JOURNAL OF COMBINATORIAL THEORY, Series A **79**, 315–325 (1997)
ARTICLE NO. TA972790

Ultra Logconcave Sequences and Negative Dependence*

Thomas M. Liggett

University of California, Los Angeles, Los Angeles, California

Communicated by the Managing Editors

Received January 28, 1997

We prove that the convolution of two ultra-logconcave sequences is ultra-logconcave. This was conjectured recently by Pemantle and implies that a natural negative dependence property is preserved under the operation of "joining" families of exchangeable Bernoulli random variables. © 1997 Academic Press

1. INTRODUCTION AND STATEMENT OF RESULTS

There is a well established and very useful theory of positive dependence for families of random variables, much of which is related to the FKG theorem (Fortuin, Kastelyn, and Ginibre [2]). Areas of application in which this arises include reliability theory, statistical mechanics, percolation, and interacting particle systems. The monographs [3, 9] explain the connection to the latter two fields, for example.

The corresponding theory of negative dependence is potentially just as useful, but is far less well developed. One simple reason for the greater subtlety of negative dependence is the fact that every random variable is positively dependent with itself, but is only negatively dependent with itself if it is constant. There have been a number of attempts to develop a comprehensive theory of negative dependence over the years, and negative dependence has been used in proving various limit theorems in probability theory. Rather than try to provide a complete account of this, we have merely listed some of the relevant papers in the references. In a recent unpublished manuscript [12], Pemantle tries to put various competing concepts of negative dependence together into a cohesive framework. The present paper arose out of his attempt.

We begin with some terminology. A nonnegative sequence a_i, $i \geqslant 0$, is said to be *logconcave* (LC) if the set of indices of nonzero terms forms an interval of integers, and

$$a_i^2 \geqslant a_{i-1} a_{i+1}, \qquad i \geqslant 1.$$

* Preparation of this paper was supported in part by NSF Grant DMS-94-00644.

0097-3165/97 $25.00

Following Pemantle, say that a_i is *ultra-logconcave of order m* (ULC(m)) if $a_i = 0$ for $i > m$ and the sequence

$$\frac{a_i}{\binom{m}{i}}$$

is logconcave. Note that this condition can be rewritten as

$$i(m-i)\, a_i^2 \geqslant (i+1)(m-i+1)\, a_{i-1} a_{i+1},$$

and in this form it is clear what the definition of ULC(∞) should be:

$$i a_i^2 \geqslant (i+1)\, a_{i-1} a_{i+1}.$$

This is then equivalent to the logconcavity of the sequence $i!\, a_i$. It is clear from the definitions that ULC(m) implies ULC(m + 1) for each m.

Random variables $(X_i, 1 \leqslant i \leqslant m)$ are said to be Bernoulli if they take only the values 0 and 1. The collection is said to be exchangeable if the joint distribution of the family is invariant under permutations of the indexes, i.e.,

$$P(X_1 = i_1, ..., X_m = i_m)$$

depends on $i_1, ..., i_m$ only through their sum $\sum_j i_j$.

In his paper, Pemantle considers a number of properties of a finite collection of Bernoulli random variables $X = (X_i, 1 \leqslant i \leqslant m)$ that express in one way or another a property of negative dependence. One of these properties is the "negative" FKG condition: If $i = (i_1, ..., i_m)$ and $j = (j_1, ..., j_m)$ are in $\{0, 1\}^m$ and $i \vee j$ and $i \wedge j$ are defined coordinatewise, then

$$P(X = i \vee j)\, P(X = i \wedge j) \leqslant P(X = i)\, P(X = j).$$

(The hypothesis of the FKG theorem is this statement with the opposite inequality.) Pemantle proved certain implications among these properties and showed that most of them are equivalent if $(X_i, 1 \leqslant i \leqslant m)$ is exchangeable. In the exchangeable case, these properties are also equivalent to the statement that the "rank" sequence

$$a_i^* = P\left(\sum_{j=1}^{m} X_j = i\right)$$

is ULC(m). (This is immediate for the negative FKG condition.) We will take this to be the definition of negative dependence of the exchangeable

sequence $(X_i, 1 \leqslant i \leqslant m)$. As a consequence of exchangeability, negative dependence is equivalent to the logconcavity of the sequence

$$a_i = P(X_1 = \cdots = X_i = 1, X_{i+1} = \cdots = X_m = 0) = \frac{a_i^*}{\binom{m}{i}}.$$

(Pemantle speculates that the ULC property is characteristic of negative dependence more generally, but its relationship to non-exchangeable sequences is less clear.)

