Duality for a Class of Nondifferentiable Mathematical Programming Problems in Complex Space

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1. INTRODUCTION

Craven and Mond [1] have given Fritz John type necessary conditions for a class of nonlinear programming problems in complex space over polyhedral cones. Here we derive the necessary and sufficient conditions for the static minimax problems in complex space which are extensions of the corresponding real space conditions of [4]. These conditions are then used to extend some duality results of [2] for a general class of nondifferentiable programming problems to complex space over arbitrary polyhedral cones.

The complex minimax problem that we consider seeks to choose \( \zeta \in S^0 = \{ \zeta \in c^{2n} : -g(\zeta) \in S \} \) which minimizes

\[
 f(\zeta) = \sup_{\eta \in W} \text{Re} \, \phi(\zeta, \eta),
\]

where \( \zeta = (z, \bar{z}), \eta = (w, \bar{w}) \) for \( z \in C^n, w \in C^m, \phi(\cdot, \cdot) : C^{2n} \times C^{2m} \to C \) is analytic with respect to \( \zeta \), \( W \) is a specified compact subset in \( C^{2m} \), \( S \) is the polyhedral cone in \( C^p \), and \( g : C^{2n} \to C^p \) is analytic.

To derive the necessary conditions, we shall need the following lemma [1].

**Lemma.** Let \( B \in C^{p \times q} \), \( v \in C^p \), \( w \in C^q \), and \( S \subset C^p \) be a convex polyhedral cone with nonempty interior.

Then exactly one of the following two systems has a solution:

(i) \(-Bw \in \text{Int} S\), or

(ii) \( Bhv = 0, 0 \neq v \in S^*\), where \( S^* \) is the polar cone of \( S \).
2. NECESSARY AND SUFFICIENT CONDITIONS FOR THE STATIC COMPLEX MINIMAX PROBLEM

For $\zeta = (z, \bar{z}) \in S^0$, we define

$$W(\zeta) = \{ \eta \in W / \Re \phi(\zeta, \eta) = \sup_{\zeta \in W} \Re \phi(\zeta, \zeta) \},$$

and note that $W(\zeta)$ is compact and nonempty.

Let $Q = \{ [z ; 1] \in C^{2n} / z_2 = \bar{z}_1 \}$ and the polar cone of $Q$ [5] is $Q^* = \{ [\bar{z} ; 1] \in C^{2n} / z_2 = -\bar{z}_1 \}$. For each $\eta \in W$, the function $\phi(\cdot, \cdot) : C^{2n} \times C^{2m} \to C$, and $g : C^{2n} \to C^p$ are differentiable with respect to $\zeta$ if

$$\phi(z, \bar{z}; \eta) - \phi(z_0, \bar{z}_0; \eta) = \nabla_z \phi(z_0, \bar{z}_0; \eta)(z - z_0) + \nabla_{\bar{z}} \phi(z_0, \bar{z}_0; \eta)(\bar{z} - \bar{z}_0) + O(|z - z_0|)$$

and

$$g(z, \bar{z}) - g(z_0, \bar{z}_0) = \nabla_z g(z_0, \bar{z}_0)(z - z_0) + \nabla_{\bar{z}} g(z_0, \bar{z}_0)(\bar{z} - \bar{z}_0) + O(|z - z_0|),$$

where $\nabla_z \phi$, $\nabla_{\bar{z}} \phi$, $\nabla_z g$, and $\nabla_{\bar{z}} g$ denote, respectively, the vectors of partial derivatives of $\phi$ and $g$ with respect to $z$ and $\bar{z}$. Further $O(|z - z_0|)/|z - z_0| \to 0$ as $z \to z_0$. Notice that $f$ is not differentiable, in general.

**Theorem 1.** Let $\phi(\cdot, \cdot) : C^{2n} \times C^{2m} \to C$ be differentiable with respect to $\zeta$ for each $\eta \in W$, $g : C^{2n} \to C^p$ be differentiable with respect to $\zeta$, and let $S \subset C^p$ be a polyhedral cone with nonempty interior. Let $\zeta^0$ be a solution to the minimax problem.

