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On the representation of fuzzy rules

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Abstract

In fuzzy logic, connectives have a meaning that, can frequently be known through the use of these connectives in a given context. This implies that there is not a universal-class for each type of connective, and because of that several continuous t-norms, continuous t-conorms and strong negations, are employed to represent, respectively, the *and*, the *or*, and the *not*. The same happens with the case of the connective If/then for which there is a multiplicity of models called T-conditionals or implications. To reinforce that there is not a universal-class for this connective, four very simple classical laws translated into fuzzy logic are studied.

Keywords: Fuzzy T-conditional; Fuzzy implication; Boolean laws; Orthomodular laws

1. On fuzzy rules

The typical simplest rule appearing in fuzzy logic is "If x is P, then y is Q", with x in a universe of discourse X (usually an interval in the real line $\mathbb{R}$), y in another universe Y (also usually an interval in $\mathbb{R}$), P an imprecise predicate on X, and Q another imprecise predicate on Y. This rule is to be represented by means of a numerical function $J : [0,1] \times [0,1] \to [0,1]$ such that $J(\mu_P(x), \mu_Q(y))$ captures the meaning of the rule, that is, the use of it made by who is uttering the conditional statement "If x is P, then y is Q". It is well known that such use, or meaning, is sometimes captured by several models coming from ortholattices, Boolean algebras, and orthomodular lattices.

The first goal of functions J is to represent the meaning of the rule and, for this, J can be chosen as being or being not an implication function. This election depends on the character of the conditional statement "If x is P, then y is Q".

The second goal of functions J is to allow inferences. For this, J should verify some directive concerning the inference to be done. For example, if it is a forward inference, J does verify the inequality $T(a, J(a,b)) \leqslant b$ for

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all a, b in $[0, 1]$, and a continuous t-norm T. This idea comes from the inequality $p \cdot (p \rightarrow q) \leqslant q$ in lattices that compacts the rule of *Modus Ponens* "$p \rightarrow q,\ p : q$" (see [6]). In this case, J is called a T-conditional. Provided the inference is backwards, J does verify the inequality $T(N(b), J(a, b)) \leqslant N(a)$ for all a, b in [0,1], a continuous t-norm T, and a strong negation N corresponding to *not* (recall that $N : [0, 1] \rightarrow [0, 1]$ is said to be a strong negation when it is decreasing and involutive). This time the inequality comes from the case of ortholattices where *Modus Tollens* "$p \rightarrow q$, not q : not p" is compacted by $q' \cdot (p \rightarrow q) \leqslant p'$. A complete study about modus ponens and modus tollens can be found in [15].

Among the models quoted above the most widely considered are the following (see for instance [8,13]):

(a) **Mamdani–Larsen operators:** $J(\mu_P(x), \mu_Q(y)) = T(\mu_P(x), \mu_Q(y))$, with T a continuous t-norm without zero-divisors.
(b) **R-implication operators:** $J(\mu_P(x), \mu_Q(y)) = \sup\{z \in [0, 1] | T(z, \mu_P(x)) \leqslant \mu_Q(y)\}$, with a continuous t-norm T.
(c) **S-implication operators:** $J(\mu_P(x), \mu_Q(y)) = S(N(\mu_P(x)), \mu_Q(y))$, with S a continuous t-conorm and N a strong negation.[1]
(d) **QM-operators:** $J(\mu_P(x), \mu_Q(y)) = S(N(\mu_P(x)), T(\mu_P(x), \mu_Q(y)))$, with S a continuous t-conorm, N a strong negation and T a continuous t-norm.
(e) **D-operators:** $J(\mu_P(x), \mu_Q(y)) = S(\mu_Q(y), T(N(\mu_P(x)), N(\mu_Q(y))))$, with $S,\ N$ and T like in (d).

Model (a) comes from the conditional in lattices $p \rightarrow q = p \cdot q$, that is not an implication in the usual sense. However, we include them in our study because this model is frequently used to model fuzzy rules in fuzzy control. The main reason is because Mamdani–Larsen operators are always T_1-conditionals for any t-norm T_1, that is, they satisfy $T_1(x, J(x, y)) \leqslant y$ for all $x, y \in [0, 1]$. This condition is important because when it is satisfied, the inference rule of Modus Ponens is guaranteed. Note that all other models only satisfy the previous condition for some adequate t-norms T_1.

Model (b) comes from $p \rightarrow q = p' + q$, the typical implication in Boolean algebras that is only a conditional on them but not in other ortholattices or De Morgan algebras. Model (c) comes from the equality $p' + q = \sup\{z | p \cdot z \leqslant q\}$ that only holds in Boolean algebras. Models (d) and (e) come, respectively, from the Sasaki hook $p \rightarrow q = p' + p \cdot q$, and the Dishkant hook $p \rightarrow q = q + p' \cdot q'$, that are conditionals in orthomodular lattices and collapse, on Boolean algebras, with $p' + q$. In fuzzy logic only models (b) and (c) are implications and models (b)–(e) coincide, when restricted to crisp sets μ_P, μ_Q with the classical implication: $\mu'_P + \mu_Q = \max \cdot ((1 - \mathrm{i}d) \cdot \mu_P \times \mu_Q)$. For more details see [3] for R- and S-implications and [12] for Q- and D-operators. See also the recent survey [13].

Another interesting point lies in the study of these models of implications in the framework of discrete settings. In particular, operators used in model (a), defined on a finite chain, are characterized in [9]. In the same framework, those operators used in models (b)–(c) are developed in [10] and those used in models (d)–(e) in [11].

A particular type of these operators is useful if it reflects what rules mean, that is, how the linguistic statement "If x is P, then y is Q" is actually used for what concerns the equivalence between meaning and use (see [17]). Notice that, in principle, if rules are not correctly represented, the problem that will be solved is not the one posed, but another corresponding to the same rules but used in a different way.

Representing the statement "x is P" by α, and "y is Q" by β, the protoform rule "If α, then β" ($\alpha \rightarrow \beta$) will mean some (not necessarily functionally expressible) model involving α and β. For example, let $\mathbf{L} = (L, \cdot, +, ', 0, 1)$ be an ortholattice, then $\alpha \rightarrow \beta$ could mean:

$$\alpha' + \beta, \quad \text{or} \quad \alpha' + \alpha \cdot \beta, \quad \text{or} \quad \alpha \cdot \beta, \quad \text{or} \quad \begin{cases} \alpha' + \beta & \text{if } \alpha \cdot \beta = 0 \\ \alpha \cdot \beta & \text{otherwise} \end{cases}, \ldots$$

Only after this protoform is established in accordance with the use of $\alpha \rightarrow \beta$, can function J be searched for. Notice that if the rule $\alpha \rightarrow \beta$ means $\alpha' + \beta$, both models (b) and (c) are candidates for its representation.

