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Representations of certain non-rational vertex operator algebras of affine type

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Abstract

In this paper we study a series of vertex operator algebras of integer level associated to the affine Lie algebra $A_\ell^{(1)}$. These vertex operator algebras are constructed by using the explicit construction of certain singular vectors in the universal affine vertex operator algebra $N_l(n-2,0)$ at the integer level. In the case $n=1$ or $l=2$, we explicitly determine Zhu's algebras and classify all irreducible modules in the category $\mathcal{O}$. In the case $l=2$, we show that the vertex operator algebra $N_2(n-2,0)$ contains two linearly independent singular vectors of the same conformal weight.

Keywords: Vertex operator algebra; Generalized Verma module; Singular vectors; Affine Kac–Moody algebra; Zhu's algebra; Category $\mathcal{O}$

1. Introduction

Let $\hat{\mathfrak{g}}$ be the affine Lie algebra associated with the finite-dimensional simple Lie algebra $\mathfrak{g}$. Then, on the generalized Verma module $N(k,0)$, exists the natural structure of a vertex operator algebra (cf. [10,14,16,19–21]). In the representation theory of affine Lie algebras it is important to study annihilating ideals of highest weight representations. This problem is related with the ideal lattice of the vertex operator algebra $N(k,0)$ (cf. [13]). Every $\hat{\mathfrak{g}}$-submodule I of $N(k,0)$ becomes an ideal in the vertex operator algebra $N(k,0)$, and on the

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quotient $N(k,0)/I$, there exists the structure of a vertex operator algebra. Let $N^1(k,0)$ be the maximal ideal in $N(k,0)$. Then $L(k,0) = N(k,0)/N^1(k,0)$ is a simple vertex operator algebra.

It is particularly important to study ideals in $N(k,0)$ generated by one singular vector. If k is a positive integer, then $N^1(k,0)$ is generated by the singular vector $e_\theta(-1)^{k+1}\mathbf{1}$ (cf. [14,21]). In the case when k is an admissible rational number, the maximal ideal is also generated by one singular vector, but the expression for this singular vector is more complicated (cf. [1,4,22]).

By using explicit formulas for singular vectors which involve the powers of determinants, a family of ideals were constructed in [2]. The corresponding quotient vertex operator algebras give new examples of vertex operator algebras of affine type which are different from $N(k,0)$ and $L(k,0)$. So it is an interesting problem to investigate their representation theory.

In this paper we shall study the (non-simple) vertex operator algebras associated to the affine Lie algebra $A_l^{(1)}$ of integer level. We shall classify all irreducible highest weight representations of the vertex operator algebra $V_{l,1}$ associated to a level -1 representation for $A_l^{(1)}$. We use the methods developed in [1] and [22]. We also classify the finite-dimensional irreducible representations of corresponding Zhu's algebra $A(V_{l,1})$. We prove an interesting fact that every finite-dimensional irreducible $A(V_{l,1})$-module has all 1-dimensional weight spaces. Therefore Zhu's algebra $A(V_{l,1})$ provides a natural framework for studying irreducible representations having 1-dimensional weight spaces. These representations appeared in [7] in the context of the representation theory of Lie algebras. So our results show their importance in the theory of vertex operator algebras.

We shall also extend the construction of singular vectors from [2]. In particular, we shall present new explicit formulas for singular vectors in the case of affine Lie algebra $A_2^{(1)}$ at integer levels. We show that the vertex operator algebra $N_2(n-2,0)$ contains two linearly independent singular vectors $v_{2,n}$ and $\widetilde{v_{2,n}}$ of conformal weight $3n$. Moreover, there is an automorphism Ψ of the vertex operator algebra $N_2(n-2,0)$ such that $\Psi(v_{2,n}) = \widetilde{v_{2,n}}$. These singular vectors generate ideals $J_2(n-2,0)$ and $\widetilde{J_2}(n-2,0)$ whose top levels are irreducible modules with all 1-dimensional weight spaces. Let $V_{2,n}$ and $\widetilde{V_{2,n}}$ be the corresponding quotient vertex operator algebras. It turns out that these vertex operator algebras are isomorphic. We show the irreducible representations of $V_{2,n}$ and $\widetilde{V_{2,n}}$ in the category $\mathcal{O}$ are parameterized by the zeros of the same polynomial $P \in \mathbb{C}[x,y]$.

Therefore, for every $n \in \mathbb{N}$, we have a new vertex operator algebra, which is a certain quotient of $N_2(n-2,0)$, for which we know all irreducible highest weight modules. The irreducible highest weight modules for that vertex operator algebra describe a new infinite class of $A_2^{(1)}$-modules, which includes integrable highest weight $A_2^{(1)}$-modules of level $n-2$.

We believe that the categories of representations which appear in this paper have interesting characters, and we hope to study the corresponding fusion rules.

In this paper, $\mathbb{Z}_+$ and $\mathbb{N}$ denote the sets of nonnegative integers and positive integers, respectively.

2. Vertex operator algebras associated to affine Lie algebras

In this section we review certain results about vertex operator algebras associated to affine Lie algebras.

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (cf. [6,11,12]). We shall assume that

$$V = \coprod_{n \in \mathbb{Z}_+} V_n, \quad \text{where } V_n = \{a \in V \mid L(0)a = na\}.$$

For $a \in V_n$ we shall write $\mathrm{wt}(a) = n$. Following [23], we define bilinear maps $* : V \times V \to V$ and $\circ : V \times V \to V$ as follows. For any homogeneous $a \in V$ and for any $b \in V$, let

$$a \circ b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}\, a}}{z^2} Y(a, z)b,$$

$$a * b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt}\, a}}{z} Y(a, z)b$$

and extend to $V \times V \to V$ by linearity. Denote by $O(V)$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$. The multiplication $*$ induces the multiplication on $A(V)$ and $A(V)$ has a structure of an associative algebra. There is a one-to-one correspondence between the equivalence classes of the irreducible $A(V)$-modules and the equivalence classes of the irreducible $\mathbb{Z}_+$-graded weak V-modules [23].

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. The affine Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$ is the vector space $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ equipped with the usual bracket operation and the canonical central element c (cf. [15]). Let $h^\vee$ be the dual Coxeter number of $\hat{\mathfrak{g}}$. Let $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ be the corresponding triangular decomposition of $\hat{\mathfrak{g}}$.

Let U be a $\mathfrak{g}$-module, and let $k \in \mathbb{C}$. Let $\hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes t\mathbb{C}[t]$ act trivially on U and c as the scalar multiplication operator k. Considering U as a $\mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+$-module, we have the induced $\hat{\mathfrak{g}}$-module (so-called *generalized Verma module*)

$$N(k, U) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+)} U.$$

For a fixed $\mu \in \mathfrak{h}^*$, denote by $V(\mu)$ the irreducible highest-weight $\mathfrak{g}$-module with highest-weight μ. We shall use the notation $N(k, \mu)$ to denote the $\hat{\mathfrak{g}}$-module $N(k, V(\mu))$. Denote by $N^1(k, \mu)$ the maximal proper submodule of $N(k, \mu)$ and by $L(k, \mu) = N(k, \mu)/N^1(k, \mu)$ the corresponding irreducible $\hat{\mathfrak{g}}$-module.

For $k \neq -h^\vee$, $N(k, 0)$ has the structure of vertex operator algebra (cf. [14] and [21], see also [10,16,19,20]). The associative algebra $A(N(k, 0))$ is canonically isomorphic to $U(\mathfrak{g})$ [14].

Let $J = U(\hat{\mathfrak{g}})v$ be the $\hat{\mathfrak{g}}$-submodule of $N(k, 0)$ generated by the singular vector v. Then J is an ideal in $N(k, 0)$ and the quotient $N(k, 0)/J$ is also a vertex operator algebra. We present the method from [1,4,21,22] for the classification of irreducible $A(N(k, 0)/J)$-modules from the category $\mathcal{O}$ by solving certain systems of polynomial equations.

