J. Math. Anal. Appl. 378 (2011) 109–116

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The Riemannian product structures of spacelike hypersurfaces with constant k-th mean curvature in the de Sitter spaces

Jiancheng Liu *, Yaowen Gao

College of Mathematics and Information Science, Northwest Normal University, Lanzhou 730070, China

ARTICLE INFO

Article history:
Received 17 January 2010
Available online 20 January 2011
Submitted by Steven G. Krantz

Keywords:
Riemannian product structure
Spacelike hypersurface
de Sitter space
k-th mean curvature

ABSTRACT

In this paper, we investigate complete spacelike hypersurfaces in the de Sitter space $S_1^{n+1}(c)$ with constant k-th mean curvature and two distinct principal curvatures one of which is simple. We obtain some characterizations of the Riemannian product $\mathbb{H}^1(c_1) \times \mathbb{S}^{n-1}(c_2)$ or $\mathbb{H}^{n-1}(c_1) \times \mathbb{S}^1(c_2)$ in the de Sitter space $S_1^{n+1}(c)$.

© 2011 Elsevier Inc. All rights reserved.

1. Introduction and main result

Let $M_1^{n+1}(c)$ be an $(n+1)$-dimensional Lorentzian manifold of constant curvature c, which we call a *Lorentzian space form*. Then a Lorentzian space form $M_1^{n+1}(c)$ is said to be *a de Sitter space* $S_1^{n+1}(c)$, *a Lorentz–Minkowski space* L^{n+1} or *an anti-de Sitter space* $H_1^{n+1}(c)$ respectively, according to its sectional curvature $c > 0$, $c = 0$ or $c < 0$. A hypersurface M in a Lorentzian space form $M_1^{n+1}(c)$ is said to be *spacelike* if the induced metric on M from that of $M_1^{n+1}(c)$ is positive definite.

The study of spacelike hypersurfaces in Lorentzian space forms has been of substantial interest from both physical and mathematical points of view, and has been under very extensive study by many geometricians. From the physical one, that interest became clear when Lichnerowicz showed that the Cauchy problem of the Einstein equation with initial conditions on a spacelike hypersurface with vanishing mean extrinsic curvature has a particularly nice form, reducing to a linear differential system of first order and to a non-linear second order elliptic differential equation. Also, it turns out that the knowledge of spacelike hypersurfaces in de Sitter spaces can give information about the causal structure in this interesting class of spacetimes.

From the geometric point of view, it is seen that a complete spacelike hypersurface of a Lorentz–Minkowski space L^{n+1} possesses a remarkable Bernstein property in the maximal case by E. Calabi [7], S.Y. Cheng and S.T. Yau [10]. The initial step for the study of spacelike hypersurfaces in de Sitter space is due to A.J. Goddard [11], that conjectured that *every complete spacelike hypersurface in* $S_1^{n+1}(c)$ *with constant mean curvature must be totally umbilical.* Since Goddard's conjecture has been completely settled (cf. [2,19] and [21] for details), most of the research interest turns to the study of hypersurfaces in $S_1^{n+1}(c)$ with constant scalar curvature instead of constant mean curvature. Especially, the interest focuses on characterizing the totally umbilical properties of such hypersurfaces. A classical result due to Q.-M. Cheng and S. Ishikawa [9] states that the totally round spheres are the only compact spacelike hypersurfaces in $S_1^{n+1}(c)$ with constant normalized scalar curvature $R < c$. For a more closely study related to the complete spacelike hypersurfaces in $S_1^{n+1}(c)$ with constant scalar curvature, we refer to [5,6,16–18,23] and the references therein.

* Corresponding author.
E-mail addresses: liujc@nwnu.edu.cn (J. Liu), ldxyg1984@163.com (Y. Gao).

0022-247X/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2011.01.035

Recently, another important and popular problem is to describe the Riemannian product structure of spacelike hypersurfaces in de Sitter space $S_1^{n+1}(c)$ with constant scalar curvature and with two distinct principal curvatures. For instance, Z.-J. Hu et al. [12] and S.-C. Shu [22] proved that the spacelike hypersurface in de Sitter space $S_1^{n+1}(c)$ with constant scalar curvature and with two distinct principal curvatures whose multiplicities are greater than 1 is isometric to the Riemannian product $\mathbb{H}^k(c_1)\times\mathbb{S}^{n-k}(c_2)$, $1<k<n-1$. Here, $\mathbb{H}^k(c_1)$ is a k-dimensional hyperbolic space with constant sectional curvature $c_1<0$ and $\mathbb{S}^{n-k}(c_2)$ is well known as an $(n-k)$-dimensional sphere with constant sectional curvature c_2. c_1 and c_2 are related by $\frac{1}{c_1}+\frac{1}{c_2}=\frac{1}{c}$. Furthermore, Z.-J. Hu et al. [12] investigated the similar problem for the case of the multiplicity of one of the principal curvatures is $n-1$, and proved the following theorem.

