Artin’s L-functions and one-dimensional characters

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Received 19 January 2006; revised 1 February 2006
Available online 29 September 2006
Communicated by Gebhard Böckle

Abstract

Let \( K/\mathbb{Q} \) be a finite Galois extension with the Galois group \( G \), let \( \chi_1, \ldots, \chi_r \) be the irreducible non-trivial characters of \( G \), and let \( \mathcal{A} \) be the \( \mathbb{C} \)-algebra generated by the Artin L-functions \( L(s, \chi_1), \ldots, L(s, \chi_r) \). Let \( \mathcal{B} \) be the subalgebra of \( \mathcal{A} \) generated by the L-functions corresponding to induced characters of non-trivial one-dimensional characters of subgroups of \( G \). We prove: (1) \( \mathcal{B} \) is of Krull dimension \( r \) and has the same quotient field as \( \mathcal{A} \); (2) \( \mathcal{B} = \mathcal{A} \) iff \( G \) is \( M \)-group; (3) the integral closure of \( \mathcal{B} \) in \( \mathcal{A} \) equals \( \mathcal{A} \) iff \( G \) is quasi-\( M \)-group.

MSC: 11R42

Keywords: Artin L-function; \( M \)-group; Quasi-\( M \)-group

Let \( K/\mathbb{Q} \) be a finite Galois extension with the Galois group \( G \), and let \( \chi \) be a non-trivial irreducible character of \( G \). Artin’s conjecture predicts that the L-function \( L(s, \chi, K/\mathbb{Q}) \) is holomorphic in the whole complex plane [1, p. 105].

Let \( \chi_1, \ldots, \chi_r \) be the irreducible non-trivial characters of \( G \). The corresponding L-functions \( L(s, \chi_1), \ldots, L(s, \chi_r) \) are algebraically independent over \( \mathbb{C} \) [2, Corollary 4, p. 183)]. Let \( \mathcal{A} := \mathbb{C}[L(s, \chi_1), \ldots, L(s, \chi_r)] \) be the \( \mathbb{C} \)-algebra generated by the meromorphic functions \( L(s, \chi_1), \ldots, L(s, \chi_r) \). It is isomorphic to the algebra of polynomials in \( r \) variables over \( \mathbb{C} \). Let \( \mathcal{O}(\mathbb{C}) \) be the \( \mathbb{C} \)-algebra of holomorphic functions in \( \mathbb{C} \). Artin’s conjecture is:

\[ \mathcal{A} \subseteq \mathcal{O}(\mathbb{C}). \]
Let $S$ be the set of all subgroups of $G$. For a subgroup $H \in S$ let $\hat{H}_0^1$ be the set of all non-trivial one-dimensional complex characters of $H$, that is, the set of all non-constant group homomorphisms of $H$ in the multiplicative group $\text{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. For a subgroup $H \in S$ and a character $\varphi \in \hat{H}_0^1$ let $\varphi^G$ be the induced character of $G$. The Artin $L$-function $L(s, \varphi^G, K/\mathbb{Q})$ is holomorphic, being equal to the Hecke $L$-function $L(s, \varphi, K_2/K_1)$ of the abelian extension $K_2/K_1$, $K_1$ the fixed field of $H$, $K_2$ the fixed field of $\ker \varphi \subseteq H$, so:

$$B \subseteq \mathcal{H} \subseteq \mathcal{A},$$

where $B$ is the $\mathbb{C}$-subalgebra of $\mathcal{A}$ generated by the functions $L(s, \varphi^G, K/\mathbb{Q})$, $H \in S$, $\varphi \in \hat{H}_0^1$, and $\mathcal{H} := \mathcal{A} \cap \mathcal{O}(\mathbb{C})$. How large is the algebra $B$?

**Theorem 1.** The finitely generated $\mathbb{C}$-algebra $B$ is of Krull dimension $r$. The quotient field of $B$ equals the quotient field of $A$.