(A) Then there exist a positive integer $s$, scalars $\lambda_i \geq 0$, $i = 1, 2, \ldots, s$, $0 \neq u \in S^*$, and vectors $\eta_i \in W(\zeta^0)$, $i = 1, 2, \ldots, s$ such that

$$\sum_{i=1}^{s} \lambda_i \nabla_z \phi(\zeta^0, \eta_i) + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\zeta^0, \eta_i) + u^T \nabla_z g(\zeta^0) + u^H \nabla_{\bar{z}} g(\zeta^0) = 0, \quad (1)$$

$$\Re u^H g(\zeta^0) = 0. \quad (2)$$

**Proof.** Equation (2) can be written as

$$\frac{1}{2} u^T g(\zeta^0) + \frac{1}{2} u^H g(\zeta^0) = 0.$$

If (A) does not hold, then $\forall \eta_i \in W(\zeta^0)$, $i = 1, 2, \ldots, s$, there exist no $\lambda_i$, $i = 1, 2, \ldots, s$, and $u$ satisfying (1) and (2). Let
Thus, for all \( \eta_i \), the system

\[
M = \begin{pmatrix}
\nabla_z \phi(\zeta^0, \eta_1) + \overline{\nabla_z \phi(\zeta^0, \eta_1)} & 0 \\
\nabla_z \phi(\zeta^0, \eta_2) + \overline{\nabla_z \phi(\zeta^0, \eta_2)} & 0 \\
\vdots & \vdots \\
\nabla_z \phi(\zeta^0, \eta_s) + \overline{\nabla_z \phi(\zeta^0, \eta_s)} & 0 \\
\frac{\nabla_z g(\zeta^0)}{\nabla_z g(\zeta^0)} & \frac{g(\zeta^0)}{g(\zeta^0)}
\end{pmatrix}
\]

has no solution whenever

\[
M^H \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_s \\
u \\
\bar{u}
\end{bmatrix} = 0
\]

where \( \tilde{S}^* = \{ \tilde{w} : w \in S^* \} \). By the lemma, for all \( \eta_i \in \mathcal{W}(\zeta^0), i = 1, 2, \ldots, s \), there exist \( p \) and \( q \) (depending on \( \eta_i \)) satisfying

\[
-M \begin{bmatrix}
p \\
q
\end{bmatrix} \subset \text{int}[R_+^s \times (S^* \times \tilde{S}^*) \cap Q],
\]

Thus, there exists a solution to the system

\[
\text{Re}[\nabla_z \phi(\zeta^0, \eta_i) p + \overline{\nabla_z \phi(\zeta^0, \eta_i)} \bar{p}] < 0, \quad i = 1, 2, \ldots, s, 
\]

\[
\nabla_z g(\zeta^0) p + g(\zeta^0) q = -\alpha - r, 
\]

\[
\overline{\nabla_z g(\zeta^0)} p + \overline{g(\zeta^0)} q = -\bar{\beta} + \bar{r}, 
\]

where \( \alpha, \beta \in \text{int} S \) and \( r \in \mathbb{C}^p \). Taking the conjugate of relation (5) and adding to (4)

\[
\nabla_z g(\zeta^0) p + \overline{\nabla_z g(\zeta^0)} \bar{p} + g(\zeta^0)(q + \bar{q}) = -\alpha - \beta \in -\text{int} S.
\]
Now by making use of the differentiability of \( g \) with respect to \( \zeta \), it follows from Theorem 1 [1] that \((z_0 + tp, \bar{z}_0 + t\bar{p})\) is feasible for the minimax problem for sufficiently small \( t \), \( 0 < t < R_+ \).