[1] Extensions of this kind of implications have been considered in [16] under the name of improper S-implications. Some examples are included there illustrating the use of these new operators.

Many works in fuzzy logic deal with logical formalisms studying the correctness and completeness of the corresponding formal fuzzy system, and proving that, from this point of view, R-implications are the best suited (see [1,4,5]).

These syntactically oriented researchers in fuzzy logic advocate mainly using R-implications to represent rules. This is a position that avoids taking into account the "meaning" of rules and requires posing the problems in a way not always reflecting what is actually stated by, for example, the experts giving the rules. With the selection of R-implications only one of the possible meanings of the rules is taken into account.

Since quantum logic broke by the first time the operator $\alpha \to \beta = \alpha' + \beta$, this paper presents three classical laws involving the symbol "$\to$" that are in between Boolean and quantum logics. It appears that R-implications seem to be well suited when the law is only typical of Boolean logic, but not when the law also holds in quantum logic. In addition, there is a case in which the law does hold in Boolean and in quantum logic but no one of the operators stated before are useful in fuzzy logic.

2. Preliminaries

We will suppose the reader to be familiar with basic results concerning t-norms and t-conorms that can be found for instance in [7]. We only recall here some definitions and results that we will specially use.

Definition 1 [7, Definition 11.3]. A function $N : [0,1] \to [0,1]$ is called a *strong negation* if it is non-increasing, and $N(N(x)) = x$ for all $x \in [0,1]$.

Of course, all strong negations do verify $N(0) = 1$ and $N(1) = 0$.

Proposition 2 [14], see also [3, Theorem 1.1]. *A function $N : [0,1] \to [0,1]$ is a strong negation if and only if there exists a strictly increasing function $\varphi : [0,1] \to [0,1]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$ such that*

$$N(x) = N_\varphi(x) = \varphi^{-1}(1 - \varphi(x)) \quad \text{for all } x \in [0,1].$$

Definition 3 [7, Definition 2.9]. A t-norm T is an *Archimedean t-norm* if for each $(x,y) \in]0,1[^2$ there is an $n \in \mathbb{N}$ such that $x_T^{(n)} < y$ where $x_T^{(n)}$ is defined recursively by

$$x_T^{(1)} = x, \quad x_T^{(n)} = T\left(x, x_T^{(n-1)}\right) \quad \text{for all } n \geqslant 2.$$

Archimedean t-conorms are defined dually, and in the continuous case, it is known that a t-norm T (t-conorm S) is Archimedean if and only if $T(x,x) < x$ $(S(x,x) > x)$ for all $x \in]0,1[$ (see [7, Theorem 2.12]).

Definition 4 [7, Definition 2.13]. A t-norm T is called

- *Strict* if it is continuous and strictly monotone.
- *Nilpotent* if it is continuous and for each $x \in]0,1[$ there is some $n \in \mathbb{N}$ such that $x_T^{(n)} = 0$.

Definition 5 [7, Definitions 3.25 and 3.39]. Let T (S) be a continuous t-norm (t-conorm). An *additive generator* of T (S) is a continuous strictly decreasing (increasing) function $t : [0,1] \to [0,+\infty]$ $(s : [0,1] \to [0,+\infty])$ with $t(1) = 0$ $(s(0) = 0)$ such that for all $(x,y) \in [0,1]^2$ we have

$$T(x,y) = t^{-1}(\min(t(0), t(x) + t(y))), \quad (S(x,y) = s^{-1}(\min(s(1), s(x) + s(y)))).$$

Note that, when exist, additive generators are uniquely determined up to a positive multiplicative constant. On the other hand, all continuous Archimedean t-norms are either strict or nilpotent as it is stated in the following proposition.

Proposition 6 [7, Corollary 5.5]. *A function $S : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-conorm if and only if has a (continuous) additive generator, i.e., there is a continuous, strictly increasing function $s : [0,1] \to [0,\infty]$ with $s(0) = 0$ such that*

$$S(x,y) = s^{-1}(\min(s(1), s(x) + s(y)))$$

for all $x, y \in [0,1]$. *The t-conorm* S *is strict when* $s(1) = \infty$ *and nilpotent when* $s(1) < \infty$.

A dual characterization can be given for Archimedean t-norms which additive generators are strictly decreasing with $f(1) = 0$. On the other hand, when a t-conorm S is nilpotent one can take the additive generator f such that $f(1) = 1$ which is called the *normalized* generator of S and will be represented here by $\varphi : [0,1] \to [0,1]$. In this case, N_φ is a strong negation and S satisfies $S(a,b) = 1$ if and only if $b \geqslant N_\varphi(a)$.

Given any binary operator F on [0,1] and any increasing bijection $\varphi : [0,1] \to [0,1]$, we denote by F_φ the φ-conjugate of F given by

$$F_\varphi(x,y) = \varphi^{-1}(F(\varphi(x), \varphi(y))) \quad \text{for all } x, y \in [0,1].$$

With this notation, a nilpotent t-conorm can be always written as the φ-conjugate of the Łukasiewicz t-conorm $W^*(x,y) = \min(1, x+y)$, where φ is the normalized generator of S, that is $S = W^*_\varphi$.

