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GENERALIZED INVERSES OF MATRICES OVER A FINITE FIELD

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For a given $m \times n$ matrix A of rank r over a finite field F, the number of generalized inverses, of reflexive generalized inverses, of normalized generalized inverses, and of pseudoinverses of A are determined by elementary methods. The more difficult problem of determining which $m \times n$ matrices A of rank r over F have normalized generalized inverses and which have pseudoinverses is solved. Moreover, the number of such matrices which possess normalized generalized inverses and the number which possess pseudoinverses are found.

1. Introduction

Moore [8, 9] generalized the notion of the inverse of an $n \times n$ matrix to include generalized inverses of $m \times n$ matrices of arbitrary rank r over the real and the complex fields. Penrose [11] applied generalized inverses of matrices to solutions of simultaneous linear equations. Rohde [12, 13] distinguished four different generalized inverses of a given $m \times n$ matrix A of rank r over the complex field (see Section 2). Pearl [10] considered the existence of the various generalized inverses of a given $m \times n$ matrix of rank r over an arbitrary field F under an arbitrary involutory automorphism $-: F \to F$. Kim [7] has found, for a given $m \times n$ matrix of rank r over a finite field, the number of reflexive generalized inverses of A.

2. Notation and preliminaries

Let F be a field with an involutory automorphism $-$. Let $M_{n,m}(F)$ denote the set of all $n \times m$ matrices over F. If $A = (a_{ij}) \in M_{n,m}(F)$, then $A^* = (a^*_{ij}) \in M_{m,n}(F)$, where $a^*_{ij} = \bar{a}_{ji}$.

Definition 2.1. Let $A \in M_{m,n}(F)$ and $A = (a_{ij})$. Any X in $G(m, n, F) = \{X \in M_{n,m}(F): AXA = A\}$ will be called a *generalized inverse* of A and will be denoted by $A^g = X$. Any X in $R(m, n, F) = \{X \in G(m, n, F): XAX = X\}$ will be called a *reflexive generalized inverse* of A and will be denoted by $A^r = X$. Any X

in $N(m, n, F) = \{X \in R(m, n, F) : (AX)^* = AX\}$ will be called a *normalized generalized inverse* of A and will be denoted by $A^n = X$. Any X in $P(m, n, F) = \{X \in N(m, n, F) : (XA)^* = XA\}$ will be called a *pseudoinverse* of A and will be denoted by $A\dagger = X$.

GF (q^2) will denote a finite field of cardinality q^2, where $q = p^y$, p a prime, y a positive integer. The symbol $-$ will denote an involutory field automorphism of GF (q^2) given by $\bar{a} = a^q$. Then GF (q) is the fixed subfield of GF (q^2) relative to the automorphism $-$. If q is odd and g is any generator of the multiplicative group of GF (q), let $w = g^{(q+1)/2}$. Then GF $(q^2) = \{c + dw : c, d \in \text{GF}(q)\}$ and $\overline{c + dw} = c + dw^q = c - dw$. If q is even, let w denote any primitive element of GF (q^2). Then GF $(q^2) = \{c + dw : c, d \in \text{GF}(q)\}$, and if $a = c + dw$, then $\bar{a} = c + dw^q$ and $a\bar{a} = c^2 + (w + w^q)cd + w^{q+1}d^2 \in \text{GF}(q)$.

$\mathcal{V}_c(q^i)$ will denote the vector space of c-tuples $\chi = (x_1, x_2, \ldots, x_c)$ over GF (q^i), $i = 1, 2$. If h is a Hermitian scalar product on $\mathcal{V}_c(q^2) \times \mathcal{V}_c(q^2)$ and if $\mathcal{B}$ is any ordered basis for $\mathcal{V}_c(q^2)$, then there exist elements h_{ij} in GF (q^2) such that $h(\chi, \chi) = \sum_i^c \sum_j^c h_{ij} x_i \bar{x}_j = \chi H \chi^*$, where $H = (h_{ij})$ is the $c \times c$ Hermitian matrix of the Hermitian form on $\mathcal{V}_c(q^2)$ relative to $\mathcal{B}$ defined by h and where * represents the conjugate, transpose.