Theorem 1.1. *(See [12].) Let M^n $(n\geqslant 3)$ be an n-dimensional complete spacelike hypersurface in $S_1^{n+1}(1)$ with two distinct principal curvatures.*

(i) *Assume that M^n has constant scalar curvature $n(n-1)R$ and that the multiplicity of one of the principal curvatures is $n-1$, then $R<\frac{n-2}{n}$. Moreover, if we assume that $R\neq 0$ and that the squared length S of the second fundamental form of M^n satisfies*

$$S\geqslant(n-1)\frac{n-2-nR}{n-2}+\frac{n-2}{n-2-nR},$$

then M^n is isometric either to the Riemannian product $\mathbb{H}^1(\frac{-nR}{n-2-nR})\times\mathbb{S}^{n-1}(\frac{nR}{n-2})$ for $R>0$ or to the Riemannian product $\mathbb{H}^{n-1}(\frac{nR}{n-2})\times\mathbb{S}^1(\frac{-nR}{n-2-nR})$ for $R<0$.

(ii) *Assume that M^n has constant scalar curvature $n(n-1)R$, $R>0$, and that the multiplicity of one of the principal curvatures is $n-1$. If, in addition, the squared length S of the second fundamental form of M^n satisfies*

$$S\leqslant(n-1)\frac{n-2-nR}{n-2}+\frac{n-2}{n-2-nR},$$

then M^n is isometric to the Riemannian product $\mathbb{H}^1(\frac{-nR}{n-2-nR})\times\mathbb{S}^{n-1}(\frac{nR}{n-2})$.

It is well known that the k-th mean curvatures H_k, for $k=1,\dots,n$, of a given hypersurface in Lorentzian space forms are the natural generalizations of mean curvature for $k=1$ and normalized scalar curvature for $k=2$ (up to a constant), for the details, see Section 2. Therefore, it is also a natural and interesting thing to characterize the totally umbilical properties and to investigate the Riemannian product structures of spacelike hypersurfaces in a Lorentzian space form $M_1^{n+1}(c)$ with constant k-th mean curvature H_k for some $k\in[1,n]$. We refer reader to [3,4,15] and the references therein for that things of characterizing the totally umbilical properties of such hypersurfaces.

In this paper, we will focus our attention on studying the Riemannian product structures for spacelike hypersurfaces in the de Sitter space $S_1^{n+1}(c)$ with constant k-th mean curvature and two distinct principal curvatures whose multiplicities are $n-1$ and 1, respectively. In fact, we will prove the following result.

Theorem 1.2. *Let M^n $(n\geqslant 3)$ be an n-dimensional complete spacelike hypersurface in the de Sitter space $S_1^{n+1}(c)$ with constant k-th mean curvature H_k (>0) for some $k\in[2,n]$ and with two distinct principal curvatures one of which is simple. Assume that $H_k^{\frac{2}{k}}\neq c$. If the squared length S of the second fundamental form of M^n in $S_1^{n+1}(c)$ satisfies*

$$S\geqslant(n-1)t_0^{\frac{2}{k}}+c^2t_0^{-\frac{2}{k}}$$

or

$$S\leqslant(n-1)t_0^{\frac{2}{k}}+c^2t_0^{-\frac{2}{k}},$$

then M^n is isometric to the Riemannian product $\mathbb{H}^1(c_1)\times\mathbb{S}^{n-1}(c_2)$ or $\mathbb{H}^{n-1}(c_1)\times\mathbb{S}^1(c_2)$, where $c_1<0$, $c_2>0$ and $\frac{1}{c_1}+\frac{1}{c_2}=\frac{1}{c}$, t_0 is the positive real root of the equation $P_{H_k}(t)\equiv ckt^{\frac{k-2}{k}}+(n-k)t-nH_k=0$ for $t>0$.

Remark 1. We will show in Lemma 3.2 that the equation $P_{H_k}(t)\equiv ckt^{\frac{k-2}{k}}+(n-k)t-nH_k=0$ for $t>0$ has actually only one positive root when $H_k>0$ for any $k\geqslant 2$. The motivation of constructing such a function $P_{H_k}(t)$ of t will be explained in Remark 3.

Remark 2. Let $k=2$, $c=1$, then $H_2=1-R$ (here R is the normalized scalar curvature). The assumption $H_k^{\frac{2}{k}}\neq c$ reduces to $H_2\neq 1$, equivalently, $R\neq 0$. Also for $k=2$ and $c=1$, the only one root of the equation $P_{H_k}(t)\equiv ckt^{\frac{k-2}{k}}+(n-k)t-nH_k=0$ is $t_0=\frac{n(1-R)-2}{n-2}$. We remark that $t_0>0$ because of $R<\frac{n-2}{n}$ by Theorem 1.1. At that time, it is easy to check that $(n-1)t_0^{\frac{2}{k}}+c^2t_0^{-\frac{2}{k}}=(n-1)\frac{n-2-nR}{n-2}+\frac{n-2}{n-2-nR}$. So the assumption $S\geqslant(n-1)t_0^{\frac{2}{k}}+c^2t_0^{-\frac{2}{k}}$ or $S\leqslant(n-1)t_0^{\frac{2}{k}}+c^2t_0^{-\frac{2}{k}}$ in Theorem 1.2 reduces to $S \geqslant (n-1)\frac{n-2-nR}{n-2} + \frac{n-2}{n-2-nR}$ or $S \leqslant (n-1)\frac{n-2-nR}{n-2} + \frac{n-2}{n-2-nR}$, respectively. Therefore, our main Theorem 1.2 for $k=2$ reduces to Theorem 1.1.