3. The case with the classical law $a \to (b \to a) = 1$

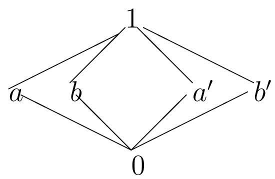
Note that in a Boolean algebra, by taking $a \to b = a' + b$, is

$$a \to (b \to a) = a' + (b' + a) = a + a' + b = 1 + b = 1.$$

Hence $a \to (b \to a) = 1$ is a law of Boolean algebras with $a \to b = a' + b$. Nevertheless, the expression $a \to (b \to a) = 1$ is not a law in any lattice if taking the conditional $a \to b = a \cdot b$, since then

$$a \to (b \to a) = a \cdot (b \cdot a) = a \cdot b \neq 1.$$

In orthomodular lattices $a \to (b \to a) = 1$ is not a law with the conditional $\to_Q$ given by $a \to_Q b = a' + a \cdot b$. To see this, it is enough to consider the orthomodular lattice:



where

$$a \to_Q (b \to_Q a) = a \to_Q (b' + b \cdot a) = a \to_Q b' = a' + a \cdot b' = a' \neq 1.$$

On the contrary, $a \to (b \to a) = 1$ is a law with respect to the conditional $\to_D$ given by $a \to_D b$, since

$$\begin{aligned} a \to_D (b \to_D a) &= a \to_D (a + b' \cdot a') = (a + b' \cdot a') + a' \cdot (a + b' \cdot a')' = a + a' \cdot b' + a' \cdot (a' \cdot (a+b)) \\ &= a + a' \cdot (a+b) + a' \cdot b' \end{aligned}$$

but now using the orthomodular property

$$y + x \cdot y' = x \quad \text{for all } y \leqslant x \tag{1}$$

for the values $y = a$ and $x = a + b$, we obtain

$$a \to_D (b \to_D a) = a + a' \cdot (a+b) + a' \cdot b' = a + b + a' \cdot b' = (a+b) + (a+b)' = 1.$$

Hence, $a \to (b \to a) = 1$ is a law that holds in orthomodular lattices, depending on the used conditional.

3.1. *The law* $\mu \to (\sigma \to \mu) = 1$ *in fuzzy logic*

Let us now investigate what happens with this law in fuzzy logic when we take as conditionals the five possible models stated in the introduction. To do this, we study the functional equation

$$J(a, J(b, a)) = 1 \quad \text{for all } a, b \in [0, 1], \tag{2}$$

where J is an implication function given by one of the five possible models. We begin with an example showing the behavior of some particular implications, one in each one of the five types considered.

Example 7. Let J_0 be the Mamdani–Larsen operator given by $J_0(a, b) = \min(a, b)$. Let J_1 and J_2 be the R-implication derived from the minimum (*Gödel implication*) and the S-implication derived from the maximum and the negation $N(a) = 1 - a$ (Kleene–Dienes implication), respectively. That is

$$J_1(a, b) = \begin{cases} 1 & \text{if } a \leqslant b \\ b & \text{if } a > b \end{cases}, \quad J_2(a, b) = \max(1 - a, b).$$

Finally, let J_3 and J_4 be, respectively, the QM-operator and the D-operator derived from $S = \max$, $T = \min$ and $N(a) = 1 - a$. That is

$$J_3(a, b) = \max(1 - a, \min(a, b)), \quad \text{and} \quad J_4(a, b) = \max(b, \min(1 - a, 1 - b)).$$

In these cases, we have that Eq. (2):

- Is not a law with J_0 since $J_0(a, J_0(b, a)) = \min(a, b) \neq 1$.
- Is a law with J_1 since, when $b \leqslant a$ we have $J_1(a, J_1(b, a)) = J_1(a, 1) = 1$, whereas when $b > a$ we have $J_1(a, J_1(b, a)) = J_1(a, a) = 1$.
- Is not a law neither with J_2, nor with J_3, nor with J_4 since in these three cases we have, for instance

$$J(0.6, J(0.7, 0.6)) = J(0.6, 0.6) = 0.6.$$

Now, we deeply analyze Eq. (2). We begin by proving that in the case of Mamdani–Larsen operators there are no solutions of this law.

Proposition 8. *Let $J = T$ the Mamdani–Larsen operator given by a t-norm T. Then, the functional equation $T(a, T(b, a)) = 1$ for all $a, b \in [0, 1]$ is never satisfied.*

Proof. This is obvious since $T(a, T(b, a)) \leqslant a < 1$ for all $a, b \in [0, 1[$. □

Now we deal with R-implications defined, from a left-continuous t-norm T, by

$$J_T(a, b) = \sup\{z \in [0, 1] | T(a, z) \leqslant b\} \quad \text{for all } a, b \in [0, 1].$$

In this case, all implications J_T are solutions of the law. In fact, this is a well known property about R-implications, but we include here the proof for the sake of completeness.

Proposition 9. *Let T be a left-continuous t-norm and J_T the corresponding R-implication. Then, the functional equation $J_T(a, J_T(b, a)) = 1$ for all $a, b \in [0, 1]$ is always satisfied.*

Proof. Since $b \leqslant 1$ we have

$$J_T(b, a) \geqslant J_T(1, a) = \sup\{z \in [0, 1] | T(1, z) = z \leqslant a\} = a$$

for all $a \in [0, 1]$. But then

$$J_T(a, J_T(b, a)) = \sup\{z \in [0, 1] | T(a, z) \leqslant J_T(b, a)\} = 1. \qquad \square$$

In the case of S-implications defined from a continuous t-conorm S and a strong negation N by

$$J_{S,N}(a, b) = S(N(a), b) \quad \text{for all } a, b \in [0, 1]$$

we can characterize which S-implications are solutions of the law. Note that in this case Eq. (2) can be written as

$$S(N(a), S(N(b), a)) = 1 \quad \text{for all } a, b \in [0, 1]. \tag{3}$$

Then,

Proposition 10. *Let S be a continuous t-conorm, N a strong negation and $J_{S,N}$ the corresponding S-implication. Then, the functional Eq. (3) is satisfied if and only if, S is a nilpotent t-conorm, say $S = W^*_\varphi$ and $N \geqslant N_\varphi$.*

Proof. Suppose first that the law is satisfied. Then, we have

$$S(N(a), S(N(b), a)) = 1 \quad \text{for all } a, b \in [0,1]$$

and taking $b = 1$ in this equation we obtain $S(N(a), a) = 1$ for all $a \in [0,1]$ and this is equivalent to be S a nilpotent t-conorm $S = W^*_\varphi$ with $N \geqslant N_\varphi$. Conversely, if S and N satisfy the conditions in the proposition, we have $S(N(a), a) = 1$ for all $a \in [0,1]$ and then,

$$S(N(a), S(N(b), a)) \geqslant S(N(a), a) = 1 \quad \text{for all } a, b \in [0,1]. \qquad \square$$

We deal now with QM-operators. That is, given a strong negation N, a continuous t-conorm S and a continuous t-norm T, the corresponding QM-operator is given by

$$J_Q(a, b) = S(N(a), T(a, b)) \quad \text{for all } a, b \in [0,1].$$

In this case, we have not found a complete characterization of the corresponding functional equation, that can be written as

$$S(N(a), T(a, S(N(b), T(b, a)))) = 1 \quad \text{for all } a, b \in [0,1], \tag{4}$$

but we can give several necessary conditions as well as sufficient ones.

