Differentiability of a class of semi-linear hyperbolic systems with respect to a parameter

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Abstract

If a parameter is contained in a hyperbolic semi-linear equation, then the solution to the equation will depend on the parameter. It is proved by using the theory of semi-groups of bounded operators that the solution continuously depends on the parameter and is Fréchet continuously differentiable, under certain mild assumptions.

Keywords: \( C_0 \)-semigroup; Fréchet differentiable; Semi-linear hyperbolic system

1. Introduction

If a parameter is contained in an equation, the study of the continuous dependence and differentiability of the solution to the equation with respect to the parameter is a topic very meaningful, which has important applications in the research of mathematical modelling, optimization and automatic control.

For the case where the equation is a system of nonlinear ordinary differentiable equations and the parameter is a constant vector, the research results can be found in [1], when the parameter is a time-varying vector, see [5].
For the case where the equation is a linear elliptic or parabolic equation of second order and the parameter is a vector-valued function having spatial variables as its arguments, the articles [6,7] show that the solution to the equation is Fréchet or Gâteaux continuously differentiable with respect to the parameter, using the method of a priori estimation for solutions of partial differential equations.

In the present article, semi-linear hyperbolic equations of second order, which contain as the parameter a vector-valued function with spatial variables as its arguments, are investigated. Under a set of mild assumptions, based on the research work in [8], it is proved, by using the method of semi-groups of strongly continuous operators, that the solutions of equations possess continuous dependence and Fréchet differentiability with respect to the parameter. Our main results are stated in Section 2. In Section 3, firstly, the problem is transformed into a Cauchy problem of semi-linear equation in a Hilbert space, then, it is proved that the principal operator generates a $C_0$-group of strongly continuous bounded Unitary operators, and then the main results of the article are proved by use of the research work on the differentiability of $C_0$-semigroups [8].

For shortness, only mixed initial-boundary value problems and the Fréchet differentiability are considered in this article, the results regarding Cauchy problems and the Gâteaux differentiability are omitted here.

Notations:

$\|\cdot\|_k, \langle\cdot,\cdot\rangle_k$ the norm and inner product of Sobolev space $H^k(\Omega)(L^2(\Omega))$, if $k = 0$;

$\mathbb{R}^+, \mathbb{R}_0^+$ the sets $\{t \in \mathbb{R}; t > 0\}$, $\{t \in \mathbb{R}; t \geq 0\}$, respectively;

$\mathcal{D}(T)$ the domain of operator $T$;

$T^*$ the adjoint operator of operator $T$;

$C^k(X, Y)$ the space of $k$th-degree Fréchet continuously differentiable functions from $X$ to $Y$.

2. Problem statement and main results

For briefness, we only consider problems with the Dirichlet boundary value, corresponding results can be obtained for Cauchy problems and other initial-boundary value problems.

Given a second-order semi-linear hyperbolic equation

$$
\begin{aligned}
\begin{cases}
\frac{\partial^2 u}{\partial t^2} = A(q)u + f(x, u; q), & (x, t) \in \Omega \times (0, T), \\
u|_{\partial\Omega} = 0, \\
u(x, 0) = u_0(x), & \frac{\partial u(x, 0)}{\partial t} = u_1(x), & x \in \Omega,
\end{cases}
\end{aligned}
$$

(2.1)

where the operator

$$
\mathcal{A}(q) \equiv \sum_{i=1}^{m} \left[ \partial_i \left( a_{ij}(x; q) \partial_j \cdot \right) + b_i(x; q) \partial_i \cdot \right] + C(x; q),
$$

satisfies the uniform ellipticity condition:

$$
\exists \nu, \mu > 0, \text{ s.t. } \nu |\xi|^2 \leq \sum_{i,j=1}^{m} a_{ij}(x; q) \xi_i \xi_j \leq \mu |\xi|^2, \text{ a.e. } x \in \Omega, \xi \in \mathbb{R}^m,
$$

(2.2)

the symmetry condition:

$$
a_{ij}(x; q) = a_{ji}(x; q), \text{ a.e. } x \in \Omega, i, j = 1, \ldots, m,
$$

(2.3)
and the boundedness condition:

\[ \exists L > 0, \quad \text{s.t.} \quad |b_i(x; q)|, |c(x; q)| \leq L, \quad \text{a.e. } x \in \Omega, \quad i = 1, \ldots, m, \quad (2.4) \]

here, \( \partial_i \equiv \partial / \partial x_i \), \( \Omega \subset \mathbb{R}^n \) is a bounded domain, \( \partial \Omega \) is the boundary of \( \Omega \), \( q \in Q_{ad} \subset \mathbb{R} \) is the parameter, \( Q_{ad} \) is the set of admissible parameters (with a nonempty interior part \( Q_{ad}^0 \)), and \( Q \) is a Banach space. The constants \( \mu, \nu \), and \( L \) in above may depend on parameter \( q \), i.e., \( \nu = \nu(q), \mu = \mu(q) \) and \( L = L(q) \).