Remark 3. The well-known standard models (see U.H. Ki et al. [14]) $\mathbb{H}^m(c_1) \times \mathbb{S}^{n-m}(c_2)$, $m = 1, 2, \ldots, n-1$, are the complete hypersurfaces with nonzero constant k-th mean curvature in the de Sitter space $S_1^{n+1}(c)$, where $c_1 < 0$, $c_2 > 0$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$. We note that $\mathbb{H}^m(c_1) \times \mathbb{S}^{n-m}(c_2)$ has two distinct principal curvatures $\sqrt{c - c_1}$ with multiplicity m and $\sqrt{c - c_2}$ with multiplicity $n-m$. In particular, hyperbolic cylinder $\mathbb{H}^1(c_1) \times \mathbb{S}^{n-1}(c_2)$ or spherical cylinder $\mathbb{H}^{n-1}(c_1) \times \mathbb{S}^1(c_2)$ has two distinct principal curvatures one of which is simple. Without loss of generality, we may put $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda$, $\lambda_n = \mu$. Notice that $\lambda \neq 0$, $\mu \neq 0$ and $\lambda\mu = c$, then the squared length S^* of the second fundamental form of $\mathbb{H}^1(c_1) \times \mathbb{S}^{n-1}(c_2)$ or $\mathbb{H}^{n-1}(c_1) \times \mathbb{S}^1(c_2)$ in $S_1^{n+1}(c)$ is

$$S^* = (n-1)\lambda^2 + \mu^2 = (n-1)\left(\lambda^k\right)^{\frac{2}{k}} + c^2\left(\lambda^k\right)^{-\frac{2}{k}}.$$

Solving μ from Eq. (2.7) and substituting into the formula $\lambda\mu = c$ yields

$$ck\lambda^{k-2} + (n-k)\lambda^k - nH_k = 0.$$

Putting $t = \lambda^k$, the above equation becomes

$$ckt^{\frac{k-2}{k}} + (n-k)t - nH_k = 0.$$

This explains where the function $P_{H_k}(t) = ckt^{\frac{k-2}{k}} + (n-k)t - nH_k$ of t in our main Theorem 1.2 arises from.

Remark 4. The similar Riemannian product results for spacelike hypersurfaces in an anti-de Sitter space $H_1^{n+1}(c)$ have been obtained by L. Cao and G. Wei in [8], Y. Jin Suh and G. Wei in [13].

2. Preliminaries

Let M^n be an n-dimensional spacelike hypersurface of $S_1^{n+1}(c)$. We choose a local field of pseudo-Riemannian orthonormal frames $e_1, \ldots, e_{n+1}$ in $S_1^{n+1}(c)$ with dual coframe $\omega_1, \ldots, \omega_{n+1}$, such that, at each point of M^n, $e_1, \ldots, e_n$ are tangent to M^n and e_{n+1} is the unit timelike normal vector. Then the structure equations of $S_1^{n+1}(c)$ are given by

$$\mathrm{d}\omega_A = \sum_{B=1}^{n+1} \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,\ \varepsilon_i = 1,\ \varepsilon_{n+1} = -1,$$

$$\mathrm{d}\omega_{AB} = \sum_{C=1}^{n+1} \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D=1}^{n+1} K_{ABCD}\omega_C \wedge \omega_D,$$

$$K_{ABCD} = c\varepsilon_A\varepsilon_B(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

Restricted to M^n, then $\omega_{n+1} = 0$ and there are, by Cartan's lemma, symmetric functions h_{ij} such that $\omega_{in+1} = \sum_j h_{ij}\omega_j$ (here and in the sequel, we use the convention for the range of indices: $1 \leqslant i, j, \ldots \leqslant n$). This gives the second fundamental form of M^n, $B = \sum_{i,j} h_{ij}\omega_i\omega_j$ with squared length $S = \sum_{i,j} h_{ij}^2$. The mean curvature H is defined by $H = \frac{1}{n}\sum_i h_{ii}$.

From all of which, we obtain the structure equations of M^n

$$\mathrm{d}\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.1}$$

$$\mathrm{d}\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.2}$$

and the Gauss equation

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.3}$$

Let h_{ijk} denotes the covariant derivative of h_{ij}, then we have

$$\sum_k h_{ijk}\omega_k = \mathrm{d}h_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \tag{2.4}$$

The Codazzi equation is

$$h_{ijk} = h_{ikj}. \tag{2.5}$$

Let H_k, $1 \leqslant k \leqslant n$, be the k-th mean curvatures of the spacelike hypersurface M^n in $S_1^{n+1}(c)$, which are defined by

$$\binom{n}{k} H_k = \sigma_k(\lambda_1, \ldots, \lambda_n), \tag{2.6}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, and $\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \lambda_{i_1} \cdots \lambda_{i_k}$, $1 \leqslant k \leqslant n$, be the normalized symmetric functions of principal curvatures $\lambda_1, \ldots, \lambda_n$. In particular, when $k = 1$, $H_1 = H$ is nothing but the mean curvature of M^n, which is the main extrinsic curvature of the hypersurface. H_n defines the Gauss–Kronecker curvature of M^n. On the other hand, H_2 defines a geometric quality which is related to the (intrinsic) scalar curvature of M^n. Indeed, a straightforward calculation by using Gauss equation of M^n, we can show that the scalar curvature of M^n is $n(n-1)(c - H_2)$, in other words, its normalized scalar curvature is $c - H_2$. We refer reader to [3] for details.

