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Biplanes (79, 13, 2) with Involutory Automorphism

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We show that each (79, 13, 2) biplane admitting an involutory automorphism is isomorphic to one of the two designs constructed by Aschbacher. © 1992 Academic Press, Inc.

1. Introduction and Preliminary Results

This paper is a contribution to the investigation of possible automorphism groups of biplanes with parameters (79, 13, 2). This is the largest set of parameters for which a biplane is known to exist. In [1], Aschbacher constructs two such designs with automorphism group of order $2 \cdot 5 \cdot 11$ which are dual. It is an open question whether these designs are, up to isomorphism, the only biplanes with these parameters.

If G is a group of automorphism of a (79, 13, 2) biplane $\mathscr{D}$ and p is a prime divisor of $|G|$, then $p \in \{2, 3, 5, 11, 13\}$ (see [1]). Further, various authors have shown that if $p > 3$ then $\mathscr{D}$ is an Aschbacher biplane. See [5] for the case $p = 5$. The case $p = 11$ is handled by V. Ćepulić and M. Essert in a paper which has not yet appeared; see also [6]. Finally, the case $p = 13$ is eliminated in [6]. Thus it remains to investigate the cases $p = 2$ and 3.

In this paper we consider the case where $\mathscr{D}$ admits an involutory automorphism. We show that each such design is an Aschbacher design. The approach is quite similar to that of [4], but is much more difficult because of a far larger number of possible orbital structures. Therefore the computing time is increased by a factor of 1200.

Our algorithm consists of two steps. The fundamental idea goes back to Janko and van Trung ([3]). In some aspects we follow the presentation and notation of Ćepulić ([2]). We build at first all possible orbital structures $\mathscr{S}$ of $\mathscr{D}$, and after that the biplanes $\mathscr{D}$ themselves by "indexing" the "big points" of $\mathscr{S}$. So we begin by recalling some basic definitions and facts related to the first step.

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2. Basic Notions Concerning Tactical Decompositions

Let $\mathscr{D} = (\mathscr{P}, \mathscr{B}, I)$ be an incidence structure with point set $\mathscr{P}$, line set $\mathscr{B}$, and incidence relation $I \subseteq \mathscr{P} \times \mathscr{B}$. For $P \in \mathscr{P}$, $x \in \mathscr{B}$ denote

$$\langle P \rangle = \{ y \in \mathscr{B} \mid (P, y) \in I \}, \qquad |P| = |\langle P \rangle|,$$
$$\langle x \rangle = \{ Q \in \mathscr{P} \mid (Q, x) \in I \}, \qquad |x| = |\langle x \rangle|.$$

Definition 1. A *symmetric* (v, k, λ)*-block design*, $v, k, \lambda \in N$, is an incidence structure $\mathscr{D} = (\mathscr{P}, \mathscr{B}, I)$ such that:

(i) $|\mathscr{P}| = |\mathscr{B}| = v = k(k-1)/\lambda + 1$

(ii) $|x| = |P| = k$, for all $x \in \mathscr{B}$, $P \in \mathscr{P}$

(iii) $|\langle x \rangle \cap \langle y \rangle| = |\langle P \rangle \cap \langle Q \rangle| = \lambda$, for all $x, y \in \mathscr{B}$, $x \neq y$, $P, Q \in \mathscr{P}$, $P \neq Q$.

The conditions (iii) we call the *consistence conditions.*

For two symmetric designs $\mathscr{D}_1 = (\mathscr{P}_1, \mathscr{B}_1, I_1)$ and $\mathscr{D}_2 = (\mathscr{P}_2, \mathscr{B}_2, I_2)$ an *isomorphism* from $\mathscr{D}_1$ onto $\mathscr{D}_2$ is a bijection which maps points onto points and lines onto lines preserving the incidence. An isomorphism from $\mathscr{D}$ onto $\mathscr{D}$ is an *automorphism* of $\mathscr{D}$. Similarly, *dual isomorphisms* and *dual automorphisms* are such bijections which map points onto lines and lines onto points and preserve the incidences. In the following we shall use the term *design* for symmetric block designs.

Let $(\mathscr{P}, \mathscr{B}, I)$ be a (v, k, λ)-design and ρ an automorphism of $\mathscr{D}$, that is $\langle \rho \rangle \leqslant \operatorname{Aut} \mathscr{D}$. For $x \in \mathscr{B}$, $P \in \mathscr{P}$ we denote with $x\rho$, $P\rho$ the ρ-images of P and x, and with $x\langle \rho \rangle$, $P\langle \rho \rangle$ the ρ-orbits of x and P, respectively. By a known result the number of point orbits equals the number of line orbits. Denoting this number with t and the corresponding orbits with $\mathscr{B}_i$, $\mathscr{P}_r$, $1 \leqslant i, r \leqslant t$, we have

$$\mathscr{B} = \bigsqcup_{i=1}^{t} \mathscr{B}_i, \qquad \mathscr{P} = \bigsqcup_{r=1}^{t} \mathscr{P}_r, \tag{1}$$

where $\mathscr{B}_i = x_i \langle \rho \rangle$, $\mathscr{P}_r = P_r \langle \rho \rangle$ for some $x_i \in \mathscr{B}$, $P_r \in \mathscr{P}$, $1 \leqslant i, r \leqslant t$. We use the symbol $\bigsqcup$ for the disjoint union of sets.

Denote $|\mathscr{B}_i| = \Omega_i$, $|\mathscr{P}_r| = \omega_r$. From (1) and the Definition 1. (i), it follows immediately that

$$\sum_{i=1}^{t} \Omega_i = \sum_{r=1}^{t} \omega_r = v. \tag{2}$$

LEMMA 1. *Let* $\mathscr{D}=(\mathscr{P},\mathscr{B},I)$ *be a* (v,k,λ)*-design and* $\langle\rho\rangle\leqslant\mathrm{Aut}\,\mathscr{D}$. *Then the point orbits* $\mathscr{P}_1,\mathscr{P}_2,\ldots,\mathscr{P}_t$, *and the line orbits* $\mathscr{B}_1,\mathscr{B}_2,\ldots,\mathscr{B}_t$, *are the point classes and block classes of a tactical decomposition of* $\mathscr{D}$ *(see, e.g.,* [3]).