We denote

\[ a_{ij}(q) = a_{ij}(:, q), \quad c(q) = c(:, q), \]
\[ f(u; q) = f(:, u; q), \quad u^0 = u^0(:, ), \quad u^1 = u^1(:) \quad (2.5) \]

and assume that

(AA) \( \forall q \in Q_{ad}, \partial_k a_{ij}(q), a_{ij}(q), b_i(q), c(q) \in L^\infty(\Omega) \);

(BB) \( u^0 \in H^2(\Omega) \cap H^0_0(\Omega), u^1 \in H^1(\Omega) \);

(A1) \( a_{ij}, b_i, c \in C(Q_{ad}; L^\infty(\Omega)) \);

(A2) \( a_{ij} \in C^2(Q_{ad}; L^2(\Omega)), b_i, c \in C^1(Q_{ad}; L^\infty(\Omega)) \);

(A3) constants \( \mu(q), \nu(q), L(q) \) satisfy the local stability condition, i.e., \( \forall q_0 \in Q_{ad}^0 \), there exists \( r_0 = r_0(q_0) \) and \( \varepsilon_0 = \varepsilon_0(q_0) \) such that

\[ 0 < \nu(q_0) - \varepsilon_0 < \nu(q) < \nu(q_0) + \varepsilon_0, \quad L(q) < L(q_0) + \varepsilon_0, \]
\[ \forall q \in B(q_0; r_0) \cap Q_{ad}, \quad (2.6) \]

where \( B(q_0; r_0) \equiv \{ q \in Q; \quad \| q - q_0 \| < r_0 \} \);

(A4) \( \forall q \in Q_{ad}, u \mapsto f(u; q) \) is a continuously differentiable operator, and \( \forall u \in \mathbb{R}, f(u; :) \in C^1(Q_{ad}; L^\infty(\Omega)) \).

We have, for the well-posedness of problem (2.1), the following results:

**Theorem 2.1.** Suppose that (AA), (BB), and (A4) hold. Then \( \forall q \in Q_{ad} \), (2.1) has a unique solution \( u \in C^2(\mathbb{R}^+, L^2(\Omega)) \cap C^1(\mathbb{R}^+, H^0_0(\Omega)) \cap C(\mathbb{R}^+, H^1_0(\Omega)) \cap C^1(\mathbb{R}^+, L^2(\Omega)) \) such that \( u(\cdot, t) \in H^2(\Omega) \cap H^1_0(\Omega) \), and \( u(x, t) \) satisfies (2.1).

It is obvious that the solution depends on parameter \( q \), so we denote \( u = u(q) = u(x, t; q) \), and the result below answers how the relationship between \( u \) and \( q \) is.

**Theorem 2.2.** Suppose that (AA), (BB), (A3), and (A4) hold. Then, the solution \( u(q) \) of (2.1) is a continuous function of \( q \), i.e., \( u(\cdot, t) \in C(Q_{ad}; H^1_0(\Omega)) \), \( \forall t \in [0, T] \).

**Theorem 2.3.** Suppose that (AA), (BB), and (A1)–(A4) hold. Then the solution \( u(q) \) of (2.1) is a Fréchet continuously differentiable function of \( q \), i.e., \( u(\cdot, t) \in C^1(Q_{ad}; H^1_0(\Omega)) \), \( \forall t \in [0, T] \); moreover, its Fréchet differential \( u'(q)h \) at \( q \in Q_{ad}^0 \) along the direction of \( h \in Q \), denoted with \( \hat{u} \), is determined by the following initial-boundary value problem:

\[
\begin{cases}
\frac{\partial \hat{u}}{\partial t^2} = \left[ A(q) + \frac{\partial f}{\partial t}(x, u; q) \right] \hat{u} + A'(q)hu + \frac{\partial f}{\partial q}(x, u; q)h, \\
\hat{u} \mid_{\partial \Omega} = 0, \\
\hat{u}(x, 0) = 0, \quad \frac{\partial \hat{u}}{\partial t}(x, 0) = 0, \quad x \in \Omega,
\end{cases}
\]

in which \( u \) is determined by (2.1).
3. Proof of main results

First, we transform problem (2.1) into a Cauchy problem of an evolution equation in a Hilbert space. So, we introduce the symbols:

\[ u(t) = u(\cdot, t), \quad u(q) = u(\cdot, \cdot; q), \quad \mathcal{H} = H^1_0(\Omega) \times L^2(\Omega). \]