When the spacelike hypersurface M^n in $S_1^{n+1}(c)$ has two distinct principal curvatures λ and μ with multiplicities $n-1$ and 1, respectively, then we have from (2.6) that

$$\binom{n}{k} H_k = \binom{n-1}{k} \lambda^k + \binom{n-1}{k-1} \lambda^{k-1} \mu,$$

this implies that

$$\lambda^{k-1}\big((n-k)\lambda + k\mu\big) = nH_k. \tag{2.7}$$

In order to prove our main Theorem 1.2, we need the following lemma which can be proved by using the same method as in [20] due to T. Otsuki, see also Z.-J. Hu et al. [12].

Lemma 2.1. *Let M^n be a spacelike hypersurface in a de Sitter space $S_1^{n+1}(c)$ such that the multiplicities of the principal curvatures are all constant. Then the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than* 1, *then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of the principal vectors.*

3. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. For the sake of succinctness, we need the following four lemmas which can be viewed as the key steps in the proof of Theorem 1.2.

Lemma 3.1. *Let M^n $(n \geqslant 3)$ be an n-dimensional oriented spacelike hypersurface in $S_1^{n+1}(c)$ with constant k-th mean curvature $H_k > 0$ and two distinct principal curvatures λ and μ with multiplicities $n-1$ and* 1, *respectively. Then $\bar{w} = |\lambda^k - H_k|^{-\frac{1}{n}}$ satisfies the following ordinary differential equation of order* 2:

$$\frac{\mathrm{d}^2 \bar{w}}{\mathrm{d}s^2} + \bar{w} \frac{ck\lambda^{k-2} + (n-k)\lambda^k - nH_k}{k\lambda^{k-2}} = 0. \tag{3.1}$$

Proof. Noticing the assumption $H_k > 0$, then (2.7) ensures $\lambda \neq 0$. For $k \geqslant 2$, we can solve from (2.7) that

$$\mu = \frac{nH_k - (n-k)\lambda^k}{k\lambda^{k-1}}, \qquad \lambda - \mu = \frac{n(\lambda^k - H_k)}{k\lambda^{k-1}}. \tag{3.2}$$

Denote the integral submanifold through $x \in M^n$ corresponding to λ by $M_1^{n-1}(x)$, and write $\mathrm{d}\lambda = \sum_i \lambda_{,i}\omega_i$, $\mathrm{d}\mu = \sum_j \mu_{,j}\omega_j$, where $\lambda_{,i} = e_i(\lambda)$, $\mu_{,j} = e_j(\mu)$. Then Lemma 2.1 implies that

$$\lambda_{,1} = \lambda_{,2} = \cdots = \lambda_{,n-1} = 0 \quad \text{on } M_1^{n-1}(x). \tag{3.3}$$

Taking exterior differentiation of the first formula in (3.2), and using (3.3), we get

$$\mu_{,1} = \mu_{,2} = \cdots = \mu_{,n-1} = 0 \quad \text{on } M_1^{n-1}(x). \tag{3.4}$$

Choosing $e_1, \ldots, e_n$ such that $h_{ij} = \lambda_i \delta_{ij}$ and using (2.4), (2.5), (3.3) and (3.4), it is easy to see that

$$\sum_k h_{abk}\omega_k = \delta_{ab}\lambda_{,n}\omega_n \quad \text{for } 1 \leqslant a, b \leqslant n-1,$$

$$\sum_k h_{nnk}\omega_k = \mu_{,n}\omega_n.$$

These two formulas imply that $h_{abn} = \delta_{ab}\lambda_{,n}$ for $1 \leqslant a, b \leqslant n-1$ and $h_{nna} = 0$. Furthermore,

$$\sum_i h_{ani}\omega_i = \sum_b h_{anb}\omega_b + h_{ann}\omega_n = \sum_b \delta_{ab}\lambda_{,n}\omega_b = \lambda_{,n}\omega_a. \tag{3.5}$$

On the other hand, we know from (2.4) that

$$\sum_i h_{ani}\omega_i = \mathrm{d}h_{an} + \sum_i h_{in}\omega_{ia} + \sum_i h_{ai}\omega_{in} = (\lambda - \mu)\omega_{an}. \tag{3.6}$$