Suppose now that $(X_i, 1 \leqslant i \leqslant m)$ and $(Y_i, 1 \leqslant i \leqslant n)$ are two finite exchangeable Bernoulli sequences that are independent of one another. Construct a new exchangeable Bernoulli sequence $(Z_i, 1 \leqslant i \leqslant m+n)$ in the following way: Sample uniformly without replacement from the set $\{X_1, ..., X_m, Y_1, ..., Y_n\}$, and let Z_i be the value of the X or Y chosen at the ith trial.

THEOREM 1. *If* $(X_i, 1 \leqslant i \leqslant m)$ *and* $(Y_i, 1 \leqslant i \leqslant n)$ *are each negatively dependent, then so is* $(Z_i, 1 \leqslant i \leqslant m+n)$.

Note that

$$\sum_{i=1}^{m+n} Z_i = \sum_{i=1}^{m} X_i + \sum_{i=1}^{n} Y_i,$$

so the rank sequence of the Z's is the convolution of the rank sequence of the X's and the rank sequence of the Y's. Therefore Theorem 1 is equivalent to the following statement, which was conjectured by Pemantle. In it, m and n can be infinite.

THEOREM 2. *The convolution of a* $ULC(m)$ *sequence and a* $ULC(n)$ *sequence is* $ULC(m+n)$.

To place Theorem 2 in a different context, we note that the result that the convolution of two logconcave sequences is logconcave has been known for many years. It appears on p. 394 of Karlin (1968), for example, and is obtained there as a consequence of the elementary fact that the product of TP_2 ("totally positive of order two") matrices is TP_2. Here is a direct proof of the preservation of logconcavity under convolution. Let a_i, $0 \leqslant i \leqslant m$, and b_j, $0 \leqslant j \leqslant n$, be logconcave sequences, and let c_k, $0 \leqslant k \leqslant m+n$, be their convolution:

$$c_k = \sum_{i+j=k} a_i b_j.$$

Here and below, we take the a's and b's to be zero if their indexes are not in the ranges $[0, m]$ and $[0, n]$ respectively. Then the following expression is nonnegative because of the logconcavity assumption:

$$\sum_{i<j} (a_i a_{j-1} - a_{i-1} a_j)(b_{k-i} b_{k-j+1} - b_{k-i+1} b_{k-j}).$$

Multiply out this expression, and interchange the roles of i and j in two of the four resulting terms, to obtain the following:

$$\sum_{i \neq j} a_i a_{j-1} b_{k-i} b_{k-j+1} - \sum_{i \neq j} a_i a_{j-1} b_{k-i+1} b_{k-j}.$$

Noting that we can add the diagonal terms $i = j$ in both of these sums, it follows that this is just $c_k^2 - c_{k-1} c_{k+1}$. We provide this computation here in order to show that the proof of Theorem 2 is significantly more delicate because of the qualifier "ultra" which appears there.

Theorem 2 is an easy consequence of Theorem 3, which is stated and proved in the next section.

2. PROOFS

To state the result that leads to Theorems 1 and 2, take three nonnegative sequences a_i, b_i, u_i, $0 \leqslant i \leqslant k$, and define

$$A = \sum_{i,\, j \geqslant 0,\, i+j=k-2} (u_i + 2u_{i+1} + u_{i+2})\, a_i b_j \binom{k-2}{i},$$

$$B = \sum_{i,\, j \geqslant 0,\, i+j=k-1} (u_i + u_{i+1})\, a_i b_j \binom{k-1}{i},$$

$$C = \sum_{i,\, j \geqslant 0,\, i+j=k} u_i a_i b_j \binom{k}{i}.$$

THEOREM 3. *If a_i, b_i, u_i, $0 \leqslant i \leqslant k$, are all logconcave, then $AC \leqslant B^2$.*

Before proving Theorem 3, we will check that it implies Theorem 2.