Combining with notations (2.5), we can rewrite (2.1) as

\[
\begin{cases}
\frac{d}{dt}(u_1 u_2) = (0 I_0)(u_1 u_2) + (0 f(u; q)), \\
(u_1 u_2)|_{t=0} = (u_0 u_0).
\end{cases}
\]

(3.1)

If \( U(t) = (u_1 u_2) \) and \( U_0 = (u_0 u_0) \) are considered as elements in \( \mathcal{H} \), then we have

\[
\begin{cases}
\frac{dU}{dt} = A(q)U + F(U; q), \\
U(0) = U_0.
\end{cases}
\]

(3.2)

where

\[
A(q) = A_0(q) + B(q),
\]

\[
A_0(q) = \begin{pmatrix} 0 & I_0 \\ A_0(q) & 0 \end{pmatrix}, \quad B(q) = \begin{pmatrix} 0 & 0 \\ B(q) & 0 \end{pmatrix}, \quad F(U; q) = \begin{pmatrix} 0 \\ f(u_1; q) \end{pmatrix},
\]

\[
B(q) = \sum_{i=1}^{m} b_i(q) \partial_i \cdot + [\nu + c(q)] \cdot,
\]

\[
A_0(q) = \sum_{i=1}^{m} \partial_i(a_{ij}(q) \partial_j \cdot) - \nu \cdot,
\]

(3.3)

in which \( \nu \) is the same constant in the inequality (2.2).

For any \( q \in Q_{ad} \), the domains of \( A(q), B(q), \ldots \) are

\[
\mathcal{D}(A_0(q)) = H^2(\Omega) \cap H^1_0(\Omega),
\]

\[
\mathcal{D}(B(q)) = H^1(\Omega),
\]

\[
\mathcal{D}(A_0(q)) = \mathcal{D}(A_0(q)) \times H^1(\Omega),
\]

\[
\mathcal{D}(B(q)) = H^1(\Omega) \times L^2(\Omega).
\]

(3.4)

In the following three lemmas, the condition (AA) is always assumed.

**Lemma 3.1.** \( \forall q \in Q_{ad} \), the operator \( -A_0(q) \) is self-adjoint and positive-definite in its domain, i.e.,

\[
\langle -A_0(q)u, u \rangle_0 > 0, \quad \forall u \in \mathcal{D}(A_0(q)).
\]

(3.5)

**Proof.** It is obvious that (3.5) holds true. We now prove that the operator \( -A_0(q) \) is self-adjoint. \( \forall u, v \in \mathcal{D}(A_0(q)) \subset L^2(\Omega) \), we have by Green’s formula that

\[
\langle -A_0(q)u, v \rangle = -\sum_{i,j} \int_{\Omega} a_{ij}(\partial_j u) v \cos(n, x_i) ds + \sum_{i,j} \int_{\partial \Omega} u a_{ij} \partial_i v \cos(n, x_j) ds
\]

\[
-\int_{\Omega} u \sum_{i,j} \partial_j(a_{ij} \partial_i v) dx + v \int_{\Omega} uv dx,
\]

In the following three lemmas, the condition (AA) is always assumed.
hence we have
\[\langle -A_0(q)u, v \rangle = \langle u, -A_0(q)v \rangle, \quad (3.6)\]
noting the condition (2.3) and that \(u, v \in H^1_0(\Omega)\).
Therefore
\[-A_0(q) \subset [-A_0(q)]^*.\]

Since \(C^\infty_0(\Omega) \subset \mathcal{D}(A_0(q))\) is dense in \(L^2(\Omega)\), \(-A_0(q)\) is a symmetric operator. On the other hand, by the theory of elliptic equations (see, for instance, [2, p. 155]), \(\forall g \in L^2(\Omega)\), for the homogeneous Dirichlet boundary value problem of the second-order elliptic equation
\[-A_0(q)v \equiv g,\]
there exists a unique solution \(v\), that is, the range of operator \(-A_0(q)\) is coincident with \(L^2(\Omega)\), consequently, by the theory of operators in Hilbert spaces [4, p. 107], we obtain that
\[-A_0(q) = [-A_0(q)]^*. \quad \Box\]