Also, now \( \phi \) is differentiable with respect \( \zeta \) for each \( \eta \), therefore, we have

\[
\text{Re}[\phi(z_0 + tp, \bar{z}_0 + t\bar{p}; \eta_i) - \phi(z_0, \bar{z}_0; \eta_i)] = \text{Re}[t \nabla_z \phi(z_0, \bar{z}_0; \eta_i)p + t \nabla_{\bar{z}} \phi(z_0, \bar{z}_0; \eta_i)\bar{p} + O(t)],
\]

\[i = 1, 2, \ldots, s. \tag{6}\]

Therefore, from system (3) it follows that

\[
\text{Re} \phi(z_0 + tp, \bar{z}_0 + t\bar{p}; \eta_i) < \text{Re} \phi(z_0, \bar{z}_0; \eta_i), \quad i = 1, 2, \ldots, s. \tag{7}\]

Now for each \( \eta_i \in W(\zeta^0) \), \( i = 1, 2, \ldots, s \),

\[
\text{Re} \phi(z_0, \bar{z}_0; \eta_i) = \sup_{\eta \in W} \text{Re} \phi(z_0, \bar{z}_0; \eta) = f(z_0, \bar{z}_0), \quad i = 1, 2, \ldots, s.
\]

Therefore, from (7)

\[
\text{Re} \phi(z_0 + tp, \bar{z}_0 + t\bar{p}; \eta_i) < f(z_0, \bar{z}_0), \quad i = 1, 2, \ldots, s.
\]

Now \( \eta_i \in W(\zeta^0) \subset W \), \( i = 1, 2, \ldots, s \), and \( W \) is compact.

\[
\therefore \sup_{\eta \in W} \text{Re} \phi(z_0 + tp, \bar{z}_0 + t\bar{p}) < f(z_0, \bar{z}_0)
\]

or

\[
f(z_0 + tp, \bar{z}_0 + t\bar{p}) < f(z_0, \bar{z}_0).
\]

This contradicts the assumption that \( \zeta^0 \) is a solution to the minimax problem.

**Theorem 2.** Let \( \zeta^0 \in S^0 \). Let \( g(\cdot) \) be a convex function of \( \zeta \) and for every \( \eta \in W \) let \( \phi(\cdot, \eta) \) be a convex function of \( \zeta \). If there are (i) finite \( \lambda_i \geq 0 \), \( i = 1, 2, \ldots, s \), \( \sum_{i=1}^{s} \lambda_i \neq 0 \); (ii) vector \( u \in S^* \); (iii) vectors \( \eta_i \in W(\zeta^0) \), \( i = 1, 2, \ldots, s \), such that

\[
\sum_{i=1}^{s} \lambda_i \nabla_z \phi(\zeta^0, \eta_i) + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\zeta^0, \eta_i) + u^T \nabla_{\bar{z}} g(\zeta^0) + u^H \nabla_z g(\zeta^0) = 0,
\]

\[
\text{Re} u^H g(\zeta^0) = 0,
\]

then \( \zeta^0 \) is a minimax solution.

**Proof.** Suppose the conditions of the theorem are satisfied but \( \zeta^0 \) is not a minimax solution. Then there exists \( \zeta' \in S^0 \) such that

\[
\sup_{\eta \in W} \text{Re} \phi(\zeta', \eta) < \sup_{\eta \in W} \text{Re} \phi(\zeta^0, \eta). \tag{8}\]
Also, 
\[ \text{Re} \ u^H g(\zeta^0) = 0, \] (9)
and since \(-g(\zeta) \in S\) and \(u \in S^*\), therefore,
\[ \text{Re} \ u^H g(\zeta^0) \leq 0. \] (10)

