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Positive Almost Periodic Solutions of a Nonlinear Integral Equation from the Theory of Epidemics

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A generalization of a nonlinear integral equation governing the spread of certain infectious diseases with almost periodic contact rate is shown to have almost periodic positive solutions. © 1991 Academic Press, Inc.

1. INTRODUCTION

In the study of periodic outbreaks of infectious diseases such as chicken-pox, mumps, and measles [1, 7], researchers have been lead to consider the nonlinear integral equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds, \tag{1.1}$$

in which, $x(t)$ is the proportion of infectious individuals present in the population at time t, τ is the length of time an individual remains infective, and $f(t, x(t))$ is the proportion of new infective individuals per unit of time.

Since its introduction (1.1) has been studied by various authors. Using degree theoretical arguments in cones, they have been able to conclude the existence of positive solutions for sufficiently large values of the delay τ. Nussbaum [5, 6, 7] and Smith [8] in particular have demonstrated the bifurcation of positive periodic solutions from the zero solution for sufficiently large τ, provided f is periodic in t. Fink and Gatica [2] have shown the existence of nontrivial almost periodic solutions, by assuming that f is a suitable almost periodic function and τ is large.

In this note we shall simplify and extend a result of Fink and Gatica [2],

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regarding the existence of positive almost periodic solutions for (1.1) to include the more general equation

$$x(t)=\int_{t-\tau(t)}^{t} f(s, x(s))\, ds. \tag{1.2}$$

Unlike the case in which $f(\cdot, x)$ is ω-periodic, and $\tau(\cdot)$ is λ-periodic, with ω/λ a rational number, the operator

$$A[x](t) \triangleq \int_{t-\tau(t)}^{t} f(s, x(s))\, ds \tag{1.3}$$

does not have the compactness necessary for a degree theoretical argument to apply. Thus, an adaptation of a devise developed by Fink and Gatica [3] to handle (1.1) must be employed; in the process we obtain slightly more general results than they did.

2. Preliminary Results, Assumptions, and Notation

Throughout this work $\mathbb{R}$ denotes the real line. $\mathscr{C}(\mathbb{R})$ denotes the Banach space of continuous bounded functions $f\colon \mathbb{R} \to \mathbb{R}$ with the norm

$$\|f\| = \sup_{t \in \mathbb{R}} |f(t)|.$$

Recall that a function $f \in \mathscr{C}(\mathbb{R})$ is *almost periodic* if the *hull* of f,

$$H(f) \triangleq \{f_\alpha : f_\alpha(t) = f(t+\alpha),\ \alpha \in \mathbb{R}\},$$

is relatively compact in $\mathscr{C}(\mathbb{R})$. An equivalent form is that from every sequence $\alpha = \{\alpha_n\}$ one can extract a subsequence $\{\alpha_{n'}\}$ for which the limit

$$T_\alpha f \triangleq \lim_{n' \to \infty} f(t+\alpha_{n'})$$

exists uniformly on $\mathbb{R}$. If $f(t)$ is almost periodic then f has a Fourier series,

$$f(t) \sim \sum_{n=1}^{\infty} a_n e^{-i\lambda_n t},$$

where $\{\lambda_n\}$ is a countable set of λ's for which

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(s)\, e^{-i\lambda s}\, ds \neq 0.$$

The a_n's are determined by the formula

$$a_n = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(s)\, e^{-i\lambda_n s}\, ds.$$

The *module* of f, $\operatorname{mod}(f)$, is the smallest subgroup of $(\mathbb{R}, +)$ containing the set $\{\lambda_n\}$. When working with the space of almost periodic functions, $\mathrm{AP}(\mathbb{R})$, theorems such as Arzela-Ascoli's or Dini's, fail to provide information about uniform convergence on all of $\mathbb{R}$. The following result will provide us with a useful characterization of almost periodicity.

LEMMA 2.1 [2]. *A function $f \in AP(\mathbb{R})$ if and only if from every pair of sequences α and β we can extract common subsequences $\alpha' \subset \alpha$ and $\beta' \subset \beta$ for which*

$$T_{\alpha' + \beta'} f = T_{\alpha'} T_{\beta'} f \tag{2.1}$$

pointwise.

LEMMA 2.2. *If $\tau, x \in AP(\mathbb{R})$ and*

$$y(t) \triangleq \int_{t-\tau(t)}^{t} x(s)\, ds, \qquad \forall t \in \mathbb{R}$$

then $y \in AP(\mathbb{R})$.

Proof. Let $\{y(t+\alpha_n)\}$, $\{x(t+\alpha_n)\}$ and $\{\tau(t+\alpha_n)\}$ be sequences of translates corresponding, respectively, to $y(t)$, $x(t)$, and $\tau(t)$. Since $x(t)$ and $\tau(t)$ are almost periodic, there exists a common subsequence $\{\alpha'_n\} \subset \{\alpha_n\}$ such that

$$x(t+\alpha_{n'}) \to z(t) \qquad \text{and} \qquad \tau(t+\alpha_{n'}) \to \beta(t)$$

uniformly on $\mathbb{R}$. Then

$$\begin{aligned}
&\left| y(t+\alpha_{n'}) - \int_{t-\beta(t)}^{t} z(s)\, ds \right| \\
&\quad = \left| \int_{t-\tau(t+\alpha_{n'})}^{t} x(s+\alpha_{n'})\, ds - \int_{t-\beta(t)}^{t} z(s)\, ds \right| \\
&\quad \leqslant \left| \int_{t-\tau(t+\alpha_{n'})}^{t} [x(s+\alpha_{n'}) - z(s)]\, ds \right| + \left| \int_{t-\tau(t+\alpha_{n'})}^{t-\beta(t)} z(s)\, ds \right| \\
&\quad \leqslant \|x(\cdot+\alpha_{n'}) - z(\cdot)\|\, \|\tau\| + \|z(\cdot)\|\, \|\tau(\cdot+\alpha_{n'}) - \beta(\cdot)\|.
\end{aligned}$$

Hence

$$y(t+\alpha_{n'}) \to \int_{t-\beta(t)}^{t} z(s)\, ds$$

uniformly on $\mathbb{R}$. This shows that the hull of y is compact in the uniform topology of $\mathscr{C}(\mathbb{R})$ and y is almost periodic. ∎

LEMMA 2.3. *If $\tau, b \in AP(\mathbb{R})$ and α is a given sequence, then*

$$\sup_{t\in\mathbb{R}} \int_{t-\tau(t)}^{t} b(s)\, ds < 1 \Rightarrow \sup_{t\in\mathbb{R}} \int_{t-T_\alpha \tau(t)}^{t} T_\alpha b(s)\, ds < 1. \tag{2.2}$$

Proof. After extracting a subsequence of α, if necessary, we may assume that

$$\tau(t+\alpha_n) \to T_\alpha \tau(t) \qquad \text{and} \qquad b(t+\alpha_n) \to T_\alpha b(t),$$

uniformly on $\mathbb{R}$. If $\varepsilon > 0$ is such that

$$\int_{t-\tau(t)}^{t} b(s)\, ds + \varepsilon < 1,$$

then

$$\int_{t-\tau(t+\alpha_n)}^{t} b(s+\alpha_n)\, ds < 1 - \varepsilon, \qquad n = 1, 2, \ldots .$$

By the proof of Lemma 2.2

$$\int_{t-\tau(t+\alpha_n)}^{t} b(s+\alpha_n)\, ds \to \int_{t-T_\alpha\tau(t)}^{t} T_\alpha b(s)\, ds,$$

and the lemma is proved. ∎

DEFINITION 2.4. A bounded function $a: \mathbb{R} \to \mathbb{R}^+$ is said to be of the class $\mathscr{J}$ provided:

(j) $a(t) \leqslant a(s)$ if $-\infty < t \leqslant s \leqslant 0$; and

(jj) $a(t) \geqslant a(s)$ if $0 < t \leqslant s < +\infty$.