Proof. Let $x\in\mathscr{B}_i$, $P\in\mathscr{P}_r$. Then $|\langle x\rangle\cap\mathscr{P}_r|=|\langle x\rangle g\cap\mathscr{P}_r g|=|\langle xg\rangle\cap\mathscr{P}_r|$ for all $g\in\langle\rho\rangle$. As $x\langle\rho\rangle=\mathscr{B}_i$, we see that $|\langle x\rangle\cap\mathscr{P}_r|=\mu_{ir}$ depends on $\mathscr{B}_i$ and $\mathscr{P}_r$ only. Similarly, $|\langle P\rangle\cap\mathscr{B}_i|=\Gamma_{ir}$ does not depend on the choice of P. Each line of $\mathscr{B}_i$ contains exactly μ_{ir} points from $\mathscr{P}_r$, and each point of $\mathscr{P}_r$ lies in exactly Γ_{ir} lines of $\mathscr{B}_i$. Thus, a partition of $\mathscr{P}$ into t point orbits and a partition of $\mathscr{B}$ into t line orbits give a tactical decomposition of $\mathscr{D}$, with the corresponding t by t "*multiplicity matrices*" $\mathscr{S}=[\mu_{ir}]$ and $\mathscr{O}=[\Gamma_{ir}]$, the remaining parameters of decomposition being $|\mathscr{B}_i|=\Omega_i$, $|\mathscr{P}_r|=\omega_r$, $1\leqslant i,\ r\leqslant t$.

In the following we state some important relations among the parameters of our tactical decomposition. By counting the incidences of lines in $\mathscr{B}_i$ and points in $\mathscr{P}_r$ in two ways, we obtain

$$\Omega_i\mu_{ir}=\omega_r\Gamma_{ir},\qquad 1\leqslant i,\ r\leqslant t. \tag{3}$$

From Definition 1 (ii) it follows that

$$\begin{aligned}&\sum_{r=1}^{t}\mu_{ir}=k && 1\leqslant i\leqslant t,\\ &\sum_{i=1}^{t}\Gamma_{ir}=\sum_{i=1}^{t}\frac{\Omega_i}{\omega_r}\mu_{ir}=k, && 1\leqslant r\leqslant t.\end{aligned} \tag{4}$$

Let $P\in\mathscr{P}$ and $\mathscr{P}_r=P\langle\rho\rangle$. Denote the points of $\mathscr{P}_r$ with $P_r,P_r\rho,\ldots,P_r\rho^{|\rho|-1}$ or, abbreviated in a customary manner, as $r_0,r_1,\ldots,r_{|\rho|-1}$. In this context one often speaks about r as about "*big point*" (also: *orbital number*), which is supplied with indices. Now, for each orbit $\mathscr{P}_r=\{r_0,r_1,\ldots,r_{|\rho|-1}\}$ the automorphism group $\langle\rho\rangle$ is represented as a permutation group on the indices $0,1,\ldots,|\rho|-1$. The same holds for the line orbits.

From the rows of a multiplicity matrice $[\mu_{ir}]$, denoted with $[\mu_{ir}]_i$, we derive so-called *orbital lines*, denoted with $\hat{x}_i$, as the multisets $\langle\hat{x}_i\rangle$ consisted of big points: inside a multiset $\langle\hat{x}_i\rangle$ a big point r occurs μ_{ir} times. So we call μ_{ir} the *multiplicity* of the big point r inside the orbital line $\hat{x}_i$. The sets containing all big points and orbital lines we denote with $\hat{\mathscr{P}}$ and $\hat{\mathscr{B}}$, respectively.

LEMMA 2. *Let $\mathscr{D}=(\mathscr{P}, \mathscr{B}, I)$ be a (v, k, λ)-design, and $\langle\rho\rangle \leqslant \operatorname{Aut} \mathscr{D}$. We assume other notation to be as stated above. It holds*:

$$\begin{aligned} \sum_{r=1}^{t} \mu_{ir} \Gamma_{jr} &= \lambda \Omega_j + \delta_{ij}(k-\lambda), \\ \sum_{i=1}^{t} \Gamma_{ir} \mu_{is} &= \lambda \omega_s + \delta_{rs}(k-\lambda). \end{aligned} \qquad (*)$$

for all $1 \leqslant i, j \leqslant t$, $1 \leqslant r, s \leqslant t$, δ_{ij}, δ_{rs} *being the correspondent Kronecker symbols.*

Proof. See, e.g., [2].

DEFINITION 2. We denote

$$\begin{aligned} [\hat{x}_i, \hat{x}_j] &\equiv \sum_{r=1}^{t} \mu_{ir} \Gamma_{jr} = \sum_{r=1}^{t} \frac{\Omega_j}{\omega_r} \mu_{ir} \mu_{jr}, \qquad 1 \leqslant i, j \leqslant t, \\ [\hat{T}_r, \hat{T}_s] &\equiv \sum_{i=1}^{t} \Gamma_{ir} \mu_{is} = \sum_{i=1}^{t} \frac{\Omega_i}{\omega_r} \mu_{ir} \mu_{is}, \qquad 1 \leqslant r, s \leqslant t, \end{aligned} \qquad (**)$$

and call these expressions the *game products*.