If h is a Hermitian scalar product of rank k on $\mathcal{V}_n(q^2) \times \mathcal{V}_n(q^2)$, it may be seen in the text by Jacobson [6, p.153], for example that there exists an ordered basis $(\nu_1, \ldots, \nu_k, \zeta_1, \ldots, \zeta_{n-k})$ of $\mathcal{V}_n(q^2)$ such that the matrix of h relative to this basis is the diagonal matrix $D = D[b_1, \ldots, b_k, 0, \ldots, 0]$, where $0 \neq b_i = h(\nu_i, \nu_i)$, $i = 1, \ldots, k$.

Carlitz and Hodges [1] use a theorem by Dickson [2, p.46] to show that if q^2 is odd, there exists a basis $(\omega_1, \ldots, \omega_k, \zeta_1, \ldots, \zeta_{n-k})$ of $\mathcal{V}_n(q^2)$ such that the matrix of h relative to this basis is

$$\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix},$$

where I_k is the $k \times k$ identity matrix.

Suppose q^2 is even. Since each b_i in the matrix D above is a Hermitian element of GF (q^2), choose element $c_i \in \text{GF}(q)$ such that $c_i^2 = b_i \in \text{GF}(q)$. Then $c_i \bar{c}_i = b_i$. Hence, there exists an ordered basis $(\omega_1, \ldots, \omega_k, \zeta_1, \ldots, \zeta_{n-k})$ such that the matrix of h relative to this basis is

$$\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}.$$

If $\mathcal{S}$ is a subspace of $\mathcal{V} = \mathcal{V}_c(q^2)$, subspace $\mathcal{S}^\perp = \{\chi \in \mathcal{V} : h(\chi, \sigma) = 0$ for all $\sigma \in \mathcal{S}\}$. The *radical* of subspace $\mathcal{S}$ is the subspace Rad $\mathcal{S} = \mathcal{S} \cap \mathcal{S}^\perp$. A subspace $\mathcal{S}$ of $\mathcal{V}$ is said to be *nonisotropic, isotropic,* or *totally isotropic* according as Rad$(\mathcal{S})$ is $\{0\}$, is not $\{0\}$, or is $\mathcal{S}$, respectively. The Hermitian scalar product h is said to be *nondegenerate*, or be of *full rank* if Rad $\mathscr{V}=\{0\}$. Subspaces $\mathscr{S}_1$ and $\mathscr{S}_2$ of $\mathscr{V}$ are said to be *h-equivalent* if and only if there exists a linear isomorphism U of $\mathscr{S}_1$ onto $\mathscr{S}_2$ such that $h(\chi,\eta)=h(U(\chi),U(\eta))$ for all $\chi,\eta\in\mathscr{S}_1$. Also, if U defines an h-equivalence of $\mathscr{V}$, then U is said to be a *unitary transformation* on $\mathscr{V}$.

Throughout this paper, $|\mathscr{S}|$ will denote the cardinality of the set $\mathscr{S}$, and $\rho(A)$ will denote the rank of matrix A.

3. The generalized inverses of a given matrix

We shall prove the following theorem:

Theorem 3.1. *Let* $A\in M_{m,n}(\mathrm{GF}(q))$ *of rank* $r=\rho(A)$. *Then*

$$|G(m,n,\mathrm{GF}(q))|=q^{nm-r^2} \tag{1}$$

$$|R(m,n,\mathrm{GF}(q))|=q^{r(m+n-2r)}. \tag{2}$$

Let $A\in M_{m,n}(\mathrm{GF}(q^2))$ *of rank* $r=\rho(A)$. *Then*

$$|N(m,n,\mathrm{GF}(q^2))|=\begin{cases} q^{2r(n-r)} & \text{if } r=\rho(A^*A),\\ 0 & \text{if } r>\rho(A^*A).\end{cases} \tag{3}$$

$$|P(m,n,\mathrm{GF}(q^2))|=\begin{cases} 1 & \text{if } r=\rho(A^*A)=\rho(AA^*),\\ 0 & \text{otherwise.}\end{cases} \tag{4}$$

