G$ in a normed linear space $E$ the metric projection $P : E \to 2^G$ is called **pointwise Lipschitzian** at $f_0 \in E$, if $P(f_0) = \{g_{f_0}\}$ and if there exists a number $K > 0$ such that for each $f \in E$ and each $g_f \in P(f)$

$$\|g_{f_0} - g_f\| \leq K \|f_0 - f\|.$$

We say $P : E \to G$ is **pointwise Lipschitzian** if $P$ is pointwise Lipschitzian at each $f_0 \in F$.

The following lemma, which is due to Cheney [4, p. 82], shows that pointwise Lipschitzian continuity of the metric projection follows from strong unicity properties:

3.10. **Lemma.** Let $G$ be a set in a normed linear space $E$. If $g_0 \in E$ is a strongly unique element of best approximation of an element $f_0 \in E$ then the metric projection $P : E \to 2^G$ is pointwise Lipschitzian at $f_0 \in E$.

Using Lemma 3.10 we immediately get from Corollary 3.4 and Corollary 3.5 the following statements on pointwise Lipschitzian metric projections:

3.11 **Corollary.** Let $G$ be a nonempty set in $E = L_1$ (respectively in $E = L_2(T, m)$).

1. If $G$ is a finite-dimensional Chebyshev subspace of $E$ then the metric projection $P : E \to G$ is pointwise Lipschitzian.
2. If $G$ is a one-dimensional convex Chebyshev set in $E$. Then the metric projection $P : E \to G$ is pointwise Lipschitzian.
3. If $G$ is a finite-dimensional convex semi-Chebyshev cone of $E$ then for each $f_0 \in E$ with $0 \in P(f_0)$ the metric projection $P : E \to G$ is pointwise Lipschitzian at $f_0$. 


A direct proof of statement (1) in Corollary 3.11 has been given by Freud [8] for $E = C[a, b]$.

3.12. COROLLARY. In Theorem 2.11 the metric projection $P: E \rightarrow 2^g$ is pointwise Lipschitzian at $f$.

Corollary 3.12 has been proved by Schumaker [14].

REFERENCES