Proof of Theorem 2. Suppose that $\{a_i^*, 0 \leqslant i \leqslant m\}$ and $\{b_i^*, 0 \leqslant i \leqslant n\}$ are ULC(m) and ULC(n) respectively, and let $\{c_i^*, 0 \leqslant i \leqslant m+n\}$ be their convolution. (We will write the proof down in the case that both m and n

are finite; simple modifications cover the other cases.) Let a_i, b_i, c_i be the corresponding sequences without the $*$'s:

$$a_i = \frac{a_i^*}{\binom{m}{i}}, \qquad b_i = \frac{b_i^*}{\binom{n}{i}}, \qquad c_i = \frac{c_i^*}{\binom{m+n}{i}}.$$

Then

$$c_k = \sum_{i+j=k} \frac{\binom{m}{i}\binom{n}{j}}{\binom{m+n}{k}} a_i b_j, \qquad 0 \leqslant k \leqslant m+n.$$

We want to prove that the logconcavity of the a's and b's implies the logconcavity of the c's. Fix a value of k for which we want to prove $c_{k-1}^2 \geqslant c_k c_{k-2}$. Put

$$u_i = \frac{\binom{m}{i}\binom{n}{k-i}}{\binom{m+n}{k}\binom{k}{i}} = \frac{\binom{m+n-k}{m-i}}{\binom{m+n}{m}}, \tag{1}$$

so that $c_k = C$. Using the rightmost expression in (1) and the property of binomial coefficients

$$\binom{N}{M} = \binom{N-1}{M} + \binom{N-1}{M-1},$$

it follows that

$$u_i + u_{i+1} = \frac{\binom{m+n-k+1}{m-i}}{\binom{m+n}{m}},$$

$$u_i + 2u_{i+1} + u_{i+2} = \frac{\binom{m+n-k+2}{m-i}}{\binom{m+n}{m}},$$

so that $c_{k-1} = B$ and $c_{k-2} = A$. Therefore, Theorem 2 will follow from Theorem 3 once we have shown that u_i is logconcave. But looking at the

right side of (1), we see that this is a consequence of the logconcavity of the sequence $\binom{N}{M}$ as a function of M, which is easy to check directly.

Proof of Theorem 3. Write

$$B^2 - AC = \sum_{i,j} \Bigg[(u_i + u_{i+1})\, a_i b_{k-i-1} \binom{k-1}{i} (u_j + u_{j+1})\, a_j b_{k-j-1} \binom{k-1}{j}$$
$$- u_i a_i b_{k-i} \binom{k}{i} (u_j + 2u_{j+1} + u_{j+2})\, a_j b_{k-j-2} \binom{k-2}{j} \Bigg].$$

(The limits on the i and j in this sum come from the usual convention that $\binom{N}{M} = 0$ unless $0 \leqslant M \leqslant N$.) This can be rearranged so that it becomes

$$\sum_{i,j} u_i u_j \Bigg[a_i a_j b_{k-i-1} b_{k-j-1} \binom{k-1}{i}\binom{k-1}{j}$$
$$+ 2a_i a_{j-1} b_{k-i-1} b_{k-j} \binom{k-1}{i}\binom{k-1}{j-1}$$
$$+ a_{i-1} a_{j-1} b_{k-i} b_{k-j} \binom{k-1}{i-1}\binom{k-1}{j-1}$$
$$- a_i a_j b_{k-i} b_{k-j-2} \binom{k}{i}\binom{k-2}{j}$$
$$- 2a_i a_{j-1} b_{k-i} b_{k-j-1} \binom{k}{i}\binom{k-2}{j-1}$$
$$- a_i a_{j-2} b_{k-i} b_{k-j} \binom{k}{i}\binom{k-2}{j-2} \Bigg]. \tag{2}$$

We need to show that (2) is nonnegative whenever u_i is logconcave. This logconcavity implies that

$$u_i^2 \geqslant u_{i-1} u_{i+1} \geqslant u_{i-2} u_{i+2} \cdots \quad \text{and} \quad u_i u_{i+1} \geqslant u_{i-1} u_{i+2} \geqslant u_{i-2} u_{i+3} \cdots .$$

Therefore, it is enough to show that the sum of the coefficients of $u_i u_j$ in (2) for i, j satisfying $i + j = q$, $|i - j| \leqslant l$, is nonnegative for all choices of integers $q \geqslant l \geqslant 0$ (which we may assume have the same parity).

