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On the Existence of Positive Solutions of Quasilinear Elliptic Boundary Value Problems

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We establish the existence of positive solutions to a class of quasilinear anisotropic problems which have either sublinear or superlinear nonlinearity. With a, b nonnegative constants and α, β positive constants, one example is

[illegible]

If $b-a<1$ (sublinear case), then for each $\lambda \in [0, \infty)$, (1) has a solution. On the other hand, if $b-a>1$ (superlinear case), then there exists a $\lambda^* > 0$ such that for $0 \leqslant \lambda < \lambda^*$, (1) has at least one solution, and for $\lambda > \lambda^*$ no solution exists.

1. INTRODUCTION

There is a considerable amount of knowledge about positive solutions for semilinear elliptic problems of the form

$$\begin{cases} \Delta u + \lambda f(u) = 0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

Here Ω is a bounded smooth domain in R^N, f is a given nonlinear function, and λ is a non-negative constant. The existence and multiplicity results of these bifurcation problems are well described in P. L. Lions' survey in 1982 [14].

The Laplacian operator in the above equation arises, because the medium in the model we are interested in is isotropic in all directions. In some applications, we need to pursue the anisotropic case.

Recently there is an interesting application involving such an anisotropic case. It comes from the study of unsteady small disturbance equations by Canic and Keyfitz [3]. They focused on the self-similar solution for these

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hyperbolic conservation equations in two dimensional domains, and derived an associated anisotropic singular elliptic equation:

$$\begin{cases} ((u-\rho)\, u_\rho + u/2)_\rho + u_{\eta\eta} = 0, \\ u|_{\partial\Omega} = g. \end{cases} \tag{3}$$

A survey found that the amount of literature devoted to the nonlinear anisotropic medium is slim, whether the elliptic equations are singular or not. At about the same time as the Canic and Keyfitz's study, Choi, Lazer, and McKenna [4] proved the existence of a positive solution for the following singular quasilinear boundary value problem in a convex domain:

$$\begin{cases} u^a u_{xx} + u^b u_{yy} + p(x) = 0, \qquad x \in \Omega \subset R^2, \\ \mathrm{u}|_{\partial\Omega} = 0, \end{cases} \tag{4}$$

where $p \in C^\alpha(\bar{\Omega})$ with some $0 < \alpha < 1$, is bounded and positive, with $a > b \geqslant 0$. Physically, it represents a kind of diffusive anisotropy. Uniqueness has been proven in [15] with some further restrictions on the geometry for Ω.

Later, Choi and McKenna [5] extended the above result to cover the multi-dimensional cases:

$$\begin{cases} \sum_{i=1}^{N} u^{a_i} \dfrac{\partial^2 u}{\partial x_i^2} + p(x) = 0, \qquad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{5}$$

where p is the same as in (4), with $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_N \geqslant 0$. It also includes some partial results for non-convex domains.

Another application involves numerical studies of the nonlinear heat equation in an anisotropic medium, [2]. The self-similar solution also reduces to an elliptic equation of the form:

$$(u^a u_x)_x + (u^b u_y)_y + C_1 x u_x + C_2 y u_y + u^c - \frac{u}{c-1} = 0, \tag{6}$$

where all parameters are given constants. It is noted that one source term is u^c, which introduces another kind of nonlinearity into the equation.

In this paper, we will study the effect of nonlinear source terms in an anisotropic diffusive medium. The focus will be on a class of equations which does not exhibit singular behavior for the diffusion coefficients. This allows a comparison with known results on Eq. (2), and serves as a first step to studying the singular problem.

More precisely, we will study

$$\begin{cases} \sum_{i=1}^{N} A_i(u) \dfrac{\partial^2 u}{\partial x_i^2} + \lambda f(u, x) = 0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{7}$$

With appropriate assumptions on A_i and f to be specified later in this section, we get some numerical results which are qualitatively similar to those of the semilinear elliptic problems. However, the proof is quite different and required some interesting refinement of techniques. Here we present the first results of our investigation.

First we list some assumptions about Ω, the coefficients $A_i(u)$, and f. Different theorems will require different combinations of such assumptions.

(D) $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$ of class $C^{2+\alpha}$ where $0 < \alpha < 1$.

(C) For all $1 \leqslant i \leqslant N$, the A_i's are in C^1, and there exists a $\delta > 0$ such that $A_i \geqslant \delta > 0$. Furthermore, for a nonnegative constant s_1,

$$\lim_{t \to \infty} A_1(t)\, t^{-s_1} = C_1,$$

where C_1 is a positive constant.

(C1) For all $t > 0$, $A_1'(t) \geqslant 0$.

(C2) For all $t > 0$, $(A_i(t)/A_1(t))' \leqslant 0$, $i = 2, ..., N$.

(H) $f \in C^1$, and there exists an $\varepsilon > 0$ such that for all $x \in \bar{\Omega}$, $f(\cdot, x) \geqslant \varepsilon > 0$.

(H1) (Sublinear f) For all $x \in \bar{\Omega}$, $\lim_{t \to \infty} f(t, x)\, t^{-(s_1+1)} = 0$.

(H2) (Superlinear f) For all $x \in \bar{\Omega}$, $\lim_{t \to \infty} f(t, x)\, t^{-(s_1+1)} = \infty$.

Remark. For assumption (C2) to hold, one can deduce that the growth rate of A_i, $i = 2, ..., N$, will be no faster than that of A_1.

In Section 2, we will show that when the conditions (C), (C1), (H), and (H1) are satisfied, that is, when f/A_1 is sublinear in the sense that

$$\lim_{t \to \infty} \frac{f(t, x)}{A_1(t)\, t} = 0, \tag{8}$$

the following theorem holds.

THEOREM 1. *Let Ω satisfy* (D). *Consider Eq.* (7) *with conditions* (C), (C1), (H), *and* (H1) *satisfied. Then for each $\lambda > 0$, there exists at least one positive solution $u \in C^2(\bar{\Omega})$.*

This will be proven by using the Schauder Fixed Point Theorem.