A bounded function $b: \mathbb{R} \to \mathbb{R}^+$ is of the class $\mathscr{K}$ if

(k) $b(t) \geqslant b(s)$ if $-\infty < t \leqslant s \leqslant 0$; and

(kk) $b(t) \leqslant b(s)$ if $0 < t \leqslant s < +\infty$.

3. Existence of Solutions

Let $f\colon \mathbb{R}\times\mathbb{R}^+ \to \mathbb{R}^+$ and $\tau\colon \mathbb{R}\to\mathbb{R}^+$. Throughout this paper, f and τ are assumed to satisfy one (or more) of the following conditions:

H$_1$: $f(\cdot, x)$ is almost periodic uniformly for x in compact subsets of $\mathbb{R}$ and $\tau(\cdot)$ is almost periodic on $\mathbb{R}$;

H$_2$: There exists an almost periodic function $a_\infty : \mathbb{R}\to\mathbb{R}^+$,

$$\sup_{t\in\mathbb{R}} \int_{t-\tau(t)}^{t} a_\infty(s)\, ds < 1 \tag{3.1}$$

such that

$$\limsup_{x\to+\infty} \frac{f(t,x)}{x} \leqslant a_\infty(t) \tag{3.2}$$

uniformly for $t\in\mathbb{R}$; and

H$_3$: $f(t,\cdot)$ is continuously differentiable with $f_x(t,\eta)$ uniformly continuous for $(t,\eta)\in\mathbb{R}\times[0,\eta_0]$ for some $\eta_0>0$, $f_x(t,\cdot)\geqslant 0$ with $f(t,0)\equiv 0$ and

$$\inf_{t\in\mathbb{R}} \int_{t-\tau(t)}^{t} f_x(s,0)\, ds > 1. \tag{3.3}$$

It is well known that if f satisfies **H**$_1$ and possesses the smoothness required by **H**$_3$, then $f_x(s,0)$ is almost periodic and it has positive mean value [2]; that is to say

$$\lim_{\tau\to+\infty} \frac{1}{\tau}\int_{t-\tau}^{t} f_x(s,0)\, ds \triangleq \mathbf{M}\{f_x\} > 0 \tag{3.4}$$

uniformly in t. Hence,

$$\inf_{t\in\mathbb{R}} \int_{t-\tau(t)}^{t} f_x(s,0)\, ds > 1 \tag{3.5}$$

provided

$$\tau(t)\geqslant\tau_0, \qquad \forall t\in\mathbb{R}$$

for some τ_0 sufficiently large.

A close inspection of assumptions **H**$_3$ shows that the delay term, $\tau(t)$, in

(1.2) is bounded away from zero; indeed, one sees that (3.3) forces τ to satisfy

$$\tau(t) > \frac{1}{\max_{s \in \mathbb{R}} f_x(s, 0)}. \tag{3.6}$$

In addition to $\mathbf{H}_1$, $\mathbf{H}_2$, and $\mathbf{H}_3$, we shall also need the following

$\mathbf{H}_1^*$: f is continuous on $\mathbb{R} \times \mathbb{R}^+$; τ is continuous on $\mathbb{R}$ and there exist positive numbers ω and λ such that $f(t+\omega, x) = f(t, x)$ and $\tau(t+\lambda) = \tau(t)$ for all $t \in \mathbb{R}$.

$\mathbf{H}_2^*$: There exists a continuous class $\mathscr{J}$ function $a_\infty : \mathbb{R} \to \mathbb{R}^+$,

$$\sup_{t \in \mathbb{R}} \int_{t-\tau(t)}^{t} a_\infty(s)\, ds < 1 \tag{3.1$'$}$$

such that

$$\limsup_{x \to +\infty} \frac{f(t, x)}{x} \leqslant a_\infty(t) \tag{3.2$'$}$$

uniformly for $t \in \mathbb{R}$.

$\mathbf{H}_2^\bullet$:

$$\lim_{x \to +\infty} \frac{f(t, x)}{x} = 0 \tag{3.2$''$}$$

uniformly for $t \in \mathbb{R}$.

$\mathbf{H}_3^*$: $f(t, \cdot)$ is continuously differentiable with $f_x(t, \eta)$ uniformly continuous for $(t, \eta) \in \mathbb{R} \times [0, \eta_0]$ for some $\eta_0 > 0$, and $f(t, 0) \equiv 0$.

Remark 3.1. It should be pointed out that our condition $\mathbf{H}_2$ is weaker than Fink and Gatica's condition $\mathbf{H}_2$ [3].

We are now in position to state the main results.

THEOREM 1. *Let f and τ satisfy $\mathbf{H}_1$, $\mathbf{H}_2$, and $\mathbf{H}_3$, then* (1.2) *has an almost periodic solution $\bar{x}_{f,\tau}$, such that*

$$\inf_{t \in \mathbb{R}} \bar{x}_{f,\tau}(t) > 0. \tag{3.7}$$

Furthermore, if $\operatorname{mod}(\tau) \subset \operatorname{mod}(f)$, *then*

$$\operatorname{mod}(\bar{x}_{f,\tau}) \subset \operatorname{mod}(f). \tag{3.8}$$

THEOREM 2. *Let f and τ satisfy* $\mathbf{H}_1$ *and* $\mathbf{H}_3^*$. *Then there exists $\tau^* > 0$ for which Eq.* (1.2) *does not have continuous nonnegative solutions of arbitrarily small norm, other than $x(t) \equiv 0$, if*

$$\sup_{t \in \mathbb{R}} \tau(t) < \tau^*.$$

THEOREM 3. *Let f and τ satisfy* $\mathbf{H}_1^*$ *and* $\mathbf{H}_2^*$. *If for some $a > 0$ and a continuous class $\mathscr{K}$ function $b(t)$ for which*

$$\inf_{t \in \mathbb{R}} \int_{t-\tau(t)}^{t} b(s)\, ds > a, \qquad \forall t \in \mathbb{R} \tag{3.9}$$

f is such that

$$f(t, x) \geqslant b(t), \qquad \forall t \in \mathbb{R},\ x \geqslant a \tag{3.10}$$

then (1.2) *has a nontrivial continuous solution, $x_{f,\tau}$, whose infimum satisfies*

$$\inf_{t \in \mathbb{R}} x_{f,\tau}(t) \geqslant a.$$

When $\tau(t) \equiv \tau_0$, and $b(t)$ is periodic, more may be concluded about the solutions of Eq. (1.1).