Comparing (3.5) with (3.6), it follows that

$$\omega_{an} = \frac{\lambda_{,n}}{\lambda - \mu}\omega_a. \tag{3.7}$$

According to the structure equations of M^n, we get from (3.7) that

$$\mathrm{d}\omega_n = 0. \tag{3.8}$$

Since the multiplicities of λ and μ are constant, their eigenspaces are completely integrable. Notice that $\nabla_{e_n} e_n = -\sum_a \omega_{na}(e_n)e_a = 0$ (here ∇ is the Levi-Civita connection of M^n), the integral curves corresponding to μ are geodesics, and they are orthogonal trajectories of the family of the integral submanifolds corresponding to λ. Let s be the arc length of the geodesic corresponding to μ. Taking into count of (3.8), we may put $\omega_n = \mathrm{d}s$ (cf. [20]). Thus, we may consider $\lambda = \lambda(s)$ to be locally a function of s, consequently, $\mathrm{d}\lambda = \lambda_{,n}\mathrm{d}s$ by (3.3). Substituting into (3.7) and using (3.2), then

$$\omega_{an} = \frac{\mathrm{d}\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s}\omega_a. \tag{3.9}$$

Taking exterior differentiation of (3.9), we derive

$$\begin{aligned}
\mathrm{d}\omega_{an} &= \frac{\mathrm{d}^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s^2}\,\mathrm{d}s \wedge \omega_a + \frac{\mathrm{d}\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s}\,\mathrm{d}\omega_a \\
&= -\frac{\mathrm{d}^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s^2}\omega_a \wedge \mathrm{d}s + \frac{\mathrm{d}\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s}\left(\sum_{b=1}^{n-1}\omega_{ab} \wedge \omega_b + \omega_{an} \wedge \omega_n\right) \\
&= \left\{-\frac{\mathrm{d}^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s^2} + \left[\frac{\mathrm{d}\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s}\right]^2\right\}\omega_a \wedge \mathrm{d}s \\
&\quad + \frac{\mathrm{d}\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s}\sum_{b=1}^{n-1}\omega_{ab} \wedge \omega_b.
\end{aligned} \tag{3.10}$$

On the other hand, using (2.1), (2.2), (2.3) and (3.9), a standard computation gives

$$\begin{aligned}
\mathrm{d}\omega_{an} &= \sum_{b=1}^{n-1}\omega_{ab} \wedge \omega_{bn} - \frac{1}{2}\sum_{k,l}R_{ankl}\omega_k \wedge \omega_l \\
&= \sum_{b=1}^{n-1}\omega_{ab} \wedge \omega_{bn} + (\lambda\mu - c)\omega_a \wedge \omega_n \\
&= \frac{\mathrm{d}\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s}\sum_{b=1}^{n-1}\omega_{ab} \wedge \omega_b + (\lambda\mu - c)\omega_a \wedge \mathrm{d}s.
\end{aligned} \tag{3.11}$$

Comparing (3.10) and (3.11), we obtain

$$\frac{\mathrm{d}^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s^2} - \left\{\frac{\mathrm{d}\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s}\right\}^2 + (\lambda\mu - c) = 0. \tag{3.12}$$

Using (3.12) and (3.2), for $\bar{w}(s) = |\lambda^k - H_k|^{-\frac{1}{n}}$, $s \in (-\infty, +\infty)$, a straightforward calculation finishes the proof of Lemma 3.1. □

Lemma 3.2. *Let $P_{H_k}(t) = ckt^{\frac{k-2}{k}} + (n-k)t - nH_k$, $t > 0$, be the same as in Theorem 1.2, where $c > 0$, $k \geqslant 2$ and $H_k > 0$ is constant. Then $P_{H_k}(t)$ is a strictly monotone increasing function of t and has unique positive root, denoted by t_0. Moreover,*

(1) *$t_0 < H_k$ when $H_k^{\frac{2}{k}} < c$.*

(2) *$t_0 > H_k$ when $H_k^{\frac{2}{k}} > c$.*

Proof. Since $\frac{\mathrm{d}P_{H_k}(t)}{\mathrm{d}t} = c(k-2)t^{-\frac{2}{k}} + (n-k) > 0$ for $t > 0$, it follows that $P_{H_k}(t)$ is a strictly monotone increasing function of t and $\lim_{t\to+\infty} P_{H_k}(t) = +\infty$.

It is obvious that $\lim_{t\to 0^+} P_{H_k}(t) = -nH_k < 0$ for $k > 2$. When $k = 2$, the normalized scalar curvature $R = c - H_2$ as we pointed out in Section 2. On the other hand, Theorem 1.1 asserts that $R < \frac{n-2}{n}c$, equivalently, $H_2 > \frac{2}{n}c$. Hence, $\lim_{t\to 0^+} P_{H_2}(t) = 2c - nH_2 < 0$. In summary, we conclude that $\lim_{t\to 0^+} P_{H_k}(t) < 0$ for $k \geqslant 2$. Therefore, according to the continuous property of $P_{H_k}(t)$, we infer that $P_{H_k}(t) = 0$ has only one positive root.