Let v' be the image of the vector v in Zhu's algebra $A(N(k, 0)) \cong U(\mathfrak{g})$. Denote by ${}_L$ the adjoint action of $U(\mathfrak{g})$ on $U(\mathfrak{g})$ defined by $X_L f = [X, f]$ for $X \in \mathfrak{g}$ and $f \in U(\mathfrak{g})$. Let R be a $U(\mathfrak{g})$-submodule of $U(\mathfrak{g})$ generated by the vector v' under the adjoint action. Clearly, R is an irreducible highest-weight $U(\mathfrak{g})$-module. Let R_0 be the zero-weight subspace of R.

The next proposition follows from [1,4,21,22]:

Proposition 2.1. *Let $V(\mu)$ be an irreducible highest weight $U(\mathfrak{g})$-module with the highest-weight vector v_μ, for $\mu \in \mathfrak{h}^*$. The following statements are equivalent*:

(1) *$V(\mu)$ is an $A(N(k,0)/J)$-module,*
(2) *$RV(\mu)=0$,*
(3) *$R_0 v_\mu = 0$.*

Let $r \in R_0$. Clearly there exists the unique polynomial $p_r \in S(\mathfrak{h})$ such that

$$r v_\mu = p_r(\mu) v_\mu.$$

Set $\mathcal{P}_0 = \{ p_r \mid r \in R_0\}$. We have:

Corollary 2.2. *There is a one-to-one correspondence between*

(1) *irreducible $A(N(k,0)/J)$-modules from the category $\mathcal{O}$,*
(2) *weights $\mu \in \mathfrak{h}^*$ such that $p(\mu)=0$ for all $p \in \mathcal{P}_0$.*

3. Simple Lie algebra of type A_l

Let $\mathfrak{g}$ be the simple Lie algebra of type A_l. The root system of $\mathfrak{g}$ is given by

$$\Delta = \{\epsilon_i - \epsilon_j \mid 1 \leqslant i, j \leqslant l+1,\ i \neq j\}$$

with $\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_l = \epsilon_l - \epsilon_{l+1}$ being a set of simple roots. The highest root is $\theta = \epsilon_1 - \epsilon_{l+1}$. We fix the root vectors and coroots as in [2] and [9]. For any positive root $\alpha \in \Delta_+$ denote by e_α and f_α the root vectors corresponding to α and $-\alpha$, respectively. Denote by h_α the corresponding coroots, and by $h_i = h_{\alpha_i}$, for $i = 1, \ldots, l$. Let $\omega_1, \ldots, \omega_l$ be the fundamental weights for A_l.

We recall the following well-known and important fact about the weight spaces of $\mathfrak{g}$-modules $V(n\omega_1)$ and $V(n\omega_l)$.

Proposition 3.1. *All weight spaces of irreducible $\mathfrak{g}$-modules $V(n\omega_1)$ and $V(n\omega_l)$, for $n \in \mathbb{Z}_+$, are 1-dimensional.*

Remark 3.2. One can show that every irreducible finite-dimensional $\mathfrak{g}$-module having all 1-dimensional weight spaces is isomorphic to either $V(n\omega_1)$ or $V(n\omega_l)$ for certain $n \in \mathbb{Z}_+$. In what follows we shall see that the category of irreducible modules with 1-dimensional weight spaces provides all irreducible finite-dimensional modules for certain Zhu's algebra.

4. Singular vectors in the vertex operator algebras $N_l(n-2,0)$ for $n \in \mathbb{N}$

Let $\hat{\mathfrak{g}}$ be the affine Lie algebra of type $A_l^{(1)}$. For $k \in \mathbb{C}$, denote by $N_l(k,0)$ the generalized Verma module associated to $\hat{\mathfrak{g}}$ of level k.

We shall recall some facts about singular vectors in the vertex operator algebra $N_l(n-2,0)$ for $n \in \mathbb{N}$. We know that $N_l(n-2,0)$ is not a simple vertex operator algebra. If $n \geqslant 2$, it contains the maximal ideal $N_l^1(n-2,0)$ which is generated by the singular vector $e_\theta(-1)^{n-1}\mathbf{1}$. In the case $l=1$, one can show that the maximal ideal is an irreducible $\hat{\mathfrak{sl}}_2$-module. Therefore, there are no other ideals in $N_l^1(n-2,0)$. But for $l \geqslant 2$, the situation is very much different. We have non-trivial ideals which are different from $N_l^1(n-2,0)$. These ideals are generated by singular vectors in $N_l^1(n-2,0)$. Some non-trivial ideals were constructed in [2] in the case $l \geqslant 3$.

We have:

Theorem 4.1. *(See [2, Theorem 5.1].) Assume that $l, n \in \mathbb{N}$, $l \geqslant 3$.*

(1) *For every $n \in \mathbb{N}$, vector*

$$v_{l,n} = \big(e_{\epsilon_1-\epsilon_{l+1}}(-1)e_{\epsilon_2-\epsilon_l}(-1) - e_{\epsilon_2-\epsilon_{l+1}}(-1)e_{\epsilon_1-\epsilon_l}(-1)\big)^n \mathbf{1}$$

is a non-trivial singular vector in $N_l(n-2,0)$.

(2) *Assume that $n \geqslant 2$. Then the maximal submodule $N_l^1(n-2,0)$ of $N_l(n-2,0)$ is reducible.*

Remark 4.2. Note that we constructed singular vectors in the case $l \geqslant 3$. So the construction from [2] cannot be applied directly in the case $l = 2$. In [9], A. Feingold and I. Frenkel presented an explicit realization of the vacuum $A_l^{(1)}$-module of level -1. Let us denote this module by $\widetilde{L}_l(-1,0)$. $\widetilde{L}_l(-1,0)$ carries the structure of a vertex operator algebra (this vertex operator algebra was also studied in [17]), and it is a non-trivial quotient of the vertex operator algebra $N_l(-1,0)$. By a simple analysis of characters, one can show that in the case $l \geqslant 3$, $v_{\ell,1}$ is a unique, up to scalar factor, singular vector in $N_l(-1,0)$ with conformal weight 2. We omit the details. In the case $l = 2$, we have

$$\begin{aligned} \operatorname{ch} \widetilde{L}_2(-1,0) &= 1 + 8q + 44q^2 + 172q^3 + o\big(q^4\big), \\ \operatorname{ch} N_2(-1,0) &= 1 + 8q + 44q^2 + 192q^3 + o\big(q^4\big) \end{aligned} \tag{4.1}$$

which indicates that there are no singular vectors of conformal weight 2, and that there should exist at least one singular vector of conformal weight 3 in $N_2(-1,0)$. In Section 6 we shall find two linearly independent singular vectors of conformal weight 3. Moreover, we shall extend Theorem 4.1 in the case $l = 2$.

5. Vertex operator algebras associated to affine Lie algebras of type $A_l^{(1)}$ on level -1 for $l \geqslant 3$

In the special case $n = 1$, Theorem 4.1 implies that vector

$$v_{l,1} = e_{\epsilon_1-\epsilon_{l+1}}(-1)e_{\epsilon_2-\epsilon_l}(-1)\mathbf{1} - e_{\epsilon_2-\epsilon_{l+1}}(-1)e_{\epsilon_1-\epsilon_l}(-1)\mathbf{1} \tag{5.1}$$

is a singular vector in $N_l(-1,0)$.

Define the ideal $J_{l,1}(-1,0)$ in the vertex operator algebra $N_l(-1,0)$ by $J_{l,1}(-1,0) = U(\hat{\mathfrak{g}})v_{l,1}$, and denote the corresponding quotient vertex operator algebra by $V_{l,1}$, i.e., $V_{l,1} = N_l(-1,0)/J_{l,1}(-1,0)$.

Then [2, Theorem 5.4]:

Proposition 5.1. *The associative algebra $A(V_{l,1})$ is isomorphic to the algebra $U(\mathfrak{g})/I_{l,1}$, where $I_{l,1}$ is the two-sided ideal of $U(\mathfrak{g})$ generated by*

$$v'_{l,1} = e_{\epsilon_1-\epsilon_{l+1}}e_{\epsilon_2-\epsilon_l} - e_{\epsilon_2-\epsilon_{l+1}}e_{\epsilon_1-\epsilon_l}.$$

The $U(\mathfrak{g})$-submodule R of $U(\mathfrak{g})$ generated by the vector $v'_{l,1}$ under the adjoint action is isomorphic to $V(\omega_2+\omega_{l-1})$. In the next lemma we determine the dimension of R_0.