So, write the sum of the coefficients of $u_i u_j$ in (2) for i, j satisfying $i + j = q$, $|i - j| \leqslant l$, as $D_+ + 2D_0 + D_-$, where

$$D_+ = \sum_{\substack{i+j=q \\ |i-j|\leqslant l}} \left[a_i a_j b_{k-i-1} b_{k-j-1} \binom{k-1}{i}\binom{k-1}{j} - a_i a_j b_{k-i} b_{k-j-2} \binom{k}{i}\binom{k-2}{j} \right],$$

$$D_0 = \sum_{\substack{i+j=q \\ |i-j|\leqslant l}} \left[a_i a_{j-1} b_{k-i-1} b_{k-j} \binom{k-1}{i}\binom{k-1}{j-1} - a_i a_{j-1} b_{k-i} b_{k-j-1} \binom{k}{i}\binom{k-2}{j-1} \right],$$

$$D_- = \sum_{\substack{i+j=q \\ |i-j|\leqslant l}} \left[a_{i-1} a_{j-1} b_{k-i} b_{k-j} \binom{k-1}{i-1}\binom{k-1}{j-1} - a_i a_{j-2} b_{k-i} b_{k-j} \binom{k}{i}\binom{k-2}{j-2} \right].$$

We need to show that

$$D_+ + 2D_0 + D_- \geqslant 0. \tag{3}$$

(It is not true that each of D_+, D_0, D_- is separately nonnegative.) The idea is to manipulate each of the D's and use the logconcavity of the a's and b's to get lower bounds of the form

$$M(k, q, l) \times \text{a monomial in the } a\text{'s and } b\text{'s},$$

where $M(k, q, l)$ is an expression involving binomial coefficients, but is the same for each of the three D's. (See (7), (8), and (12) below.) The arithmetic–geometric mean inequality and the logconcavity of the a's and b's then give the nonnegativity of the sum of D's in (3).

We will begin by rewriting D_+. In the positive terms that appear in its definition, write the coefficient as

$$\binom{k-1}{i}\binom{k-1}{j} = \frac{1}{2}\binom{k-1}{i}\binom{k-1}{j}\left[\frac{k+i-j-1}{k-1} + \frac{k+j-i-1}{k-1}\right],$$

and then write the part of the sum that has the first fraction in it with the original indexes, and the part that has the second fraction with the indexes i, j replaced by $i-1$, $j+1$. Also, replace the negative terms by the ordinary average of the terms as they appear in the definition of D_+ and the ones

which would appear after the transformation $i \to j+1$, $j \to i-1$ is applied. The result is

$$
\begin{aligned}
D_+ = \frac{1}{2} \sum_{i+j=q} \Bigg[& a_i a_j b_{k-i-1} b_{k-j-1} \frac{k+i-j-1}{k-1} \binom{k-1}{i} \binom{k-1}{j} 1_{\{|i-j| \leqslant l\}} \\
& + a_{i-1} a_{j+1} b_{k-i} b_{k-j-2} \frac{k+j-i+1}{k-1} \binom{k-1}{i-1} \binom{k-1}{j+1} 1_{\{|i-j-2| \leqslant l\}} \\
& - a_i a_j b_{k-i} b_{k-j-2} \binom{k}{i} \binom{k-2}{j} 1_{\{|i-j| \leqslant l\}} \\
& - a_{i-1} a_{j+1} b_{k-i-1} b_{k-j-1} \binom{k-2}{i-1} \binom{k}{j+1} 1_{\{|i-j-2| \leqslant l\}} \Bigg].
\end{aligned}
$$

Note that all four terms appear in the above sum if $|i-j-1| \leqslant l-1$, while only the first and third appear if $2i = q-l$, $2j = q+l$, and only the second and fourth appear if $2i = q+l+2$, $2j = q-l-2$. (Recall that $i-j$, q, l all have the same parity.) With these particular values of i, j, the first and second terms agree, and the third and fourth terms agree. Therefore, factoring out a common product of binomial coefficients, we get (setting

$$
s = \frac{q+l}{2}, \qquad d = \frac{q-l}{2}
$$

to simplify the expressions)