3. BASIC PROPERTIES OF (79, 13, 2)-ORBITAL STRUCTURES

Let $\mathscr{D}=(\mathscr{P}, \mathscr{B}, I)$ be a (79, 13, 2)-biplane, and let ρ be an involutory automorphism acting on $\mathscr{D}$. By the fundamental result about involution acting on biplanes obtained by M. Aschbacher in [1], ρ fixes exactly one point on any fixed line. So ρ can operate on $\mathscr{D}$ with exactly seven fixed points (and, hence, with exactly seven fixed lines) and 36 orbits of length 2. We have in this case $t=43$, $\Omega_i=\omega_r=1$ for $1 \leqslant i$, $r \leqslant 7$; $\Omega_i=\omega_r=2$ for $8 \leqslant i$, $r \leqslant 43$. We denote, in the usual way, fixed points and lines by ∞_r and $(p_\infty)_i$, $r, i=1,2, \ldots, 7$. Thus, if $\infty_1, \ldots, \infty_7, \mathscr{P}_8, \ldots, \mathscr{P}_{43}$ and $(p_\infty)_1, \ldots, (p_\infty)_7$, $\mathscr{B}_8, \ldots, \mathscr{B}_{43}$ are the $\langle\rho\rangle$-orbits of points and lines in a defined order, we obtain the corresponding big point set $\hat{\mathscr{P}}$ and orbital line set $\hat{\mathscr{B}}$:

$$\begin{aligned} \hat{\mathscr{P}} &= \{\infty_1, \ldots, \infty_7, 8, \ldots, 43\}, \\ \hat{\mathscr{B}} &= \{(p_\infty)_1, \ldots, (p_\infty)_7, \hat{x}_8, \ldots, \hat{x}_{43}\}. \end{aligned}$$

For ρ we can write

$$\rho=(\infty_1) \cdots (\infty_7)(8_0 8_1) \cdots (43_0 43_1)$$

DEFINITION 3. Let Ω_i, ω_r, t be as stated above. Let any t by t matrix $\mathscr{S} = [\mu_{ir}]$ satisfying the conditions (3), (4), and (*) be called the (79, 13, 2)-*orbital structure* for lines with respect to $\langle \rho \rangle$, for a potential (79, 13, 2)-design $\mathscr{D}$.

For two orbital structures $\mathscr{S}_1 = [\mu'_{ir}]$ and $\mathscr{S}_2 = [\mu''_{ir}]$ with big point sets $\hat{\mathscr{P}}_1$ and $\hat{\mathscr{P}}_2$ and orbital line sets $\hat{\mathscr{B}}_1$ and $\hat{\mathscr{B}}_2$, respectively, an *isomorphism* from $\mathscr{S}_1$ onto $\mathscr{S}_2$ is a bijection σ which maps big points from $\hat{\mathscr{P}}_1$ onto big points of $\hat{\mathscr{P}}_2$, orbital lines from $\hat{\mathscr{B}}_1$ onto orbital lines of $\hat{\mathscr{B}}_2$, preserving the entries: $\mu''_{\sigma(i)\sigma(r)} = \mu'_{ir}$. If there is an isomorphism from $\mathscr{S}_1$ onto $\mathscr{S}_2$ then we say that $\mathscr{S}_1$ and $\mathscr{S}_2$ are isomorphic and write $\mathscr{S}_1 \cong \mathscr{S}_2$.

Now we try to construct all the orbital structures $\mathscr{S} = [\mu_{ir}]$ of $\mathscr{D}$ with respect to $\langle \rho \rangle$. We use the previously introduced terminology. Let $\mathscr{F}(\rho)$ be the structure consisting of orbital lines $(p_\infty)_1, \ldots, (p_\infty)_7$. Obviously, up to isomorphism, $\mathscr{F}(\rho)$ is uniquelly determined as shown in (Table I).

Denote $\hat{\mathscr{P}}_\infty = \{\infty_1, \infty_2, \ldots, \infty_7\}$, $\hat{\mathscr{P}}_1 = \{8, 9, \ldots, 28\}$, $\hat{\mathscr{P}}_2 = \{29, 30, \ldots, 43\}$. We observe that

$$\text{Aut}\, \mathscr{F}(\rho) = G_1 \times G_2 \cong S_{15} \times S_7,$$

where $G_1 = \sum_{\hat{\mathscr{P}}_2} \cong S_{15}$ is the symmetric group on $\hat{\mathscr{P}}_2$, and $G_2 = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6 \rangle \cong S_7$ is the subgroup of the symmetric group $\sum_{\hat{\mathscr{P}}_\infty \sqcup \hat{\mathscr{P}}_1}$, with generators:

$$\sigma_1 = (\infty_1\ \infty_2)(9\ 14)(10\ 15)(11\ 16)(12\ 17)(13\ 18),$$

$$\sigma_2 = (\infty_1\ \infty_3)(8\ 14)(10\ 19)(11\ 20)(12\ 21)(13\ 22),$$

$$\sigma_3 = (\infty_1\ \infty_4)(8\ 15)(9\ 19)(11\ 23)(12\ 24)(13\ 25),$$

$$\sigma_4 = (\infty_1\ \infty_5)(8\ 16)(9\ 20)(10\ 23)(12\ 26)(13\ 27),$$

$$\sigma_5 = (\infty_1\ \infty_6)(8\ 17)(9\ 21)(10\ 24)(11\ 26)(13\ 28),$$

$$\sigma_6 = (\infty_1\ \infty_7)(8\ 18)(9\ 22)(10\ 25)(11\ 27)(12\ 28).$$

TABLE I

The Structure $\mathscr{F}(\rho)$

level													
1	∞_1	8	8	9	9	10	10	11	11	12	12	13	13
2	∞_2	8	8	14	14	15	15	16	16	17	17	18	18
3	∞_3	9	9	14	14	19	19	20	20	21	21	22	22
4	∞_4	10	10	15	15	19	19	23	23	24	24	25	25
5	∞_5	11	11	16	16	20	20	23	23	26	26	27	27
6	∞_6	12	12	17	17	21	21	24	24	26	26	28	28
7	∞_7	13	13	18	18	22	22	25	25	27	27	28	28

Using (**) and (*) we have for $i>7$:

$$[\hat{x}_i, \hat{x}_i] = \sum_{r=1}^{43} \frac{\Omega_i}{\omega_r} \mu_{ir}^2 = \sum_{r=1}^{7} 2 \cdot \mu_{ir}^2 + \sum_{r=8}^{43} 1 \cdot \mu_{ir}^2 = 15. \tag{5}$$

Combined with $\sum_{r=1}^{43} \mu_{ir} = 13$, from (5) we obtain two types of nontrivial orbital lines:

(a) lines with two fixed points and 11 multiplicities equal 1 among nonfixed points, and

(b) lines without fixed points, with one multiplicity equal 2 and with 11 multiplicities equal 1, among nonfixed points.