Lemma 5.2.

$$\dim R_0 = \frac{(l-2)(l+1)}{2}.$$

Proof. In this proof we use induction on l. We use the notation $V_l(\mu)$ for the highest weight module for simple Lie algebra of type A_l, with the highest weight $\mu \in \mathfrak{h}^*$.

For $l=3$ it can be easily checked that $\dim R = 20$ and $\dim R_0 = 2$. Suppose that the claim of this lemma holds for simple Lie algebra of type A_{l-1}, $l-1 \geqslant 3$. Let $\mathfrak{g}$ be a simple Lie algebra of type A_l. Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ associated to roots $\alpha_1,\dots,\alpha_{l-1}$. Then $\mathfrak{g}'$ is a simple Lie algebra of type A_{l-1}. We can decompose $\mathfrak{g}$-module $V_l(\omega_2+\omega_{l-1})$ into the direct sum of irreducible $\mathfrak{g}'$-modules. If we denote by v the highest-weight vector of $\mathfrak{g}$-module $V_l(\omega_2+\omega_{l-1})$, then it can be easily checked that v, $f_{\epsilon_{l-1}-\epsilon_{l+1}}.v$, $(2f_{\epsilon_2-\epsilon_{l+1}} - f_{\epsilon_2-\epsilon_3}f_{\epsilon_3-\epsilon_{l+1}}).v$ and $f_{\epsilon_{l-1}-\epsilon_{l+1}}(2f_{\epsilon_2-\epsilon_{l+1}} - f_{\epsilon_2-\epsilon_3}f_{\epsilon_3-\epsilon_{l+1}}).v$ are highest-weight vectors for $\mathfrak{g}'$, which generate $\mathfrak{g}'$-modules isomorphic to $V_{l-1}(\omega_2+\omega_{l-1})$, $V_{l-1}(\omega_2+\omega_{l-2})$, $V_{l-1}(\omega_1+\omega_{l-1})$ and $V_{l-1}(\omega_1+\omega_{l-2})$, respectively. It follows from Weyl's formula for the dimension of irreducible module that $\dim V_l(\omega_2+\omega_{l-1}) = \frac{(l-2)(l+2)(l+1)^2}{4}$, $\dim V_{l-1}(\omega_2+\omega_{l-1}) = \dim V_{l-1}(\omega_1+\omega_{l-2}) = \frac{(l-2)l(l+1)}{2}$, and $\dim V_{l-1}(\omega_1+\omega_{l-1}) = (l-1)(l+1)$, since $V_{l-1}(\omega_1+\omega_{l-1})$ is the adjoint module for $\mathfrak{g}'$. We obtain

$$\begin{aligned} V_l(\omega_2+\omega_{l-1}) &\cong V_{l-1}(\omega_2+\omega_{l-1}) \oplus V_{l-1}(\omega_2+\omega_{l-2}) \oplus V_{l-1}(\omega_1+\omega_{l-1}) \\ &\quad \oplus V_{l-1}(\omega_1+\omega_{l-2}). \end{aligned}$$

Clearly, there are no zero-weight vectors for $\mathfrak{g}$ in $\mathfrak{g}'$-modules generated by highest weight vectors v and $f_{\epsilon_{l-1}-\epsilon_{l+1}}(2f_{\epsilon_2-\epsilon_{l+1}} - f_{\epsilon_2-\epsilon_3}f_{\epsilon_3-\epsilon_{l+1}}).v$. There are $l-1$ linearly independent zero-weight vectors for $\mathfrak{g}'$ in the module $V_{l-1}(\omega_1+\omega_{l-1})$, since it is isomorphic to the adjoint module for $\mathfrak{g}'$, and all those vectors have weight zero for $\mathfrak{g}$. The inductive assumption implies that there are $\frac{(l-3)l}{2}$ linearly independent zero-weight vectors for $\mathfrak{g}$ in $\mathfrak{g}'$-module $V_{l-1}(\omega_2+\omega_{l-2})$, which implies $\dim R_0 = \frac{(l-3)l}{2} + (l-1) = \frac{(l-2)(l+1)}{2}$. $\square$

Lemma 5.3. *Let*

(1) $p_{ij}(h) = h_i h_j$, *for* $i = 1,\dots,l-2$, $j-i \geqslant 2$,
(2) $q_i(h) = h_i(h_{i-1}+h_i+h_{i+1}+1)$, *for* $i = 2,\dots,l-1$.

Then $p_{ij}, q_i \in \mathcal{P}_0$.

Proof. We use induction on l. For $l=3$, one can easily obtain that

$$(-f_{\epsilon_1-\epsilon_4}f_{\epsilon_2-\epsilon_3} - f_{\epsilon_1-\epsilon_3}f_{\epsilon_2-\epsilon_4})_L v'_{3,1} \in h_1h_3 + U(\mathfrak{g})\mathfrak{n}_+ \quad \text{and}$$
$$(f_{\epsilon_1-\epsilon_4}f_{\epsilon_2-\epsilon_3})_L v'_{3,1} \in h_2(h_1+h_2+h_3+1) + U(\mathfrak{g})\mathfrak{n}_+,$$

which implies that $p_{13}, q_2 \in \mathcal{P}_0$. Suppose that the claim of this lemma holds for simple Lie algebra of type A_{l-1}, $l-1 \geqslant 3$. Let $\mathfrak{g}$ be a simple Lie algebra of type A_l. Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ associated to roots $\alpha_1, \ldots, \alpha_{l-1}$, and $\mathfrak{g}''$ the subalgebra of $\mathfrak{g}$ associated to roots $\alpha_2, \ldots, \alpha_l$. Then $\mathfrak{g}'$ and $\mathfrak{g}''$ are simple Lie algebras of type A_{l-1}. Since

$$(f_{\epsilon_1-\epsilon_3})_L v'_{l,1} = e_{\epsilon_2-\epsilon_{l+1}} e_{\epsilon_3-\epsilon_l} - e_{\epsilon_3-\epsilon_{l+1}} e_{\epsilon_2-\epsilon_l} \quad \text{and}$$
$$(-f_{\epsilon_{l-1}-\epsilon_{l+1}})_L v'_{l,1} = e_{\epsilon_1-\epsilon_l} e_{\epsilon_2-\epsilon_{l-1}} - e_{\epsilon_1-\epsilon_{l-1}} e_{\epsilon_2-\epsilon_l} = v'_{l-1,1},$$

we get the corresponding vectors for $\mathfrak{g}''$ and $\mathfrak{g}'$, respectively. Using the inductive assumption for $\mathfrak{g}'$, we get that $p_{ij} \in \mathcal{P}_0$ for $i = 1, \ldots, l-3$, $j - i \geqslant 2$, and $q_i \in \mathcal{P}_0$ for $i = 2, \ldots, l-2$. And using the inductive assumption for $\mathfrak{g}''$, we get that $p_{ij} \in \mathcal{P}_0$ for $i = 2, \ldots, l-2$, $j - i \geqslant 2$, and $q_i \in \mathcal{P}_0$ for $i = 3, \ldots, l-1$. Also, one can easily verify that

$$(-f_{\epsilon_1-\epsilon_{l+1}} f_{\epsilon_2-\epsilon_l} - f_{\epsilon_2-\epsilon_{l+1}} f_{\epsilon_1-\epsilon_l})_L v'_{l,1} \in h_1 h_l + U(\mathfrak{g})\mathfrak{n}_+,$$

which implies that $p_{1l} \in \mathcal{P}_0$. Thus $p_{ij} \in \mathcal{P}_0$ for $i = 1, \ldots, l-2$, $j - i \geqslant 2$, and $q_i \in \mathcal{P}_0$ for $i = 2, \ldots, l-1$, and the claim of lemma is proved. □