On the other hand, when (C), (C2), (H), and (H2) hold, that is, when f/A_1 is a superlinear nonlinearity,

$$\lim_{t\to\infty} \frac{f(t,x)}{A_1(t)\,t} = \infty, \tag{9}$$

then we can have the following:

THEOREM 2. *Let Ω satisfy* (D). *Consider Eq.* (7) *with conditions* (C), (C2), (H), *and* (H2) *satisfied. Then there exists $\lambda^* \in (0, \infty)$ such that*

(1) *for each λ satisfying $0 < \lambda < \lambda^*$,* (7) *has at least one positive solution in $C^2(\bar{\Omega})$.*

(2) *for $\lambda > \lambda^*$, no classical solution exists.*

In the proof of Theorem 2, since this elliptic problem is quasilinear, some techniques associated with the semilinear problems are unavailable. In fact, the monotone iteration technique [16] fails in the elliptic case. We have to bypass this difficulty by studying an associated parabolic problem. We show the existence of a global solution for the parabolic equation, and as time $t \to \infty$, the limiting solution solves the elliptic problem.

Remark. For notational convenience, we will write $f(u, x) \equiv f(u)$.

2. SUBLINEAR CASE

We will fix λ in this section. Rewrite our equation as follows:

$$\begin{cases} u_{x_1x_1} + \sum_{i=2}^{N} \frac{A_i(u)}{A_1(u)} u_{x_i x_i} + \lambda \frac{f(u)}{A_1(u)} = 0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{10}$$

First, we will construct an upper solution. Without loss of generality, we can assume that Ω lies inside the strip $[0, L] \times (\mathbb{R})^{N-1}$ for some $L > 0$. Let $\psi_1 > 0$ be the first eigenfunction which satisfies

$$\psi_{1x_1x_1} + \beta_1 \psi_1 = 0, \tag{11}$$

with corresponding eigenvalue $\beta_1 > 0$ in the slightly extended domain $[-1, L+1]$ with zero boundary condition.

LEMMA 1. *Assume conditions* (D), (C), (C1), (H), *and* (H1) *hold. Then there exists a* $K_1 \geqslant 1$, *such that* $\psi \equiv K_1 \psi_1$ *is an upper solution of* (10).

Proof. Since condition (H1) holds, given any $\varepsilon > 0$, there exists a constant $M_\varepsilon > 0$ such that for all $t \geqslant 0$,

$$\frac{f(t)}{A_1(t)} \leqslant \varepsilon t + M_\varepsilon.$$

Let $K_1 \geqslant 1$. Define $\psi \equiv K_1 \psi_1$ and

$$\delta \equiv \min_{x \in [0, L]} \psi_1(x) > 0.$$

Choosing ε such that $\lambda\varepsilon = \beta_1/2$ and K_1 sufficiently large, we have

$$\begin{aligned} \psi_{x_1 x_1} + \lambda \frac{f(\psi)}{A_1(\psi)} &\leqslant -\beta_1 \psi + \lambda\varepsilon\psi + \lambda M_\varepsilon \\ &\leqslant -(\beta_1/2)\, K_1 \delta + \lambda M_\varepsilon \\ &< 0. \end{aligned}$$

Thus ψ is an upper solution for (10) because $\psi|_{\partial\Omega} > 0$.

With this ψ, define a set

$$S = \{u \in C(\bar{\Omega}) \mid 0 \leqslant u \leqslant \psi \text{ on } \bar{\Omega},\ u|_{\partial\Omega} = 0\},$$

which is closed, bounded, and convex. Now, we are ready to prove Theorem 1 using the Schauder fixed point theorem.

Proof of Theorem 1. Define a map $T\colon S \to S$ such that for all $u \in S$, $Tu \equiv w$, which satisfies

$$\begin{cases} w_{x_1 x_1} + \displaystyle\sum_{i=2}^{N} \frac{A_i(u)}{A_1(u)} w_{x_i x_i} + \lambda \frac{f(u)}{A_1(w)} = 0, \\ w|_{\partial\Omega} = 0. \end{cases} \tag{12}$$

Observe that $A_i(t)/A_1(t)$ is continuous and positive for $t \in [0, \|\psi\|_{L^\infty}]$. So for $i = 2, ..., N$, there exist positive constants θ and Θ such that for all $u \in S$,

$$0 < \theta \leqslant A_i(u)/A_1(u) \leqslant \Theta < \infty,$$

which gives uniform ellipticity for equation (12). This is only a semilinear equation, which allows the application of a monotone iteration method. Using the monotonicity on A_1, and $u \in S$,

$$\begin{aligned} \psi_{x_1x_1} + \lambda \frac{f(u)}{A_1(\psi)} &\leqslant -\beta_1 \psi + \lambda \frac{f(u)}{A_1(u)} \\ &\leqslant -\beta_1 \psi + \lambda \varepsilon u + \lambda M_\varepsilon \\ &\leqslant -\beta_1 \psi + \lambda \varepsilon \psi + \lambda M_\varepsilon \\ &< 0 \end{aligned}$$

by the calculations in Lemma 1. Hence, with ψ being an upper solution and zero being a lower solution, the existence and uniqueness of the solution $w \in W^{2,q}(\Omega) \cap C(\bar{\Omega})$ for any $q > N$ is obtained. Moreover, $w \in S$ by construction (see Lemma 1 in [5]). Thus T is well defined and T maps S into S.

Next, we want to show that T is compact. For $u \in S$, since $w \in S$,

$$\left\| \lambda \frac{f(u)}{A_1(w)} \right\|_{L^n(\Omega)} \leqslant C.$$

Together with the uniform ellipticity, we have the global Holder estimate (Corollary 9.29 in [7]):

$$\|w\|_{C^\alpha(\bar{\Omega})} \leqslant C,$$

for some $\alpha \in (0, 1)$. So, T is compact.