THEOREM 4. *If, in addition to* $\mathbf{H}_2^{*}$, *f is ω-periodic and satisfies*

$\mathbf{H}_6$: *There exists $a > 0$ and a continuous periodic function $b\colon \mathbb{R} \to \mathbb{R}^+$, $b(t) \not\equiv 0$, for which*

$$f(t, x) \geqslant b(t), \qquad \forall t \in \mathbb{R},\ x \geqslant a, \tag{3.11}$$

then there exists $m > 0$ such that for each $\tau \geqslant m$ Eq. (1.1) *has a nontrivial ω-periodic solution $x(t)$ whose infimum satisfies*

$$\inf_{t \in \mathbb{R}} x(t) \geqslant a.$$

Remark 3.2. Unlike Theorem 1, where the monotonicity of f in its second variable provides the necessary tools to conclude the almost periodicity of maximal solutions of (1.2) (see [3]), we have not been able to conclude the almost periodicity of solutions under the assumptions of Theorem 3.

4. Proof of Theorem 1

The proof of Theorem 1 is modeled after Fink and Gatica's proof of a similar result. Let

$$\mathscr{P} \triangleq \{x \in \mathrm{AP}(\mathbb{R}) : x(t) \geqslant 0 \text{ for all } t \in \mathbb{R}\}.$$

Lemma 4.1. *Let f and τ be as in Theorem* 1, *then* (1.2) *has a continuous bounded solution $x(t)$. Furthermore,*

$$x(t) = \lim_{n \to \infty} x_n(t),$$

where $\{x_n(t)\} \subset \mathscr{P}$ is a uniformly bounded increasing sequence.

Proof. Let $A: \mathscr{C}(\mathbb{R}) \to \mathscr{C}(\mathbb{R})$ be the operator defined by

$$A[x](t) \triangleq \int_{t-\tau(t)}^{t} f(s, x(s))\, ds. \tag{4.1}$$

Because of $\mathbf{H}_1$, $\mathbf{H}_3$, and Lemma 1.1

$$A(\mathscr{P}) \subset \mathscr{P}. \tag{4.2}$$

Furthermore,

$$\begin{aligned}\int_{t-\tau(t)}^{t} f(s, x)\, ds &= \int_{t-\tau(t)}^{t} [f(s, x) - f(s, 0)]\, ds \\ &= \int_{t-\tau(t)}^{t} \left[\int_0^x f_x(s, u)\, du\right] ds \\ &= \int_0^x \left[\int_{t-\tau(t)}^{t} f_x(s, u)\, ds\right] du. \end{aligned} \tag{4.3}$$

Given $\varepsilon > 0$, by the uniform continuity of $f_x(\cdot, \cdot)$ on $\mathbb{R} \times [0, b_0]$, we can conclude the existence of $\delta = \delta(\varepsilon) > 0$ for which

$$|f_x(s, u) - f_x(s, 0)| \leqslant \frac{\varepsilon}{\|\tau\|},$$

for $0 \leqslant u < \delta$ and $s \in \mathbb{R}$. Hence, for $0 \leqslant x < \delta$

$$\begin{aligned}
&\int_0^x \left[\int_{t-\tau(t)}^t f_x(s,u)\,ds\right] du \\
&\quad \geqslant \int_0^x \left[\int_{t-\tau(t)}^t f_x(s,0)\,ds\right] du - x\,\frac{\varepsilon}{\|\tau\|}\,\tau(t) \\
&\quad \geqslant \left[\int_{t-\tau(t)}^t f_x(s,0)\,ds-\varepsilon\right] x \\
&\quad \geqslant \left[\inf_{t\in\mathbb{R}} \int_{t-\tau(t)}^t f_x(s,0)\,ds-\varepsilon\right] x. \qquad (4.4)
\end{aligned}$$

For ε sufficiently small, we then conclude that

$$\int_{t-\tau(t)}^t f(s,x)\,ds \geqslant x \qquad (4.5)$$

for $0 \leqslant x < \delta$. Let $x_0(t) \equiv \delta/2$. Clearly, the sequence

$$x_n(t) \triangleq A[x_{n-1}](t), \qquad n=1,2,...,$$

is increasing. Furthermore, given $\mu > 0$ there exists $x_0^* > 0$ such that

$$f(t,x) \leqslant [a_\infty(t)+\mu]\,x, \qquad \forall x \geqslant x_0^*.$$

Choosing μ small enough, and using (3.1) it is possible to conclude that

$$\sup_{t\in\mathbb{R}} \int_{t-\tau(t)}^t f(s,x)\,ds \leqslant x, \qquad \forall x \geqslant x_0^* \qquad (4.6)$$

and then

$$\frac{\delta}{2} \equiv x_0(t) \leqslant x_1(t) \leqslant x_2(t) \leqslant \cdots \leqslant x_0^*. \qquad (4.7)$$

By Lebesgue's Monotone Convergence theorem, the pointwise limit

$$\lim_{n\to\infty} x_n(t) \triangleq x(t)$$

is a continuous solution of

$$x(t) = \int_{t-\tau(t)}^t f(s,x(s))\,ds. \qquad (4.8)$$

Finally, from (4.7)

$$\inf_{t \in \mathbb{R}} x(t) \geqslant \frac{\delta}{2} > 0, \tag{4.9}$$

concluding the proof of Lemma 4.1. ▮

To prove that (1.2) has a nontrivial almost periodic solution, we proceed as in [3]. Hereafter, x_0^* shall be exclusively used to denote the greatest lower bound of those x's for which

$$f(t, y) \leqslant [a_\infty(t) + \varepsilon]\ y$$

for all $y \geqslant x$.

The following two lemmata are trivial adaptations of Fink and Gatica's results [3, Lemmas 1 and 2]. We include a few details for the reader's convenience.

LEMMA 4.2. *Let f and τ satisfy $\mathbf{H}_1$, $\mathbf{H}_2$, and $\mathbf{H}_3$. Then there is a solution $\bar{x}_{f,\tau}$ of (1.2),*

$$\frac{\delta}{2} \leqslant \bar{x}_{f,\tau}(t) \leqslant x_0^*, \qquad \forall t \in \mathbb{R}$$

such that

$$x(t) \leqslant \bar{x}_{f,\tau}(t), \qquad \forall t \in \mathbb{R}$$

for any solution $x(t)$ of (1.2) whose range is contained in $[\delta/2, x_0^*]$.