Proposition 11. *Let S be a continuous t-conorm, N a strong negation, T a continuous t-norm and J_Q the corresponding QM-operator. Then,*

(*i*) *If Eq. (4) is satisfied, S must be a nilpotent t-conorm $S = W^*_\varphi$ with $N \geqslant N_\varphi$. Moreover, T must satisfy $T(a, a) \geqslant N_\varphi(N(a))$ for all $a \in [0,1]$ (which in particular implies that T must have a trivial zero region, that is, $T(a, b) > 0$ for all $a, b > 0$).*

(*ii*) *If the above conditions hold and, in addition, $T(a, b) \geqslant \varphi^{-1}(\varphi(a) - \varphi(N(b)))$ for all $a, b \in [0,1]$ such that $N(b) < a$, then Eq. (4) holds.*

Proof. Taking $b = 1$ in Eq. (4), we obtain

$$S(N(a), T(a, a)) = 1 \quad \text{for all } a \in [0,1], \tag{5}$$

which implies $S(N(a), a) = 1$ and consequently, $S = W^*_\varphi$ with $N \geqslant N_\varphi$. Now, Eq. (5) also implies that $T(a, a) \geqslant N_\varphi(N(a))$ for all $a \in [0,1]$ and then (i) is proved.

To prove (ii) note that if

$$S(N(b), T(b, a)) \geqslant a \quad \text{for all } a, b \in [0,1] \tag{6}$$

then we obtain

$$S(N(a), T(a, S(N(b), T(b, a)))) \geqslant S(N(a), T(a, a)) = 1$$

and thus Eq. (4) holds. Note however that, when $a \leqslant N(b)$, condition (6) is trivially satisfied. Consequently, to ensure that Eq. (4) holds, it is enough to have

$$S(N(b), T(b, a)) \geqslant a \quad \text{for all } N(b) < a.$$

But this is equivalent to:

$$\varphi^{-1}(\varphi(N(b)) + \varphi(T(b, a))) \geqslant a \iff T(a, b) \geqslant \varphi^{-1}(\varphi(a) - \varphi(N(b))).$$

Thus, the proposition is proved. $\square$

Example 12

(i) Note that taking for instance $S = W^*_\varphi$ with $N \geqslant N_\varphi$ and T the t-norm minimum, we obtain solutions of Eq. (4). Moreover, if we take $N = N_\varphi$ we have $N_\varphi(N(a)) = a$ for all $a \in [0,1]$ and then $T = \min$ is the only possible t-norm for which the corresponding J_Q satisfies Eq. (4).

(ii) Analogously if we take $S = W^*_\varphi$ with $N > N_\varphi$ and T any t-norm such that $T(a,a) \geqslant N_\varphi(N(a))$ for all $a \in [0,1]$ and T greater than or equal to the N_φ-dual of S in points (a,b) such that $N(b) < a$, we obtain again solutions of Eq. (4). Effectively, since in this case we have, for $N(b) < a$,

$$\varphi^{-1}(\varphi(a) - \varphi(N(b))) < \varphi^{-1}(\varphi(a) + \varphi(b) - 1) \leqslant T(a,b).$$

(iii) Taking S to be the Łukasiewicz t-conorm, T and N such that $N(a) \geqslant 1 - T(a,a)$ for all $a \in [0,1]$ and $N(b) \geqslant a - T(a,b)$ for all $a, b \in [0,1]$, we obtain solutions of Eq. (4).

(iv) More particularly, taking S to be the Łukasiewicz t-conorm, T the t-norm product and N a strong negation such that $N(a) \geqslant 1 - a^2$ for all $a \in [0,1]$, we obtain a solution of Eq. (4).

Let us finally deal with D-operators. Given a strong negation N, a continuous t-conorm S and a continuous t-norm T, the corresponding D-operator is given by

$$J_D(a,b) = S(b, T(N(a), N(b))) \quad \text{for all } a, b \in [0,1]$$

and the functional Eq. (2) is written as:

$$S(S(a, T(N(b), N(a))), T(N(a), N(S(T(N(b), N(a)), a)))) = 1 \tag{7}$$

for all $a, b \in [0,1]$. This case is quite similar to the above one and we can give a result analogous to Proposition 11.

Proposition 13. *Let S be a continuous t-conorm, N a strong negation, T a continuous t-norm and J_D the corresponding D-operator. If Eq. (7) is satisfied, S must be a nilpotent t-conorm $S = W^*_\varphi$ with $N \geqslant N_\varphi$. Moreover, T must satisfy $T(a,a) \geqslant N_\varphi(N(a))$ for all $a \in [0,1]$.*

Proof. Just taking $b = 1$ in Eq. (7) we obtain $S(T(N(a), N(a)), a) = 1$ for all $a \in [0,1]$ and the proof follows as in part (i) of Proposition 11. □

In this case, we have not found any sufficient condition but note that we have solutions by taking simply $T = \min$ as it is stated in the following example.

Example 14. Note that taking for instance $S = W^*_\varphi$ with $N \geqslant N_\varphi$ and T the t-norm minimum, we obtain solutions of Eq. (7). Moreover, as in the case of QM-operators, if we take $N = N_\varphi$, $T = \min$ is the only possible t-norm for which the corresponding J_D satisfies Eq. (7).

In conclusion, the law $a \to (b \to a) = 1$, that is typical of Boolean algebras and of orthomodular lattices with the conditional $a \to_D b$, holds in fuzzy logic with R-implications. Moreover, it should be noticed that there are a lot of cases for which S-implications also work, and the same happens for Q- and D-operators.

4. The case with the classical law $(a \to a') \to a = a$

Note that in a Boolean algebra, by taking $a \to b = a' + b$, is

$$(a \to a') \to a = (a' + a') \to a = a' \to a = a + a = a$$

and $(a \to a') \to a = a$ is a law. But also in orthomodular lattices with the conditionals $a \to_Q b = a' + a \cdot b$, or $a \to_D b = b + a' \cdot b'$ is a law, since then

$$(a \to_Q a') \to_Q a = (a' + a \cdot a') \to_Q a = a' \to_Q a = a + a' \cdot a = a$$

and

$$(a \to_D a') \to_D a = (a' + a' \cdot a) \to_D a = a' \to_D a = a + a \cdot a' = a.$$

On the other hand, in ortholattices with the conditional $a \to b = a \cdot b$, is not a law because $(a \to a') \to a = (a \cdot a') \to a = 0 \to a = 0$.