Next we show that T is continuous. Let u_n, u be in S, and $u_n \to u$ in $C(\bar{\Omega})$. Define $w_n \equiv Tu_n$ and $w \equiv Tu$. We will show that w_n converges to w in $C(\bar{\Omega})$. Consider the difference of (12) at w_n and w:

$$\begin{aligned} &(w - w_n)_{x_1x_1} + \sum_{i=2}^{N} \frac{A_i(u_n)}{A_1(u_n)} (w - w_n)_{x_i x_i} + \lambda f(u) \left\{ \frac{1}{A_1(w)} - \frac{1}{A_1(w_n)} \right\} \\ &= \sum_{i=2}^{N} \left\{ \frac{A_i(u_n)}{A_1(u_n)} - \frac{A_i(u)}{A_1(u)} \right\} w_{x_i x_i} + \lambda \frac{1}{A_1(w_n)} \{ f(u_n) - f(u) \}. \end{aligned}$$

Hence there exists $h_i \in L^\infty(\Omega)$, $h \in L^\infty(\Omega)$, and positive $g \in L^\infty(\Omega)$ such that

$$\begin{aligned} &(w - w_n)_{x_1 x_1} + \sum_{i=2}^{N} \frac{A_i(u_n)}{A_1(u_n)} (w - w_n)_{x_i x_i} - \lambda f(u)\, g(x)(w - w_n) \\ &= \sum_{i=2}^{N} h_i(x)(u_n - u)\, w_{x_i x_i} + \lambda \frac{h(x)(u_n - u)}{A_1(w_n)}. \end{aligned}$$

Since the equation is uniform elliptic, we can apply the Aleksandrov Maximum Principle (Theorem 9.1 in [7]) to conclude

$$\begin{aligned}
\|w-w_n\|_{L^\infty} &\leqslant C\left\|\left\{\sum_{i=2}^{N} h_i(x)\, w_{x_i x_i}+\lambda\frac{h(x)}{A_1(w_n)}\right\}(u-u_n)\right\|_{L^N}\\
&\leqslant C\left\{\|w\|_{W^{2,N}}\sum_{i=2}^{N}\|h_i\|_{L^\infty}+\left\|\frac{h}{A_1(0)}\right\|_{L^N}\right\}\|u-u_n\|_{L^\infty}\\
&\leqslant C\,\|u-u_n\|_{L^\infty}.
\end{aligned}$$

Hence T is continuous. By the Schauder Fixed Point Theorem, T has a fixed point u in the set S, that is, $Tu=u\in S$. We can apply L^q theory to this u so that $u\in W^{2,q}(\Omega)$ for all $q>1$. Sobolev imbedding gives $u\in C^{1+\alpha}(\bar{\Omega})$. Since f is smooth, we apply the Schauder estimate to get $u\in C^{2+\alpha}(\bar{\Omega})$. This completes the proof of the Theorem 1. ▮

3. SUPERLINEAR CASE

Define $S=\{\lambda\in[0,\infty)$: (7) has a nonnegative solution in $C^2(\Omega)\cap C(\bar{\Omega})\}$. We note that the global Holder estimate (Corollary 9.29 of [7]) followed by the Schauder estimate ensure that a classical solution of (7) is automatically in $C^2(\bar{\Omega})$. We will show that S is a bounded interval of the form $[0,\lambda^*)$ or $[0,\lambda^*]$ for some $\lambda^*>0$.

3.1. *S Bounded*

LEMMA 2. *Let* (D), (C), (C2), (H), *and* (H2) *hold. If* λ *is sufficiently small then* $\lambda\in S$

Proof. Clearly, when $\lambda=0$, $u=0$ solves (7). So, $0\in S$. Take $\alpha\in(0,1)$, and define $X\equiv\{u\in C^{2+\alpha}(\bar{\Omega}) : u|_{\partial\Omega}=0\}$. Define F: $X\times\mathbb{R}\to C^\alpha(\bar{\Omega})$ such that for any $(u,\lambda)\in X\times\mathbb{R}$,

$$F(u,\lambda)\equiv\sum_{i=1}^{N} A_i(u)\frac{\partial^2 u}{\partial x_i^2}+\lambda f(u).$$

F is Frechet differentiable and for any $v\in X$,

$$F_u(0,0)\,v=\sum_{i=1}^{N} A_i(0)\frac{\partial^2 v}{\partial x_i^2}.$$

Hence $F_u(0,0)$ has a bounded inverse. Therefore by the implicit function theorem, there exists a $\lambda_0>0$ such that (7) has a solution u_λ for every

$\lambda \in [0, \lambda_0)$. Since $A_i(0) > 0$ and $f(0) > 0$, reducing λ_0 if necessary, we can assume $A_i(u_\lambda) > 0$ and $f(u_\lambda) > 0$. Hence

$$\sum_{i=1}^{N} A_i(u_\lambda) \frac{\partial^2 u_\lambda}{\partial x_i^2} = -\lambda f(u_\lambda)$$
$$< 0.$$

Thus u_λ is positive by the maximum principle. So, $\lambda \in S$ for all $0 \leqslant \lambda < \lambda_0$. ▮

LEMMA 3. *Let* (D), (C), (C2), (H), *and* (H2) *hold. Then S is bounded.*

Proof. Let $\lambda \in S$. As we remarked earlier, the corresponding solution u is in $C^2(\bar{\Omega})$. Thus

$$u_{x_1x_1} + \sum_{i=2}^{N} \frac{A_i(u)}{A_1(u)} u_{x_i x_i} + \lambda \frac{f(u)}{A_1(u)} = 0. \tag{13}$$

Since A_i/A_1 is nonincreasing,

$$\frac{A_i(u)}{A_1(u)} u_{x_i x_i} = \left(\frac{A_i(u)}{A_1(u)} u_{x_i}\right)_{x_i} - \frac{\partial}{\partial u}\left(\frac{A_i(u)}{A_1(u)}\right) u_{x_i}^2$$
$$\geqslant \left(\frac{A_i(u)}{A_1(u)} u_{x_i}\right)_{x_i}.$$

Moreover,

$$\max_t (A_i(t)/A_1(t)) = A_i(0)/A_1(0) \equiv M_i < \infty,$$

for all $2 \leqslant i \leqslant N$.