Also, for \(\eta_i \in W(\zeta^0)\), we have
\[ \text{Sup} \ \text{Re} \ \phi(\zeta^0, \eta_i) = \text{Re} \ \phi(\zeta^0, \eta_i), \quad i = 1, 2, ..., s, \] (11)
and
\[ \text{Re} \ \phi(\zeta', \eta_i) \leq \text{Sup} \ \text{Re} \ \phi(\zeta', \eta), \quad i = 1, 2, ..., s, \] (12)
therefore
\[ \text{Re} \ \phi(\zeta', \eta_i) < \text{Re} \ \phi(\zeta^0, \eta_i), \quad i = 1, 2, ..., s, \] (13)
where (13) follows by making use of (8), (11), and (12). Multiplying each equation in (13) by \(\lambda_i\), summing up for values of \(i = 1, 2, ..., s\), and using (9) and (10) we have
\[ \text{Re} \left[ \sum_{i=1}^{s} \lambda_i \phi(\zeta', \eta_i) + u^H g(\zeta') \right] < \text{Re} \left[ \sum_{i=1}^{s} \lambda_i \phi(\zeta^0, \eta_i) + u^H g(\zeta^0) \right], \] (14)
where the strict inequality follows from the fact that
\[ \lambda_i \geq 0, \quad i = 1, 2, ..., s, \quad \text{and} \quad \sum_{i=1}^{s} \lambda_i \neq 0. \]

From convexity assumptions,
\[ \text{Re} \left[ \phi(\zeta^0, \eta_i) - \phi(\zeta^0, \eta_i) - \nabla_z \phi(\zeta^0, \eta_i)(\zeta' - \zeta^0) - \nabla_{\bar{z}} \phi(\zeta^0, \eta_i)(\zeta' - \zeta^0) \right] \geq 0, \] (15)
\[ \text{Re} \left[ g(\zeta^0) - g(\zeta^0) - g'(\zeta^0)(\zeta' - \zeta^0) \right] \geq 0, \] (16)
where \(g'(\zeta^0)\) is the matrix whose components are partial derivatives of \(g\) at \(\zeta^0\) with respect to \(z\) and \(\bar{z}\).

Multiplying (15) by \(\lambda_i\), summing up for values of \(i = 1, 2, ..., s\), and (16) by \(u^H\), and adding, we have
\[ \text{Re} \left[ \sum_{i=1}^{s} \lambda_i \phi(\zeta', \eta_i) + u^H g(\zeta') - \sum_{i=1}^{s} \lambda_i \phi(\zeta^0, \eta_i) - u^H g(\zeta^0) \right] \]
\[ \geq \left\{ \left[ \sum_{i=1}^{s} \lambda_i \nabla_z \phi(\zeta^0, \eta_i) + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\zeta^0, \eta_i) \right]ight. \]
\[ \left. + u^T \nabla_{\bar{z}} g(\zeta^0) + u^H \nabla_{\bar{z}} g(\zeta^0) \right\} (\zeta' - \zeta^0). \] (17)
From the given conditions, the right-hand side of (17) is zero. Consequently, (17) and (14) contradict each other. Hence, $\zeta^0$ is a minimax solution.

Next we prove the sufficient condition without making use of convexity assumptions of Theorem 2. But here we assume $\phi(\cdot, \eta)$ for each $\eta \in W$ and $g(\cdot)$ to be twice Fréchet differentiable with respect to $\zeta$ on $C^{2n} \times C^{2m}$ and $C^{2n}$, respectively. Let

$$
\psi_i(\zeta, \eta_i) = \Re \phi(\zeta, \eta_i).
$$

Denote by $\langle \psi_i \rangle$ the matrix

$$
\langle \psi_i \rangle = \begin{pmatrix}
\psi_{iz} & \psi_{iz2} \\
\psi_{iz1} & \psi_{iz2}
\end{pmatrix}.
$$

Similarly, we can define $\langle g \rangle$ and $\langle \bar{g} \rangle$.