Lemma 5.4. *Polynomials $p_{ij} \in \mathcal{P}_0$ for $i = 1, \ldots, l-2$, $j - i \geqslant 2$ and $q_i \in \mathcal{P}_0$ for $i = 2, \ldots, l-1$ form a basis for the vector space $\mathcal{P}_0$.*

Proof. Lemma 5.3 implies that $p_{ij} \in \mathcal{P}_0$ for $i = 1, \ldots, l-2$, $j - i \geqslant 2$ and $q_i \in \mathcal{P}_0$ for $i = 2, \ldots, l-1$ are $\frac{(l-2)(l+1)}{2}$ linearly independent polynomials in the set $\mathcal{P}_0$. It follows from Lemma 5.2 that $\dim \mathcal{P}_0 = \frac{(l-2)(l+1)}{2}$. Thus, these polynomials form a basis for $\mathcal{P}_0$. □

Proposition 5.5. *The set*

$$\{V(t\omega_1) \mid t \in \mathbb{C}\} \cup \{V(t\omega_l) \mid t \in \mathbb{C}\} \cup \bigcup_{i=1}^{l-1} \{V(t\omega_i + (-1-t)\omega_{i+1}) \mid t \in \mathbb{C}\}$$

provides the complete list of irreducible $A(V_{l,1})$-modules from the category $\mathcal{O}$.

Proof. It follows from Corollary 2.2 that the highest weights $\mu \in \mathfrak{h}^*$ of irreducible $A(V_{l,1})$-modules $V(\mu)$ are in one-to-one correspondence with solutions of the system of polynomial equations

$$h_i h_j = 0, \quad \text{for } i = 1, \ldots, l-2, \ j - i \geqslant 2, \tag{5.2}$$
$$h_i(h_{i-1} + h_i + h_{i+1} + 1) = 0, \quad \text{for } i = 2, \ldots, l-1. \tag{5.3}$$

Clearly, $h_i = 0$, for $i = 1, \ldots, l$ is a solution of the system corresponding to the highest weight $\mu = 0$. Suppose that $h_i \neq 0$, for some $i \in \{2, \ldots, l-1\}$. Then (5.2) implies that $h_j = 0$, for $j \neq i-1, i, i+1$. From relations

$$h_{i-1} h_{i+1} = 0,$$
$$h_{i-1} + h_i + h_{i+1} + 1 = 0,$$

we get the solutions corresponding to highest weights $\mu = t\omega_{i-1} + (-1-t)\omega_i$ and $\mu = t\omega_i + (-1-t)\omega_{i+1}$. If $h_1 \neq 0$, then (5.2) implies that $h_j = 0$, for $j = 3, \ldots, l$. Relation (5.3) then implies that

$$h_2(h_1 + h_2 + 1) = 0,$$

and we get the solutions corresponding to highest weights $\mu = t\omega_1$ and $\mu = t\omega_1 + (-1-t)\omega_2$. Similarly, if $h_l \neq 0$, we get the solutions corresponding to highest weights $\mu = t\omega_l$ and $\mu = t\omega_{l-1} + (-1-t)\omega_l$. $\square$

Corollary 5.6. *The set*

$$\{V(t\omega_1) \mid t \in \mathbb{Z}_+\} \cup \{V(t\omega_l) \mid t \in \mathbb{Z}_+\}$$

provides the complete list of irreducible finite-dimensional $A(V_{l,1})$-modules. Moreover, every irreducible finite-dimensional $A(V_{l,1})$-module has all 1-dimensional weight spaces for $\mathfrak{h}$.

It follows from Zhu's theory that:

Theorem 5.7. *The set*

$$\{L(-1, t\omega_1) \mid t \in \mathbb{C}\} \cup \{L(-1, t\omega_l) \mid t \in \mathbb{C}\} \cup \bigcup_{i=1}^{l-1} \{L(-1, t\omega_i + (-1-t)\omega_{i+1}) \mid t \in \mathbb{C}\}$$

provides the complete list of irreducible weak $V_{l,1}$-modules from the category $\mathcal{O}$.

Remark 5.8. Theorem 5.7 gives the classification of irreducible objects in the category of weak $V_{l,1}$-modules that are in category $\mathcal{O}$ as $\hat{\mathfrak{g}}$-modules. We will show that this category is not semi-simple. Consider the following $\mathfrak{g}$-modules

$$M_t = \frac{M(t\omega_1)}{\sum_{i=2}^{l} U(\mathfrak{g}) f_{\epsilon_i - \epsilon_{i+1}} v_{t\omega_1}},$$

for $t \in \mathbb{Z}_+$, where $M(t\omega_1)$ denotes the Verma $\mathfrak{g}$-module with the highest weight $t\omega_1$ and the highest-weight vector $v_{t\omega_1}$. It can be easily verified that M_t is a highest-weight $A(V_{l,1})$-module, which is not irreducible. Using Zhu's theory we obtain a highest weight module for the vertex operator algebra $V_{l,1}$, which is not irreducible. Therefore we get an example of a weak $V_{l,1}$-module from the category $\mathcal{O}$, which is not completely reducible.

6. Vertex operator algebras associated to affine Lie algebra of type $A_2^{(1)}$ on levels $n-2$, for $n \in \mathbb{N}$

Let $\mathfrak{g}$ be the simple Lie algebra of type A_2, and $\hat{\mathfrak{g}}$ the affine Lie algebra associated to $\mathfrak{g}$. In this section we study the vertex operator algebra $N_2(n-2, 0)$ associated to $\hat{\mathfrak{g}}$ on level $n-2$, for $n \in \mathbb{N}$.

We shall extend the construction of singular vectors from [2] in the case of the affine Lie algebra $A_2^{(1)}$. We will present the formulas for singular vectors. These vectors generate ideals in vertex operator algebras $N_2(n-2,0)$. It will be important for our construction that top levels of these ideals are irreducible finite-dimensional $\mathfrak{g}$-modules with all 1-dimensional weight spaces.

Theorem 6.1. *The vector*

$$v_{2,n}=\sum_{t=0}^{2n}\frac{1}{t!}e_{\epsilon_1-\epsilon_3}(-1)^t e_{\epsilon_2-\epsilon_3}(-1)^{2n-t} f_{\epsilon_1-\epsilon_2}(0)^t e_{\epsilon_1-\epsilon_2}(-1)^n\mathbf{1}$$

is a singular vector in $N_2(n-2,0)$, for $n\in\mathbb{N}$.

Proof. Since

$$\begin{aligned}&e_{\epsilon_1-\epsilon_2}(0).e_{\epsilon_1-\epsilon_3}(-1)^t e_{\epsilon_2-\epsilon_3}(-1)^{2n-t} f_{\epsilon_1-\epsilon_2}(0)^t e_{\epsilon_1-\epsilon_2}(-1)^n\mathbf{1}\\&\quad=-(2n-t)e_{\epsilon_1-\epsilon_3}(-1)^{t+1} e_{\epsilon_2-\epsilon_3}(-1)^{2n-t-1} f_{\epsilon_1-\epsilon_2}(0)^t e_{\epsilon_1-\epsilon_2}(-1)^n\mathbf{1}\\&\qquad+t(2n-t+1)e_{\epsilon_1-\epsilon_3}(-1)^t e_{\epsilon_2-\epsilon_3}(-1)^{2n-t} f_{\epsilon_1-\epsilon_2}(0)^{t-1} e_{\epsilon_1-\epsilon_2}(-1)^n\mathbf{1}\end{aligned}$$

for $t\in\{0,1,\ldots,2n\}$, one can easily obtain that $e_{\epsilon_1-\epsilon_2}(0).v_{2,n}=0$. Similarly, one can check that $e_{\epsilon_2-\epsilon_3}(0).v_{2,n}=0$ and $f_\theta(1).v_{2,n}=0$. □

Theorem 6.1 proves that for $n\geqslant 2$, the maximal submodule $N_2^1(n-2,0)$ is reducible.