Let $\psi \in C^2(\bar{\Omega})$, $\psi \geqslant 0$, and $\psi|_{\partial\Omega} = 0$. On integration by parts on Eq. (13) twice, we have

$$0 = \int \left\{u_{x_1 x_1} + \sum_{i=2}^{N} \frac{A_i(u)}{A_1(u)} u_{x_i x_i} + \lambda \frac{f(u)}{A_1(u)}\right\} \psi$$
$$\geqslant \int \left\{u_{x_1 x_1} + \sum_{i=2}^{N} \left(\frac{A_i(u)\, u_{x_i}}{A_1(u)}\right)_{x_i} + \lambda \frac{f(u)}{A_1(u)}\right\} \psi$$
$$= \int u \left\{\psi_{x_1 x_1} + \sum_{i=2}^{N} \left(\frac{A_i(u)\, \psi_{x_i}}{A_1(u)}\right)_{x_i}\right\} + \int \lambda \frac{f(u)}{A_1(u)} \psi.$$

We now pick $\psi > 0$ such that it is the first eigenfunction satisfying

$$\begin{cases} \psi_{x_1 x_1} + \sum_{i=2}^{N} \left(\dfrac{A_i(u)\, \psi_{x_i}}{A_1(u)} \right)_{x_i} + \beta\psi = 0, \\ \psi|_{\partial\Omega} = 0, \end{cases} \tag{14}$$

with eigenvalue $\beta > 0$. Thus

$$\beta \equiv \min_{\psi \in H_0^1} \frac{\int \psi_{x_1^2} + \int \sum_{i=2}^{N} (A_i(u)/A_1(u))\, \psi_{x_i^2}}{\int \psi^2}$$

$$\leqslant \min_{\psi \in H_0^1} \frac{\int \psi_{x_1^2} + \sum_{i=2}^{N} M_i \int \psi_{x_1^2}}{\int \psi^2} \equiv \bar{\lambda} < \infty,$$

where $\bar{\lambda}$ is independent of u.

Because of condition (H2) and $f(0)/A_1(0) > 0$, we can find a $\delta > 0$ such that for $t \geqslant 0$,

$$\frac{f(t)}{A_1(t)} \geqslant \delta t.$$

Hence with these ψ and δ, the previous inequality becomes

$$0 \geqslant \int \{ -\beta + \lambda\delta \} u\psi.$$

Thus we get $\delta\lambda \leqslant \beta$, which leads to

$$\lambda \leqslant \frac{\bar{\lambda}}{\delta}.$$

The proof of the lemma is now complete. ∎

By this lemma, $\sup S \equiv \lambda^*$ exists, so that there is no solution for $\lambda > \lambda^*$.

3.2. *Local Existence of a Parabolic Problem*

Let $\bar{\lambda} \in S$ with a corresponding solution $\bar{u}$. We claim that for all λ where $\lambda \in (0, \bar{\lambda})$, λ is in the set S. We will establish this claim in the next two subsections.

Because of the technical difficulties in the elliptic problem, we will look at the problem via an associated parabolic problem and then later we will take the limit of the parabolic solution as $t \to \infty$, and show that this limit is the solution of the elliptic problem.

Given any $\Gamma > 0$, we define

- $Q_\Gamma \equiv \Omega \times [0, \Gamma]$
- $\partial Q_\Gamma \equiv \{\partial\Omega \times (0, \Gamma)\} \cup \{\Omega \times 0\}$, which is the parabolic boundary of Q_Γ.

LEMMA 4 (Local Existence). *Let assumptions* (D), (C), (C2), (H), *and* (H2) *hold. For* $\lambda \in [0, \bar{\lambda})$, *the problem*

$$\begin{cases} \dfrac{\partial v}{\partial t} = \sum_{i=1}^{N} A_i(v) \dfrac{\partial^2 v}{\partial x_i^2} + \lambda f(v), \\ v|_{\partial Q_\Gamma} = 0, \end{cases} \tag{15}$$

has a unique solution $v \in C^{2+\alpha, (2+\alpha)/2}(Q_\Gamma) \cap C^{\alpha, \alpha/2}(\bar{Q}_\Gamma)$, *where* $0 < \alpha < 1$ *for some* $\Gamma > 0$.

Proof. Take any $\Gamma \in (0, 1]$, and define a set

$$S = \{v \in C^{\alpha, \alpha/2}(\bar{Q}_\Gamma) : \|v\|_{C^{\alpha, \alpha/2}(\bar{Q}_\Gamma)} \leqslant 1, v|_{\partial Q_\Gamma} = 0\}$$

for some $0 < \alpha < 1$. Define

$$T: S \to S$$

such that for any $v \in S$, Tv satisfies

$$\frac{\partial Tv}{\partial t} = \sum_{i=1}^{N} A_i(v) \frac{\partial^2 Tv}{\partial x_i^2} + \lambda f(v).$$

Since the set S is bounded, and for $i = 1, ..., N$, A_i's are positive and continuous, so there exist positive constants θ and Θ such that for each $i = 1, ..., N$ and for any $v \in S$

$$0 < \theta \leqslant A_i(v) \leqslant \Theta < \infty.$$

Hence we have a uniform bound on the ellipticity constants for the above equation.

Since the equation is linear with zero boundary and initial conditions, so there exists a unique solution $Tv \in W_q^{2,1}(Q_\Gamma)$ for any $q > 1$ (see Theorem 9.1 of [12]) with $Tv|_{\partial Q_\Gamma} = 0$.

Because $\lambda f(v)$ has a uniform L^∞ bound and the equation has uniform modulus of continuity of $A_i(v)$ for all $v \in S$, standard L^q estimate (Theorem 9.1 of [12]) on the above equation yields

$$\|Tv\|_{W_q^{2,1}} \leqslant C \|\lambda f(v)\|_{L^q} \leqslant M_1, \tag{16}$$

where M_1 is independent of $\Gamma \in (0, 1]$. Choosing sufficiently large q, we get $Tv \in C(\bar{Q}_\Gamma)$, and

$$\|Tv\|_{C^{1+\alpha,(1+\alpha)/2}(\bar{Q}_\Gamma)} \leqslant M, \tag{17}$$

as a result of embedding (Lemma 3.3 of [12]). Furthermore the interior regularity bootstrap ensures that the solution Tv is in $C^2(Q_\Gamma) \cap C(\bar{Q}_\Gamma)$. It is noted that (17) holds uniformly for $\Gamma \in (0, 1]$ with the same M.