Note that $P_{H_k}(H_k) = kH_k^{\frac{k-2}{k}}(c - H_k^{\frac{2}{k}})$, so $P_{H_k}(H_k) < 0$ for $H_k^{\frac{2}{k}} > c$ and $P_{H_k}(H_k) > 0$ for $H_k^{\frac{2}{k}} < c$. Since $P_{H_k}(t)$ is a strictly monotone increasing function of t and t_0 is a positive root of $P_{H_k}(t) = 0$, henceforth, $t_0 > H_k$ for $H_k^{\frac{2}{k}} > c$ and $t_0 < H_k$ for $H_k^{\frac{2}{k}} < c$. This completes the proof of Lemma 3.2. □

Lemma 3.3. *Let M^n be the same hypersurface as in Theorem* 1.2 *with constant k-th mean curvature $H_k > 0$ and with two distinct principal curvatures λ and μ. Suppose that the multiplicity of λ is $n-1$. If $H_k^{\frac{2}{k}} \neq c$, then $\lambda^k \neq H_k$. Moreover*:

(1) *If $H_k^{\frac{2}{k}} > c$, then $\lambda^k > H_k$.*
(2) *If $H_k^{\frac{2}{k}} < c$, then $\lambda^k < H_k$.*

Proof. Because of $\lambda \neq \mu$, it follows from (3.2) that $\lambda^k \neq H_k$.

Since $\lambda \neq 0$ by the assumption $H_k > 0$ and Eq. (2.7), choosing the appropriate orientation of M^n, we may suppose that $\lambda > 0$ on M^n. According to the definition of $P_{H_k}(t)$, we can rewrite (3.1) as

$$\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}s^2} + \bar{w}\frac{P_{H_k}(\lambda^k)}{k\lambda^{k-2}} = 0. \tag{3.13}$$

(1) When $H_k^{\frac{2}{k}} > c$, Lemma 3.2 asserts $H_k < t_0$. Suppose now, by contradiction, that $\lambda^k < H_k$, then $\lambda^k < t_0$. Consequently, $P_{H_k}(\lambda^k) < 0$ and $\frac{\mathrm{d}^2\bar{w}(s)}{\mathrm{d}s^2} > 0$ by (3.13). Thus $\frac{\mathrm{d}\bar{w}}{\mathrm{d}s}$ is a strictly monotone increasing function of s and has at most one zero point for $s \in (-\infty, +\infty)$. If $\frac{\mathrm{d}\bar{w}(s)}{\mathrm{d}s}$ has no zero point in $(-\infty, +\infty)$, then $\bar{w}(s)$ is a monotone function of s in $(-\infty, +\infty)$. If $\frac{\mathrm{d}\bar{w}(s)}{\mathrm{d}s}$ has exactly one zero point s_0 in $(-\infty, +\infty)$, then $\bar{w}(s)$ is a monotone function of s in $(-\infty, s_0]$ and $[s_0, +\infty)$. Notice that $\bar{w}(s)$ is bounded by definition, so both $\lim_{s\to-\infty} \bar{w}(s)$ and $\lim_{s\to+\infty} \bar{w}(s)$ exist and

$$\lim_{s\to-\infty} \frac{\mathrm{d}\bar{w}(s)}{\mathrm{d}s} = \lim_{s\to+\infty} \frac{\mathrm{d}\bar{w}(s)}{\mathrm{d}s} = 0.$$

This is impossible because $\frac{\mathrm{d}\bar{w}(s)}{\mathrm{d}s}$ is a strictly monotone increasing function of s. Therefore, it must be $\lambda^k > H_k$ when $H_k^{\frac{2}{k}} > c$.

(2) When $H_k^{\frac{2}{k}} < c$, then $H_k > t_0$ by Lemma 3.2. Suppose now, by contradiction, that $\lambda^k > H_k$, then $\lambda^k > t_0$. Consequently, $P_{H_k}(\lambda^k) > 0$ and $\frac{\mathrm{d}^2\bar{w}(s)}{\mathrm{d}s^2} < 0$ by (3.13). Thus $\frac{\mathrm{d}\bar{w}}{\mathrm{d}s}$ is a strictly monotone decreasing function of s. This will lead to a contradiction by taking the similar argument as in case (1), so it must be $\lambda^k < H_k$ when $H_k^{\frac{2}{k}} < c$. This completes the proof of Lemma 3.3. □

Lemma 3.4. *For $t > 0$, $k \geqslant 2$ and H_k a positive constant. Let*

$$f(t) = \frac{1}{k^2 t^{\frac{2k-2}{k}}}\left\{(n-1)k^2t^2 + \left((n-k)t - nH_k\right)^2\right\}.$$

Then $f(t_0) = (n-1)t_0^{\frac{2}{k}} + c^2 t_0^{-\frac{2}{k}}$, where t_0 is the only positive root of the equation $P_{H_k}(t) = 0$ obtained as in Lemma 3.2. *Also $f(t)$ is an increasing* (*resp. decreasing*) *function of t for $t \geqslant H_k$* (*resp.* $0 < t \leqslant H_k$).

Proof. Using the fact $P_{H_k}(t_0) = 0$, it is easy to verify $f(t_0) = (n-1)t_0^{\frac{2}{k}} + c^2 t_0^{-\frac{2}{k}}$.