Define the ideal $J_2(n-2,0)$ in the vertex operator algebra $N_2(n-2,0)$ by

$$J_2(n-2,0)=U(\hat{\mathfrak{g}})v_{2,n},$$

and denote the corresponding quotient vertex operator algebra by $V_{2,n}$, i.e., $V_{2,n}=N_2(n-2,0)/J_2(n-2,0)$.

It follows from Zhu's theory that:

Proposition 6.2. *The associative algebra $A(V_{2,n})$ is isomorphic to the algebra $U(\mathfrak{g})/I_{2,n}$, where $I_{2,n}$ is the two-sided ideal of $U(\mathfrak{g})$ generated by*

$$v'_{2,n}=\sum_{t=0}^{2n}\frac{1}{t!}\bigl(f^t_{\epsilon_1-\epsilon_2}\bigr)_L\bigl(e^n_{\epsilon_1-\epsilon_2}\bigr)e^{2n-t}_{\epsilon_2-\epsilon_3}e^t_{\epsilon_1-\epsilon_3}.$$

The $U(\mathfrak{g})$-submodule R of $U(\mathfrak{g})$ generated by the vector $v'_{2,n}$ under the adjoint action is isomorphic to $V(3n\omega_2)$. Proposition 3.1 gives that all weight spaces of R are 1-dimensional. In particular, we have $\dim R_0=1$. The following lemma will be proved in Section 7.

Lemma 6.3. *Let*

$$\begin{aligned}p(h)=h_1(h_1-1)\cdot\ldots\cdot(h_1-n+1)h_2(h_2-1)\cdot\ldots\cdot(h_2-n+1)\\\cdot(h_1+h_2+1)(h_1+h_2)\cdot\ldots\cdot(h_1+h_2-n+2).\end{aligned}\tag{6.1}$$

Then $p\in\mathcal{P}_0$.

Since $\dim \mathcal{P}_0 = \dim R_0 = 1$, Corollary 2.2 implies that

Proposition 6.4. *The set*

$$\begin{aligned}&\{V(t\omega_1 + m\omega_2) \mid t \in \mathbb{C},\ m \in \{0,1,\dots,n-1\}\}\\ &\quad\cup \{V(m\omega_1 + t\omega_2) \mid t \in \mathbb{C},\ m \in \{0,1,\dots,n-1\}\}\\ &\quad\cup \{V(t\omega_1 + (-1-t+m)\omega_2) \mid t \in \mathbb{C},\ m \in \{0,1,\dots,n-1\}\}\end{aligned}$$

provides the complete list of irreducible $A(V_{2,n})$-modules from the category $\mathcal{O}$.

In the case $n = 1$, we get

Corollary 6.5. *The set*

$$\{V(t\omega_1) \mid t \in \mathbb{Z}_+\} \cup \{V(t\omega_2) \mid t \in \mathbb{Z}_+\}$$

provides the complete list of irreducible finite-dimensional $A(V_{2,1})$-modules. Moreover, every irreducible finite-dimensional $A(V_{2,1})$-module has all 1-dimensional weight spaces for $\mathfrak{h}$.

It follows from Proposition 6.4 and Zhu's theory that

Theorem 6.6. *The set*

$$\begin{aligned}&\{L(n-2, t\omega_1 + m\omega_2) \mid t \in \mathbb{C},\ m \in \{0,1,\dots,n-1\}\}\\ &\quad\cup \{L(n-2, m\omega_1 + t\omega_2) \mid t \in \mathbb{C},\ m \in \{0,1,\dots,n-1\}\}\\ &\quad\cup \{L(n-2, t\omega_1 + (-1-t+m)\omega_2) \mid t \in \mathbb{C},\ m \in \{0,1,\dots,n-1\}\}\end{aligned} \tag{6.2}$$

provides the complete list of irreducible weak $V_{2,n}$-modules from the category $\mathcal{O}$.

Specially, for $n = 1$, we get an analogue of Theorem 5.7 for the case $A_2^{(1)}$:

Corollary 6.7. *The set*

$$\{L(-1, t\omega_1) \mid t \in \mathbb{C}\} \cup \{L(-1, t\omega_2) \mid t \in \mathbb{C}\} \cup \{L(-1, t\omega_1 + (-1-t)\omega_2) \mid t \in \mathbb{C}\}$$

provides the complete list of irreducible weak $V_{2,1}$-modules from the category $\mathcal{O}$.

Let Ψ be the automorphism of the vertex operator algebra $N_2(n-2, 0)$ of order two which is lifted from the automorphism

$$e_{\alpha_1} \mapsto e_{\alpha_2}, \qquad e_{\alpha_2} \mapsto e_{\alpha_1}, \qquad f_{\alpha_1} \mapsto f_{\alpha_2}, \qquad f_{\alpha_2} \mapsto f_{\alpha_1}$$

of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$. It is also an automorphism of the affine Lie algebra $\hat{\mathfrak{g}}$. In a different context, this automorphism was also considered in [3] and [5].

For any $\hat{\mathfrak{g}}$-module M, let $\Psi(M)$ by the $\hat{\mathfrak{g}}$-module obtained by applying the automorphism Ψ. The proof of the following lemma is obvious.

Lemma 6.8.

(i) *If v is a non-trivial singular vector in $N_2(n-2,0)$, then $\Psi(v)$ is also a non-trivial singular vector in $N_2(n-2,0)$.*
(ii) *The set* (6.2) *is Ψ-invariant.*

The previous lemma implies that the vector

$$\widetilde{v_{2,n}} = \Psi(v_{2,n}) = \sum_{t=0}^{2n} \frac{(-1)^t}{t!} e_{\epsilon_1-\epsilon_3}(-1)^t e_{\epsilon_1-\epsilon_2}(-1)^{2n-t} f_{\epsilon_2-\epsilon_3}(0)^t e_{\epsilon_2-\epsilon_3}(-1)^n \mathbf{1}$$

is a singular vector in $N_2(n-2,0)$.

Define the ideal $\widetilde{J_2}(n-2,0) = U(\hat{\mathfrak{g}})\widetilde{v_{2,n}}$ in $N_2(n-2,0)$ and denote the corresponding quotient vertex operator algebra by $\widetilde{V_{2,n}}$. By using Theorem 6.6 and Lemma 6.8 we get the following result.

Theorem 6.9.

(i) *For every $n \in \mathbb{N}$, the vector $\widetilde{v_{2,n}} = \Psi(v_{2,n})$ is a non-trivial singular vector in $N_2(n-2,0)$. In particular, the vertex operator algebra $N_2(n-2,0)$ contains two linearly independent singular vectors of conformal weight $3n$.*
(ii) *The vertex operator algebras $V_{2,n}$ and $\widetilde{V_{2,n}}$ are isomorphic, and $\Psi|_{V_{2,n}} : V_{2,n} \to \widetilde{V_{2,n}}$ is the corresponding isomorphism.*
(iii) *The set* (6.2) *provides all irreducible weak $\widetilde{V_{2,n}}$-modules from the category $\mathcal{O}$.*

Remark 6.10. For $n = 1$ we obtain two singular vectors in $N_2(-1,0)$ of conformal weight 3:

$$v_{2,1} = e_{\epsilon_2-\epsilon_3}(-1)^2 e_{\epsilon_1-\epsilon_2}(-1)\mathbf{1} - e_{\epsilon_1-\epsilon_3}(-1)e_{\epsilon_2-\epsilon_3}(-1)h_1(-1)\mathbf{1} - e_{\epsilon_1-\epsilon_3}(-1)^2 f_{\epsilon_1-\epsilon_2}(-1)\mathbf{1},$$
$$\widetilde{v_{2,1}} = e_{\epsilon_1-\epsilon_2}(-1)^2 e_{\epsilon_2-\epsilon_3}(-1)\mathbf{1} + e_{\epsilon_1-\epsilon_3}(-1)e_{\epsilon_1-\epsilon_2}(-1)h_2(-1)\mathbf{1} - e_{\epsilon_1-\epsilon_3}(-1)^2 f_{\epsilon_2-\epsilon_3}(-1)\mathbf{1}.$$

Since there are no singular vectors in $N_2(-1,0)$ of conformal weight 2, we showed that the maximal ideal $N_2^1(-1,0)$ is not generated by a single singular vector. It is an interesting problem to describe the structure of $N_2^1(-1,0)$.