Proof. As in [3] let

$$\mathscr{S} = \{x \in \mathscr{C}(\mathbb{R}) \colon x \text{ solves Eq. (1.2) and } x(t) \in [\delta/2, x_0^*],\ t \in \mathbb{R}\}$$

and

$$z(t) = \sup_{x \in S} x(t).$$

Then

$$A[z](t) = \int_{t-\tau(t)}^{t} f(s, z(s))\, ds \geqslant \int_{t-\tau(t)}^{t} f(s, x(s))\, ds = x(t)$$

for each $x \in \mathscr{S}$. Thus $A[z](t) \geqslant z(t)$, and as in Lemma 4.1, $A^k[z]$ converges to an element $\bar{x}_{f,\tau}$ of $\mathscr{S}$. Hence, $\bar{x}_{f,\tau}(t) = A[\bar{x}_{f,\tau}](t)$ is then the maximal solution. ▮

LEMMA 4.3. *Let f, τ, g, and μ satisfy* $\mathbf{H}_1$, $\mathbf{H}_2$, *and* $\mathbf{H}_3$. *Let*

$$x(t)=A_{f,\tau}[x](t)\triangleq\int_{t-\tau(t)}^{t} f(s, x(s))\,ds \tag{4.10}$$

and

$$y(t)=A_{g,\mu}[y](t)\triangleq\int_{t-\mu(t)}^{t} g(s, y(s))\,ds \tag{4.11}$$

have solutions in $\mathscr{S}$, *both* $A_{f,\tau}$ *and* $A_{g,\mu}$ *mapping* $\{x: 0\leqslant x(t)\leqslant x_0^*\}$ *into itself. If* $f(t, x)\leqslant g(t, x)$ *and* $\tau(t)\leqslant\mu(t)$ *for all* $t\in\mathbb{R}$ *and* $0\leqslant x\leqslant x_0^*$, *then the maximal solution* $\bar{x}_{f,\tau}$ *of* (4.10) *is less or equal to the maximal solution* $\bar{x}_{g,\mu}$ *of* (4.11).

Proof. As in [3],

$$\begin{aligned}
A_{g,\mu}[\bar{x}_{f,\tau}](t) &\geqslant \int_{t-\mu(t)}^{t} f(s, \bar{x}_{f,\tau}(s))\,ds \\
&\geqslant \int_{t-\tau(t)}^{t} f(s, \bar{x}_{f,\tau}(s))\,ds + \int_{t-\mu(t)}^{t-\tau(t)} f(s, \bar{x}_{f,\tau}(s))\,ds \\
&\geqslant \bar{x}_{f,\tau}(t).
\end{aligned}$$

Therefore, the iteration $A_{g,\mu}^k[\bar{x}_{f,\tau}](t)$ converges to a solution $y(t)$ of (4.11) and

$$\bar{x}_{f,\tau}(t)\leqslant y(t)\leqslant \bar{x}_{g,\mu}(t).$$

This completes the proof of the lemma. ∎

To complete this sequence of preparatory lemmas, we now prove

LEMMA 4.4. *Let f, τ satisfy* $\mathbf{H}_1$ *and* $\mathbf{H}_2$, *let ϕ be such that* $0\leqslant\phi(t)\leqslant x_0^*$. *If α is a sequence for which* $T_\alpha f(t, x)$, $T_\alpha\tau(t)$, $T_\alpha a_\infty(t)$ *exist uniformly on* $\mathbb{R}$, *and* $T_\alpha\phi(t)$ *exists uniformly on compact subsets of* $\mathbb{R}$, *then*

$$T_\alpha \int_{t-\tau(t)}^{t} f(s, \phi(s))\,ds = \int_{t-T_\alpha\tau(t)}^{t} T_\alpha f(s, T_\alpha\phi(s))\,ds, \tag{4.12}$$

$$\limsup_{x\to\infty} \frac{T_\alpha f(t, x)}{x} = T_\alpha a_\infty(t) \qquad \textit{uniformly for } t\in\mathbb{R} \tag{4.13}$$

and

$$\sup_{t\in\mathbb{R}} \int_{t-T_\alpha\tau(t)}^{t} T_\alpha a_\infty(s)\,ds < 1. \tag{4.14}$$

Proof. The proof of (4.12) follows from estimates similar to those obtained in the proof of Lemma 2.2 (see Appendix). To prove (4.13), let ε be an arbitrary positive real number, x a sufficient large real number and pick a positive integer n such that

$$T_\alpha f(t, x) \leqslant f(t+\alpha_n, x) + \varepsilon$$

and

$$T_\alpha a_\infty(t) \geqslant a_\infty(t+\alpha_n) - \varepsilon,$$

for $t \in \mathbb{R}$. Then,

$$\begin{aligned} T_\alpha f(t, x) &\leqslant [a_\infty(t+\alpha_n) + \varepsilon] x + \varepsilon \\ &\leqslant [T_\alpha a_\infty(t) + 2\varepsilon]\, x + \varepsilon \end{aligned}$$

for $x \geqslant x_0^*$ and all $t \in \mathbb{R}$. Since ε was arbitrary, (4.13) is proved. The proof of (4.14) is a consequence of Lemma 2.3. ∎

Remark 4.1. An immediate consequence of Lemma 4.4 is the estimate

$$\int_{t - T_\alpha \tau(t)}^{t} T_\alpha f(s, \phi(s))\, ds \leqslant x_0^*, \tag{4.15}$$

if $0 \leqslant \phi(t) \leqslant x_0^*$ for all $t \in \mathbb{R}$.

We now conclude the proof of the theorem. First, for f and τ, let $\bar{x}_{f,\tau}(t)$ denote the maximal solution of (1.2) whose existence is guaranteed by Lemma 4.2. Then, for a given $\delta > 0$

$$\begin{aligned} &f^\alpha(t, x) \triangleq T_\alpha f(t, x) \leqslant f(t+\alpha_n, x) + \delta \\ &\tau^\alpha(t) \triangleq T_\alpha \tau(t) \leqslant \tau(t+\alpha_n) + \delta \end{aligned}$$

for all $n \geqslant N$ and all $t \in \mathbb{R}$. Now, if $0 \leqslant \phi(t) \leqslant x_0^*$

$$\begin{aligned} &\int_{t - \tau(t+\alpha_n) - \delta}^{t} [f(s+\alpha_n, \phi(s)) + \delta]\, ds \\ &\quad \leqslant \delta(\delta + \sup_{\mu \in H\{\tau\}} \|\mu\|) + \|f(\cdot, x_0^*)\|\, \delta \\ &\quad\quad + \left[\int_{t-\tau(t)}^{t} a_\infty(s)\, ds + \delta\right] x_0^* \leqslant x_0^*, \end{aligned}$$

for appropriate values of δ. From Lemma 4.3, and the previous calculations, we see that

$$\bar{x}_{f^\alpha,\tau^\alpha}(t) \leqslant \bar{x}_{f_n^\delta,\tau_n^\delta}(t),$$

where $f_n^\delta(t,x) = f(t+\alpha_n, x) + \delta$ and $\tau_n^\delta(t) = \tau(t+\alpha_n) + \delta$. Also, formula (4.12) tells us that $T_\alpha \bar{x}_{f,\tau} t$ solves

$$x(t) = \int_{t-\tau^\alpha(t)}^{t} f^\alpha(s, x(s))\, ds,$$

hence,

$$T_\alpha \bar{x}_{f,\tau}(t) \leqslant \bar{x}_{f^\alpha,\tau^\alpha}(t).$$

Further use of Lemma 4.3 yields

$$\bar{x}_{f\alpha,\tau^\alpha}(t) \leqslant \bar{x}_{f_n^\delta,\tau_n^\delta}(t) \leqslant \bar{x}_{f_n^0,\tau_n^0}(t) + \mu.$$