**Lemma 3.2.** \(\forall q \in Q_{ad}\), the operator \(-A_0(q)\) is skew-self-adjoint, i.e.,
\[[-A_0(q)]^* = -A_0(q).\]

**Proof.** It follows from Lemma 3.1 that there exists a self-adjoint dense-definite operator \(R\) such that \(-A_0(q) = R^2\). Introduce into \(H^1_0(\Omega)\) the inner product:
\[\langle u, v \rangle_H = \langle Ru, Rv \rangle_0, \quad u, v \in H^1_0(\Omega), \quad (3.7)\]
and then we have: \(\forall u \in C^\infty_0(\Omega) \subset L^2(\Omega),\)
\[\langle u, u \rangle_H = \langle R^2u, u \rangle_0 = \langle -A_0(q)u, u \rangle_0 = \sum_{i, j=1}^m \langle a_{ij}(q) \partial_j u, \partial_i u \rangle + \nu \langle u, u \rangle. \quad (3.8)\]

By the uniform ellipticity condition (2.2):

\[\nu \|u\|^2_H \leq \langle u, u \rangle_H \leq \mu \|u\|^2_1, \quad (3.9)\]

which shows that \(\|\cdot\|_H\) and \(\|\cdot\|_1\) are equivalent norms, we may denote with \(H\) the completion space of \(C^\infty_0(\Omega)\) by the norm \(\|\cdot\|_H\), then we have obviously that \(H\) is equivalent to \(H^1_0(\Omega)\), thus, \(\mathcal{H} = H \times L^2(\Omega)\).

Furthermore, observe the operator \(A_0(q)\) on \(\mathcal{H}\):
\[\forall U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{D}(A_0(q)) \subset \mathcal{H},\]

\[\langle A_0(q)U, V \rangle_{\mathcal{H}} = \langle u_2, v_1 \rangle_H + \langle A_0(q)u_1, v_2 \rangle_0 = \langle Ru_2, Rv_1 \rangle_0 - \langle R^2u_1, v_2 \rangle_0 = -\langle Ru_1, Rv_2 \rangle_0 - \langle u_2, A_0(q)v_1 \rangle_0 = \langle U, -A_0(q)V \rangle_{\mathcal{H}}, \quad (3.10)\]

so we have
\[-A_0(q) \subset [-A_0(q)]^*.\]

Since it is obvious that the operator \(-A_0(q)\) is dense-definite, it suffices to prove that the operators \(-A_0(q)\) and \([A_0(q)]^*\) have the same domain. We proceed as follows:
∀ \( V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{D}(\mathcal{A}_0(q)^\ast) \), the linear functional \( g \) determined by \( V \),
\[
g(U) = \langle \mathcal{A}_0(q)U, V \rangle_{\mathcal{H}}, \quad U \in \mathcal{D}(\mathcal{A}_0(q)),
\]
is bounded, hence, by (3.10), the linear functional \( g_1 \) determined by \( v_1 \),
\[
g_1(u_2) = \langle Ru_2, Rv_1 \rangle_0, \quad u_2 \in H,
\]
is also bounded. Thus we have
\[
Rv_1 \in \mathcal{D}(R^\ast) = \mathcal{D}(R),
\]
and hence,
\[
v_1 \in \mathcal{D}(\mathcal{A}_0(q)).
\]
Moreover, by (3.1), the linear functional \( g_2 \) determined by \( v_2 \),
\[
g_2(u_1) = -[R^2u_1, v_2], \quad u_1 \in H^2(\Omega) \cap H^1_0(\Omega),
\]
is bounded, too, therefore we have
\[
v_2 \in \mathcal{D}(R^\ast) = \mathcal{D}(R) = H^1_0(\Omega),
\]
and hence,
\[
V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathcal{D}(\mathcal{A}_0(q)) = \mathcal{D}(\mathcal{A}_0(q)) = \mathcal{D}(\mathcal{A}_0(q)^\ast).
\]

Lemma 3.3. ∀ \( q \in Q_{ad} \), the operator \( \mathcal{A}_0(q) \) and \( \mathcal{A}(q) \) respectively generate \( C_0 \)-groups of bounded unitary operators (a unitary operator \( T \) means a linear operator \( T \) that satisfies \( T^* = T^{-1} \)).