7. Proof of Lemma 6.3

The following relations hold in $U(\mathfrak{g})$:

$$\big(f_\alpha^m\big)_L\big(e_\alpha^k\big) \in (-1)^k m! \binom{k}{2k-m} f_\alpha^{m-k} \cdot (h_\alpha + k - m) \cdot \ldots \cdot (h_\alpha - k + 1) + U(\mathfrak{g})e_\alpha,$$
$$\text{for } 2k \geqslant m \geqslant k \text{ and } \forall \alpha \in \Delta_+; \tag{7.1}$$

$$\big(f_\alpha^m\big)_L\big(e_\alpha^k\big) \in (-1)^m k \cdot \ldots \cdot (k-m+1)(h_\alpha - k + 1) \cdot \ldots \cdot (h_\alpha - k + m) \cdot e_\alpha^{k-m}$$
$$+ U(\mathfrak{g})e_\alpha^{k-m+1}, \quad \text{for } m \leqslant k \text{ and } \forall \alpha \in \Delta_+; \tag{7.2}$$

$$e_{\alpha}^{m} f_{\alpha}^{k} \in m \cdot \ldots \cdot (m-k+1)(h_{\alpha}-m+k) \cdot \ldots \cdot (h_{\alpha}-m+1) \cdot e_{\alpha}^{m-k} + U(\mathfrak{g}) e_{\alpha}^{m-k+1},$$
$$\text{for } m \geqslant k \text{ and } \forall \alpha \in \Delta_{+}; \tag{7.3}$$

$$e_{\epsilon_1-\epsilon_3}^{m} f_{\epsilon_1-\epsilon_2}^{k} \in k \cdot \ldots \cdot (k-m+1) f_{\epsilon_1-\epsilon_2}^{k-m} e_{\epsilon_2-\epsilon_3}^{m} + U(\mathfrak{g}) e_{\epsilon_1-\epsilon_3}, \quad \text{for } m \leqslant k; \tag{7.4}$$

$$\left(x^{m}\right)_{L}\left(y^{k}\right) = k \cdot \ldots \cdot (k-m+1) y^{k-m}[x,y]^{m}, \quad \text{for } x, y \in \mathfrak{g} \text{ such that } \big[[x,y],y\big]=0,$$
$$\big[x,[x,y]\big]=0 \text{ and } m \leqslant k. \tag{7.5}$$

We claim that

$$\left(f_{\epsilon_1-\epsilon_3}^{n} f_{\epsilon_2-\epsilon_3}^{n}\right)_{L} v_{2,n}' \in (-1)^{n}(n!)^{2} p(h) + U(\mathfrak{g})\mathfrak{n}_{+},$$

with $p(h)$ given by relation (6.1).

It can easily be checked that for $t > n$,

$$\left(f_{\epsilon_1-\epsilon_3}^{n} f_{\epsilon_2-\epsilon_3}^{n}\right)_{L}\left[\left(f_{\epsilon_1-\epsilon_2}^{t}\right)_{L}\left(e_{\epsilon_1-\epsilon_2}^{n}\right) e_{\epsilon_2-\epsilon_3}^{2n-t} e_{\epsilon_1-\epsilon_3}^{t}\right] \in U(\mathfrak{g})\mathfrak{n}_{+}. \tag{7.6}$$

Let $t \leqslant n$. Then

$$\begin{aligned}
&\left(f_{\epsilon_1-\epsilon_3}^{n} f_{\epsilon_2-\epsilon_3}^{n}\right)_{L}\left[\left(f_{\epsilon_1-\epsilon_2}^{t}\right)_{L}\left(e_{\epsilon_1-\epsilon_2}^{n}\right) e_{\epsilon_2-\epsilon_3}^{2n-t} e_{\epsilon_1-\epsilon_3}^{t}\right] \\
&\quad = \sum_{\substack{(m_1,m_2,m_3)\in\mathbb{Z}_{+}^{3} \\ m_1+m_2+m_3=n}} \sum_{\substack{(k_1,k_2,k_3)\in\mathbb{Z}_{+}^{3} \\ k_1+k_2+k_3=n}} \binom{n}{m_1, m_2, m_3}\binom{n}{k_1, k_2, k_3}\left(f_{\epsilon_1-\epsilon_3}^{m_1} f_{\epsilon_2-\epsilon_3}^{k_1} f_{\epsilon_1-\epsilon_2}^{t}\right)_{L}\left(e_{\epsilon_1-\epsilon_2}^{n}\right) \\
&\qquad \cdot \left(f_{\epsilon_1-\epsilon_3}^{m_2} f_{\epsilon_2-\epsilon_3}^{k_2}\right)_{L}\left(e_{\epsilon_1-\epsilon_3}^{t}\right) \cdot \left(f_{\epsilon_1-\epsilon_3}^{m_3} f_{\epsilon_2-\epsilon_3}^{k_3}\right)_{L}\left(e_{\epsilon_2-\epsilon_3}^{2n-t}\right).
\end{aligned}$$

Using relations (7.1)–(7.5) one can easily check that terms which are not in $U(\mathfrak{g})\mathfrak{n}_{+}$ are obtained only for $m_1 + k_1 = 0$, $m_2 + k_2 \leqslant t$ and $m_3 + k_3 \geqslant 2n - t$. Denoting $m_2 = m$ and $k_2 = k$, we get

$$\begin{aligned}
&\left(f_{\epsilon_1-\epsilon_3}^{n} f_{\epsilon_2-\epsilon_3}^{n}\right)_{L}\left[\left(f_{\epsilon_1-\epsilon_2}^{t}\right)_{L}\left(e_{\epsilon_1-\epsilon_2}^{n}\right) e_{\epsilon_2-\epsilon_3}^{2n-t} e_{\epsilon_1-\epsilon_3}^{t}\right] \\
&\quad \in \sum_{m=0}^{t} \sum_{k=0}^{t-m} \binom{n}{m}\binom{n}{k}\left(f_{\epsilon_1-\epsilon_2}^{t}\right)_{L}\left(e_{\epsilon_1-\epsilon_2}^{n}\right) \cdot \left(f_{\epsilon_1-\epsilon_3}^{m} f_{\epsilon_2-\epsilon_3}^{k}\right)_{L}\left(e_{\epsilon_1-\epsilon_3}^{t}\right) \\
&\qquad \cdot \left(f_{\epsilon_1-\epsilon_3}^{n-m} f_{\epsilon_2-\epsilon_3}^{n-k}\right)_{L}\left(e_{\epsilon_2-\epsilon_3}^{2n-t}\right) + U(\mathfrak{g})\mathfrak{n}_{+}.
\end{aligned} \tag{7.7}$$

Using relations (7.1), (7.2) and (7.5) we obtain

$$\begin{aligned}
&\left(f_{\epsilon_1-\epsilon_3}^{n-m} f_{\epsilon_2-\epsilon_3}^{n-k}\right)_{L}\left(e_{\epsilon_2-\epsilon_3}^{2n-t}\right) = (2n-t) \cdot \ldots \cdot (n+m-t+1)\left(f_{\epsilon_2-\epsilon_3}^{n-k}\right)_{L}\left(f_{\epsilon_1-\epsilon_2}^{n-m} e_{\epsilon_2-\epsilon_3}^{n+m-t}\right) \\
&\quad = (2n-t) \cdot \ldots \cdot (n+m-t+1) \sum_{a=0}^{n-k} \binom{n-k}{a}\left(f_{\epsilon_2-\epsilon_3}^{a}\right)_{L}\left(f_{\epsilon_1-\epsilon_2}^{n-m}\right) \cdot \left(f_{\epsilon_2-\epsilon_3}^{n-k-a}\right)_{L}\left(e_{\epsilon_2-\epsilon_3}^{n+m-t}\right) \\
&\quad \in (2n-t) \cdot \ldots \cdot (n+m-t+1) \sum_{a=0}^{t-k-m} \binom{n-k}{a}(-1)^{a}(n-m) \cdot \ldots \cdot (n-m-a+1)
\end{aligned}$$

$$\cdot f_{\epsilon_1-\epsilon_2}^{n-m-a} f_{\epsilon_1-\epsilon_3}^{a} \cdot (-1)^{n+m-t}(n-k-a)! \binom{n+m-t}{n+2m-2t+k+a} f_{\epsilon_2-\epsilon_3}^{t-k-m-a}$$
$$\cdot (h_{\epsilon_2-\epsilon_3}+m-t+k+a) \cdot \ldots \cdot (h_{\epsilon_2-\epsilon_3}-n-m+t+1) + U(\mathfrak{g})\mathfrak{n}_+.$$