Since $Tv = 0$ at $t = 0$, by choosing Γ small enough, we have

$$\begin{aligned}\|Tv\|_{L^\infty(\bar{Q}_\Gamma)} &\leqslant t^{(1+\alpha)/2} \|Tv\|_{C^{1+\alpha,(1+\alpha)/2}(\bar{Q}_\Gamma)} \\ &\leqslant \Gamma^{(1+\alpha)/2} M \\ &\leqslant 1/4.\end{aligned}$$

For $0 \leqslant t_1 \leqslant t_2 \leqslant \Gamma$,

$$\begin{aligned}\frac{|Tv(x, t_2) - Tv(x, t_1)|}{|t_2 - t_1|^{\alpha/2}} &\leqslant |t_2 - t_1|^{1/2} \|Tv\|_{C^{1+\alpha,(1+\alpha)/2}(\bar{Q}_\Gamma)} \\ &\leqslant \Gamma^{1/2} M \\ &\leqslant 1/4.\end{aligned}$$

In addition, since $\partial Tv/\partial x \in C^{\alpha,\alpha/2}$, for any x_1, x_2 in Ω,

$$\begin{aligned}\frac{|Tv(x_2, t) - Tv(x_1, t)|}{|x_2 - x_1|^{\alpha}} &\leqslant \left\|\frac{\partial Tv}{\partial x}\right\|_{L^\infty(\bar{Q}_\Gamma)} |x_2 - x_1|^{1-\alpha} \\ &\leqslant t^{\alpha/2} \|Tv\|_{C^{1+\alpha,(1+\alpha)/2}(\bar{Q}_\Gamma)} |x_2 - x_1|^{1-\alpha} \\ &\leqslant C\Gamma^{\alpha/2} M \\ &\leqslant 1/4.\end{aligned}$$

Therefore, T maps S into S for sufficiently small Γ.

Next we will show that T is a contraction map. Let v and w be in S. If we look at the difference between the two equations,

$$\begin{aligned}\frac{\partial(Tw - Tv)}{\partial t} = {} & \sum_{i=1}^{N} A_i(w) \frac{\partial^2(Tw - Tv)}{\partial x_i^2} \\ & + \sum_{i=1}^{N} \{A_i(w) - A_i(v)\} \frac{\partial^2 Tv}{\partial x_i^2} + \lambda \{f(w) - f(v)\}.\end{aligned}$$

With A_i's and f are in C^1, there exist $h_i \in L^\infty$ and $g \in L^\infty$ such that

$$A_i(w) - A_i(v) = h_i(x)(w - v),$$

and

$$\lambda\{f(w)-f(v)\}=\lambda g(x)(w-v).$$

Hence the difference becomes

$$\frac{\partial(Tw-Tv)}{\partial t}$$
$$=\sum_{i=1}^{N} A_i(w)\frac{\partial^2(Tw-Tv)}{\partial x_i^2}+\sum_{i=1}^{N} h_i(x)(w-v)\frac{\partial^2 Tv}{\partial x_i^2}+\lambda g(x)(w-v),$$

and

$$(Tw-Tv)|_{\partial Q_\Gamma}=0.$$

It is noted that $A_i(w)$ has a uniform Holder norm bound for $w\in S$. Therefore for any $q>1$, we can apply L^q estimate,

$$\begin{aligned}
\|Tw-Tv\|_{W_q^{2,1}(Q_\Gamma)} &\leqslant C\left\|(w-v)\left\{\sum_{i=1}^{N} h_i(x)\,Tv_{x_ix_i}+\lambda g(x)\right\}\right\|_{L^q(Q_\Gamma)}\\
&\leqslant C\left\{\sum_{i=1}^{N}\|h_i\|_{L^\infty(Q_\Gamma)}\|Tv\|_{W_q^{2,1}(Q_\Gamma)}+\lambda\,\|g\|_{L^q(Q_\Gamma)}\right\}\|w-v\|_{L^\infty(Q_\Gamma)}\\
&\leqslant C\,\|w-v\|_{L^\infty(Q_\Gamma)}
\end{aligned}$$

using (16), where C is independent of $\Gamma\in(0,1]$. Thus, choosing q large enough, using Lemma 3.3 in [12] with $\delta=\sqrt{\Gamma}$,

$$\begin{aligned}
\|Tw-Tv\|_{C^{\alpha,\alpha/2}(\bar{Q}_\Gamma)} \leqslant{}& C\sqrt{\Gamma}^{2-\alpha-(n+2)/q}\,\|Tw-Tv\|_{W_q^{2,1}(Q_\Gamma)}\\
&+C\sqrt{\Gamma}^{-\alpha-(n+2)/q}\,\|Tw-Tv\|_{L_q(Q_\Gamma)}.
\end{aligned}$$

Since $Tw=Tv=0$ at $t=0$,

$$\begin{aligned}
|Tw-Tv| &\leqslant \int_0^t\left|\frac{\partial}{\partial\tau}(Tw-Tv)(x,\tau)\right|d\tau\\
&\leqslant\left(\int_0^\Gamma\left|\frac{\partial}{\partial\tau}(Tw-Tv)\right|^q d\tau\right)^{1/q}\Gamma^{1-1/q}.
\end{aligned}$$

Thus,

$$\int_0^\Gamma \int_\Omega |Tw - Tv|^q \, dx\, dt \leqslant \left(\int_\Omega dx \int_0^\Gamma dt \int_0^\Gamma \left| \frac{\partial}{\partial \tau} (Tw - Tv) \right|^q d\tau \right) \Gamma^{q-1}$$
$$\leqslant C\Gamma^q \| Tw - Tv \|^q_{W_q^{2,1}(Q_\Gamma)}.$$

Since $\Gamma \leqslant 1$, and by the above L^q estimate, the inequality becomes

$$\begin{aligned}\| Tw - Tv \|_{C^{\alpha, \alpha/2}(\bar{Q}_\Gamma)} &\leqslant C\Gamma^{1-\alpha/2-(n+2)/2q} \| Tw - Tv \|_{W_q^{2,1}(Q_\Gamma)} \\ &\quad + C\Gamma^{1-\alpha/2-(n+2)/2q} \| Tw - Tv \|_{W_q^{2,1}(Q_\Gamma)} \\ &\leqslant C \| Tw - Tv \|_{W_q^{2,1}(Q_\Gamma)} \\ &\leqslant C \| w - v \|_{L^\infty(Q_\Gamma)} \\ &\leqslant C\Gamma^{\alpha/2} \| w - v \|_{C^{\alpha, \alpha/2}(\bar{Q}_\Gamma)}.\end{aligned}$$

By choosing Γ small enough, T is a contraction map. Thus there exists a unique $v \in S$ satisfying $Tv = v$. Hence, $v \in C^{\alpha, \alpha/2}(\bar{Q}_\Gamma)$. We can bootstrap to get a classical solution, and then apply the interior Schauder estimate to get $v \in C^{2+\alpha, (2+\alpha)/2}(Q_\Gamma)$. Hence v solves (15). ▌

3.3. *Proof of Theorem* 2

We have proved the existence of the classical solution in the parabolic problem for a small time interval. In this section, we will finish the proof of Theorem 2. This is achieved by showing the global existence of a solution in (15). Then as $t \to \infty$, we will establish that the parabolic solution settles down to a steady state solution of (7).