4.1. The law $(\mu \to \mu') \to \mu = \mu$ in fuzzy logic

This law in fuzzy logic becomes the functional equation

$$J(J(a, N(a)), a) = a \quad \text{for all } a \in [0, 1], \tag{8}$$

where J is an implication function and N is a strong negation. We will study again the five possible models of implications stated in the preliminaries. In this case, we have the following example.

Example 15. With the notations in Example 7, and taking $N(a) = 1 - a$ we have that Eq. (8):

- Is not a law with J_0 since $J_0(J_0(a, 1-a), a) = \min(1-a, a) \neq a$ for all $a > 1/2$.
- Is not a law with J_1 since, for instance $J_1(J_1(0.6, 1-0.6), 0.6) = J_1(0.4, 0.6) = 1 \neq 0.6$.
- Is a law with J_2 because

$$J_2(J_2(a, 1-a), a) = \max(1 - \max(1-a, 1-a), a) = \max(a, a) = a.$$

- Is also a law with J_3 because

$$J_3(J_3(a, 1-a), a) = J_3(\max(1-a, \min(a, 1-a)), a) = J_3(1-a, a) = \max(a, \min(1-a, a)) = a.$$

- Is again a law with J_4 because

$$J_4(J_4(a, 1-a), a) = J_4(\max(1-a, \min(1-a, a)), a) = J_4(1-a, a) = \max(a, \min(a, 1-a)) = a.$$

We begin the study of Eq. (8) by proving that in the cases of Mamdani–Larsen operators and R-implications, there are no solutions of this law.

Proposition 16. *Let $J = T$ the Mamdani–Larsen operator given by a t-norm T. Then, the functional equation $T(T(a, N(a)), a) = a$ for all $a \in [0, 1]$ is never satisfied.*

Proof. This is obvious since taking $a \in [0, 1]$ such that $N(a) < a$, we have

$$T(T(a, N(a)), a) \leqslant T(N(a), a) \leqslant N(a) < a$$

for all these values. □

Proposition 17. *Let T be a continuous t-norm and J_T the corresponding R-implication. Then, the functional equation $J_T(J_T(a, N(a)), a) = a$ for all $a \in [0, 1]$ is never satisfied.*

Proof. Note that when $a \leqslant N(a)$ we have $J_T(J_T(a, N(a)), a) = J_T(1, a) = a$. However, we will prove in two steps that when $N(a) < a$ this is not true:

- Suppose first that T has an idempotent $b \in]0, 1[$. In this case taking a such that $N(a) < b < a < 1$, we obtain:

$$J_T(J_T(a, N(a)), a) = J_T(N(a), a) = 1 \neq a.$$

- If T has no non-trivial idempotents, then T must be Archimedean with additive generator $t : [0, 1] \to [0, +\infty]$, and J_T is given by (see for instance [3, Theorem 1.16]):

$$J_T(a, b) = t^{-1}(\max(0, t(b) - t(a))) \quad \text{for all } a, b \in [0, 1].$$

Thus, for $N(a) < a < 1$ we have $J_T(a, N(a)) = t^{-1}(t(N(a)) - t(a))$. Now, when $a \to 1$, we obtain $J_T(a, N(a)) \to 0$ and consequently there is some a such that $N(a) < a < 1$ and $J_T(a, N(a)) \leqslant a$. But then $J_T(J_T(a, N(a)), a) = 1 \neq a$. □

For S-implications, we obtain solutions of the functional equation but only when the t-conorm is the maximum.

Proposition 18. *Let S be a continuous t-conorm, N a strong negation and $J_{S,N}$ the corresponding S-implication. Then, $J_{S,N}$ satisfies Eq. (8) if and only if, $S = \max$.*

Proof. In this case, the functional Eq. (8) becomes:

$$S(N(S(N(a),N(a))),a) = a \quad \text{for all } a \in [0,1]. \tag{9}$$

It is clear that when $S = \max$ this equation holds. To prove the converse, assume that Eq. (9) holds and suppose that S has an Archimedean ordinal summand on $[c,d]$ with $0 \leqslant c < d \leqslant 1$. If there is some $a \in [c,d[$ such that $N(S(N(a),N(a))) > c$, since $N(S(N(a),N(a))) \leqslant a$ for all $a \in [0,1]$, we have $c < N(S(N(a),N(a))) \leqslant a < d$ and then for this value:

$$S(N(S(N(a),N(a))),a) > a$$

obtaining a contradiction. Thus, $N(S(N(a),N(a))) \leqslant c$ for all $a \in [c,d[$ and consequently $S(N(a),N(a)) \geqslant N(c) > N(a)$ for all $a \in]c,d[$. That is, S is Archimedean in $]N(d),N(c)[$.

On the other hand, if S was Archimedean in an interval $[r,s]$ with $r < N(d)$ or $s > N(c)$, the same reasoning could be used to prove that S should be Archimedean in the interval $]N(s),N(r)[$ that strictly contains c or d, reaching a contradiction. Consequently, S is Archimedean in $[N(d),N(c)]$ with $S(N(a),N(a)) \geqslant N(c)$ for all $N(a) \in]N(d),N(c)[$ contradicting the continuity of S.

Thus, S cannot have any Archimedean summand and so $S = \max$. □

In the case of QM-operators Eq. (8) can be written as

$$S(N(S(N(a),T(a,N(a)))),T(S(N(a),T(a,N(a))),a)) = a \tag{10}$$

for all $a \in [0,1]$. We have not been able to find a complete characterization of the previous functional equation, but a lot of solutions can be presented.

Proposition 19. *Let S be a continuous t-conorm, N a strong negation, T a continuous t-norm and J_Q the corresponding QM-operator*:

(*i*) *If $S = \max$, then Eq. (10) holds for any strong negation N and for any t-norm T.*
(*ii*) *If $T(a,N(a)) = 0$ for all $a \in [0,1]$, Eq. (10) holds for any strong negation N and for any t-conorm S.*

Proof. In both cases, the proof is a simple checking. □

In the case of D-operators, Eq. (8) can be written as

$$S(a,T(N(a),N(S(N(a),T(N(a),a))))) = a \tag{11}$$

for all $a \in [0,1]$. Analogously to the case of Q-operators, we can find a lot of solutions of this equation.