Similarly, using relations (7.2) and (7.5) one can obtain that

$$\begin{aligned}
&\left(f_{\epsilon_1-\epsilon_3}^{m} f_{\epsilon_2-\epsilon_3}^{k}\right)_L \left(e_{\epsilon_1-\epsilon_3}^{t}\right) \\
&\quad \in t \cdot \ldots \cdot (t-k+1) \sum_{b=0}^{m} \Bigg[\binom{m}{b} (-1)^b b \cdot \ldots \cdot (b-k+1) \\
&\qquad \cdot f_{\epsilon_2-\epsilon_3}^{b} e_{\epsilon_1-\epsilon_2}^{k-b} \cdot (-1)^{m-b}(t-k) \cdot \ldots \cdot (t-k-m+b+1) \\
&\qquad \cdot (h_{\epsilon_1-\epsilon_3}-t+k+1) \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_3}-t+k+m-b) e_{\epsilon_1-\epsilon_3}^{t-k-m+b} + U(\mathfrak{g}) e_{\epsilon_1-\epsilon_3}^{t-k-m+b+1} \Bigg].
\end{aligned}$$

Using relations (7.3) and (7.4) one can easily check that the terms in expression

$$\left(f_{\epsilon_1-\epsilon_3}^{m} f_{\epsilon_2-\epsilon_3}^{k}\right)_L \left(e_{\epsilon_1-\epsilon_3}^{t}\right) \cdot \left(f_{\epsilon_1-\epsilon_3}^{n-m} f_{\epsilon_2-\epsilon_3}^{n-k}\right)_L \left(e_{\epsilon_2-\epsilon_3}^{2n-t}\right)$$

which are not in $U(\mathfrak{g})\mathfrak{n}_+$ are obtained only for $b=0$. It follows that

$$\begin{aligned}
&\left(f_{\epsilon_1-\epsilon_3}^{m} f_{\epsilon_2-\epsilon_3}^{k}\right)_L \left(e_{\epsilon_1-\epsilon_3}^{t}\right) \cdot \left(f_{\epsilon_1-\epsilon_3}^{n-m} f_{\epsilon_2-\epsilon_3}^{n-k}\right)_L \left(e_{\epsilon_2-\epsilon_3}^{2n-t}\right) \\
&\quad \in (-1)^m t \cdot \ldots \cdot (t-k-m+1) e_{\epsilon_1-\epsilon_2}^{k} (h_{\epsilon_1-\epsilon_3}-t+k+1) \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_3}-t+k+m) \\
&\qquad \cdot e_{\epsilon_1-\epsilon_3}^{t-k-m} \cdot (2n-t) \cdot \ldots \cdot (n+m-t+1) \\
&\qquad \cdot \sum_{a=0}^{t-k-m} \binom{n-k}{a} (-1)^a (n-m) \cdot \ldots \cdot (n-m-a+1) f_{\epsilon_1-\epsilon_2}^{n-m-a} f_{\epsilon_1-\epsilon_3}^{a} \\
&\qquad \cdot (-1)^{n+m-t}(n-k-a)! \binom{n+m-t}{n+2m-2t+k+a} f_{\epsilon_2-\epsilon_3}^{t-k-m-a} \\
&\qquad \cdot (h_{\epsilon_2-\epsilon_3}+m-t+k+a) \cdot \ldots \cdot (h_{\epsilon_2-\epsilon_3}-n-m+t+1) + U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned} \tag{7.8}$$

It follows from relations (7.3) and (7.4) that

$$\begin{aligned}
&e_{\epsilon_1-\epsilon_3}^{t-k-m} f_{\epsilon_1-\epsilon_3}^{a} f_{\epsilon_1-\epsilon_2}^{n-m-a} f_{\epsilon_2-\epsilon_3}^{t-k-m-a} \in (t-k-m) \cdot \ldots \cdot (t-k-m-a+1) \\
&\quad \cdot (h_{\epsilon_1-\epsilon_3}-t+k+m+a) \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_3}-t+k+m+1) \\
&\quad \cdot (n-m-a) \cdot \ldots \cdot (n-t+k+1) f_{\epsilon_1-\epsilon_2}^{n-t+k} (t-k-m+a)! \\
&\quad \cdot h_{\epsilon_2-\epsilon_3} \cdot \ldots \cdot (h_{\epsilon_2-\epsilon_3}-t+k+m+a+1) + U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned} \tag{7.9}$$

Using relations (7.1), (7.2), (7.8) and (7.9) we get

$$\begin{aligned}
&\big(f^{t}_{\epsilon_1-\epsilon_2}\big)_L\big(e^{n}_{\epsilon_1-\epsilon_2}\big)\cdot\big(f^{m}_{\epsilon_1-\epsilon_3}f^{k}_{\epsilon_2-\epsilon_3}\big)_L\big(e^{t}_{\epsilon_1-\epsilon_3}\big)\cdot\big(f^{n-m}_{\epsilon_1-\epsilon_3}f^{n-k}_{\epsilon_2-\epsilon_3}\big)_L\big(e^{2n-t}_{\epsilon_2-\epsilon_3}\big)\\
&\quad\in(-1)^n n\cdot\ldots\cdot(n-t+1)(h_{\epsilon_1-\epsilon_2}-n+1)\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_2}-n+t)t!\\
&\qquad\cdot(h_{\epsilon_1-\epsilon_3}-n+1)\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_3}-n+m)\cdot(2n-t)\cdot\ldots\cdot(n+m-t+1)(n-m)!\\
&\qquad\cdot h_{\epsilon_1-\epsilon_2}\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_2}-n+t-k+1)h_{\epsilon_2-\epsilon_3}\cdot\ldots\cdot(h_{\epsilon_2-\epsilon_3}-n-m+t+1)\\
&\qquad\cdot\sum_{a=0}^{t-k-m}\binom{n-k}{a}(-1)^a(n-k-a)!\binom{n+m-t}{n+2m-2t+k+a}\\
&\qquad\cdot(h_{\epsilon_1-\epsilon_3}-n+m+a)\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_3}-n+m+1)+U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}\tag{7.10}$$

Clearly

$$\begin{aligned}
&\sum_{a=0}^{t-k-m}\binom{n-k}{a}(-1)^a(n-k-a)!\binom{n+m-t}{n+2m-2t+k+a}\\
&\qquad\cdot(h_{\epsilon_1-\epsilon_3}-n+m+a)\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_3}-n+m+1)\\
&\quad=(n-k)!\sum_{a=0}^{t-k-m}\binom{-h_{\epsilon_1-\epsilon_3}+n-m-1}{a}\binom{n+m-t}{t-k-m-a}\\
&\quad=(n-k)!\binom{-h_{\epsilon_1-\epsilon_3}+2n-t-1}{t-k-m}.
\end{aligned}\tag{7.11}$$