But

$$\begin{aligned}
\bar{x}_{f_n^0,\tau_n^0}(t) &= \int_{t-\tau(t+\alpha_n)}^{t} f(s+\alpha_n, \bar{x}_{f_n^0,\tau_n^0}(s))\, ds \\
&\leqslant \int_{t-\tau(t+\alpha_n)}^{t} f(s+\alpha_n, \bar{x}_{f,\tau}(s+\alpha_n))\, ds \\
&= \int_{t+\alpha_n-\tau(t+\alpha_n)}^{t+\alpha_n} f(s, \bar{x}_{f,\tau}(s))\, ds \\
&= \bar{x}_{f,\tau}(t+\alpha_n).
\end{aligned}$$

Thus

$$T_\alpha \bar{x}_{f,\tau}(t) \leqslant \bar{x}_{T_\alpha f, T_\alpha \tau}(t) \leqslant \bar{x}_{f,\tau}(t+\alpha_n),$$

and

$$T_\alpha \bar{x}_{f,\tau}(t) = \bar{x}_{T_\alpha f, T_\alpha \tau}(t).$$

The almost periodicity of $\bar{x}_{f,\tau}(t)$ now follows from Lemma 2.1. To prove the module containment (3.8), let $\{\alpha'\} \subset \{\alpha\}$ be such that $T_{\alpha'} f = f$ and $T_{\alpha'} \tau = \tau$. Then

$$T_{\alpha'} \bar{x}_{f,\tau} = \bar{x}_{T_{\alpha'} f, T_{\alpha'} \tau} = \bar{x}_{f,\tau}.$$

Now (3.8) follows from [2, Theorem 4.5]. ∎

Remark 4.2. A dual argument, starting with $x_0(t) \equiv x_0^*$, may be used to obtain an almost periodic minimal solution $\underline{x}_{f,\tau}$ whose range is contained in $[\delta/2, x_0^*]$.

A simple modification of the previous argument will apply in case $f: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$, $\tau: \mathbb{R} \to \mathbb{R}^+$, and $\hat{x} \geqslant 0$ are such that

$\mathbf{G}_1$: $f(t, x)$ is monotone nondecreasing on $[\hat{x}, +\infty)$, i.e., if $x, y \in \mathscr{C}(\mathbb{R})$ and $\hat{x} \leqslant x(t) \leqslant y(t)$, then $f(t, x(t)) \leqslant f(t, y(t))$;

$\mathbf{G}_2$: f is uniformly continuous on $\mathbb{R} \times [\hat{x}, \hat{x} + \eta_0]$ for some $\eta_0 > 0$; and

$\mathbf{G}_3$:

$$\inf_{t \in \mathbb{R}} \int_{t-\tau(t)}^{t} f(s, \hat{x})\, ds > \hat{x}.$$

THEOREM 4.5. *Let f, τ, and $\hat{x}$ be such that $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{G}_1$, $\mathbf{G}_2$, and $\mathbf{G}_3$ are satisfied. Then Eq.* (1.2) *has an almost periodic solution $\bar{x}_{f,\tau}$ whose infimum is such that*

$$\inf_{t \in \mathbb{R}} \bar{x}_{f,\tau}(t) \geqslant \hat{x}.$$

Sketch of Proof. Let $\varepsilon > 0$ be such that

$$\inf_{t \in \mathbb{R}} \int_{t-\tau(t)}^{t} f(s, \hat{x})\, ds - \varepsilon \|\tau\| \geqslant \hat{x}.$$

Then, by $\mathbf{G}_2$, there exists $\delta > 0$ such that

$$|f(t, x(t)) - f(t, \hat{x})| < \varepsilon, \qquad \forall t \in \mathbb{R}$$

if $\|x - \hat{x}\| < \delta$. Hence

$$\begin{aligned} \int_{t-\tau(t)}^{t} f(s, x(s))\, ds &\geqslant \int_{t-\tau(t)}^{t} f(s, \hat{x})\, ds - \varepsilon \|\tau\| \\ &\geqslant \inf_{t \in \mathbb{R}} \int_{t-\tau(t)}^{t} f(s, \hat{x})\, ds - \varepsilon \|\tau\| \\ &\geqslant \hat{x}. \end{aligned}$$

Now the proof of Theorem 1, with $x_0(t) \equiv \hat{x}$, applies. ∎

5. Proof of Theorem 2

Let us begin with the following observation (see [3])

PROPOSITION 5.1. *Let f satisfy $\mathbf{H}_1$ and $\mathbf{H}_3^*$. Then, there exists $\tau^*>0$ for which* (1.1) *does not have continuous nonnegative solutions of arbitrarily small norm, other than $x(t)\equiv 0$, if $\tau<\tau^*$.*

Proof. Let $\varepsilon=\|f_x(\cdot,0)\|+1$, and

$$\phi(\tau)=\sup_{t\in\mathbb{R}}\int_{t-\tau}^{t}[f_x(s,0)+\varepsilon]\,ds. \tag{5.1}$$

Because of our assumptions on f, the function $\phi\colon\mathbb{R}^+\to\mathbb{R}^+$ is well defined, and

- $\phi(0)=0$;
- ϕ is bounded below by an increasing function; and
- $\phi(\tau)\to+\infty$ as $\tau\to+\infty$.

Since

$$\left|\int_{t-\tau}^{t}f_x(s,0)\,ds-\int_{t-\mu}^{t}f_x(s,0)\,ds\right|\leqslant\|f_x(\cdot,0)\|\;|\tau-\mu|$$

for each $t\in\mathbb{R}$, it follows that

- ϕ is continuous on $\mathbb{R}^+$.

If not, there exist τ_0, a sequence $\{\tau_k\}$,

$$\lim_{k\to\infty}\tau_k=\tau_0,$$

and a positive number δ for which

$$\sup_{t\in\mathbb{R}}\int_{t-\tau_0}^{t}[f_x(s,0)+\varepsilon]\,ds+\delta\leqslant\sup_{r\in\mathbb{R}}\int_{r-\tau_k}^{r}[f_x(s,0)+\varepsilon]\,ds.$$

Then

$$\int_{t-\tau_0}^{t}[f_x(s,0)+\varepsilon]\,ds+\frac{\delta}{2}\leqslant\int_{r_k-\tau_k}^{r_k}[f_x(s,0)+\varepsilon]\,ds,\qquad\forall t\in\mathbb{R},$$

for some r_k. Hence,

$$\frac{\delta}{2} \leqslant \left| \int_{r_k - \tau_0}^{r_k} [f_x(s, 0) + \varepsilon]\, ds - \int_{r_k - \tau_k}^{r_k} [f_x(s, 0) + \varepsilon]\, ds \right|$$
$$\leqslant (\|f(\cdot, 0)\| + \varepsilon) |\tau_k - \tau_0|,$$

a contradiction.