Proposition 20. *Let S be a continuous t-conorm, N a strong negation, T a continuous t-norm and J_D the corresponding D-operator*:

(*i*) *If $S = \max$, then Eq. (11) holds for any strong negation N and for any t-norm T.*
(*ii*) *If $T(a,N(a)) = 0$ for all $a \in [0,1]$, Eq. (11) holds for any strong negation N and for any t-conorm S.*

Proof. Straightforward. □

As a conclusion note that, although $(a \to a') \to a = a$ is a law in Boolean algebras as well as in orthomodular lattices, in fuzzy logic no solutions of the corresponding functional equation can be found for Mamdani–Larsen, or for R-implications. On the other hand, the law holds for S-implications only when $S = \max$, whereas there are a lot of solutions for Q- and D-operators. This shows that the most usual models are not a good option when this law is required, being best suited the less usual ones: Q- and D-operators.

5. The case with the classical law $(a \to a') \to a' = 1$

Note that in a Boolean algebra, by taking $a \to b = a' + b$,

$$(a \to a') \to a' = (a' + a') \to a' = a' \to a' = a + a' = 1$$

and $(a \to a') \to a' = 1$ is a law. But also in orthomodular lattices with the conditionals $a \to_Q b = a' + a \cdot b$, or $a \to_D b = b + a' \cdot b'$ is a law, since then

$$(a \to_Q a') \to_Q a' = (a' + a \cdot a') \to_Q a' = a' \to_Q a' = a + a' \cdot a' = 1$$

and

$$(a \to_D a') \to_D a' = (a' + a' \cdot a) \to_D a' = a' \to_D a' = a' + a \cdot a = 1.$$

On the other hand, in ortholattices with the conditional $a \to b = a \cdot b$, is not a law because $(a \to a') \to a' = (a \cdot a') \to a' = 0 \to a' = 0$.

5.1. The law $(\mu \to \mu') \to \mu' = 1$ in fuzzy logic

In fuzzy logic this law will be clarified by the analysis of the functional equation

$$J(J(a, N(a)), N(a)) = 1 \quad \text{for all } a \in [0, 1], \tag{12}$$

where J is an implication function and N a strong negation.

Example 21. Following again the notations in Example 7, and taking $N(a) = 1 - a$ we have that Eq. (12) fails in all cases since:

- With J_0 we have $J_0(J_0(a, 1-a), 1-a) = \min(a, 1-a) \neq 1$ for all a.
- With J_1, we have for instance

$$J_1(J_1(0.5, 1-0.5), 1-0.5) = J_1(1, 0.5) = 0.5 \neq 1.$$

- With J_2, we have

$$J_2(J_2(a, 1-a), 1-a) = \max(1 - \max(1-a, 1-a), 1-a) = \max(a, 1-a) \neq 1 \quad \text{for all } a \neq 0, 1.$$

- With J_3 and J_4, we have in both cases

$$J(J(0.6, 1-0.6), 1-0.6) = J(0.4, 0.4) = 0.6 \neq 1.$$

In this case, the situation is worse than the others as we can see in the following results.

Proposition 22. *There are no solutions of the Eq. (12) neither when J is a Mamdani–Larsen operator, nor when J is an R- nor an S-implication.*

Proof. Let us prove it case by case:

- If $J = T$, then we have $T(T(a, N(a)), N(a)) \leqslant T(N(a), N(a)) \leqslant N(a)$ for all $a \in [0, 1]$ and then Eq. (12) does not hold.
- If $J = J_T$ for a continuous t-norm T, taking a such that $a \leqslant N(a) < 1$, we obtain:

$$J_T(J_T(a, N(a)), N(a)) = J_T(1, N(a)) = N(a) \neq 1.$$

- Finally, when $J = J_{S,N}$ for a continuous t-conorm S and a strong negation N, Eq. (12) becomes

$$S(N(S(N(a), N(a))), N(a)) = 1 \quad \text{for all } a \in [0, 1]. \tag{13}$$

Since $N(S(N(a), N(a))) \leqslant a$, if S, N satisfy the above equation $S(a, N(a)) = 1$ for all $a \in [0,1]$ and consequently S should be $S = W^*_\varphi$ and $N \geqslant N_\varphi$. But then, taking s the fixed point of N, we have $S(s,s) = 1$ and

$$S(N(S(N(s), N(s))), N(s)) = S(N(S(s,s)), s) = S(0,s) = s \neq 1$$

obtaining a contradiction. Thus, there are no solutions of Eq. (12) neither for S-implications. □

In the case of QM-operators, Eq. (12) becomes

$$S(N(S(N(a), T(a, N(a)))), T(S(N(a), T(a, N(a))), N(a))) = 1 \tag{14}$$

for all $a \in [0,1]$. And for D-operators it can be written as

$$S(N(a), T(a, N(S(N(a), T(N(a), a))))) = 1 \tag{15}$$

for all $a \in [0,1]$.

Proposition 23. *Let N be a strong negation, S a continuous t-conorm and T a continuous t-norm. If Eq. (14) holds then $S = W^*_\varphi$ and $N \geqslant N_\varphi$. The same happens if Eq. (15) holds.*

Proof. Note that for all $a \in [0,1]$ we have

$$N(S(N(a), T(a, N(a)))) \leqslant a \quad \text{and} \quad T(S(N(a), T(a, N(a))), N(a)) \leqslant N(a)$$

and therefore, if Eq. (14) holds we have $1 \leqslant S(a, N(a))$ and the result follows.

On the other hand, if Eq. (15) holds we similarly obtain $S(N(a), a) = 1$ and the result also follows trivially. □

Remark 24. In this case, we have not found any solution neither for Eq. (14) nor for Eq. (15). Note for instance that, with the conditions of the previous proposition, $T = \min$ is not a solution and neither t-norms with $T(a, N(a)) = 0$.

As a conclusion note that the law $(a \to a') \to a' = 1$, true in Boolean algebras as well as in orthomodular lattices, does not hold in fuzzy logic for Mamdani–Larsen, nor for R-, nor for S-implications. Moreover, it seems to be no solutions for Q- nor for D-operators. That is, when this law is required, no one of the five models of implications are suitable, showing that even new models for implications should be welcome in fuzzy logic.