Analogously, we apply: $[\hat{T}_r, \hat{T}_r] = 2\omega_r + 11$ for $1 \leqslant r \leqslant 43$. Taking into account that $\sum_{i=1}^{7} (\Omega_i/\omega_r)\, \mu_{ir}^2$ is already determined by $\mathscr{F}(\rho)$ we obtain

$$\sum_{i=8}^{43} \mu_{ir}^2 = \begin{cases} 6, & 1 \leqslant r \leqslant 7, \\ 11, & 8 \leqslant r \leqslant 28, \\ 15, & 29 \leqslant r \leqslant 43. \end{cases} \tag{6}$$

We also count $\sum_{i=1}^{43} \Gamma_{ir} = \sum_{i=1}^{7} (\Omega_i/\omega_r)\, \mu_{ir} + \sum_{i=8}^{43} (\Omega_i/\omega_r)\, \mu_{ir} = 13$, thus obtaining

$$\sum_{i=8}^{43} \mu_{ir} = \begin{cases} 6, & 1 \leqslant r \leqslant 7, \\ 11, & 8 \leqslant r \leqslant 28, \\ 13, & 29 \leqslant r \leqslant 43. \end{cases} \tag{7}$$

From (6) and (7) we conclude for the nontrivial part of $[\mu_{ir}]$, consisting of the rows $[\mu_{ir}]_i$ with $i>7$: the first seven columns have six units, the columns $8 \leqslant r \leqslant 28$ have 11 units, and the columns $29 \leqslant r \leqslant 43$ have one entry equal to two and 11 units, the remaining entries being zero in all the considered cases.

All the above conclusions imply that inside an orbital structure $\mathscr{S}$ there are 21 orbital lines of type (a) and 15 orbital lines of type (b). Each type (a) orbital line contains a pair of fixed points $\infty_r \infty_s$, $r, s \in \{1, 2, ..., 7\}$, $r<s$. Note that a pair $\infty_r \infty_s$ cannot appear twice inside $\mathscr{S}$, since the game product $[\hat{T}_r, \hat{T}_s]$ would exceed $\lambda\omega_s = 2$. In type (b) orbital lines big points 29, 30, ..., 43 appear with multiplicity 2. If we denote the set containing type (a) orbital lines with $\hat{\mathscr{B}}_1$, and the set containing type (b) orbital lines with $\hat{\mathscr{B}}_2$, we can set

$$\hat{\mathscr{P}} = \hat{\mathscr{P}}_\infty \sqcup \hat{\mathscr{P}}_1 \sqcup \hat{\mathscr{P}}_2, \qquad \hat{\mathscr{B}} = \mathscr{F}(\rho) \sqcup \hat{\mathscr{B}}_1 \sqcup \hat{\mathscr{B}}_2.$$

4. Construction of (79, 13, 2)-Orbital Structures

In the following we shall assume for orbital structures $\mathscr{S}$ to be written as sets or orbital lines $\hat{x}_i$ represented as sequences of their $k = 13$ big points from $\langle \hat{x}_i \rangle$. Without loss of generality we assume that the first seven levels of $\mathscr{S}$ coincide with $\mathscr{F}(\rho)$. Next, we introduce canonical form of $\mathscr{S}$.

Definition 4. Let $\hat{x} \in \hat{\mathscr{B}}_1 \sqcup \hat{\mathscr{B}}_2$. Then there is a unique sequence $\tilde{x}$ of length k consisted of big points from $\langle \hat{x} \rangle$, such that

$$\tilde{x}(1) = \infty_r, \quad \tilde{x}(2) = \infty_s, \quad r < s, \qquad \text{for} \quad \hat{x} \in \hat{\mathscr{B}}_1,$$

$$\tilde{x}(1) = \tilde{x}(2), \qquad \text{for} \quad \hat{x} \in \hat{\mathscr{B}}_2,$$

and big point sequence $\tilde{x}(3), \tilde{x}(4), ..., \tilde{x}(13)$ is ordered lexicographically. The sequence $\tilde{x}$ will be called the *canonical form of* $\hat{x}$.

Obviously, each canonical line $\hat{x}$ is uniquely determined within an orbital structure by its beginning pair $(\tilde{x}(1), \tilde{x}(2))$. So we can establish an correspondence $l : \hat{x} \to l(\hat{x})$ between the orbital lines from the set $\hat{\mathscr{B}}_1 \sqcup \hat{\mathscr{B}}_2$ and their ordinal numbers $8, 9, ..., 43$: if a sequence $\tilde{x}(1), \tilde{x}(2)$ corresponding to $\hat{x} \in \hat{\mathscr{B}}_1 \sqcup \hat{\mathscr{B}}_2$ precedes lexicographically a sequence $\tilde{y}(1), \tilde{y}(2)$ corresponding to $\hat{y} \in \hat{\mathscr{B}}_1 \sqcup \hat{\mathscr{B}}_2$, we set $l(\hat{x}) < l(\hat{y})$. The number $l(\hat{x})$ will be called the *orbital level* of $\hat{x}$. A line $\hat{x} \in \hat{\mathscr{B}}_1 \sqcup \hat{\mathscr{B}}_2$ with the orbital level l we denote with $\hat{x}_l$.

In the following we shall identify $\tilde{x}$ with $\hat{x}$.

Definition 5. Let $\mathscr{S}$ be an orbital structure of $\mathscr{D}$. Then there is a unique sequence $\tilde{\mathscr{S}}$ of the length $t - 7 = 36$ consisted of canonical lines $\hat{x}_r \in \hat{\mathscr{B}}_1 \sqcup \hat{\mathscr{B}}_2$, such that the corresponding sequence consisted of orbital levels of $\hat{x}_r$ is ordered lexicographically. The sequence $\tilde{\mathscr{S}}$ will be called the *canonical form of* $\mathscr{S}$.

In our further explanation we deal with canonical structures $\tilde{\mathscr{S}}$ only, and identify $\tilde{\mathscr{S}}$ with $\mathscr{S}$.