If τ^* is defined by

$$\tau^* = \inf\{\tau : \phi(\tau) = 1\}$$

then

$$\phi(\tau) < 1, \qquad \forall \tau < \tau^*,$$

and the conclusion of the proposition follows from the estimate

$$\begin{aligned} \int_{t-\tau}^{t} f(s, x(s))\, ds &= \int_{t-\tau}^{t} \int_0^{x(s)} f_x(s, \eta)\, d\eta\, ds \\ &\leqslant \int_{t-\tau}^{t} [f_x(s, 0) + \varepsilon]\, x(s)\, ds \\ &\leqslant \|x\|\, \phi(\tau), \end{aligned}$$

which is satisfied by all nonnegative continuous functions of sufficiently small norm. ∎

Let τ^* be as in Proposition 5.1. If $\tau : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$\sup_{t \in \mathbb{R}} \tau(t) < \tau < \tau^*.$$

Then

$$\begin{aligned} \int_{t-\tau(t)}^{t} f(s, x(s))\, ds &\leqslant \int_{t-\tau(t)}^{t} [f_x(s, 0) + \varepsilon]\, x(s)\, ds \\ &\leqslant \|x\| \int_{t-\tau(t)}^{t} [f_x(s, 0) + \varepsilon]\, ds \\ &\leqslant \|x\| \int_{t-\tau}^{t} [f_x(s, 0) + \varepsilon]\, ds \\ &\leqslant \|x\|\, \phi(\tau), \end{aligned}$$

and the proof of the theorem is completed. ∎

Minor adjustments in the proof of Proposition 7.1 are needed to prove the following.

THEOREM 5.2. *Let f and τ satisfy* $\mathbf{H}_1$. *If, in addition, f is such that*

$\mathbf{H}_7$: *For some continuous bounded function $a_0\colon \mathbb{R} \to \mathbb{R}^+$,*

$$\limsup_{x \to 0^+} \frac{f(t, x)}{x} \leqslant a_0(t)$$

uniformly for $t \in \mathbb{R}$, then, there exists $\tau^* > 0$ for which (1.2) *does not have continuous nonnegative solutions of arbitrarily small norm, if*

$$\sup_{t \in \mathbb{R}} \tau(t) < \tau^*.$$

Proof. It is sufficient to define

$$\phi(\tau) = \sup_{t \in \mathbb{R}} \int_{t-\tau}^{t} [a_0(s) + \varepsilon]\, ds$$

with ε an arbitrarily chosen positive number. ∎

6. PROOF OF THEOREM 3

Fundamental to the proof of Theorem 3 is the following result

PROPOSITION 6.1 [9]. *Assume f and τ satisfy* $\mathbf{H}_2$ *and* $\mathbf{H}_4$, *with*

$$\frac{\omega}{\lambda} = \frac{p}{q}, \qquad p, q \in \mathbb{N}.$$

If in addition, there exist $a > 0$ and a nonnegative continuous function $b(t)$ for which

$$\inf_{t \in \mathbb{R}} \int_{t-\tau(t)}^{t} b(s)\, ds > a, \qquad \forall t \in \mathbb{R} \tag{6.1}$$

and

$$f(t, x) \geqslant b(t), \qquad \forall t \in \mathbb{R},\ x \geqslant a \tag{6.2}$$

then (1.2) *has a nontrivial $q\omega$-periodic solution $x(t)$ such that*

$$\inf_{t \in \mathbb{R}} x(t) > a. \tag{6.3}$$

With this proposition in mind, we now define a sequence of approximate problems,

$$x(t) = \int_{t-\tau_n(t)}^{t} f_n(s, x(s))\, ds, \tag{6.4}$$

where $\tau_n\colon \mathbb{R} \to \mathbb{R}^+$ and $f_n\colon \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ are defined by

$$f_n(t, x) = f\left(\frac{\omega}{\omega_n} t, x\right), \qquad \tau_n(t) = \tau\left(\frac{\lambda}{\lambda_n} t\right)$$

with ω_n and λ_n satisfying

(i) $1 < (\lambda/\lambda_n) < (\omega/\omega_n)$, $\lim_{n\to\infty} \omega_n = \omega$, $\lim_{n\to\infty} \lambda_n = \lambda$; and

(ii) $\omega_n/\lambda_n = p_n/q_n$, $p_n, q_n \in \mathbb{N}$.

LEMMA 6.2. *Let $b\colon \mathbb{R} \to \mathbb{R}^+$ be a continuous class $\mathscr{K}$ function. For each $n \in \mathbb{N}$ let*

$$b_n(t) = b\left(\frac{\omega}{\omega_n} t\right).$$

Then, for sufficiently large n,

$$\inf_{t\in\mathbb{R}} \int_{t-\tau(t)}^{t} b(s)\, ds > a \Rightarrow \inf_{t\in\mathbb{R}} \int_{t-\tau_n(t)}^{t} b_n(s)\, ds > a. \tag{6.5}$$

Proof. Let $p > 0$ be such that

$$\int_{t-\tau(t)}^{t} b(s)\, ds \geqslant a + p, \qquad \forall t \in \mathbb{R}.$$

For $t \in \mathbb{R}$ and $n \in \mathbb{N}$ let $t_n = (\lambda/\lambda_n) t$, then

$$\begin{aligned}
\int_{t-\tau_n(t)}^{t} b_n(s)\, ds &= \int_{t-\tau_n(t)}^{t} b\left(\frac{\omega}{\omega_n} s\right) ds \\
&\geqslant \int_{t-\tau_n(t)}^{t} b\left(\frac{\lambda}{\lambda_n} s\right) ds \\
&\geqslant \frac{\lambda_n}{\lambda} \int_{t_n-\tau(t_n)}^{t_n} b(r)\, dr \\
&\geqslant \frac{\lambda_n}{\lambda}(a + p) \\
&> a
\end{aligned}$$

provided λ_n is such that

$$\frac{a}{a+p} < \frac{\lambda_n}{\lambda} < 1. \quad \blacksquare$$

LEMMA 6.3. *Let $a\colon \mathbb{R} \to \mathbb{R}^+$ be a continuous class $\mathcal{J}$ function. If*

$$a_n(t) = a\left(\frac{\omega}{\omega_n} t\right), \qquad n \in \mathbb{N}$$

then,

$$\sup_{t \in \mathbb{R}} \int_{t-\tau_n(t)}^{t} a_n(s)\, ds \leqslant \sup_{t \in \mathbb{R}} \int_{t-\tau(t)}^{t} a(s)\, ds. \tag{6.6}$$

Proof. Let t_n be as in the proof of the previous lemma. Then

$$\begin{aligned} \int_{t-\tau_n(t)}^{t} a_n(s)\, ds &\leqslant \int_{t-\tau_n(t)}^{t} a\left(\frac{\lambda}{\lambda_n} s\right) ds \\ &\leqslant \frac{\lambda_n}{\lambda} \int_{t_n-\tau(t_n)}^{t_n} a(s)\, ds \\ &\leqslant \sup_{t \in \mathbb{R}} \int_{t-\tau(t)}^{t} a(s)\, ds. \quad \blacksquare \end{aligned}$$