Proof of Theorem 2. Let $\bar{\lambda} \in S$ with a corresponding solution $\bar{u}$. If we can show that for all $\lambda \in (0, \bar{\lambda})$, λ is in the set S, then we finish the proof of the Theorem. Let $v(x, t)$ be the solution of (15) for $\lambda \in (0, \bar{\lambda})$. Thus, $\bar{u}$ is an upper solution for the parabolic problem:

$$\sum_{i=1}^N A_i(\bar{u}) \frac{\partial^2 \bar{u}}{\partial x_i^2} + \lambda f(\bar{u}) < 0.$$

So,

$$\begin{aligned}\frac{\partial (v - \bar{u})}{\partial t} &> \sum_{i=1}^N A_i(v) \frac{\partial^2 (v - \bar{u})}{\partial x_i^2} + \sum_{i=1}^N \{A_i(v) - A_i(\bar{u})\} \frac{\partial^2 \bar{u}}{\partial x_i^2} + \lambda \{ f(v) - f(\bar{u})\} \\ &= \sum_{i=1}^N A_i(v) \frac{\partial^2 (v - \bar{u})}{\partial x_i^2} + \left\{ \sum_{i=1}^N h_i(x) \frac{\partial^2 \bar{u}}{\partial x_i^2} + \lambda g(x) \right\} (v - \bar{u}),\end{aligned}$$

with

$$(v-\bar{u})|_{\partial Q_\Gamma} \leqslant 0.$$

Since h_i, g, and $\partial^2\bar{u}/\partial x_i^2$ are in L^∞ for all $1 \leqslant i \leqslant N$, and both v and $\bar{u}$ are in $C^2(Q_\Gamma) \cap C(\bar{Q}_\Gamma)$, we can employ the classical maximum principle to conclude

$$v - \bar{u} < 0 \qquad \text{on } Q_\Gamma.$$

By a similar argument, $v \geqslant 0$. Thus we have an L^∞ bound on v, independent of Γ. Hence $\lambda f(v) \in L^\infty$ and Eq. (15) is uniform elliptic. By the global Holder estimate due to Krylov and Safonov [11], and Ladyzenskaja and Ural'ceva [13], there exists an $\alpha \in (0, 1)$ such that $v \in C^{\alpha, \alpha/2}(\bar{Q}_\Gamma)$ with a quantitive bound independent of Γ. For any fixed $0 < \delta < \Gamma$, bootstrap using the Schauder estimate to get $v \in C^{2+\alpha, 1+\alpha/2}(\bar{Q}_\Gamma \backslash \bar{Q}_\delta)$ with bounds independent of Γ. Therefore the solution must exist globally in time.

We now show that $\partial v(x, t)/\partial t \geqslant 0$. Let $v^h(t) \equiv v(x, t+h)$, so that

$$\frac{\partial v^h}{\partial t} = \sum_{i=1}^{N} A_i(v^h) \frac{\partial^2 v^h}{\partial x_i^2} + \lambda f(v^h),$$

and

$$\frac{\partial v}{\partial t} = \sum_{i=1}^{N} A_i(v) \frac{\partial^2 v}{\partial x_i^2} + \lambda f(v).$$

For fixed $h > 0$, we consider the difference between these two equations,

$$\frac{\partial (v^h - v)}{\partial t} = \sum_{i=1}^{N} A_i(v) \frac{\partial^2 (v^h - v)}{\partial x_i^2} + \left\{ \sum_{i=1}^{N} h_i(x) \frac{\partial^2 v^h}{\partial x_i^2} + \lambda g(x) \right\} (v^h - v),$$

for some h_i and g in L^∞, and $\partial^2 v^h/\partial x_i^2$ is in L^∞ for all $1 \leqslant i \leqslant N$. Define

$$w_h(x, t) \equiv \frac{v^h - v}{h},$$

which satisfies the zero boundary condition, with initial condition

$$w_h(x, 0) = \frac{v^h(x, 0) - v(x, 0)}{h} = \frac{v^h(x, 0)}{h} > 0.$$

The maximum principle then gives

$$w_h(x, t) > 0,$$

and hence for all $x \in \bar{\Omega}$ and t,

$$\frac{\partial v(x, t)}{\partial t} \geqslant 0.$$

Therefore, by the monotonicity of v and $v < \bar{u}$ for all t, we have

$$\lim_{t \to \infty} v(x, t) \equiv u_\lambda(x)$$

in a pointwise sense for all $x \in \Omega$. Since $v \in C^{2+\alpha, (2+\alpha)/2}(\bar{\Omega} \times [\delta, \infty))$ with a uniform $C^{2+\alpha, (2+\alpha)/2}$ norm bound independent of t, hence, there exists a subsequence $\{v(x, t_n)\}$ that converges in $C^2(\bar{Q})$. But v converges to $u_\lambda(x)$ pointwise. So by the uniqueness of this limit, $v(x, t)$ must converge to $u_\lambda(x)$ in $C^2(\bar{Q})$, not only in a subsequence t_n but as $t \to \infty$.

With this convergency, it is easy to check

$$\lim_{T \to \infty} \frac{1}{T} \int_\delta^T v(x, t)\, dt = u_\lambda(x),$$

and

$$\lim_{T \to \infty} \frac{1}{T} \int_\delta^T f(v(x, t))\, dt = f(u_\lambda(x)),$$

for all $x \in \Omega$.