Definition 6. Let $\hat{x}, \hat{y} \in \hat{\mathscr{B}}_1 \sqcup \hat{\mathscr{B}}_2$ be two canonical lines corresponding to a same orbital level, i.e., $l(\hat{x}) = l(\hat{y})$. Then $\hat{x}$ *precedes* $\hat{y}$, $\hat{x} \preccurlyeq \hat{y}$, if $\tilde{x}$ precedes $\tilde{y}$ lexicographically. As usual $\hat{x} \prec \hat{y}$ will stand for $\hat{x} \preccurlyeq \hat{y}$ and $\hat{x} \neq \hat{y}$.

Definition 7. Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be orbital structures and $\tilde{\mathscr{S}}_1$ and $\tilde{\mathscr{S}}_2$ their canonical forms. We define that $\mathscr{S}_1$ *precedes* $\mathscr{S}_2$, $\mathscr{S}_1 \preccurlyeq \mathscr{S}_2$, if $\tilde{\mathscr{S}}_1$ precedes $\tilde{\mathscr{S}}_2$

in terms of the canonical precedence of their orbital lines. As usual $\mathscr{S}_1 \prec \mathscr{S}_2$ will stand for $\mathscr{S}_1 \preccurlyeq \mathscr{S}_2$ and $\mathscr{S}_1 \neq \mathscr{S}_2$.

LEMMA 3. *An orbital structure* $\mathscr{S} = [\mu_{ir}]$ *of* $\mathscr{D}$ *can be represented by block matrices* $[\mu_{ir}] = [N_{mn}]$, $m, n \in \{1, 2, 3\}$, *where*

N_{11} *is the* 7 *by* 7 *identity matrix* I_7,

N_{12} *is the* 7 *by* 21 *matrice*:

$$N_{12} = \begin{bmatrix} 2&2&2&2&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 2&0&0&0&0&0&2&2&2&2&2&0&0&0&0&0&0&0&0&0&0 \\ 0&2&0&0&0&0&2&0&0&0&0&2&2&2&2&0&0&0&0&0&0 \\ 0&0&2&0&0&0&0&2&0&0&0&2&0&0&0&2&2&2&0&0&0 \\ 0&0&0&2&0&0&0&0&2&0&0&0&2&0&0&2&0&0&2&2&0 \\ 0&0&0&0&2&0&0&0&0&2&0&0&0&2&0&0&2&0&2&0&2 \\ 0&0&0&0&0&2&0&0&0&0&2&0&0&0&2&0&0&2&0&2&2 \end{bmatrix}$$

N_{13} *is the* 7 *by* 15 *zero matrix*,

$N_{21} = \frac{1}{2} N_{12}^T$,

N_{22} *is a* 21 *by* 21 *incidence matrix with exactly six units in each row and exactly six units in each column*,

N_{23} *is a* 21 *by* 15 *incidence matrix with exactly five units in each row and exactly seven units in each column*,

$N_{31} = N_{13}^T$,

N_{32} *is a* 15 *by* 21 *incidence matrix with exactly seven units in each row and exactly five units in each column*,

$N_{33} = 2I_{15} + N_{33}^*$, *where*

N_{33}^* *is a* 15 *by* 15 *incidence matrix with exactly four units in each row and exactly four units in each column.*

Proof. From the game products $[\hat{x}_i, \hat{x}_j]$ and $[\hat{T}_r, \hat{T}_s]$, defined by (**) and (*), by a some rearrangement one obtains

$$\sum_{r=8}^{43} \mu_{ir}\mu_{jr} = 2(2-f), \qquad 1 \leqslant i, j \leqslant 48, \quad i \neq j, \tag{8}$$

$$\sum_{i=8}^{43} \mu_{ir}\mu_{is} = \omega_r\omega_s - \frac{1}{2} \sum_{i=1}^{7} \mu_{ir}\mu_{is}, \qquad i \leqslant r, s \leqslant 48, \quad r \neq s, \tag{9}$$

where $f \in \{0, 1\}$ is the number of common fixed points from the orbital lines $\hat{x}_i, \hat{x}_j$.

Applying (8) with fixed $i \in \{8, 9, ..., 28\}$, and changing $j \in \{1, 2, ..., 7\}$, one obtains, in a similar manner as in [4], a system consisting of seven equations. Adding up these equations and dividing by 4 we obtain

$$\sum_{j=8}^{28} \mu_{ij} = 6, \qquad 8 \leqslant i \leqslant 28.$$

Thus, the block matrix N_{22} has six units in each row. Analogously, applying (9) with fixed $s \in \{8, 9, ..., 28\}$ and changing $r \in \{1, 2, ..., 7\}$ one obtains: $\sum_{r=8}^{28} \mu_{rs} = 6$, for $8 \leqslant s \leqslant 28$, and thus N_{22} has six units in each column. The rest of the proof is analogous.

We also observe that the application of (9) with $r = 1$, $s \in \{29, 30, ..., 43\}$ gives the equations:

$$\sum_{i=8}^{13} \mu_{is} = 2, \qquad 29 \leqslant s \leqslant 43. \tag{10}$$

Thus, by selecting the rows $i = 8, 9, ..., 13$ and the columns $r = 29, 30, ..., 43$ of $[\mu_{ir}]$ one produces a 6 by 15 incidence matrix with five units in each row and two units in each column. We denote this matrice by $M = [\nu_{ir}]$.