Proof. (4) follows from [8] and [10, Theorem 1]. (2) is clear from [7]. We consider (1). For A, there exist two nonsingular matrices P and Q over $\mathrm{GF}(q)$ such that

$$PAQ=\begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}=K_r,$$

where I_r denotes the $r\times r$ identity matrix. Let $X\in G(m,n,\mathrm{GF}(q))$ and let

$$Y=Q^{-1}XP^{-1}=\begin{pmatrix} Y_1 & Y_2\\ Y_3 & Y_4\end{pmatrix}.$$

From $AXA=A$, we obtain $K_rYK_r=K_r$ and $Y_1=I_r$. We see that Y_2, Y_3, and Y_4 are arbitrary. Thus, we find that $|G(m,n,\mathrm{GF}(q))|=q^{mn-r^2}$ since Y_2 is $r\times(m-r)$, Y_3 is $(n-r)\times r$, and Y_4 is $(n-r)\times(m-r)$.

Consider (3). If $r>\rho(A^*A)$, then $|N(m,n,\mathrm{GF}(q^2))|=0$ by [10, Corollary 1]. We suppose $\rho(A^*A)=r=\rho(A)$. For A, there exist nonsingular matrices P in $M_{m,m}(\mathrm{GF}(q^2))$ and Q in $M_{n,n}(\mathrm{GF}(q^2))$ such that $PAQ=K_r$. Let $X\in N(m,n,\mathrm{GF}(q^2))$ and let

$$Y=Q^{-1}XP^{-1}=\begin{pmatrix} Y_1 & Y_2\\ Y_3 & Y_4\end{pmatrix}$$

Then we can obtain that $K_rYK_r = K_r$, $Y_1 = I_r$, $Y_4 = Y_3Y_2$ and $PP^*(K_rY)^*(PP^*)^{-1} = K_rY$. By letting

$$(PP^*)^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

we show with little difficulty that $Y_2^*B_1 = B_3$. We show that B_1 is nonsingular (see [10, pp.574–575]). Thus, we have $Y_2 = (B_3B_1^{-1})^*$. $Y_3 \in M_{r,n-r}(\mathrm{GF}(q^2))$ is arbitrary, and, hence, we infer that $|N(m, n, \mathrm{GF}(q^2))| = q^{2r(n-r)}$. This proves the theorem.

Thus, as has been indicated by Pearl [10], the methods of Section 3 are field independent and certainly free of the theory of Hermitian forms. However, as will be seen in Section 4, the more difficult problem of determining which matrices over GF (q^2) have normalized generalized inverses and which have pseudionverses will be resolved by methods peculiar to finite fields. Our methods invoke the classical theory of Hermitian forms over GF (q^2).

4. Matrices A such that A^n and $A\dagger$ exist

Pearl [10, Theorem 1] proved that $A \in M_{m,n}(\mathrm{GF}(q^2))$ with $r = \rho(A)$ has a normalized generalized inverse if and only if $r = \rho(A^*A)$ and has a pseudoinverse if and only if $r = \rho(A^*A) = \rho(AA^*)$. We let $\mathscr{A}(m, n, r, q^2) = \{A \in M_{m,n}(\mathrm{GF}(q^2)) : r = \rho(A) = \rho(A^*A)\}$ and let $\mathscr{B}(m, n, r, q^2) = \{A \in \mathscr{A}(m, n, r, q^2) : r = \rho(AA^*)\}$.

We require in this section the cardinality (see [14, p. 33], for example)

$$|\mathscr{U}_k(q^2)| = q^{(k^2-k)/2} \prod_{i=1}^{k} (q^i - (-1)^i) \tag{5}$$

of $\mathscr{U}_k(q^2)$, the unitary subgroup of the general linear group in $M_{k,k}(\mathrm{GF}(q^2))$. Also, we require the widely known cardinality

$$|\mathscr{S}(r, k, q^2)| = \prod_{i=0}^{r-1} (q^{2k} - q^{2i}) \tag{6}$$

of $\mathscr{S}(r, k, q^2) = \{A \in M_{r,k}(\mathrm{GF}(q^2)) : r = \rho(A)\}$.