**Theorem 3.** Let $\zeta^0 \in S^0$. If there exist (i) a finite positive integer $s$, (ii) scalars $\lambda_i \geq 0$, $i = 1, 2, \ldots, s$ with $\sum_{i=1}^{s} \lambda_i \neq 0$, (iii) $u \in S^*$, and (iv) vectors $\eta_i \in W(\zeta^0)$, $i = 1, 2, \ldots, s$, such that

$$
\sum_{i=1}^{s} \lambda_i \nabla_{\zeta} \phi(\zeta^0, \eta_i) + \sum_{i=1}^{s} \lambda_i \nabla_{\zeta} \phi(\zeta^0, \eta_i) + u^T \nabla_{\zeta} g(\zeta^0) + u^H \nabla_{\zeta} g(\zeta^0) = 0, \quad (18)
$$

and

$$
\Re u^H g(\zeta^0) = 0, \quad (19)
$$

and

$$
\sum_{i=1}^{s} \lambda_i \langle \psi_i \rangle + u^H \langle g \rangle + u^T \langle \bar{g} \rangle \text{ is positive semidefinite,} \quad (20)
$$

for all $\zeta \in C^{2n}$. Then $\zeta^0$ is a minimax solution.

**Proof.** Consider any $\zeta' \in S^0$. Then

$$
\Re \left[ \sum_{i=1}^{s} \lambda_i \phi(\zeta', \eta_i) + u^H g(\zeta') - \sum_{i=1}^{s} \lambda_i \phi(\zeta^0, \eta_i) - u^H g(\zeta^0) \right]
$$

$$
= \Re \left[ \sum_{i=1}^{s} \lambda_i \nabla_{\zeta} \phi(\zeta^0, \eta_i) + \sum_{i=1}^{s} \lambda_i \nabla_{\zeta} \phi(\zeta^0, \eta_i)
$$

$$
+ u^T \nabla_{\zeta} g(\zeta^0) + u^H \nabla_{\zeta} g(\zeta^0) \right] (\zeta' - \zeta^0)
$$

$$
+ \frac{1}{2}(\zeta' - \zeta^0)^H \left[ \sum_{i=1}^{s} \lambda_i \langle \psi_i \rangle + u^H \langle g \rangle + u^T \langle \bar{g} \rangle \right] (\zeta' - \zeta^0) = 0, \quad (21)
$$
where $\zeta^2 = \alpha \zeta^0 + (1 - \alpha) \zeta'$, $0 \leq \alpha \leq 1$. Using (18)-(20) in (21) we obtain
\[\text{Re} \left[ \sum_{i=1}^{s} \lambda_i \phi(\zeta', \eta_i) - \sum_{i=1}^{s} \lambda_i \phi(\zeta^0, \eta_i) \right] \geq \text{Re} \ u^H g(\zeta'). \quad (22)\]

Now $\text{Re} \ u^H g(\zeta') \leq 0$ as $u \in S^*$ and $-g(\zeta') \in S$, therefore from (22), we obtain
\[\text{Re} \ \sum_{i=1}^{s} \lambda_i \phi(\zeta', \eta_i) \geq \text{Re} \ \sum_{i=1}^{s} \lambda_i \phi(\zeta^0, \eta_i). \quad (23)\]

But
\[\text{Re} \ \phi(\zeta^0, \eta_i) = \sup_{\eta \in W} \text{Re} \ \phi(\zeta^0, \eta), \quad i = 1, 2, ..., s \quad (24)\]

and
\[\sup_{\eta \in W} \text{Re} \ \phi(\zeta', \eta) \geq \text{Re} \ \phi(\zeta', \eta_i), \quad i = 1, 2, ..., s. \quad (25)\]

Hence, by making use of (23)-(25), we have
\[\sum_{i=1}^{s} \lambda_i \sup_{\eta \in W} \text{Re} \ \phi(\zeta', \eta) \geq \sum_{i=1}^{s} \lambda_i \sup_{\eta \in W} \text{Re} \ \phi(\zeta^0, \eta)\]

or
\[\sup_{\eta \in W} \text{Re} \ \phi(\zeta', \eta) \geq \sup_{\eta \in W} \text{Re} \ \phi(\zeta^0, \eta).\]

Hence, $\zeta^0$ is a minimax solution.