Denote $\hat{x}_l^{(\infty)}$, $\hat{x}_l^{(1)}$, and $\hat{x}_l^{(2)}$ the "*sublines*" of an orbital line $\hat{x}_l$, defined by:

$$\langle \hat{x}_l^{(\infty)} \rangle = \langle \hat{x}_l \rangle \cap \hat{\mathscr{P}}_{\infty}, \qquad \langle \hat{x}_l^{(1)} \rangle = \langle \hat{x}_l \rangle \cap \hat{\mathscr{P}}_1, \qquad \langle \hat{x}_l^{(2)} \rangle = \langle \hat{x}_l \rangle \cap \hat{\mathscr{P}}_2. \tag{11}$$

Obviously, we have

$$\langle \hat{x}_i \rangle = \langle \hat{x}_i^{(\infty)} \rangle \sqcup \langle \hat{x}_i^{(1)} \rangle \sqcup \langle \hat{x}_i^{(2)} \rangle. \tag{12}$$

By solving the system of equations considered in the proof of Lemma 3, we actually search for lines $\hat{x}_i$ of the i-th level which are consistent with all the fixed lines $\hat{x}_j$, $1 \leqslant j \leqslant 7$. A decomposition of $\hat{x}_i$ given by (12) implies the fact that possible solutions of $\hat{x}_i$ can be obtained by "combining" the components $\hat{x}_i^{(1)}$ and $\hat{x}_i^{(2)}$. One obtains 167 possible solutions for sublines $\hat{x}_i^{(1)}$ and $\binom{15}{5}$ for sublines $\hat{x}_i^{(2)}$, for all $i \in \{8, 9, ..., 28\}$. So we can compute the number N_1 of possibilities for $\hat{x}_l$ on the level l, for $8 \leqslant l \leqslant 28$:

$$N_1 = 167 \cdot \binom{15}{5} = 501{,}501.$$

In a similar manner we obtain 465 distinct sublines $\hat{x}_l^{(1)}$ for $29 \leqslant l \leqslant 43$. The number N_2 of possibilities for $\hat{x}_l$ on the level l, for $29 \leqslant l \leqslant 43$, equals:

$$N_2 = 465 \cdot \binom{14}{4} = 465{,}465.$$

Now we can sketch an algorithm for constructing all nonisomorphic orbital structures $\mathscr{S}$ in the canonical form.

ALGORITHM—STEP 1. We build the *partial orbital structures*, level by level. A partial scheme of lth level, denoted $S(l)$, is any l by t matrix satisfying the consistence conditions (*) for rows, and not violating the consistence conditions (*) for columns. Let $S^{(l)}$ be the set of all possible partial schemes $S(l)$. In our case the sets $S^{(1)}, S^{(2)}, ..., S^{(7)}$ are trivial. We construct $S^{(l)}$ from $S^{(l-1)}$, $8 \leqslant l \leqslant 43$, by joining to each $S(l-1) \in S^{(l-1)}$ all possible canonical lines $\hat{x}_l$. Let the consistencies among $S(l-1)$ and some $\hat{x}_l$ be satisfied. Then we include $S(l) = S(l-1) \cup \hat{x}_l$ into $S^{(l)}$ if it cannot be eliminated by finding a scheme $S(l)\,\sigma$ isomorphic to $S(l)$, which precedes $S(l)$. We try to reach the elimination by means of automorphisms $\sigma \in \operatorname{Aut} \mathscr{F}(\rho)$. If $S(l)\,\sigma \prec S(l)$ (in terms of the precedence of partial schemes considered as parts of the whole orbital structures $\mathscr{S}$) $S(l)$ is omitted. In this way we ensure the elimination of a lot of isomorphic orbital structures, retaining only those among them which are first in terms of the defined precedence.

Let $\hat{S}(l)$ be the set of lines of a scheme $S(l)$, i.e., $\hat{S}(l) = \{\hat{x}_i \mid i = 8, 9, ..., l\}$. Then a decomposition of $\hat{x}_i$ given by (12) enables us to consider the subline sets: $\hat{S}_1(l) = \{\hat{x}_i^{(1)} \mid i = 8, 9, ..., l\}$ and $\hat{S}_2(l) = \{\hat{x}_i^{(2)} \mid i = 8, 9, ..., l\}$. We denote the corresponding substructures by $S_1(l)$ and $S_2(l)$, respectively.

(I) *Generating new schemes* $S(l)$ *for* $8 \leqslant l \leqslant 13$. For a chosen subline $\hat{x}_l^{(1)}$ we generate one by one $\binom{15}{5}$ the lexicographically ordered sublines $\hat{x}_l^{(2)}$. When a first subline $\hat{x}_l^{(2)}$ appears such that the whole line $\hat{x}_l$ ($\langle \hat{x}_l \rangle = \langle \hat{x}_l^{(\infty)} \rangle \sqcup \langle \hat{x}_l^{(1)} \rangle \sqcup \langle \hat{x}_l^{(2)} \rangle$) is consistent with the corresponding scheme $S(l-1)$, a new scheme $S(l)$ is produced. All forthcoming consistent schemes $\bar{S}(l)$ with the same $\hat{x}_l^{(1)}$ are isomorphic to $S(l)$, since Eq. (10) and Lemma 3 of [4] imply the existence of such an automorphism $\delta \in G_1$ which maps the scheme $\bar{S}_2(l)$ onto $S_2(l)$. Therefore we interrupt generating the remaining sublines $\hat{x}_l^{(2)}$ and repeat the same procedure with the next subline $\hat{x}_l^{(1)}$, exhausting all 167 possibilities.

(II) *Elimination of isomorphic schemes for* $8 \leqslant l \leqslant 14$. We search for a $\sigma \in G_2$ which gives $S_1(l)\,\sigma \prec S_1(l)$. When such a σ is found, $S(r)$ is omitted, because from Eq. (10) and Lemma 3 of [4] there follows the existence of such an automorphism $\delta \in G_1$ which maps the scheme $S_2(l)\,\sigma$ onto $S_2(l)\,\sigma\delta = S_2(l)$, i.e., $S\sigma\delta \prec S$.

(III) *Generating new schemes for* $l \geqslant 14$. We must examine all 501,501 possibilities for $\hat{x}_l^{(1)}$ on each level $14 \leqslant l \leqslant 28$, and all 465,465 possibilities for $\hat{x}_l^{(1)}$ on each level $29 \leqslant l \leqslant 43$.

(IV) *Elimination of isomorphic schemes for* $14 \leqslant l \leqslant 28$:

(a) We use $G_t \subseteq G_1$: the set of transpositions over $\hat{\mathscr{P}}_2$, $|G_t| = \binom{15}{5} = 105$.