6. The case with the classical law $(a \to a \cdot b) \to a = a$

Note that in a Boolean algebras, by taking $a \to b = a' + b$, is

$$(a \to a \cdot b) \to a = (a' + a \cdot b) \to a = (a' + a \cdot b)' + a = a \cdot (a \cdot b)' + a = a$$

and $(a \to a \cdot b) \to a = a$ is a law. But also in orthomodular lattices with the conditionals $a \to_Q b = a' + a \cdot b$, or $a \to_D b = b + a' \cdot b'$ is a law, since then

$$(a \to_Q a \cdot b) \to_Q a = (a' + a \cdot b)' + (a' + a \cdot b) \cdot a$$

but, applying the orthomodular property (1) with $x = a$ and $y = (a' + a \cdot b)' = a \cdot (a \cdot b)' \leqslant a$, we obtain

$$(a \to_Q a \cdot b) \to_Q a = a.$$

With respect to the conditional $\to_D$ we have

$$a \to_D a \cdot b = a \cdot b + a' \cdot (a \cdot b)' = a \cdot b + a'$$

and consequently

$$(a \to_D a \cdot b) \to_D a = a + a' \cdot (a \cdot b + a')' = a + a' \cdot (a \cdot (a \cdot b)') = a + 0 = a.$$

On the other hand, in ortholattices with the conditional $a \to b = a \cdot b$, is not a law because $(a \to a \cdot b) \to a = a \cdot b \neq a$.

6.1. The law $(\mu \to \mu \cdot \sigma) \to \mu = \mu$ in fuzzy logic

In fuzzy logic this law becomes the functional equation

$$J(J(a, T_0(a,b)), a) = a \quad \text{for all } a \in [0,1], \tag{16}$$

where J is an implication function and T_0 is a continuous t-norm. In this case the situation is similar to the previous one.

Example 25. Following again the notations in Example 7, and taking $T_0 = \min$ we have that Eq. (16):

- Fails with J_0 since $J_0(J_0(a, \min(a,b)), a) = \min(a,b) \neq a$ for all $b < a$.
- Fails also with J_1 since for instance

  $$J_1(J_1(0.6, \min(0.6, 0.4)), 0.6) = J_1(0.4, 0.6) = 1 \neq 0.6.$$

- Is a law with J_2 because

  $$J_2(J_2(a, \min(a,b)), a) = \max(1 - \max(1-a, \min(a,b)), a) = \max(\min(a, 1-\min(a,b)), a) = a.$$

- Fails with J_3 since for instance

  $$J_3(J_3(0.7, \min(0.7, 0.5)), 0.7) = J_3(0.5, 0.7) = 0.5 \neq 0.7.$$

- Is a law with J_4 since, if $a \leqslant b$, $J_4(J_4(a, \min(a,b)), a)$ coincides with

  $$J_4(J_4(a,a), a) = J_4(\max(a, 1-a), a) = \max(a, \min(a, 1-a)) = a,$$

whereas, if $b < a$ we have $1 - a < 1 - b$ and then $J_4(J_4(a, \min(a,b)), a)$ coincides with

$$\begin{aligned} J_4(\max(b, 1-a), a) &= \max(a, \min(1 - \max(b, 1-a), 1-a)) = \max(a, \min(1-b, a, 1-a)) \\ &= \max(a, \min(1-a, a)) = a. \end{aligned}$$

Proposition 26. *Let J be an implication function and T_0 a continuous t-norm. Then,*

(*i*) *There are no solutions of the Eq. (16) neither when J is a Mamdani–Larsen operator, nor when J is an R-implication.*
(*ii*) *When $J = J_{S,N}$ is an S-implication, it satisfies Eq. (16) if and only if $S = \max$.*

Proof. To prove (i) note that when $J = T$ is a Mamdani–Larsen operator, taking $b = 0$, we obtain

$$T(T(a, T_0(a,0)), a) = T(T(a,0), a) = T(0,a) = 0 \neq a.$$

When $J = J_T$ is an R-implication, take also $b = 0$ and $a \in]0,1[$ such that $T(a,a) \neq 0$. We have $J_T(a,0) = \sup\{z \in [0,1] | T(a,z) = 0\} \leqslant a$ and then

$$J_T(J_T(a, T_0(a,0)), a) = J_T(J_T(a,0), a) = 1 \neq a.$$

To prove (ii), note that when $J = J_{S,N}$ is an S-implication, Eq. (16) can be written as

$$S(N(S(N(a), T_0(a,b))), a) = a \quad \text{for all } a \in [0,1]. \tag{17}$$

Now, since $a \geqslant N(S(N(a), T_0(a,b)))$, it is clear that $S = \max$ satisfies Eq. (17). Conversely, if $J_{S,N}$ satisfies (17), taking $b = 0$, we obtain

$$a = S(N(S(N(a), T_0(a,0))), a) = S(a,a)$$

and consequently, S must be the maximum. □

On the other hand, for Q- and D-operators, Eq. (16) can be written, respectively, by

$$S(N(S(N(a), T(a, T_0(a,b)))), T(S(N(a), T(a, T_0(a,b))), a)) = a \tag{18}$$

and

$$S(a, T(N(S(T_0(a,b), T(N(a), N(T_0(a,b))))), N(a))) = a \tag{19}$$

for all $a, b \in [0,1]$. In the first case, we obtain a complicated functional equation that has no immediate solutions. For instance, we have the following result.

Proposition 27. *Let N be any strong negation and take* $T = T_0 = \min$. *Then, there is no continuous t-conorms S satisfying* (18).

Proof. Suppose that a t-conorm S satisfies Eq. (18). Taking $b = 0$ in this equation, we obtain

$$S(a, \min(N(a), a)) = a \quad \text{for all } a \in [0,1]. \tag{20}$$

Now, if s is the fixed point of N, for all $a \leqslant s$ we have $a \leqslant N(a)$ and (20) implies $S(a,a) = a$ for all $a \leqslant s$. By continuity of S we have $S(a,b) = \max(a,b)$ for all a, b with $\min(a,b) \leqslant s \leqslant \max(a,b)$.

At the same time, taking a, b in (18) such that $N(a) < N(b) < s < b < a$, we have $S(N(a), b) = b$ and then Eq. (18) becomes

$$a = S(N(S(N(a), b)), \min(S(N(a), b), a)) = S(N(b), \min(b, a)) = b$$

which is a contradiction. Thus, the proposition is proved. □

On the contrary, for D-operators we can give some solutions as stated in the following proposition.