3. Duality Theorems

Here we prove some duality theorems for a class of nondifferentiable programming problems in complex space. We consider the primal problem (P):

Minimize $\text{Re} \ f(\zeta)$

Subject to $\zeta \in S^0$.

Let $Y$ be the set of triplets $(s, \lambda, \eta^0)$, where $s$ is a finite positive integer, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_s)$ be an $s$-dimensional vector with $\lambda_i > 0$, $i = 1, 2, ..., s$; $\sum_{i=1}^{s} \lambda_i = 1$; and $\eta^0 = (\eta_1, \eta_2, ..., \eta_s)$ is an $ms$-dimensional vector such that $\eta_i \in W(\zeta)$, $i = 1, 2, ..., s$, for some $\zeta \in S^0$. For each $(s, \lambda, \eta^0) \in Y$, we define

$X(s, \lambda, \eta^0) = (\zeta, U) \in C^{2n} \times C^{2p},$
where \( U = (u, \bar{u}) \in C^2p \) satisfying \( \eta_i \in W(\zeta), i = 1, 2, \ldots, s, u \in S^* \) and
\[
\sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\zeta, \eta_i) + \sum_{i=1}^{s} \lambda_i \nabla_{z} \phi(\zeta, \eta_i) + u^T \nabla_{z} g(\zeta) + u^{\bar{u}} \nabla_{\bar{z}} g(\zeta) = 0.
\]

Dual of problem (P) is defined (D):
\[
\text{Maximize Sup Re}[f(\zeta) + u^H g(\zeta)],
\]
\[(s, \lambda, \eta') \in Y(\zeta, U) \in X(s, \lambda, \eta').\]

If for a triplet \((s, \lambda, \eta')\) in \(Y\) the set \(X(s, \lambda, \eta')\) is empty, we define the supremum over it to be \(-\infty\).

In the next theorem, we establish the duality relationship between problems (P) and (D) under the convexity condition imposed on the functions \(\phi\) and \(g\). It is assumed \(\phi(\cdot, \eta)\) is a convex function of \(\zeta\) for each \(\eta\) and \(g\) is a convex function of \(\zeta\).

**Theorem 4.** Let \(\zeta \in S^0\) be an optimal solution to problem (P). Then there exists \((s, \lambda, \eta)\) in \(Y\) and \(\hat{U} \in C^2p\) with \((\zeta, \hat{U}) \in X(s, \lambda, \eta)\) such that \((s, \lambda, \eta)\) and \((\zeta, \hat{U})\) give an optimal solution to problem (D). Furthermore, the two problems (P) and (D) have the same extremal values.

**Proof.** Since \(\zeta\) is an optimal solution to problem (P). Therefore, from Theorem 2 it follows that there exist (i) a finite positive integer \(s\), (ii) nonnegative scalars \(\lambda_1, \lambda_2, \ldots, \lambda_s\) with \(\sum_{i=1}^{s} \lambda_i \neq 0\), (iii) \(\theta \neq u^0 \in S^*\) such that
\[
\sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\zeta, \eta_i) + \sum_{i=1}^{s} \lambda_i \nabla_{z} \phi(\zeta, \eta_i) + u^0 \nabla_{z} g(\zeta) + u^{\bar{u}} \nabla_{\bar{z}} g(\zeta) = 0,
\]
\[\text{Re } u^{0\bar{u}} g(\zeta) = 0.\]

Let \(\alpha = \sum_{i=1}^{s} \lambda_i\). Then \((s, \alpha^{-1} \lambda, \eta) \in Y\) and \((\zeta, \alpha^{-1} U^0) \in X(s, \alpha^{-1} \lambda, \eta)\), where \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)\). Put \(\hat{\lambda} = \alpha^{-1} \lambda\) and \(\hat{U} = \alpha^{-1} U^0\). We first prove that \((\zeta, \hat{U})\) attains the maximum of the following problem:

\[
\text{Maximize } \text{Re } [f(\zeta) + u^H g(\zeta)]
\]
\[
\text{Subject to } (\zeta, U) \in X(s, \hat{\lambda}, \eta).
\]