Having taken care of these preparatory details, we now turn our attention to the actual proof of the theorem. Let us first note that f_n (resp. τ_n) is ω_n-periodic (resp. λ_n-periodic). By Proposition 6.1, Lemma 6.2, and Lemma 6.3, Eq. (6.4) has (for sufficiently large n) a $q_n\omega_n$-periodic solution, $x_n(t)$, satisfying

$$\inf_{t \in \mathbb{R}} x_n(t) \geqslant a. \tag{6.7}$$

Clearly,

$$\limsup_{x \to +\infty} \frac{f_n(t, x)}{x} \leqslant a_{\infty,n}(t)$$

with $a_{\infty,n}(t) = a_\infty((\omega/\omega_n) t)$, uniformly for $t \in \mathbb{R}$. Hence

$$f_n(t, x) \leqslant [a_{\infty,n}(t) + \varepsilon]\, x + \beta, \qquad \forall x \geqslant 0, \tag{6.8}$$

where

$$\beta = \sup_{t \in \mathbb{R},\, 0 \leqslant x \leqslant x_0^*} f(t, x).$$

Lemma 6.3, in conjunction with (6.8), yields the uniform boundedness of

$$\{x_n(t): n \geqslant N\}.$$

Also,

$$\begin{aligned}|x_n(t)-x_n(s)| \leqslant & \left|\int_{t-\tau_n(t)}^{t} [f_n(r, x_n(r)) - f(r, x_n(r))]\, dr\right| \\ & + \left|\int_{t-\tau_n(t)}^{t-\tau(t)} f(r, x_n(r))\, dr\right| + \left|\int_{t}^{s} f(r, x_n(r))\, dr\right| \\ & + \left|\int_{s-\tau(s)}^{t-\tau(t)} f(r, x_n(r))\, dr\right| + \left|\int_{s-\tau(s)}^{s-\tau_n(s)} f(r, x_n(r))\, dr\right| \\ & + \left|\int_{s-\tau_n(s)}^{s} [f(r, x_n(r)) - f_n(r, x_n(r))]\, dr\right|.\end{aligned}$$

Using the uniform continuity on compact subsets of both τ and f we conclude that $\{x_n(t)\}$ is equicontinuous on compact subsets of $\mathbb{R}$. The conclusion of the theorem is now consequence of (6.7) and the Arzela–Ascoli's theorem. ∎

7. Proof of Theorem 4

As it was observed in Section 3, if $b(t) \not\equiv 0$, its mean

$$\mathbf{M}\{b\} \triangleq \lim_{\tau \to \infty} \frac{1}{\tau} \int_{t-\tau}^{t} b(s)\, ds > 0$$

uniformly in t. Consequently, there exists $m > 0$ for which

$$\inf_{t \in \mathbb{R}} \int_{t-\tau}^{t} b(s)\, ds \geqslant a$$

for all $\tau \geqslant m$. Hence

$$\inf_{t \in \mathbb{R}} \int_{t-\tau}^{t} f(s, x(s))\, ds \geqslant a \tag{7.1}$$

for all $\tau \geqslant m$ and each continuous function $x(t)$ whose infimum satisfies

$$\inf_{t \in \mathbb{R}} x(t) \geqslant a.$$

Let $\mathscr{P}_\omega$ be the cone of nonnegative ω-periodic functions on $\mathbb{R}$, and $A_\tau : \mathscr{P}_\omega \to \mathscr{P}_\omega$ be the completely continuous operator

$$A_\tau[x](t) \triangleq \int_{t-\tau}^{t} f(s, x(s))\, ds.$$

Because of $\mathbf{H}_2^\bullet$

$$A_\tau[x](t) \leqslant \|x\|\, \varepsilon\tau + \beta\tau,$$

where

$$\beta = \max_{a \leqslant x \leqslant x_0^*,\, t \in \mathbb{R}} f(t, x). \tag{7.2}$$

Let us choose $R > x_0^*$ so that

$$R\varepsilon\tau + \beta\tau < R$$

and define

$$\Omega_a^R = \{x \in \mathscr{P}_\omega : a \leqslant x(t) \leqslant R,\ \forall t \in \mathbb{R}\}.$$

Clearly,

$$A_\tau(\Omega_a^R) \subset \Omega_a^R$$

and the conclusion follows from Schauder's fixed point theorem. ∎

8. Consequences

In this section we shall present a few consequences of the results previously proved and compare them with already known facts. We begin with

PROPOSITION 8.1. *Let f satisfy $\mathbf{H}_1$ and $\mathbf{H}_2^\bullet$. Assume that*

$\mathbf{H}_8$: *there exists $\hat{x} > 0$ for which $f(\cdot, \hat{x}) \not\equiv 0$, and $f(t, x)$ is non-decreasing on $[\hat{x}, +\infty)$; i.e., if $x, y \in \mathscr{C}(\mathbb{R})$ are such that $\hat{x} \leqslant x(t) \leqslant y(t)$, then $f(t, x(t)) \leqslant f(t, y(t))$.*

Then there exists τ^* such that for all $\tau \geqslant \tau^*$, Eq. (1.1) has an almost periodic solution with positive infimum.

Proof. Let

$$a = \mathbf{M}\{f(\cdot, \hat{x})\}.$$

Clearly, $a > 0$. Conditions $\mathbf{H}_1$ and $\mathbf{H}_8$ will then yield the existence of $\tau^* > 0$ for which

$$\inf_{t \in \mathbb{R}} \int_{t-\tau}^{t} f(s, \hat{x})\, ds \geqslant \hat{x}$$

if $\tau \geqslant \tau^*$. But,

$$\inf_{t \in \mathbb{R}} \int_{t-\tau}^{t} f(s, x(s))\, ds \geqslant \inf_{t \in \mathbb{R}} \int_{t-\tau}^{t} f(s, \hat{x})\, ds$$

whenever $x(t) \geqslant \hat{x}$ for all $t \in \mathbb{R}$.

Hence

$$\inf_{t \in \mathbb{R}} \int_{t-\tau}^{t} f(s, x(s))\, ds \geqslant \hat{x}.$$

Now, the proof of Theorem 1, with $\tau(t) \equiv \tau$, applies. ∎

Remark 8.1. Proposition 8.1 was proved in [3] under the stronger hypothesis:

$\mathbf{H}_3^*$: $f(t, \cdot)$ is continuously differentiable with $f_x(t, \eta)$ uniformly continuous for $(t, \eta) \in \mathbb{R} \times [0, \eta_0]$ for some $\eta_0 > 0$, $f_x(t, \cdot) \geqslant 0$ with $f(t, 0) \equiv 0$ and $f_x(t, 0) \not\equiv 0$.

PROPOSITION 8.2. *Let f and τ satisfy* $\mathbf{H}_1$, $\mathbf{H}_2$. *Assume that*

$\mathbf{H}_8$: *there exists $\hat{x} > 0$ for which $f(\cdot, \hat{x}) \not\equiv 0$, and $f(t, x)$ is nondecreasing and uniformly concave on $[\hat{x}, +\infty)$; i.e., if $\lambda \in (0, 1)$ then there is an $\eta > 0$ such that $f(t, \lambda x) \geqslant \lambda(1+\eta) f(t, x)$, $\forall t \in \mathbb{R}$ and $x \geqslant \hat{x}$; or*

$\mathbf{H}_9$: *there exists $\hat{x} > 0$ forwhich $f(\cdot, \hat{x}) \not\equiv 0$, $f(t, x)$ is nondecreasing and homogeneous on $[\hat{x}, +\infty)$; i.e., there exists an $\alpha \in (0, 1)$ such that $f(t, \lambda x) \geqslant \lambda^\alpha f(t, x)$ for $\lambda \in (0, 1)$, $t \in \mathbb{R}$, and $x \geqslant \hat{x}$.*

Then there exists τ^* such that if $\inf_{t \in \mathbb{R}} \tau(t) \geqslant \tau^*$, Eq. (1.2) *has exactly one almost periodic solution, $x_{f,\tau}$, whose infimum satisfies*

$$\inf_{t \in \mathbb{R}} x_{f,\tau}(t) \geqslant \hat{x}.$$

Proof. See [3] or [4]. ∎

APPENDIX

In this appendix we shall give a detailed proof of formula (4.12). Our proof hinges heavily on the use of subtle properties of AP($\mathbb{R}$). For the reader's convenience we survey the one that is most relevant to our purpose.