Proposition 28. *Let N be a strong negation with fixed point s, take* $T = T_0 = \min$ *and S a continuous t-conorm. Then, Eq.* (19) *holds if and only if S is such that* $S(a,a) = a$ *for all* $a \leqslant s$.

Proof. Just taking $b = 0$ in (19) we obtain again that S must satisfy Eq. (20) and consequently $S(a,a) = a$ for all $a \leqslant s$.

Conversely, if $S(a,a) = a$ for all $a \leqslant s$ we have $S(a,b) = \max(a,b)$ for all a, b with $\min(a,b) \leqslant s \leqslant \max(a,b)$ as in the previous proposition. Now, Eq. (19) can be checked by considering several cases:

- If $a \leqslant b$ then the left side of Eq. (19) becomes

  $$S(a, \min(N(S(a, N(a))), N(a))) = S(a, N(S(a, N(a))))$$

  and then, when $a \leqslant s$ the above expression is $S(a,a) = a$, whereas when $a > s$, it is $S(a, N(a)) = a$.
- If $b < a < s$, we have $S(b, N(a)) = N(a)$ and then the left side of Eq. (19) becomes

  $$S(a, \min(N(S(b, N(a))), N(a))) = \max(a, \min(a, N(a))) = a.$$

- If $b < a$ and $s \leqslant a$, the result follows similarly by distinguishing two subcases: when $b \leqslant N(a)$ and when $N(a) < b$. □

As a conclusion, the law $(a \to a \cdot b) \to a = a$ is another law for which Mamdani–Larsen operators and R-implications cannot be used, and S-implications only when $S = \max$. Even for QM-operators no simple solutions appear, whereas for D-operators we have found solutions as in Proposition 28.

7. Conclusion

In fuzzy logic, and from the very beginning, it is implicitly accepted that also connectives do have meaning, that they show different uses and that, consequently, for each particular problem there is a type of connectives that is better suited than others do. This is also the case of connective If/then, representable by means of several models some of them coming either from classical or from quantum logic. There is not a universal class for each connective, but the specific features of the given problem and, in particular, those shown by the expression of the involved knowledge, could eventually lead to the needed type of connective.

Like with *and* (t-norms), *or* (t-conorms), and *not* (strong negations), in the case of the numerical functions J representing fuzzy (If/then) rules, there is not a class of them being the universal-class. Each rule, or system of rules, deserves its own J and, to show it as clearly as possible, this paper just followed the strategy of considering four classical (boolean or quantum) very simple laws that, if needed with fuzzy sets cannot be used with the same type of functions J.[2] Namely:

- If the needed law is $\mu \to (\sigma \to \mu) = 1$, only R-implications can be used in general (and several options work for S-, Q- or D-operators).
- If the needed law is $(\mu \to \mu') \to \mu = \mu$, Mamdani–Larsen and R-implications cannot be used, S-implications only for $S = \max$, whereas a lot of Q- and D-operators can be used.
- If the needed law is $(\mu \to \mu') \to \mu' = 1$, no implications of the five models quoted in the preliminaries can be used.
- If the needed law is $(\mu \to \mu \cdot \sigma) \to \mu = \mu$, Mamdani–Larsen and R-implications cannot be used, S-implications only for $S = \max$, for QM-operators no simple solutions appear, whereas for D-operators some solutions can be found.

From all that it follows, in particular, that R-implications J_T are not the only way of representing the knowledge embedded in fuzzy (If/then) rules: R-implications are not, like the other classes of conditional functions, the universal class of fuzzy conditionals.

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References

[1] F. Esteva, L. Godo, Monoidal t-norm based logic: towards a logic for left-continuous t-norms, Fuzzy Sets and Systems 124 (3) (2001) 271–288.
[2] T. Flaminio, E. Marchioni, T-norm-based logics with an independent involutive negation, Fuzzy Sets and Systems 157 (2006) 3125–3144.
[3] J.C. Fodor, M. Roubens, Fuzzy preference modelling and multicriteria decision support, Theory and Decision Library, Series D: System theory, Knowledge Engineering and Problem Solving, Kluwer Academic Publishers, Dordrecht, 1994.
[4] S. Gottwald, A treatise on many-valued logics, Studies in Logic and Computation, vol. 9, Research Studies Press, Baldock, 2001.
[5] P. Hájek, Metamathematics of fuzzy logic, Kluwer, Dordrecht, 1998.
[6] G.M. Hardegree, Material implication in orthomodular (and Boolean) lattices, Notre Dame Journal of Formal Logic 22 (2) (1981) 163–182.
[7] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.
[8] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic. Theory and Applications, Prentice Hall, New Jersey, 1995.
[9] G. Mayor, J. Torrens, Triangular norms on discrete settings, in: E.P. Klement, R. Mesiar (Eds.), Logical, Algebraic, Analytic, and Probabilistic Aspects of Triangular Norms, Elsevier, 2005, pp. 189–230.
[10] M. Mas, M. Monserrat, J. Torrens, S-implications and R-implications on a finite chain, Kybernetika 40 (2004) 3–20.
[11] M. Mas, M. Monserrat, J. Torrens, On two types of discrete implications, International Journal of Approximate Reasoning 40 (2005) 262–279.
[12] M. Mas, M. Monserrat, J. Torrens, QL-implications versus D-Implications, Kybernetika 42 (2006) 351–366.
[13] M. Mas, M. Monserrat, J. Torrens, E. Trillas, A survey on fuzzy implication functions, IEEE Transactions on Fuzzy Systems 15 (6) (2007) 1107–1121.
[14] E. Trillas, Sobre funciones de negación en la teoría de conjuntos difusos, Stochastica III (1979) 47–60.

[2] It could be also interesting to study these properties from the point of view of formal fuzzy logic in a similar way as in [2], where it is explained how a QM-implication or a right-continuous t-conorm can be expressed in these logical formalisms.

[15] E. Trillas, C. Alsina, A. Pradera, On MPT-implication functions for fuzzy logic, Revista de la Real Academia de Ciencias. Serie A. Matemáticas 98 (2004) 259–271.
[16] T. Whalen, Interpolating between fuzzy rules using improper S-implications, International Journal of Approximate Reasoning 45 (2007) 30–42.
[17] L. Wittgenstein, Philosophical Investigations, Basil Blackwell Publisher Ltd., Oxford, 1958.