It follows from relations (7.7), (7.10) and (7.11) that

$$\begin{aligned}
&\big(f^{n}_{\epsilon_1-\epsilon_3}f^{n}_{\epsilon_2-\epsilon_3}\big)_L\big[\big(f^{t}_{\epsilon_1-\epsilon_2}\big)_L\big(e^{n}_{\epsilon_1-\epsilon_2}\big)e^{2n-t}_{\epsilon_2-\epsilon_3}e^{t}_{\epsilon_1-\epsilon_3}\big]\\
&\quad\in(-1)^n n\cdot\ldots\cdot(n-t+1)t!(h_{\epsilon_1-\epsilon_2}-n+1)\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_2}-n+t)\\
&\qquad\cdot\sum_{m=0}^{t}\binom{n}{m}(h_{\epsilon_1-\epsilon_3}-n+1)\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_3}-n+m)\\
&\qquad\cdot(2n-t)\cdot\ldots\cdot(n+m-t+1)(n-m)!h_{\epsilon_2-\epsilon_3}\cdot\ldots\cdot(h_{\epsilon_2-\epsilon_3}-n-m+t+1)\\
&\qquad\cdot\sum_{k=0}^{t-m}\binom{n}{k}h_{\epsilon_1-\epsilon_2}\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_2}-n+t-k+1)\\
&\qquad\cdot(n-k)!\binom{-h_{\epsilon_1-\epsilon_3}+2n-t-1}{t-k-m}+U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned}$$

Furthermore

$$\begin{aligned}
&\sum_{k=0}^{t-m}\binom{n}{k}h_{\epsilon_1-\epsilon_2}\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_2}-n+t-k+1)(n-k)!\binom{-h_{\epsilon_1-\epsilon_3}+2n-t-1}{t-k-m}\\
&\quad=n!h_{\epsilon_1-\epsilon_2}\cdot\ldots\cdot(h_{\epsilon_1-\epsilon_2}-n+t+1)\sum_{k=0}^{t-m}\binom{h_{\epsilon_1-\epsilon_2}-n+t}{k}\binom{-h_{\epsilon_1-\epsilon_3}+2n-t-1}{t-m-k}
\end{aligned}$$

$$= n! h_{\epsilon_1-\epsilon_2} \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_2} - n + t + 1) \binom{-h_{\epsilon_2-\epsilon_3} + n - 1}{t - m},$$

which implies

$$\begin{aligned}
&\big(f^n_{\epsilon_1-\epsilon_3} f^n_{\epsilon_2-\epsilon_3}\big)_L \big[\big(f^t_{\epsilon_1-\epsilon_2}\big)_L \big(e^n_{\epsilon_1-\epsilon_2}\big) e^{2n-t}_{\epsilon_2-\epsilon_3} e^t_{\epsilon_1-\epsilon_3}\big] \\
&\quad \in (-1)^n n! \cdot n \cdot \ldots \cdot (n - t + 1) t! \cdot h_{\epsilon_1-\epsilon_2} \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_2} - n + 1) \\
&\qquad \cdot h_{\epsilon_2-\epsilon_3} \cdot \ldots \cdot (h_{\epsilon_2-\epsilon_3} - n + 1) \sum_{m=0}^{t} \binom{n}{m} (n - m)! (-1)^{t-m} \frac{1}{(t - m)!} \\
&\qquad \cdot (h_{\epsilon_1-\epsilon_3} - n + 1) \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_3} - n + m)(2n - t) \cdot \ldots \cdot (n + m - t + 1) + U(\mathfrak{g})\mathfrak{n}_+.
\end{aligned} \tag{7.12}$$

One can easily verify that

$$\begin{aligned}
&\sum_{m=0}^{t} \binom{n}{m} (n - m)! (-1)^{t-m} \frac{1}{(t - m)!} (h_{\epsilon_1-\epsilon_3} - n + 1) \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_3} - n + m) \\
&\qquad \cdot (2n - t) \cdot \ldots \cdot (n + m - t + 1) \\
&\quad = n! \cdot (2n - t) \cdot \ldots \cdot (n + 1)(-1)^t \binom{-h_{\epsilon_1-\epsilon_3} + 2n - 1}{t}.
\end{aligned} \tag{7.13}$$

It follows from Proposition 6.2 and relations (7.6), (7.12) and (7.13) that

$$\begin{aligned}
\big(f^n_{\epsilon_1-\epsilon_3} f^n_{\epsilon_2-\epsilon_3}\big)_L v'_{2,n} \in{}& (-1)^n (n!)^2 \cdot h_{\epsilon_1-\epsilon_2} \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_2} - n + 1) \\
&\cdot h_{\epsilon_2-\epsilon_3} \cdot \ldots \cdot (h_{\epsilon_2-\epsilon_3} - n + 1) \sum_{t=0}^{n} (2n - t) \cdot \ldots \cdot (n - t + 1) \\
&\cdot (-1)^t \binom{-h_{\epsilon_1-\epsilon_3} + 2n - 1}{t}.
\end{aligned}$$

Finally

$$\begin{aligned}
&\sum_{t=0}^{n} (2n - t) \cdot \ldots \cdot (n - t + 1)(-1)^t \binom{-h_{\epsilon_1-\epsilon_3} + 2n - 1}{t} \\
&\quad = n! \cdot (-1)^n \binom{-h_{\epsilon_1-\epsilon_3} + n - 2}{n},
\end{aligned}$$

which implies

$$\begin{aligned}
\big(f^n_{\epsilon_1-\epsilon_3} f^n_{\epsilon_2-\epsilon_3}\big)_L v'_{2,n} \in{}& (-1)^n (n!)^2 \cdot h_{\epsilon_1-\epsilon_2} \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_2} - n + 1) \\
&\cdot h_{\epsilon_2-\epsilon_3} \cdot \ldots \cdot (h_{\epsilon_2-\epsilon_3} - n + 1)(h_{\epsilon_1-\epsilon_3} + 1) \cdot \ldots \cdot (h_{\epsilon_1-\epsilon_3} - n + 2),
\end{aligned}$$

and the proof is complete.

8. Conclusions

In this paper we constructed two families of isomorphic operator algebras $V_{2,n}$ and $\widetilde{V_{2,n}}$ associated to affine Lie algebra of type $A_2^{(1)}$ with positive integer levels $n-2$, for $n \geqslant 2$. These vertex operator algebras are non-simple and they are different both from $N_2(n-2,0)$ and its simple quotient $L_2(n-2,0)$ which were previously extensively studied in [8,14,18,20,21]. The class of irreducible weak $N_2(n-2,0)$-modules includes all irreducible $A_2^{(1)}$-modules of level $n-2$ from the category $\mathcal{O}$. Vertex operator algebra $L_2(n-2,0)$ has finitely many irreducible weak modules and any irreducible weak $L_2(n-2,0)$-module is an integrable highest weight $A_2^{(1)}$-module of level $n-2$. Moreover, the category of weak $L_2(n-2,0)$-modules is semisimple. By studying the representation theory of $V_{2,n}$, we obtained a new interesting subcategory of the category of $A_2^{(1)}$-modules of level $n-2$ from the category $\mathcal{O}$. The irreducible objects in that category include irreducible weak $L_2(n-2,0)$-modules, but this class is smaller than the class of irreducible weak $N_2(n-2,0)$-modules from the category $\mathcal{O}$. Irreducible weak $V_{2,n}$-modules from the category $\mathcal{O}$ are parameterized by certain lines in $\hat{\mathfrak{h}}^*$, which implies that there are infinitely many of them. Furthermore, the category of $V_{2,n}$-modules is not semisimple, which makes the representation theory of $V_{2,n}$ much different from the representation theory of $L_2(n-2,0)$.

We also gave examples of vertex operator algebras associated to affine Lie algebras whose level is neither a positive integer nor an admissible rational number. We studied representation theory of vertex operator algebras $V_{l,1}$ associated to affine Lie algebras of type $A_l^{(1)}$ with negative integer level -1, for $l \geqslant 2$. These vertex operator algebras are quotients of generalized Verma modules $N_l(-1,0)$ by certain ideals defined in [2] and in this paper. In the case $l=2$, we constructed two isomorphic vertex operator algebras $V_{2,1}$ and $\widetilde{V_{2,1}}$. We showed that irreducible weak $V_{l,1}$-modules from the category $\mathcal{O}$ are parameterized by certain lines in $\hat{\mathfrak{h}}^*$, and that the category of weak $V_{l,1}$-modules is not semisimple.

In our future work we plan to study fusion rules for the irreducible representations classified in this paper.

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