Take any \((\zeta, U)\) from \(X(s, \hat{\lambda}, \eta)\), using \(\hat{\eta} \in W(\zeta)\), the convexity of \(\phi, g\), and \(\text{Re } u^{\bar{u}} g(\zeta) = 0\).
\[
\text{Re}\left[ f(\hat{\zeta}) + \hat{u}^H g(\hat{\zeta}) - f(\zeta) - u^H g(\zeta) \right] \\
= \text{Re}\left[ \sum_{i=1}^{s} \lambda_i \phi(\zeta, \eta_i) - \hat{u}^H g(\hat{\zeta}) - \sum_{i=1}^{s} \hat{\lambda}_i \phi(\zeta, \hat{\eta}_i) - u^H g(\zeta) \right] \\
\geq \text{Re}\left[ \sum_{i=1}^{s} \lambda_i [\nabla_{\bar{z}} \phi(\zeta, \eta_i) + \nabla_{\bar{z}} \phi(\zeta, \eta_i)](\zeta - \hat{\zeta}) \right] - \text{Re} u^H g(\zeta) \\
\geq \text{Re}\left[ \sum_{i=1}^{s} \lambda_i [\nabla_{\bar{z}} \phi(\zeta, \eta_i) + \nabla_{\bar{z}} \phi(\zeta, \eta_i)] + u^T \nabla_{\bar{z}} g(\zeta) \right] \\
+ u^H \nabla_{\bar{z}} g(\zeta) \left| (\zeta - \hat{\zeta}) - \text{Re} u^H g(\hat{\zeta}) \right| \geq 0,
\]

where the last inequality follows from the fact that \((\zeta, U) \in X(\delta, \lambda, \eta)\), \(\text{Re} u^H g(\hat{\zeta}) \leq 0\). Hence, we obtain
\[
\text{Re} f(\hat{\zeta}) = \text{Re} f(\hat{\zeta}) + \hat{u}^H g(\hat{\zeta}) \geq \text{Re} f(\zeta) + u^H g(\zeta) \quad \text{for all} \quad (\zeta, U) \in X(\delta, \lambda, \eta).
\]

To complete the proof we must show any \((s, \lambda, \eta^0) \in Y\) that
\[
\sup_{(\zeta, U) \in X(s, \lambda, \eta^0)} \text{Re} f(\zeta) + u^H g(\zeta) \leq \text{Re} f(\hat{\zeta}). \tag{26}
\]

Let us assume, \(X(s, \lambda, \eta^0)\) to be nonempty. Take any \((\zeta, U) \in X(s, \lambda, \eta^0)\). By making use of \(\text{Re} f(\zeta) = \sum_{i=1}^{s} \lambda_i \text{Re} \phi(\zeta, \eta_i)\), \(\text{Re} f(\zeta) \geq \sum_{i=1}^{s} \lambda_i \text{Re} \phi(\zeta, \eta_i)\), and \(\text{Re} u^H g(\zeta) \leq 0\), we have
\[
\text{Re} f(\zeta) + u^H g(\zeta) - f(\hat{\zeta}) \leq \text{Re}\left[ \sum_{i=1}^{s} \lambda_i \phi(\zeta, \eta_i) + u^H g(\zeta) \right] \\
- \text{Re}\left[ \sum_{i=1}^{s} \lambda_i \phi(\zeta, \eta_i) + u^H g(\zeta) \right] \\
\leq -\text{Re}\left[ \left\{ \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\zeta, \eta_i) + \sum_{i=1}^{s} \lambda_i \nabla_{\bar{z}} \phi(\zeta, \eta_i) \right\} + u^T \nabla_{\bar{z}} g(\zeta) + u^H \nabla_{\bar{z}} g(\zeta) \right] \left| (\zeta - \hat{\zeta}) \right| = 0,
\]