(b) Denote with $S_1^*(14)$ and $S_2^*(14)$ the substructures obtained by selecting from the schemes $S_1(l)$ and $S_1(l)\,\sigma$, respectively, their lines with the orbital levels $\leqslant 14$. By finding a $\sigma \in G_2$ which gives $S_2^*(14) \prec S_1^*(14)$, $S(l)$ is omitted by the same argument as in case (II). We use (IV) (b) only on the level $l = 28$, since a searching for an adequate $\sigma \in G_2$ on the levels $l < 28$ consumes too much computing time.

Applying the algorithm we obtain, with the help of a computer, as the only solutions (up to isomorphism) two orbital structures: $\mathscr{S}_1$ and $\mathscr{S}_2$. This result is reached after 3700 h of continuous computing on a computer Dynatech DCS-1/320. The greatest number of schemes we observe on level 16, where we count approximately 420,000,000 (not necessarily nonisomorphic) schemes. On level 28 this number reduces to six and, after applying (IV) (b), to only two nonisomorphic schemes.

5. Final Results

Let $\mathscr{D} = (\mathscr{P}, \mathscr{B}, I)$ be a (v, k, λ)-design and $\langle \rho \rangle < \mathrm{Aut}\, \mathscr{D}$. We have shown (by Lemmas 1 and 2) that the existence of the orbital structure $\mathscr{S}$ with respect to $\langle \rho \rangle$ is the necessary condition for the existence of $\mathscr{D}$. Applying Step 1 of our algorithm we found all the possible solutions for $\mathscr{S}$. Now we try to construct the designs by "indexing" the big points of $\mathscr{S}$. This problem in the general case need not have a solution, and if a solution exists it need not be unique. We now give a brief description of our algorithm for constructing all possible solutions.

Algorithm—Step 2. Let $\mathscr{S} = [\mu_{jr}]$ be the orbital structure under consideration. For the jth row in $\mathscr{S}$ we construct lines x_j from the line orbit $\mathscr{B}_j$, by supplying orbital numbers of $\hat{x}_j$ with indices $\in \{0, 1, ..., |\rho| - 1\}$. For x_j', x_j'' corresponding to the same $\hat{x}_j$ we define: x_j' precedes x_j'', $x_j' \prec x_j''$, if the sequence of indices of big points corresponding to x_j' precedes that of x_j'' lexicographically. Among the lines of the orbit $\mathscr{B}_j$ we take out as its *representative* the first in terms of the defined precedence, thus obtaining $\tilde{x}_j$—the canonical form of x_j. In the following we identify $\tilde{x}_j$ with x_j and call it the *canonical line.* The set of all jth level canonical lines we denote $x^{(j)}$.

After finding $x^{(j)}$, we build the partial designs. A *partial design of jth level,* denoted Δ_j, is an incidence structure $(\mathscr{P}, \mathscr{B}^{(j)}, I)$ with $|\mathscr{B}^{(j)}| = j$ canonical lines, such that $|\langle x \rangle \cap \langle y \rangle| = \lambda$ for all $x, y \in \mathscr{B}^{(j)}$, $x \neq y$. With $\mathscr{D}^{(j)}$ we denote the set of all jth level partial designs Δ_j which we construct