After a directly computation, we have

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \frac{2t^{\frac{2-3k}{k}}}{k^3}\left\{\left(n^2 - 2nk + nk^2\right)t^2 + n(k-2)(n-k)H_k t + (1-k)n^2H_k^2\right\}.$$

Putting $g(t) \equiv (n^2 - 2nk + nk^2)t^2 + n(k-2)(n-k)H_k t + (1-k)n^2H_k^2$, then $g(H_k) = 0$. We remind that H_k is the only one solution of $g(t) = 0$ since $t > 0$. In view of $n^2 - 2nk + nk^2 > 0$ for $k \geqslant 2$, then $g(H_k) = 0$ implies that $g(t) \leqslant 0$ for $0 < t \leqslant H_k$ and $g(t) \geqslant 0$ for $t \geqslant H_k$. Consequently, $\frac{\mathrm{d}f(t)}{\mathrm{d}t} \leqslant 0$ (resp. $\geqslant 0$) for $0 < t \leqslant H_k$ (resp. $t \geqslant H_k$). Hence, Lemma 3.4 follows immediately. □

Now we are ready to prove our main Theorem 1.2.

Proof of Theorem 1.2. Without loss of generality, we may put $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda$, $\lambda_n = \mu$. By means of the assumption $H_k^{\frac{2}{k}} \neq c$, there are two cases, i.e. $H_k^{\frac{2}{k}} > c$ or $H_k^{\frac{2}{k}} < c$. Either of the cases ensures that $P_{H_k}(t)$ has unique positive root t_0 according to Lemma 3.2. We remind from Lemma 3.4 that $f(t_0) = (n-1)t_0^{\frac{2}{k}} + c^2 t_0^{-\frac{2}{k}}$. Meanwhile, $\lambda \neq 0$ because of the assumption $H_k > 0$ and Eq. (2.7), by choosing appropriate orientation of M^n, we may suppose $\lambda > 0$ on M^n so that $f(\lambda^k)$ is meaningful. It is easy to check that $f(\lambda^k) = (n-1)\lambda^2 + \mu^2 = S$.

If $S \geqslant (n-1)t_0^{\frac{2}{k}} + c^2 t_0^{-\frac{2}{k}}$, then

$$f(\lambda^k) = S \geqslant (n-1)t_0^{\frac{2}{k}} + c^2 t_0^{-\frac{2}{k}} = f(t_0). \tag{3.14}$$

Case (i): $H_k^{\frac{2}{k}} > c$. According to Lemmas 3.2 and 3.3, we know that $t_0 > H_k$ and $\lambda^k > H_k$. It follows from Lemma 3.4 and inequality (3.14) that $\lambda^k \geqslant t_0$. Since $P_{H_k}(t)$ is a strictly monotone increasing function of t with zero point t_0, so $P_{H_k}(\lambda^k) \geqslant 0$. This fact together with (3.13) leads to $\frac{d^2\bar{w}}{ds^2} \leqslant 0$, which implies that $\frac{d\bar{w}}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$, so does $\bar{w}(s)$ when s tends to infinity. Meanwhile, $\bar{w}(s)$ is bounded by virtue of the definition. Hence, both $\lim_{s\to-\infty} \bar{w}(s)$ and $\lim_{s\to+\infty} \bar{w}(s)$ exist and

$$\lim_{s\to-\infty} \frac{d\bar{w}(s)}{ds} = \lim_{s\to+\infty} \frac{d\bar{w}(s)}{ds} = 0.$$

Using again the monotonicity of $\frac{d\bar{w}(s)}{ds}$, we conclude that $\frac{d\bar{w}(s)}{ds} \equiv 0$ and $\bar{w}(s)$ is constant. Furthermore, λ is constant on M^n because of $\bar{w} = |\lambda^k - H_k|^{-\frac{1}{n}}$. Making use of (3.2), we also know μ is constant on M^n. Therefore, M^n is an isoparametric hypersurface. According to the congruence theorem of N. Abe et al. [1], we know that M^n is isometric to the Riemannian product $\mathbb{H}^1(c_1) \times \mathbb{S}^{n-1}(c_2)$ or $\mathbb{H}^{n-1}(c_1) \times \mathbb{S}^1(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$ and $c_2 > 0$.

Case (ii): $H_k^{\frac{2}{k}} < c$. In this case, Lemmas 3.2 and 3.3 assert $t_0 < H_k$ and $\lambda^k < H_k$. Then Lemma 3.4 and the inequality (3.14) imply that $\lambda^k \leqslant t_0$. Since $P_{H_k}(t)$ is a strictly monotone increasing function of t with zero point t_0, so $P_{H_k}(\lambda^k) \leqslant 0$. This fact together with (3.13) leads to $\frac{d^2\bar{w}}{ds^2} \geqslant 0$, which implies that $\frac{d\bar{w}}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$, so does $\bar{w}(s)$ when s tends to infinity. The same argument as in the case (i) will show that the principal curvatures λ and μ are constant on M^n, i.e. M^n is an isoparametric hypersurface. We conclude that M^n is isometric to the Riemannian product $\mathbb{H}^1(c_1) \times \mathbb{S}^{n-1}(c_2)$ or $\mathbb{H}^{n-1}(c_1) \times \mathbb{S}^1(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, $c_1 < 0$ and $c_2 > 0$.