Our method of characterizing and enumerating the rank r matrices $A \in M_{m,n}(\mathrm{GF}(q^2))$ such that A^n or $A\dagger$ exist involves the solutions in $M_{m,r}(\mathrm{GF}(q^2))$ to $T^*T = I_r$.

Lemma 4.1. *If* $\mathscr{T}(m, r, q^2) = \{T \in M_{m,r}(\mathrm{GF}(q^2)) : T^*T = I_r\}$, *then*

$$|\mathscr{T}(m, r, q^2)| = |\mathscr{U}_m(q^2)| / |\mathscr{U}_{m-r}(q^2)|, \tag{7}$$

where for $k = m$ *or* $k = m - r$, $|\mathscr{U}_k(q^2)|$ *is given by* (5).

Proof. Let $T=[\tau_1, \tau_2, \ldots, \tau_r]\in \mathcal{T}(m, r, q^2)$ and let $\mathcal{CS}$ denote column space. For $\mathcal{V}=\mathcal{V}_m(q^2)$, let h be a nondegenerate Hermitian scalar product on $\mathcal{V}\times\mathcal{V}$ such that the matrix of h relative to the ordered basis of elementary $m\times 1$ unit vectors $(\varepsilon_1, \ldots, \varepsilon_m)$ of $\mathcal{V}$ is I_m. Let $\mathcal{W}=<\varepsilon_1, \ldots, \varepsilon_r>$ and consider the linear transformation $L:\mathcal{W}\rightarrow\mathcal{CS}(T)$ such that $L(\varepsilon_i)=\tau_i$, $i=1,\ldots,r$. Then L is an h-equivalence of $\mathcal{W}$ onto $\mathcal{CS}(T)$. Since $\mathcal{W}^\perp=(\mathcal{CS}(T))^\perp=\{0\}$, each of $\mathcal{W}$ and $\mathcal{CS}(T)$ is nonisotropic and Witt's theorem [6, p. 162] applies. Hence, L can be extended to an element U_0 of the unitary group on $\mathcal{V}$. Consider the unitary group on $\mathcal{V}$ as $\mathcal{U}_m(q^2)$, the group of unitary matrices of the unitary transformations on $\mathcal{V}$ relative to the ordered basis $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$. Let $\mathcal{L}=\{U\in\mathcal{U}_m(q^2): U \text{ extends } L\}$. Also, the identity linear transformation $I:\mathcal{W}\rightarrow\mathcal{W}$ such that $I(\varepsilon_i)=\varepsilon_i$, $i=1,\ldots,r$, is an h-equivalence of $\mathcal{W}$ and, thus, can be extended to an element of $\mathcal{U}_m(q^2)$. If we let $\mathcal{I}$ denote the subgroup $\{U\in\mathcal{U}_m(q^2): U \text{ extends } I\}$ of $\mathcal{U}_m(q^2)$, it is immediate that $\mathcal{I}$ is isomorphic to $\mathcal{U}_{m-r}(q^2)$ and that $\mathcal{L}=U_0\mathcal{I}$, the coset of $\mathcal{I}$ in $\mathcal{U}_m(q^2)$ which contains U_0. Hence, $|\mathcal{L}|=|U_0\mathcal{I}|=|\mathcal{I}|=|\mathcal{U}_{m-r}(q^2)|$. Therefore, $|\mathcal{T}(m, r, q^2)|$ is given by (7), and the proof is complete.