TABLE II

An Aschbacher's Design Constructed under the Assumption of Involution Acting

Orbital level													
1	∞_1	8_0	8_1	9_0	9_1	10_0	10_1	11_0	11_1	12_0	12_1	13_0	13_1
2	∞_2	8_0	8_1	14_0	14_1	15_0	15_1	16_0	16_1	17_0	17_1	18_0	18_1
3	∞_3	9_0	9_1	14_0	14_1	19_0	19_1	20_0	20_1	21_0	21_1	22_0	22_1
4	∞_4	10_0	10_1	15_0	15_1	19_0	19_1	23_0	23_1	24_0	24_1	25_0	25_1
5	∞_5	11_0	11_1	16_0	16_1	20_0	20_1	23_0	23_1	26_0	26_1	27_0	27_1
6	∞_6	12_0	12_1	17_0	17_1	21_0	21_1	24_0	24_1	26_0	26_1	28_0	28_1
7	∞_7	13_0	13_1	18_0	18_1	22_0	22_1	25_0	25_1	27_0	27_1	28_0	28_1
8	∞_1	∞_2	8_0	19_0	20_0	24_0	27_0	28_0	29_0	30_0	31_0	32_0	33_0
9	∞_1	∞_3	9_0	15_0	17_0	25_0	26_0	27_0	29_1	34_0	35_0	36_0	37_0
10	∞_1	∞_4	13_0	16_0	17_0	21_0	22_0	23_0	30_0	31_1	34_1	38_0	39_0
11	∞_1	∞_5	12_0	14_0	18_0	22_0	23_1	24_0	32_1	35_1	36_0	40_0	41_0
12	∞_1	∞_6	11_0	14_0	15_0	20_0	25_1	28_1	38_1	39_0	40_1	42_0	43_0
13	∞_1	∞_7	10_0	16_0	18_1	19_1	21_1	26_0	33_0	37_1	41_0	42_1	43_0
14	∞_2	∞_3	10_0	11_1	16_1	22_0	24_0	28_1	34_1	35_0	38_1	41_1	42_1
15	∞_2	∞_4	11_0	12_1	15_0	21_1	22_0	27_1	33_0	36_1	37_0	40_0	43_1
16	∞_2	∞_5	10_0	13_0	14_0	21_0	25_0	26_0	29_0	30_1	36_1	39_1	42_0
17	∞_2	∞_6	9_0	13_1	18_0	19_1	23_0	26_0	31_0	34_1	35_1	40_1	43_1
18	∞_2	∞_7	9_0	12_1	17_0	20_0	23_1	25_1	32_1	37_1	38_0	39_1	41_1
19	∞_3	∞_4	8_0	12_1	18_0	20_0	25_0	26_1	30_1	31_1	40_1	41_0	42_1
20	∞_3	∞_5	12_0	13_1	15_0	16_0	19_0	28_1	31_1	32_0	37_1	39_1	43_1
21	∞_3	∞_6	8_0	10_1	18_1	21_0	23_0	27_1	32_1	33_0	36_0	38_1	39_1
22	∞_3	∞_7	11_0	13_1	14_1	17_0	23_0	24_0	29_0	30_1	33_1	40_0	43_0
23	∞_4	∞_5	9_0	10_1	17_1	18_0	20_1	28_1	29_0	33_0	35_0	38_0	43_0
24	∞_4	∞_6	9_0	13_0	14_1	16_1	24_0	27_1	29_1	32_0	37_1	41_0	42_0
25	∞_4	∞_7	8_0	11_1	14_1	19_0	26_0	28_1	32_1	34_0	35_1	36_1	39_0
26	∞_5	∞_6	8_0	11_0	17_1	19_1	22_0	25_0	30_0	33_1	34_0	37_1	41_1
27	∞_5	∞_7	8_0	9_0	15_1	21_1	24_1	27_0	31_1	34_1	38_1	40_0	42_0
28	∞_6	∞_7	10_0	12_1	15_0	16_1	20_1	22_1	29_0	30_0	31_1	35_1	36_0
29	29_0	29_1	11_0	12_0	16_1	18_1	19_0	21_1	25_0	32_1	34_1	38_0	40_1
30	30_0	30_1	9_0	11_0	15_1	18_1	21_0	23_1	28_1	32_0	35_1	37_0	42_1
31	31_0	31_1	10_0	11_0	14_1	17_1	21_0	25_1	27_0	32_1	35_0	41_0	43_1
32	32_0	32_1	10_0	13_1	15_1	17_0	20_1	22_0	26_1	33_0	34_0	40_1	42_0
33	33_0	33_1	9_0	12_0	14_0	16_1	23_0	25_1	28_0	31_1	34_0	36_1	42_1
34	34_0	34_1	10_0	12_0	14_1	18_0	20_0	24_1	27_1	30_0	37_0	39_1	43_0
35	35_0	35_1	8_0	13_0	16_0	20_1	21_1	24_0	25_1	33_1	37_0	39_1	40_1
36	36_0	36_1	11_0	13_0	17_0	18_0	19_0	20_1	24_1	31_0	37_1	38_1	42_1
37	37_0	37_1	8_0	10_0	14_0	22_1	23_0	26_1	28_1	29_1	31_0	38_0	40_0
38	38_0	38_1	12_0	13_0	14_1	15_0	19_1	26_1	27_0	30_1	33_0	35_1	41_1
39	39_0	39_1	9_0	11_1	15_0	18_1	22_0	24_1	26_1	29_0	31_0	33_1	41_0
40	40_0	40_1	9_0	10_0	16_0	17_1	19_0	27_1	28_0	30_1	36_0	39_0	41_1
41	41_0	41_1	8_0	13_1	15_0	20_1	21_0	23_1	28_0	29_1	34_1	36_1	43_0
42	42_0	42_1	8_0	12_0	17_0	19_1	22_1	23_1	27_1	29_0	35_0	39_0	43_1
43	43_0	43_1	8_0	9_1	16_1	22_0	24_1	25_1	26_0	30_1	32_0	36_0	38_0

in our procedure. For two such partial designs Δ'_j and Δ''_j we say Δ'_j *precedes* Δ''_j, $\Delta'_j \prec \Delta''_j$, if there exists some q, $q \leqslant j$, such that

(i) corresponding ith level canonical lines of Δ'_j and Δ''_j coincide for $1 \leqslant i < q$, and

(ii) qth level canonical line of Δ'_j precedes that of Δ''_j.

In our case partial designs $\mathscr{D}^{(1)}, ..., \mathscr{D}^{(7)}$ are trivial. We construct $\mathscr{D}^{(j)}$ from $\mathscr{D}^{(j-1)}$, $8 \leqslant j \leqslant 43$, in the following way: To each partial design $\Delta_{j-1} \in \mathscr{D}^{(j-1)}$ we join all possible jth level canonical lines x_j which intersect with each line of Δ_{j-1} in exactly two points. In such a way we obtain one by one potential partial designs $\Delta_j = \Delta_{j-1} \cup x_j$ of the jth level. Then we try to eliminate Δ_j by searching for a design $\Delta_j\alpha$ that is isomorphic to Δ_j, which precedes Δ_j. Denote with S the stabilizer of all $t = 43$ point orbits of $\mathscr{D}$. Obviously, $\alpha \in S$ fixes all the orbital lines of $\mathscr{S}$. Now, we include Δ_j into $\mathscr{D}^{(j)}$ if it cannot be eliminated by finding an $\alpha \in S$ such that $\Delta_j\alpha \prec \Delta_j$ in terms of the above defined precedence of partial designs. At the end of this procedure, $\mathscr{D}^{(t)}$ will be the set of all possible designs with the orbital structure $\mathscr{S}$, admitting the given automorphism group $\langle \rho \rangle$.

The described procedure is also carried out by computer. It turns out that both orbital structures $\mathscr{S}_1$ and $\mathscr{S}_2$ can be supplied by the indices in unique manner up to isomorphism. So we obtain biplane $\mathscr{D}_1$ by indexing $\mathscr{S}_1$ and biplane $\mathscr{D}_2$ by indexing $\mathscr{S}_2$ as the only solutions and conclude that they represent a pair of nonisomorphic Aschbacher's designs which are mutually dual. The biplane $\mathscr{D}_1$, lexicographically the first, is enclosed in Table II, by writing down only the $\langle \rho \rangle$-orbit representatives.

So we proved the following:

THEOREM 1. *Let $\mathscr{D}$ be a biplane* (79, 13, 2) *admitting an involutory automorphism. Then $\mathscr{D}$ is unique, up to isomorphism and duality, and represents an Aschbacher's design with the full automorphism group of the order* 110.

Together with previously mentioned results this yields

THEOREM 2. *If $\mathscr{D}$ is a biplane* (79, 13, 2) *that is not isomorphic to an Aschbacher's design, then the full automorphism group of $\mathscr{D}$ is either trivial or a* 3*-group.*

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