Finally, we need to show that $u_\lambda(x)$ solves our elliptic problem. By the convergency in $C^2(\bar{\Omega})$, for each $x \in \Omega$, and $1 \leqslant i \leqslant N$,

$$\lim_{T \to \infty} \frac{1}{T} \int_\delta^T A_i(v(x, t)) \frac{\partial^2 v(x, t)}{\partial x_i^2}\, dt = A_i(u_\lambda(x)) \frac{\partial^2 u_\lambda(x)}{\partial x_i^2}.$$

Moreover, for all $x \in \bar{\Omega}$,

$$\lim_{T \to \infty} \frac{1}{T} \int_\delta^T \frac{\partial v(x, t)}{\partial t}\, dt = \lim_{T \to \infty} \frac{v(x, T) - v(x, \delta)}{T} = \lim_{T \to \infty} \frac{v(x, T)}{T} = 0,$$

since $v(x, t) \leqslant \bar{u}(x)$ for all t.

Integrating Eq. (15) from 0 to T, we divide the resulting equation by T and take limit as $T \to \infty$, we have now constructed a solution u_λ in $C^{2+\alpha}(\bar{\Omega})$ which solves Eq. (7). ∎

4. SOME NUMERICAL RESULT AND OPEN QUESTIONS

We have carried out numerical calculations for our quasilinear anisotropic elliptic problem in a square domain $\Omega = [0, 1] \times [0, 1] \subset R^2$ with zero boundary condition. Starting with a known trivial solution when $\lambda = 0$, we find a near-by solution when λ is small using Newton's Method. Then we applied the continuation algorithm to trace the bifurcation diagram. The continuation algorithm allows us to trace the solution curve even when there is a turning point in the bifurcation diagram. Details of such numerical methods can be found in [8]. A brief outline is included below.

In solving equation $F(u, \lambda) \equiv A_1(u)\, u_{xx} + A_2(u)\, u_{yy} + \lambda f(u) = 0$ using Newton's Method, we find $u = u(\lambda)$ such that $F(u(\lambda), \lambda) = 0$. However near a turning point in a bifurcation curve, this is not a valid assumption. The continuation algorithm views u and λ as functions of arclength s along the bifurcation curve. One more equation which describes the definition of arclength is added and this will be solved together with $F(u, \lambda) = 0$ for the unknowns $u(s)$ and $\lambda(s)$. In all calculations below, we use a mesh size of $h = 1/20$ in either x or y direction, and an arclength increment of $\Delta s = 0.1$ in each continuation step. When we double the mesh size and halve the arclength increment in each step of the continuation algorithm, the bifurcation diagram remains essentially the same (less than 0.5% shift in actual numerical values).

For the sublinear case, a typical result is shown in Fig. 1. In particular, there is a solution for any $\lambda > 0$. The bifurcation diagram clearly shows that for each λ, there is a unique solution. Such uniqueness result remains open.

For the superlinear case, the situation is more complicated as we have non-existence of a solution for large λ. Examples are depicted in Figs. 2 and 3. Both bifurcation diagrams show a turning point at some $\lambda = \lambda^*$. Using $\Delta u + f(u) = 0$ as guidance, this is in accordance with the expectation of subcritical growth of nonlinearity f near infinity.

Showing the existence of a solution when $\lambda = \lambda^*$, and multiple solutions when $\lambda < \lambda^*$, are still open. Also in Fig. 2, we are not sure whether there are two solutions for small λ. If there is only one solution, it is different from the semilinear problem. This will be an interesting question.

For supercritical nonlinearity, the numerical result seems to suggest (see Fig. 4) that there is one solution when $\lambda \leqslant \lambda^*$. As $\lambda \to \lambda^*$, the solution blows up. (The blow up prevents the accurate implementation of numerics.)

Finally, one would like to study a singular anisotropic quasilinear problem with nonlinear source term

$$\begin{cases} \sum_{i=1}^{N} u^{a_i} u_{x_i x_i} + \lambda f(u, x) = 0, & x \in \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \tag{18}$$