Theorem 4.2. *Let $A\in M_{m,n}(\mathrm{GF}(q^2))$ have $\rho(A)=r$. Then $A\in\mathcal{A}(m, n, r, q^2)$ if and only if $A=TS$, where T is $m\times r$ such that $T^*T=I_r$ and where S is $r\times n$ with $\rho(S)=r$. Moreover,*

$$|\mathcal{A}(m, n, r, q^2)|=|\mathcal{S}(r, n, q^2)|\,|\mathcal{T}(m, r, q^2)|/|\mathcal{U}_r(q^2)|, \tag{8}$$

where $|\mathcal{S}(r, n, q^2)|$ is given by (6), *$|\mathcal{T}(m, r, q^2)|$ is given by* (7), *and $|\mathcal{U}_r(q^2)|$ is given by* (5).

Proof. Let $A\in M_{m,n}(\mathrm{GF}(q^2))$ have rank r. Then $A=RS_1$, where R is $m\times r$ and S_1 is $r\times n$ such that $\rho(R)=\rho(S_1)=r$ (see [3]). Now $\rho(A^*A)=r$ if and only if $\rho(R^*R)=r$. Again, for $\mathcal{V}=\mathcal{V}_m(q^2)$, let h be a nondegenerate Hermitian scalar product on $\mathcal{V}\times\mathcal{V}$ such that the matrix of h relative to the ordered basis of elementary $m\times 1$ unit vectors $(\varepsilon_1, \ldots, \varepsilon_m)$ of $\mathcal{V}$ is I_m. From Section 2, $\rho(R^*R)=r$ if and only if there exists an ordered basis $(\tau_1, \tau_2, \ldots, \tau_r)$ of $\mathcal{CS}(R)$ such that $h(\tau_i, \tau_j)=\delta_{ij}$, the Kronecker delta, and, thus, if and only if there exists a nonsingular matrix $M\in M_{r,r}(\mathrm{GF}(q^2))$ such that if $[\tau_1, \tau_2, \ldots, \tau_r]=RM=T$, then $T^*T=I_r$. Thus, $r=\rho(A^*A)=\rho(A)$ if and only if $A=RS_1=TM^{-1}S_1=TS$, where T and S satisfy the statement of Theorem 4.2.

Among the $|\mathcal{T}(m, r, q^2)|\,|\mathcal{S}(r, n, q^2)|$ pairs of matrices (T, S) in $\mathcal{T}(m, r, q^2)\times\mathcal{S}(r, n, q^2)$, the same matrix $A=TS$ in $\mathcal{A}(m, n, r, q^2)$ may be repeated many times. In fact, for (T_1, S_1), (T_2, S_2) in $\mathcal{T}(m, r, q^2)\times\mathcal{S}(r, n, q^2)$, $T_1S_1=T_2S_2$ if and only if $T_2=T_1(S_1S_2^{\dagger})$, where $I_r=T_2^*T_2=(S_1S_2^{\dagger})^*T_1^*T_1(S_1S_2^{\dagger})=(S_1S_2^{\dagger})^*(S_1S_2^{\dagger})$ and where $S_2^{\dagger}$ is any right inverse for S_2. Hence $T_1S_1=T_2S_2$ if and only if $S_1S_2^{\dagger}\in\mathcal{U}_r(q^2)$. On the other hand, if $T_1\in\mathcal{T}(m, r, q^2)$ and $U\in\mathcal{U}_r(q^2)$, so does $T_2=T_1U\in\mathcal{T}(m, r, q^2)$. Thus, $|\mathcal{A}(m, n, r, q^2)|$ is given by (8). Hence, Theorem 4.2 has been proved.

We remark here that if $\mathcal{G}(r, n, q^2) = \{G \in M_{r,n}(\mathrm{GF}(q^2)) : GG^* = I_r\}$, then $|\mathcal{G}(r, n, q^2)| = |\mathcal{T}(n, r, q^2)|$, given by (7).