$$
\begin{aligned}
D_+ = \frac{1}{2} \sum_{\substack{i+j=q \\ |i-j-1| \leqslant l-1}} \binom{k-1}{i} \binom{k-1}{j} \Bigg[& \frac{k+i-j-1}{k-1} a_i a_j b_{k-i-1} b_{k-j-1} \\
& + \frac{(k+j-i+1)\, i(k-j-1)}{(k-1)(j+1)(k-i)} a_{i-1} a_{j+1} b_{k-i} b_{k-j-2} \\
& - \frac{k(k-j-1)}{(k-1)(k-i)} a_i a_j b_{k-i} b_{k-j-2} \\
& - \frac{ki}{(k-1)(j+1)} a_{i-1} a_{j+1} b_{k-i-1} b_{k-j-1} \Bigg] \\
& + \binom{k-1}{d} \binom{k-1}{s} \Bigg[\frac{k-l-1}{k-1} a_d a_s b_{k-d-1} b_{k-s-1} \\
& - \frac{k(k-s-1)}{(k-1)(k-d)} a_d a_s b_{k-d} b_{k-s-2} \Bigg].
\end{aligned}
\tag{4}
$$

Except for a common denominator of $(k-1)(j+1)(k-i)$, the expression inside the brackets in the sum in (4) can be written as

$$\begin{aligned}
&(k+i-j-1)(j+1)(k-i)\, a_i a_j b_{k-i-1} b_{k-j-1} \\
&\quad +(k+j-i+1)\, i(k-j-1)\, a_{i-1} a_{j+1} b_{k-i} b_{k-j-2} \\
&\quad -k(k-j-1)(j+1)\, a_i a_j b_{k-i} b_{k-j-2} \\
&\quad -ki(k-i)\, a_{i-1} a_{j+1} b_{k-i-1} b_{k-j-1} \\
&= (k+j-i+1)\, i(k-j-1)[a_i a_j - a_{i-1} a_{j+1}] \\
&\quad \times [b_{k-i-1} b_{k-j-1} - b_{k-i} b_{k-j-2}] \\
&\quad +(j-i+1)\, i(j+1)[a_i a_j - a_{i-1} a_{j+1}]\, b_{k-i-1} b_{k-j-1} \\
&\quad +(j-i+1)(k-i)(k-j-1)\, a_i a_j [b_{k-i-1} b_{k-j-1} - b_{k-i} b_{k-j-2}],
\end{aligned} \tag{5}$$

which is nonnegative by the logconcavity of the a's and b's. To see this, note that in the first term on the right of (5), the coefficient is nonnegative (for $0 \leqslant i, j \leqslant k-1$, which are the only terms which appear in the sum) and the last two factors have the same sign—nonnegative if $i \leqslant j+1$ and nonpositive if $i \geqslant j+1$. To check the last two terms, consider again separately the cases $i \leqslant j+1$ and $i \geqslant j+1$. Therefore the sum in (4) is nonnegative, and we conclude that

$$\begin{aligned}
D_+ \geqslant \binom{k-1}{d}\binom{k-1}{s}\Bigg[&\frac{k-l-1}{k-1}\, a_d a_s b_{k-d-1} b_{k-s-1} \\
&- \frac{k(k-s-1)}{(k-1)(k-d)}\, a_d a_s b_{k-d} b_{k-s-2} \Bigg].
\end{aligned} \tag{6}$$

Since

$$(k-l-1)(k-d) - k(k-s-1) = d(l+1),$$

we can then use the logconcavity of the b's again in bounding below the terms on the right of (6), with the overall conclusion that

$$D_+ \geqslant \binom{k-1}{d}\binom{k-1}{s} \frac{d(l+1)}{(k-1)(k-d)}\, a_d a_s b_{k-d-1} b_{k-s-1}. \tag{7}$$

To bound D_- below, note that if we apply the following transformation to the defining expression for D_+, the result is the defining expression for D_-:

(a) Interchange the roles of the a's and b's, and then

(b) replace i by $k-i$, j by $k-j$, and q by $2k-q$.

Therefore, we may apply this transformation to (7), which gives

$$D_- \geqslant \binom{k-1}{d}\binom{k-1}{s}\frac{d(l+1)}{(k-1)(k-d)}a_{d-1}a_{s-1}b_{k-d}b_{k-s}. \tag{8}$$

In writing the bound in this form, we have used the identity

$$\binom{k-1}{s-1}\binom{k-1}{d-1}\frac{k-s}{s}=\binom{k-1}{s}\binom{k-1}{d}\frac{d}{k-d}.$$