Proof. According to the Stone’s theorem [5, p. 41], the skew-self-adjoint operator \( \mathcal{A}_0(q) \) generates on \( \mathcal{H} \) a \( C_0 \)-group of bounded unitary operators. In addition, the linear operator \( \mathcal{B}(q) \) is bounded on \( \mathcal{H} \), then, by the perturbation theorem [3, p. 79], the operator \( \mathcal{A}(q) = \mathcal{A}_0(q) + \mathcal{B}(q) \) also generates a \( C_0 \)-group of bounded unitary operators on \( \mathcal{H} \). □

Theorem 3.4. Suppose that (AA), (BB), and (A4) hold. Then there exists a unique classical solution for problem (3.2), i.e., there exists a unique \( U \in C^1(R^+; \mathcal{H}) \cap C(R^+_0; \mathcal{H}) \) such that \( U(t) \in \mathcal{D}(\mathcal{A}(q)) \) and \( U \) satisfies (3.2).

Proof. By Lemma 3.3, ∀ \( q \in Q_{ad} \), \( \mathcal{A}(q) \) generates a \( C_0 \)-group of bounded unitary operators, and the mapping \( u \mapsto f( u; q) \) is continuously differentiable, so is hence the mapping \( U \mapsto \mathcal{F}(U; q) \). Therefore it follows from [3, p. 187] that for any \( U_0 \in \mathcal{D}(\mathcal{A}(q)) \), there exists a unique classical solution for Cauchy problem (3.2). □

Lemma 3.5. Under the assumption (AA), the operators \( q \mapsto \mathcal{A}_0(q), \mathcal{B}(q), \mathcal{A}_0(q) \) and \( \mathcal{B}(q) \) are strongly continuous on \( Q_{ad} \); under the assumption (A1), these operators are strongly Fréchet continuously differentiable on the interior of \( Q_{ad} \).

(For the definitions of strong continuity and strong Fréchet continuous differentiability of the closed operators \( \mathcal{A}(q), \mathcal{B}(q) \), etc., see [8].)
**Proof.** We only prove for instance that if (A1) is satisfied then the operator \( q \mapsto A_0(q) \) is strongly Fréchet continuously differentiable. The proof for the other cases is similar and hence is omitted here.

\( \forall q \in Q^0_{ad}, \) there exists \( r > 0 \) such that the ball \( B(q; r) \), centered at \( q \) and with radius \( r \), satisfies:

\[
B(q; r) \subset Q_{ad}.
\]

\( \forall \bar{q} = q + h \in B(q; r), \) it holds by the assumption (A2) that

\[
\left\| a_{ij}(q + h) - a_{ij}(q) - a'_{ij}(q)h \right\|_\infty = o(\|h\|_Q).
\] (3.11)

Define the operator

\[
A'_0(q)h \equiv \sum_{i,j=1}^m \partial_i \left( a'_{ij}(q)h \partial_j \right)
\]

then we have the estimation that

\[
\left\| A_0(q + h)u - A_0(q)u - A'_0(q)hu \right\|_0^2 = \int_\Omega \left\{ \sum_{i,j} \partial_i \left( [a_{ij}(q + h) - a_{ij}(q) - a'_{ij}(q)h] \partial_j u \right) \right\}^2 d\Omega 
\]

\[
\leq c \int_\Omega \sum_{i,j} \left\{ \partial_i \left( [a_{ij}(q + h) - a_{ij}(q) - a'_{ij}(q)h] \partial_j u \right) \right\}^2 (\partial_j u)^2 d\Omega 
\]

\[
\leq c \int_\Omega \sum_{i,j} \left[ a_{ij}(q + h) - a_{ij}(q) - a'_{ij}(q)h \right]^2 (\partial_i \partial_j u)^2 d\Omega,
\]

hence, by (3.11),

\[
\left\| A_0(q + h)u - A_0(q)u - A'_0(q)hu \right\|_0^2 = o(\|h\|^2_0)\|u\|_2^2,
\] (3.12)

therefore, \( A_0(q) \) is strongly Fréchet continuously differentiable.

**Lemma 3.6.** Under the assumption (A3), \( \forall q_0 \in Q_{ad}, \) the \( C_0 \)-group \( J(t; q_0) \) of bounded unitary operators, generated by the operator \( A(q_0) \), satisfies the local stability condition, i.e., there exist \( M_0 > 0, \omega_0 > 0, \) and \( \varepsilon_0 > 0 \) such that

\[
\| J(t; q) \| \leq M_0 e^{\omega_0 t}, \quad t \in \mathbb{R}, \ q \in Q_{ad} \cap B(q_0; \varepsilon_0).
\] (3.13)