Theorem 4.3. *Let $A \in M_{m,n}(\mathrm{GF}(q^2))$ have $\rho(A) = r$. Then $A \in \mathcal{B}(m, n, r, q^2)$ if and only if $A = TNG$, where $T \in \mathcal{T}(m, r, q^2)$, $N \in \mathcal{S}(r, r, q^2)$, and $G \in \mathcal{G}(r, n, q^2)$. Moreover,*

$$|\mathcal{B}(m, n, r, q^2)| = |\mathcal{T}(m, r, q^2)|\,|\mathcal{S}(r, r, q^2)|\,|\mathcal{T}(n, r, q^2)|/|\mathcal{U}_r(q^2)|^2, \tag{9}$$

where each of $\mathcal{T}(m, r, q^2)$ and $\mathcal{T}(n, r, q^2)$ is given by (7), where $\mathcal{S}(r, r, q^2)$ is given by (6), and where $\mathcal{U}_r(q^2)$ is given by (5).

Proof. Let $A = TS \in \mathcal{A}(m, n, r, q^2)$, where $T \in \mathcal{T}(m, r, q^2)$ and $S \in \mathcal{S}(r, n, q^2)$. Then $r = \rho(AA^*)$ if and only if $r = \rho(SS^*)$ if and only if there exists an $r \times r$ nonsingular matrix N_1 such that if $G = N_1 S$, then $GG^* = I_r$. Hence, $\rho(AA^*) = r$ if and only if $A = TNG$, where each of T, $N = N_1^{-1}$, and G satisfies the statement of Theorem 4.3.

Consider the list of

$$|\mathcal{T}(m, r, q^2)|\,|\mathcal{S}(r, r, q^2)|\,|\mathcal{T}(n, r, q^2)|/|\mathcal{U}_r(q^2)|,$$

$m \times n$ matrices $A = TNG$, each belonging to $\mathcal{B}(m, n, r, q^2)$, where $T \in \mathcal{T}(m, r, q^2)$, $N \in \mathcal{S}(r, r, q^2)$, and $G \in \mathcal{G}(r, n, q^2)$ and where if $T_1 N_1 G_1$ and $T_2 N_2 G_2$ are in the list, then $T_1 \neq T_2 U$ for any $U \in \mathcal{U}_r(q^2)$. Now $T_1 N_1 G_1 = T_2 N_2 G_2$ in the list if and only if $T_2 = T_1(N_1 G_1 G_2^{i} N_2^{-1})$, where G_2^{i} is any right inverse for G_2 and where $U = N_1 G_1 G_2^{i} N_2^{-1} \in \mathcal{U}_r(q^2)$. That is, $T_1 N_1 G_1 = T_2 N_2 G_2$ if and only if $T_1 = T_2$ and, thus, $N_1 G_1 = N_2 G_2$. But $N_1 G_1 = N_2 G_2$ if and only if $G_2 = (N_2^{-1} N_1) G_1$, where $I_r = G_2 G_2^* = (N_2^{-1} N_1) G_1 G_1^* (N_2^{-1} N_1)^* = (N_2^{-1} N_1)(N_2^{-1} N_1)^*$. Hence each $A \in \mathcal{B}(m, n, r, q^2)$ occurs precisely $|\mathcal{U}_r(q^2)|$ times in the list. Therefore, $|\mathcal{B}(m, n, r, q^2)|$ is given by (9). The proof is complete.

The problem of finding the number $m \times n$ matrices X over $\mathrm{GF}(q^2)$ which satisfy the matrix equation $XAX^* = B$, for given Hermitian matrices A and B has received much attention [1, 4, 5, 15], and in the latter three papers, the number solutions X of rank r is given. We did not appeal to these papers, however, and preferred to develop methods of our own.

We assumed in this section that the involutory automorphism $-$ on $\mathrm{GF}(q^2)$ was not the identity (see Section 2). However, if $-$ is the identity automorphism of $\mathrm{GF}(q^2)$ ($\mathrm{GF}(q)$ could be used as well in this case), *and if q is odd*, the methods and enumerations of this section need no alteration [6, p. 162]. On the other hand, if $-$ is the identity *and q is even*, Witt's theorem must be reformulated [6, p. 162], and special methods must be devised to find $|\mathcal{A}(m, n, r, q^2)|$ and $|\mathcal{B}(m, n, r, q^2)|$. The case where the involutory automorphism $-$ of $\mathrm{GF}(q^2)$, q even, the identity automorphism will be considered in another paper by the author.

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