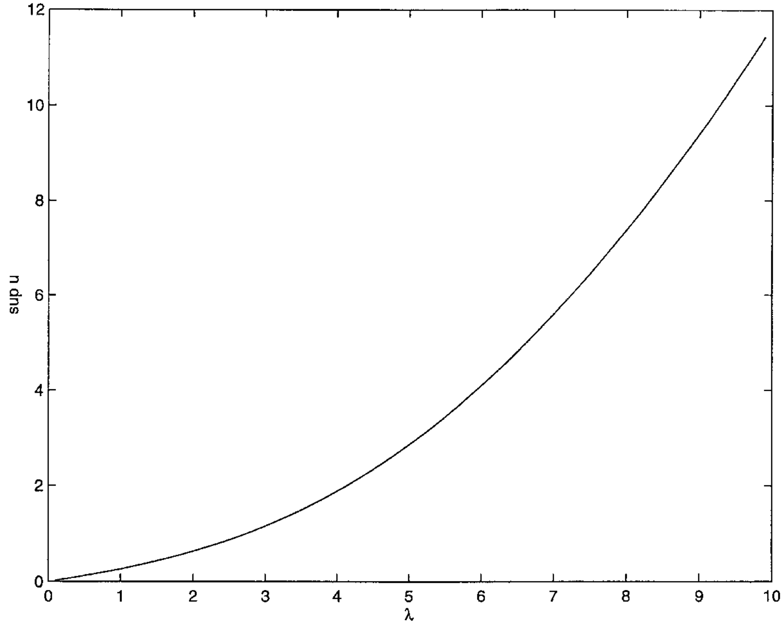

FIG. 1. $(u+1)\,u_{xx}+u_{yy}+\lambda(u+1)^{1.5}=0.$

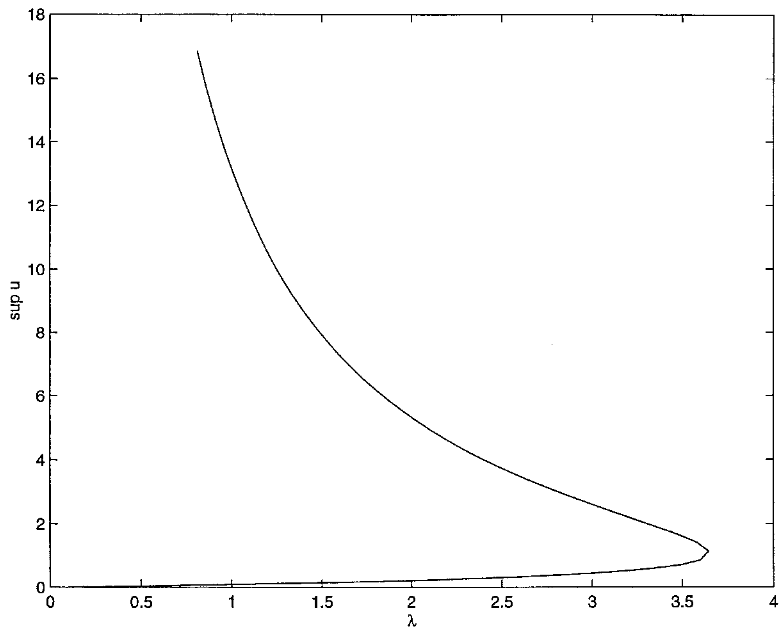

FIG. 2. $(u+1)\,u_{xx}+u_{yy}+\lambda(u+1)^{3}=0.$

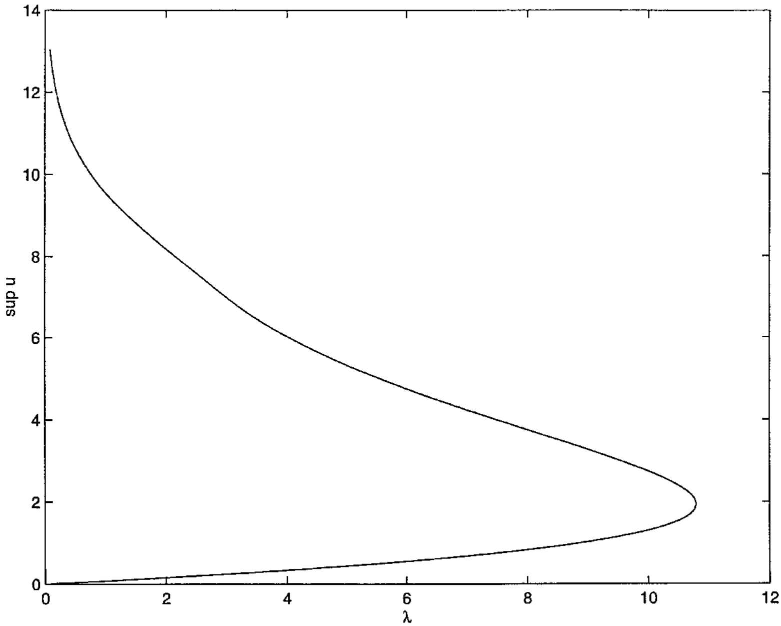

FIG. 3. $(u+1)\,u_{xx} + u_{yy} + \lambda\exp(u) = 0.$

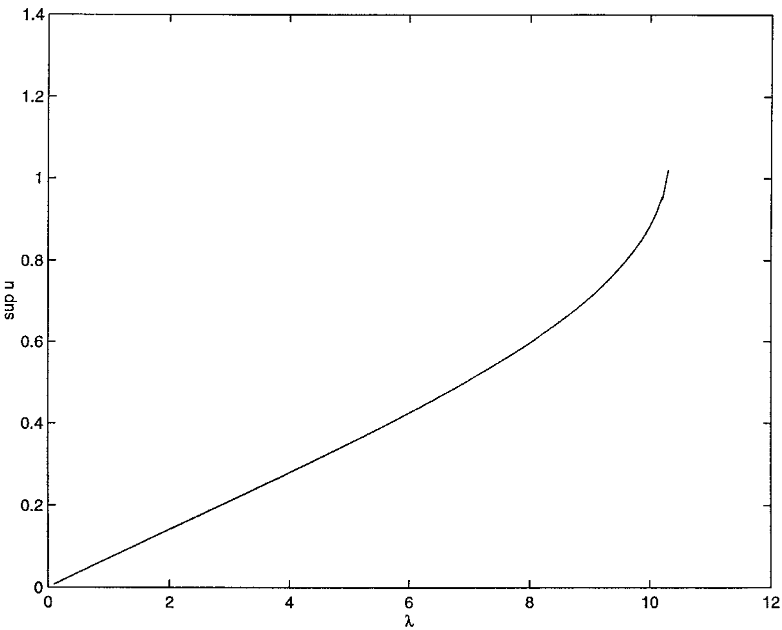

FIG. 4. $(u+1)\,u_{xx} + u_{yy} + \lambda\exp(u^2) = 0.$

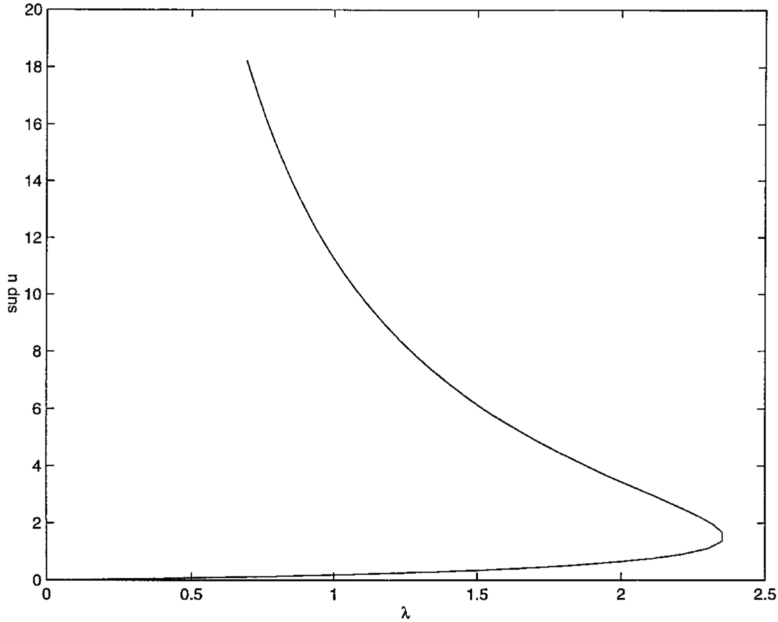

FIG. 5. $uu_{xx} + u_{yy} + \lambda(u+1)^3 = 0.$

The numerical result is similar to that in the nondegenerate case (see Fig. 5).

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