**Proof.** We prove firstly that \( A_0(q) \) is a dissipative operator, i.e.,

\[
\langle A_0(q)U, U \rangle_\mathcal{H} \leq 0, \quad \forall U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathcal{H}.
\]

It can be obtained from (3.10) that

\[
\langle A_0(q)U, U \rangle_\mathcal{H} = \langle u_2, u_1 \rangle_\mathcal{H} + \langle A_0(q)u_1, u_2 \rangle_0 = \langle Ru_2, Ru_1 \rangle_0 - \langle R^2 u_1, u_2 \rangle_0 = 0,
\]

consequently, \( A_0(q) \) is a dissipative operator.
Next, \( \forall G = (g_1, g_2) \in \mathcal{H} \), the equation

\[
\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = G = [\lambda I - A_0(q)]U = \begin{pmatrix} \lambda u_1 - u_2 \\ \lambda u_2 - A_0u_1 \end{pmatrix}
\]  
(3.14)

can be changed into

\[
u_2 = \lambda u_1 - g_1, \quad [\lambda^2 - A_0(q)]u_1 = \lambda g_1 + g_2.
\]  
(3.15)

Because the operator \( A_0(q) \) only possesses a negative discrete point spectrum, for any \( \lambda \in \mathbb{R} \), Eq. (3.15), hence (3.14), has a solution \( U \), i.e., the range of operator \( \lambda I - A_0(q) \) is \( \mathcal{H} \). According to the Lumer–Phillips theorem [3, p. 14] and Lemma 3.3, the operator \( A_0(q) \) generates a \( C_0 \)-group \( \mathcal{J}_0(t; q) \) of contracting operators, namely,

\[
\|\mathcal{J}(t; q)\| \leq 1, \quad t \in \mathbb{R}^+.
\]  
(3.16)

Moreover, calculating norms

\[
\|B(q)U\|^2 = \langle Bu_1, Bu_1 \rangle_0 \leq b^2 \|u_1\|^2_1 \leq b^2 \|U\|^2,
\]

we know by assumption \( A3 \) that the constant \( b \) depends only on \( v(q_0), \mu(q_0) \) and \( L(q_0) \). Therefore it follows from the perturbation theorem of bounded operators [3, p. 76] that the operator \( A(q) = A_0(q) + B(q) \) generates a \( C_0 \)-group \( \mathcal{J}(t; q) \) of bounded unitary operators, which satisfies

\[
\|\mathcal{J}(t; q)\| \leq e^{bt}, \quad t \in \mathbb{R}^+.
\]

**Theorem 3.7.** Suppose that \( (AA), (BB), (A3) and (A4) \) hold. Then the classical solution \( U(t; q) \) determined by (3.2) is a continuous function of \( q \), i.e., \( U(t) \in C(Q_{ad}; \mathcal{H}) \) for all \( t \in \mathbb{R}^+ \).

**Proof.** \( \forall q \in Q_{ad} \), take \( \bar{q} \in Q_{ad} \), let \( h = \bar{q} - q \) and substitute \( q \) and \( \bar{q} \) into (3.2), respectively, then we obtain \( U = U(q) \) and \( \bar{U} = U(\bar{q}) \), i.e., \( U \) satisfies (3.2) while \( \bar{U} \) satisfies

\[
\begin{aligned}
\frac{d\bar{U}}{dt} &= A(\bar{q})\bar{U} + \mathcal{F}(\bar{U}; \bar{q}), \\
\bar{U}(0) &= U_0,
\end{aligned}
\]

(3.17)

consequently the difference \( \phi \equiv \bar{U} - U \) satisfies the following:

\[
\begin{aligned}
\frac{d\phi}{dt} &= A(\bar{q})\phi + [A(\bar{q}) - A(q)]U + [\mathcal{F}(\bar{U}; \bar{q}) - \mathcal{F}(U; q)] \\
&\quad + [\mathcal{F}(U; \bar{q}) - \mathcal{F}(U; q)], \\
\phi|_{t=0} &= 0.
\end{aligned}
\]

(3.18)

It follows from Lemma 3.3 that the closed operator \( A(\bar{q}) \) generates a \( C_0 \)-group \( \mathcal{J}(t; \bar{q}) \) of bounded unitary operators, and it satisfies the estimation

\[
\|\mathcal{J}(t; \bar{q})\| \leq e^{bt}, \quad t \in \mathbb{R}^+_0,
\]