since \((\zeta, U) \in X(s, \lambda, \eta^0)\). Thus (27) is proved. From (26) and (27) it follows that \((\hat{\zeta}, \hat{U})\) and \((\hat{s}, \hat{\lambda}, \hat{\eta})\) give an optimal solution of the problem (D). Since \(f(\hat{\zeta})\) is the extremal value of problem (P), (26) implies that the two problems (P) and (D) have the same extremal values.

To prove converse duality we shall closely follow the work of [3] and make the assumptions:
(i) $\phi(\zeta, w) = K(\zeta) + w^T r(\zeta)$, where $K(\cdot)$, $r(\cdot)$ are twice Frechét differentiable functions from $C^{2n}$ into $C$ and $C^n$, respectively.

(ii) $K(\cdot)$ and $r(\cdot)$ have a convex real part with respect to $R_+$ and $R_+^m$ on $Q$, respectively. Let $g$ be convex with respect to $S$ on $Q$. Hence, we define the primal problem ($P$):

Minimize $\text{Re} f(\zeta) = \text{Maximize} \text{Re}[K(\zeta) + w^T r(\zeta)]$

Subject to $-g(\zeta) \in S,$

where $g$ is twice Frechét differentiable on $Q$. Let $h: C^n \to C^q$ be convex and Frechét differentiable and $T$ be a polyhedral cone in $C^q$. We define the constraint set $H$ as

$$H = \{w \in C^m / -h(w) \in T\}.$$ 

We assume that

$$w \in T^*, \quad H \subset T^*.$$ 

(B) We shall say that condition (B) is satisfied at $w^0 \in H$ if $v \in R^q$, $\text{Re} \{v^T h(w^0) + v^T h(w^0)\} = 0$, $\text{Re}[v^T h(w^0)] = 0$, and $v \geq 0 \Rightarrow v = 0$.

Dual to $P$ is the problem $\widetilde{P}$:

Maximize $\text{Re}[K(\zeta) + w^T r(\zeta) + y^T g(\zeta)]$

Subject to $2 \nabla_x K^R(\zeta) + w^T \nabla_x r(\zeta) + w^T \nabla_x r(\zeta) + y^T \nabla_x g(\zeta) + y^T \nabla_x g(\zeta) = 0,$

$$y \in S^*, \quad h(w) \in T,$$

where $K^R = \text{Re} K$.

**THEOREM 5.** Let $(\zeta^0, w^0, y^0)$ be an optimal solution for problem $\widetilde{P}$ and let the matrix

$$\langle D \rangle = 2\langle K^R \rangle + w^T \langle r \rangle + w^T \langle \bar{r} \rangle + y^H \langle g \rangle + y^T \langle \bar{g} \rangle$$

be nonsingular at $(\zeta^0, w^0, y^0)$ and the condition (B) is satisfied at $w^0 \in H$. Then $\zeta^0$ is optimal to the minimization problem $\widetilde{P}$ and the two extremal values of problem $\widetilde{P}$ and problem $\widetilde{D}$ are equal.

**Proof.** Since $(\zeta^0, w^0, y^0)$ is optimal for $\widetilde{D}$. The existence of this optimal solution implies the existence of the optimal solution to an equivalent problem say $\widetilde{D}'$ to $\widetilde{D}$ in real space $[3]$. Now the converse duality results of Tanimoto $[2]$ in real space are applicable to $\widetilde{D}'$ and we can find a solution
to the corresponding primal $\tilde{P}'$ in real space. Solution to $\tilde{P}'$ guarantees a solution to $\tilde{P}$ and the two extremal values of problems ($\tilde{P}$) and ($\tilde{D}$) are equal.

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**REFERENCES**