Finally, we need to get an analogous lower bound for D_0. Let D'_+ be the expression one gets by making the following transformation in the definition of D_+:

$$i \to i, \qquad j \to j-1, \qquad q \to q-1, \qquad l \to l+1. \tag{9}$$

Then the summands in D_0 and D'_+ are identical, but D'_+ has the extra term corresponding to $i-j=-l-2$, i.e., $i=(q-l-2)/2=d-1$, $j=(q+l+2)/2=s+1$. Therefore,

$$\begin{aligned} D_0 - D'_+ = &-\binom{k-1}{d-1}\binom{k-1}{s}a_{d-1}a_s b_{k-d}b_{k-s-1} \\ &+\binom{k}{d-1}\binom{k-2}{s}a_{d-1}a_s b_{k-d+1}b_{k-s-2}. \end{aligned} \tag{10}$$

Applying (6) (with the transformation (9)) gives

$$\begin{aligned} D'_+ \geqslant \binom{k-1}{d-1}\binom{k-1}{s}\Bigg[&\frac{k-l-2}{k-1}a_{d-1}a_s b_{k-d}b_{k-s-1} \\ &-\frac{k(k-s-1)}{(k-1)(k-d+1)}a_{d-1}a_s b_{k-d+1}b_{k-s-2}\Bigg]. \end{aligned} \tag{11}$$

Add (10) and (11), noting that the second terms of the two right sides cancel exactly, to get

$$D_0 \geqslant -\binom{k-1}{d}\binom{k-1}{s}\frac{d(l+1)}{(k-1)(k-d)}a_{d-1}a_s b_{k-d}b_{k-s-1}. \tag{12}$$

Combining (7), (8), and (12), we see that in order to prove (3), we need to show that

$$a_d a_s b_{k-d-1} b_{k-s-1} - 2a_{d-1} a_s b_{k-d} b_{k-s-1} + a_{d-1} a_{s-1} b_{k-d} b_{k-s} \geqslant 0. \quad (13)$$

To see this, note that the arithmetic mean of $a_d a_s b_{k-d-1} b_{k-s-1}$ and $a_{d-1} a_{s-1} b_{k-d} b_{k-s}$ is at least their geometric mean. But this geometric mean is the same as the geometric mean of $a_{d-1} a_s b_{k-s-1} b_{k-d}$ and $a_d a_{s-1} b_{k-s} b_{k-d-1}$. Since $d-1 \leqslant d$, $s-1 \leqslant s$ and $k-s-1 \leqslant k-s$, $k-d-1 \leqslant k-d$, it follows from the logconcavity of the a's and b's that this latter geometric mean is at least $a_{d-1} a_s b_{k-s-1} b_{k-d}$. This completes the proof of (13), and therefore of (3), and hence of Theorem 3.

REFERENCES

1. H. W. Block, T. H. Savits, and M. Shaked, Some concepts of negative dependence, *Ann. Probab.* **10** (1982), 765–772.
2. C. Fortuin, P. Kastelyn, and J. Ginibre, Correlation properties on some partially ordered sets, *Comm. Math. Phys.* **22** (1971), 89–102.
3. G. Grimmett, "Percolation," Springer-Verlag, New York/Berlin, 1989.
4. K. Joag-Dev, Conditional negative dependence in stochastic ordering and interchangeable random variables, *in* "Topics in Statistical Dependence," pp. 295–298, Inst. Math. Statistics, 1990.
5. K. Joag-Dev and F. Proschan, Negative association of random variables with applications, *Ann. Statist.* **11** (1983), 286–295.
6. S. Karlin, "Total Positivity," Stanford Univ. Press, Stanford, CA, 1968.
7. T. Kim and H. Seo, A note on some negative dependence notions, *Comm. Statist. Theory Methods* **24** (1995), 845–858.
8. M. T. Lee, Dependence by reverse regular rule, *Ann. Probab.* **13** (1985), 583–591.
9. T. M. Liggett, "Interacting Particle Systems," Springer-Verlag, New York/Berlin, 1985.
10. P. Matula, Some limit theorems for negatively dependent random variables, *Yokohama Math. J.* **41** (1994), 163–173.
11. C. Newman, Asymptotic independence and limit theorems for positively and negatively dependent random variables, *in* "Inequalities in Statistics and Probability," pp. 127–140, Inst. Math. Statistics, 1984.
12. R. Pemantle, Towards a theory of negative dependence, unpublished manuscript.
13. E. Samuel-Cahn, Prophet inequalities for bounded negatively dependent random variables, *Statist. Probab. Lett.* **12** (1991), 213–216.