(3.19)

where the constant \( b \) only depends on \( q \), provided that \( \|h\| \) is small enough. Thus

\[
\phi(t) = \bar{U}(t) - U(t) = \int_0^t \mathcal{F}(t; \bar{q})\left[\left[A(\bar{q}) - A(q)\right]U(\tau) + \left[\mathcal{F}(\bar{U}(\tau); \bar{q}) - \mathcal{F}(U(\tau); q)\right]
\right]
\]

\[
\quad + \left[\mathcal{F}(U(\tau); \bar{q}) - \mathcal{F}(U(\tau); q)\right]d\tau.
\]

(3.20)
When \( \|h\| \to 0 \), since \( U(t) \in D(\mathcal{A}(q)) \), \( \forall t > 0 \), it is obtained from Lemma 3.5 that
\[
\| [\mathcal{A}(\bar{q}) - \mathcal{A}(q)] U(\tau) \| = o(1) \| U(\tau) \|. \tag{3.21}
\]
Moreover, by the mean-value theorem
\[
\mathcal{F}(\bar{U}(\tau); \bar{q}) - \mathcal{F}(U(\tau); \bar{q}) = \int_0^1 \frac{\partial \mathcal{F}(U(\tau) + s\phi(\tau); \bar{q})}{\partial U} \phi(\tau) ds,
\]
we can obtain that
\[
\| \mathcal{F}(\bar{U}(\tau); \bar{q}) - \mathcal{F}(U(\tau); \bar{q}) \| \leq c_1 \| \phi(\tau) \|, \tag{3.22}
\]
and, similarly, that
\[
\| \mathcal{F}(U(\tau); \bar{q}) - \mathcal{F}(U(\tau); q) \| = O(\|h\|). \tag{3.23}
\]
Substituting (3.19) and (3.21)–(3.23) into the estimation from (3.20), we get
\[
\| \phi(t) \| \leq C_2 \int_0^t \| \phi(\tau) \| d\tau + o(1). \tag{3.24}
\]
Thus, by the Green inequality, we immediately get
\[
\| \phi(t) \| = o(1). \quad \square
\]

**Theorem 3.8.** Suppose that (AA), (BB) and (A1)–(A4) hold. Then the solution \( U(t; q) \) of Cauchy problem (3.2) is a Fréchet continuously differentiable function of \( q \), i.e., \( U(t) \in C^1(Q_{ad}; H) \), \( \forall t \in \mathbb{R}^+ \), and the Fréchet differential \( U'(t; q) h \equiv U(t) \), at \( q \in Q_{ad}^0 \) along the direction of \( h \), is determined by the following Cauchy problem:
\[
\begin{cases}
\frac{dU}{dt} = [\mathcal{A}(q) + \frac{\partial \mathcal{F}(U; q)}{\partial U}] U + \mathcal{A}'(q) h U + \frac{\partial \mathcal{F}(U; q)}{\partial q} h, \\
U(0) = 0.
\end{cases} \tag{3.25}
\]

**Proof.** By assumption (A4), \( \partial \mathcal{F}(U; q)/\partial U \) is a bounded operator, hence the operator \( \mathcal{A}(q) + \partial \mathcal{F}(U; q)/\partial U \) generates a \( C_0 \)-group of bounded unitary operators; next, the nonhomogeneous term \( t \mapsto \mathcal{A}'(q) h U(t) + [\partial \mathcal{F}(U(t); q)/\partial q] h \) is continuous, hence, the Cauchy problem (3.25) possesses a unique classical solution \( \bar{U}(t) \), and obviously \( \| \bar{U}(t) \| = O(\|h\|) \).

Let \( \bar{U}(t) \) and \( U(t) \) be determined respectively by (3.18) and (3.2), and set
\[
\eta(t) \equiv \bar{U}(t) - U(t) - \dot{U}(t).
\]
Then, by calculation (3.18)–(3.2)–(3.25), it can be obtained that
\[
\frac{d\eta}{dt} = \mathcal{A}(q) \eta(t) + [\mathcal{A}(\bar{q}) - \mathcal{A}(q) - \mathcal{A}'(q) h] \bar{U} + \mathcal{A}'(q) h [\bar{U} - U]
+ \left[ \mathcal{F}(\bar{U}; \bar{q}) - \mathcal{F}(\bar{U}; q) - \frac{\partial \mathcal{F}(\bar{U}; q)}{\partial q} h \right]
+ \left[ \frac{\partial \mathcal{F}(\bar{U}; q)}{\partial q} - \frac{\partial \mathcal{F}(U; q)}{\partial q} \right] h
+ \left[ \mathcal{F}(\bar{U}; q) - \mathcal{F}(U; q) - \frac{\partial \mathcal{F}(U; q)}{\partial U} \dot{U} \right],
\tag{3.26}
\]
\( \eta(0) = 0. \)
By the aid of the $C_0$-group $J(t; q)$ generated by $\mathcal{A}(q)$, $\eta(t)$ can be expressed as