For a function $f \in \mathrm{AP}(\mathbb{R})$, let

$$v_f(\tau) \triangleq \sup_{t \in \mathbb{R}} |f(t+\tau) - f(t)|, \tag{A.1}$$

and, if $\mathscr{F} \subset \mathrm{AP}(\mathbb{R})$ let

$$v(\tau) \triangleq \sup_{f \in \mathscr{F}} v_f(\tau). \tag{A.2}$$

DEFINITION A.1. A family $\mathscr{F} \subset \mathscr{C}(\mathbb{R})$ is said to be uniformly almost periodic (u.a.p.) if and only if

- $\mathscr{F}$ is bounded in $\mathscr{C}(\mathbb{R})$; and
- $v(\tau)$ is almost periodic.

It follows from this definition that all u.a.p. families are pre-compact in the topology of uniform convergence on compact subsets of $\mathbb{R}$. Furthermore, one can prove the following.

PROPOSITION A.2 [2]. *If $\mathscr{F}$ is a u.a.p. family then, given $\varepsilon > 0$ and a sequence α' there is a subsequence $\alpha \subset \alpha'$ and an integer $N(\varepsilon)$ so that $|f(t+\alpha_n) - f(t+\alpha_m)| < \varepsilon$ for $n, m \geqslant N(\varepsilon)$, all $t \in \mathbb{R}$ and all $f \in \mathscr{F}$.*

The following result shall also be useful to us.

PROPOSITION A.3. *A family $\mathscr{F} \subset AP(\mathbb{R})$ is compact iff it is closed and uniformly almost periodic.*

We now prove formula (4.12). Let α, f, τ, and ϕ be as in Lemma 4.4. Let

$$f^\alpha(t, x) \triangleq T_\alpha(t, x), \qquad \tau^\alpha(t) \triangleq T_\alpha \tau(t), \qquad \text{and} \qquad \phi^\alpha(t) \triangleq T_\alpha \phi(t).$$

For each $n \in \mathbb{N}$, let $t_n = t + \alpha_n$, $s_n = s + \alpha_n$. Then

$$\begin{aligned} &\left| \int_{t-\tau(t_n)}^{t} f(s_n, \phi(s_n))\, ds - \int_{t-\tau^\alpha(t)}^{t} f^\alpha(s, \phi^\alpha(s))\, ds \right| \\ &\quad \leqslant \left| \int_{t-\tau(t_n)}^{t} f(s_n, \phi(s_n))\, ds - \int_{t-\tau^\alpha(t)}^{t} f(s_n, \phi(s_n))\, ds \right| \\ &\quad\quad + \left| \int_{t-\tau^\alpha(t)}^{t} [f(s_n, \phi(s_n)) - f(s_n, \phi^\alpha(s))]\, ds \right| \\ &\quad\quad + \left| \int_{t-\tau^\alpha(t)}^{t} [f(s_n, \phi^\alpha(s)) - f^\alpha(s, \phi^\alpha(s))]\, ds \right|. \end{aligned}$$

Clearly,

$$\left| \int_{t-\tau(t_n)}^{t} f(s_n, \phi(s_n))\, ds - \int_{t-\tau^\alpha(t)}^{t} f(s_n, \phi(s_n))\, ds \right|$$
$$\leqslant \sup_{s \in \mathbb{R}, 0 \leqslant x \leqslant x_0^*} |f(s, x)|\, |\tau(t_n) - \tau^\alpha(t)|.$$

To estimate the last two terms, let us first note that, because of $\mathbf{H}_1$, for each $\varepsilon > 0$ there exists $\delta(\varepsilon)$ so that $x, y \in [0, x_0^*]$ and $|x-y| < \delta$ implies that

$$|f(t, x) - f(t, y)| < \varepsilon, \qquad \forall t \in \mathbb{R}.$$

But,

$$\lim_{n \to \infty} \phi(s + \alpha_n) = \phi^\alpha(s)$$

uniformly on $[t - \tau^\alpha(t), t]$. Therefore,

$$|f(s+\alpha_n, \phi(s+\alpha_n)) - f(s+\alpha_n, \phi^\alpha(s))| < \varepsilon$$

for each $n \geqslant N(\varepsilon)$ and each $s \in [t - \tau^\alpha(t), t]$. From this we conclude that

$$\left| \int_{t-\tau^\alpha(t)}^{t} [f(s+\alpha_n, \phi(s+\alpha_n)) - f(s+\alpha_n, \phi^\alpha(s))]\, ds \right| \leqslant \varepsilon \|\tau\|.$$

Finally, the continuity of ϕ^α on $[t - \tau^\alpha(t), t]$, in conjunction with $\mathbf{H}_1$ imply that the family

$$\mathscr{B} \triangleq \{f(\cdot, \phi^\alpha(s)) : s \in [t - \tau^\alpha(t), t]\}$$

is compact. By Proposition A.2, there exists $N(\varepsilon)$ so that

$$|f(r+\alpha_n, \phi^\alpha(s)) - f(r+\alpha_m, \phi^\alpha(s))| < \varepsilon$$

for all $n, m \geqslant N(\varepsilon)$, each $s \in [t - \tau^\alpha(t), t]$, and each $r \in \mathbb{R}$. Letting $m \to \infty$,

$$|f(r+\alpha_n, \phi^\alpha(s)) - f^\alpha(r, \phi^\alpha(s))| \leqslant \varepsilon,$$

for all $n \geqslant N(\varepsilon)$, each $r \in \mathbb{R}$, and each $s \in [t - \tau^\alpha(t), t]$. Thus

$$\left| \int_{t-\tau^\alpha(t)}^{t} [f(s+\alpha_n, \phi^\alpha(s)) - f^\alpha(s, \phi^\alpha(s))]\, ds \right| \leqslant \varepsilon \|\tau\|.$$

Putting all these estimates together

$$\left| \int_{t-\tau(t_n)}^{t} f(s_n, \phi(s_n))\, ds - \int_{t-\tau^\alpha(t)}^{t} f^\alpha(s, \phi^\alpha(s))\, ds \right|$$

$$\leqslant 2\varepsilon \|\tau\| + \sup_{t \in \mathbb{R},\, 0 \leqslant x \leqslant x_0^*} |f(s, x)|\, |\tau(t_n) - \tau^\alpha(t)|,$$

and the proof (4.12) is now complete. ∎

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