If $S \leqslant (n-1)t_0^{\frac{2}{k}} + c^2 t_0^{-\frac{2}{k}}$, then

$$f(\lambda^k) = S \leqslant (n-1)t_0^{\frac{2}{k}} + c^2 t_0^{-\frac{2}{k}} = f(t_0). \tag{3.15}$$

Applying (3.15) instead of (3.14), the same argument finishes the proof of Theorem 1.2. □

Acknowledgments

The authors wish to thank the referee for his/her carefully reading of the original manuscript and to express their gratitude for his/her suggestions which improve the paper. The first author was partially supported by National Natural Science Foundation of China (Grant No. 10871138).

References

[1] N. Abe, N. Koike, S. Yamaguchi, Congruence theorems for proper semi-Riemannian hypersurfaces in a real space forms, Yokohama Math. J. 35 (1987) 123–136.
[2] K. Akutagawa, On spacelike hypersurfaces with constant mean curvature in the de Sitter space, Math. Z. 196 (1987) 13–19.
[3] J.A. Aledo, L.J. Alías, A. Romero, Integral formulas for compact space-like hypersurfaces in de Sitter space: Applications to the case of constant higher order mean curvature, J. Geom. Phys. 31 (1999) 195–208.
[4] L.J. Alías, S.E. Koh, Remarks on compact spacelike hypersurfaces in de Sitter space with constant higher order mean curvature, J. Geom. Phys. 39 (2004) 45–49.
[5] A. Brasil Jr., A.G. Colares, On complete spacelike hypersurfaces with constant scalar curvature in the de Sitter space, An. Acad. Bras. Ci. 72 (2000) 445–452.
[6] A. Brasil Jr., A.G. Colares, O. Palmas, A gap theorem for complete constant scalar curvature hypersurfaces in the de Sitter space, J. Geom. Phys. 37 (2001) 237–250.
[7] E. Calabi, Examples of Bernstein problems for some nonlinear equations, Proc. Sympos. Pure Appl. Math. 15 (1970) 203–230.
[8] L.-F. Cao, G.-X. Wei, A new characterization of hyperbolic cylinder in anti-de Sitter space $H_1^{n+1}(-1)$, J. Math. Anal. Appl. 329 (2007) 408–414.
[9] Q.-M. Cheng, S. Ishikawa, Spacelike hypersurfaces with constant scalar curvature, Manuscripta Math. 99 (1998) 499–505.
[10] S.Y. Cheng, S.T. Yau, Maximal spacelike hypersurfaces in the Lorentz–Minkowski spaces, Ann. Math. 104 (1976) 407–419.
[11] A.J. Goddard, Some remarks on the existence of space-like hypersurfaces of constant mean curvature, Math. Proc. Cambridge Philos. Soc. 82 (1977) 489–495.
[12] Z.-J. Hu, M. Scherfner, S.-J. Zhai, On spacelike hypersurfaces with constant scalar curvature in the de Sitter space, Differential Geom. Appl. 25 (2007) 594–611.

[13] Y. Jin Suh, G.-X. Wei, Complete spacelike hypersurfaces in anti-de Sitter space $H_1^{n+1}(-1)$, Houston J. Math. 35 (2009) 93–102.
[14] U.H. Ki, H.J. Kim, H. Nakagawa, On spacelike hypersurfaces with constant mean curvature of a Lorentz space form, Tokyo J. Math. 14 (1991) 205–215.
[15] S.E. Koh, M.S. Yoo, A characterization of totally umbilical hypersurfaces in de Sitter space, J. Geom. Phys. 51 (2004) 34–39.
[16] H.-Z. Li, Global rigidity theorems of hypersurfaces, Ark. Mat. 35 (1997) 327–351.
[17] J.-C. Liu, Z.-Y. Sun, On spacelike hypersurfaces with constant scalar curvature in locally symmetric Lorentz spaces, J. Math. Anal. Appl. 364 (2010) 195–203.
[18] J.-C. Liu, L. Wei, A gap theorem for complete space-like hypersurface with constant scalar curvature in locally symmetric Lorentz spaces, Turkish J. Math. 34 (2010) 105–114.
[19] S. Montiel, An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature, Indiana Univ. Math. J. 37 (1988) 909–917.
[20] T. Otsuki, Minimal hypersurfaces in Riemannian manifold of constant curvature, Amer. J. Math. 92 (1970) 145–173.
[21] J. Ramanathan, Complete spacelike hypersurfaces of constant mean curvature in a de Sitter space, Indiana Univ. Math. J. 36 (1987) 349–359.
[22] S.-C. Shu, Complete spacelike hypersurfaces in a de Sitter space, Bull. Austral. Math. Soc. 73 (2006) 9–16.
[23] Y.-F. Zheng, Spacelike hypersurfaces with constant scalar curvature in the de Sitter spaces, Differential Geom. Appl. 6 (1996) 51–54.