$$
\eta(t) = \int_0^t J(\tau; q) \left\{ \left[ \mathcal{A}(\bar{q}) - \mathcal{A}(q) - \mathcal{A}'(q)h \right] \bar{U}(\tau) + \mathcal{A}'(q)h \left[ \bar{U}(\tau) - U(\tau) \right] 
+ \left[ \int_0^\tau \frac{\partial \mathcal{F}(\bar{U}; q + sh)}{\partial q} h ds - \frac{\partial \mathcal{F}(\bar{U}; q)}{\partial q} h \right] \right\} d\tau.
$$

(3.27)

Since $\bar{U}(\tau) \in D(\mathcal{A}(\bar{q}))$, $\tau \in \mathbb{R}_0^+$, it follows from assumption (A2) that

$$
\left\| \left[ \mathcal{A}(\bar{q}) - \mathcal{A}(q) - \mathcal{A}'(q)h \right] \bar{U}(\tau) \right\| = O(\|h\|^2),
$$

(3.28)

hence, by Theorem 3.7, as $\|h\| \to 0$,

$$
\left\| \bar{U}(\tau) - U(\tau) \right\| = o(1),
$$

thus,

$$
\left\| \mathcal{A}'(q)h \left[ \bar{U}(\tau) - U(\tau) \right] \right\| = o(\|h\|).
$$

(3.29)

Moreover, by assumption (A4) and Theorem 3.7,

$$
\left\| \int_0^1 \frac{\partial \mathcal{F}(\bar{U}(\tau); q + sh)}{\partial q} h ds - \frac{\partial \mathcal{F}(\bar{U}(\tau); q)}{\partial q} h \right\| = o(\|h\|),
$$

(3.30)

$$
\left\| \left[ \frac{\partial \mathcal{F}(\bar{U}(\tau); q)}{\partial q} - \frac{\partial \mathcal{F}(U(\tau); q)}{\partial q} \right] h \right\| = o(\|h\|),
$$

(3.31)

$$
\left\| \int_0^1 \frac{\partial \mathcal{F}(U + s(\bar{U} - U); q)}{\partial U} (\bar{U} - U) ds - \frac{\partial \mathcal{F}(U; q)}{\partial U} \bar{U} \right\|
\leq C_1 \left\| \mathcal{A}'(q)h \left[ \bar{U}(\tau) - U(\tau) \right] \right\| + o(\|h\|).
$$

(3.32)

Finally, by Lemma 3.6,

$$
\| J(t; q) \| \leq e^{bt}, \quad t \in \mathbb{R}^+.
$$

(3.33)
Substituting (3.28)–(3.33) into the estimation from (3.27), we get
\[
\|\eta(t)\| \leq C_2 \int_0^t \|\eta(\tau)\| \, d\tau + o(\|h\|),
\]
and then, by the Gronwall inequality, we obtain
\[
\|\eta(t)\| = o(\|h\|) .
\]

**Proof of Theorem 2.1.** Since (3.2) is equivalent to the system:
\[
\frac{du_1}{dt} = u_2 , \\
\frac{d^2 u_1}{dt^2} = A(q)u_1 + f(u_1; q) , \\
u_1|_{t=0} = u^0 , \\
\frac{du_1}{dt}|_{t=0} = u^1 ,
\]
if the expression of \( A(q) \) and the equality \( u_1 = u \) are substituted into (3.34), then (2.1) is obtained, hence, Theorem 2.1 can be derived from Theorem 3.4.

**Proof of Theorems 2.2 and 2.3.** Similarly as above, to obtain (2.6), it suffices to make use of Theorems 3.7 and 3.8. Indeed, (3.25) implies the system:
\[
\frac{d\dot{u}_1}{dt} = \dot{u}_2 , \\
\frac{d^2 \dot{u}_1}{dt^2} = \left[ A(q) + \frac{\partial f(u_1; q)}{\partial u_1} \right] \dot{u}_1 + A'(q)h u_1 + \frac{\partial f(u_1; q)}{\partial q} h , \\
\dot{u}_1|_{t=0} = 0 , \\
\frac{d\dot{u}_1}{dt}|_{t=0} = \dot{u}_2(0) = 0 ,
\]
while we have \( \dot{u}_1 = \dot{u} \), thus we are done.

**References**