**Proof.** Let $\chi \in \{\chi_1, \ldots, \chi_r\}$. By [3, p. 209], there exist subgroups $H_1, \ldots, H_l$ of $G$, non-trivial one-dimensional characters $\varphi_i$ of $H_i$, $i = 1, \ldots, l$, and integers $m_1, \ldots, m_l$ such that

$$\chi = m_1\varphi_1^G + \cdots + m_l\varphi_l^G.$$

It follows that

$$L(s, \chi) = L(s, \varphi_1^G)^{m_1} \cdots L(s, \varphi_l^G)^{m_l}$$

belongs to the quotient field of $B$, hence $A$ is contained in the quotient field of $B$. Since $B$ is contained in $A$ it follows that the quotient field of $B$ equals the quotient field of $A$. The Krull dimension of the finitely generated $\mathbb{C}$-algebra $B$ equals the transcendence degree of its quotient field, that is, the transcendence degree of the quotient field of $A$, which is $r$.  \(\square\)

**Definition.** A finite group $G$ is $M$-group if for every irreducible character $\chi$ of $G$ there exist a subgroup $H \subseteq G$ and a one-dimensional character $\varphi : H \to \mathbb{C}^\times$ such that

$$\chi = \varphi^G.$$

**Theorem 2.** The following assertions are equivalent:

(a) $B = A$.

(b) The Galois group $G$ is $M$-group.

**Proof.** (a) $\Rightarrow$ (b): Let $\chi \in \{\chi_1, \ldots, \chi_r\}$. Since $L(s, \chi) \in B$ there exist subgroups $H_1, \ldots, H_l$ of $G$, one-dimensional non-trivial irreducible characters $\varphi_j$ of $H_j$, $j = 1, \ldots, l$, and a polynomial

$$P(X_1, \ldots, X_l) = \sum_{i_1 \geq 0, \ldots, i_l \geq 0} a_{i_1 \ldots i_l} X_1^{i_1} \cdots X_l^{i_l} \in \mathbb{C}[X_1, \ldots, X_l]$$

such that

$$L(s, \chi) = P(L(s, \varphi_1^G), \ldots, L(s, \varphi_l^G)),$$
that is
\[
L(s, \chi) = \sum_{i_1 \geq 0, \ldots, i_l \geq 0} a_{i_1 \ldots i_l} L(s, i_1 \varphi^G_1 + \cdots + i_l \varphi^G_l).
\]

By the linear independence of L-functions corresponding to different characters [2, Theorem 1, p. 179] it follows that there exist \(i_1, \ldots, i_l\) such that
\[
\chi = i_1 \varphi^G_1 + \cdots + i_l \varphi^G_l.
\]

Since \(\chi\) is irreducible there exist \(1 \leq j \leq l\) such that
\[
\chi = \varphi^G_j,
\]
hence \(G\) is \(M\)-group.

(b) \implies (a): Let \(\chi \in \{\chi_1, \ldots, \chi_r\}\). Since \(G\) is \(M\)-group, there exist a subgroup \(H \subseteq G\) and a one-dimensional character \(\varphi: H \to \mathbb{C}^\times\) such that
\[
\chi = \varphi^G.
\]

Since \(\chi\) is not trivial, the character \(\varphi\) is not trivial, so
\[
L(s, \chi) = L(s, \varphi^G) \in B.
\]

Hence
\[
A \subseteq B. \quad \Box
\]

**Definition.** A finite group \(G\) is *quasi-M-group* if for every irreducible character \(\chi\) of \(G\) there exist a subgroup \(H \subseteq G\), a one-dimensional character \(\varphi: H \to \mathbb{C}^\times\) and a number \(k \geq 1\) such that
\[
k\chi = \varphi^G.
\]

Let \(B'\) be the integral closure of \(B\) in \(A\). It holds
\[
B \subseteq B' \subseteq H \subseteq A.
\]

**Theorem 3.** The following assertions are equivalent:

(a) \(B' = A\).

(b) The Galois group \(G\) is quasi-M-group.

**Proof.** (a) \implies (b): Let \(\chi \in \{\chi_1, \ldots, \chi_r\}\). The element \(L(s, \chi)\) of \(A\) satisfies a monic equation with coefficients in \(B\):
\[
L(s, \chi)^l + b_{l-1} L(s, \chi)^{l-1} + \cdots + b_1 L(s, \chi) + b_0 = 0,
\] (1)
$l \geq 1$, $b_0, \ldots, b_{l-1} \in \mathcal{B}$. Let $\varphi_1^G, \ldots, \varphi_m^G$ be all pairwise distinct characters of $G$ which are obtained by inducing from non-trivial linear characters of subgroups of $G$, and let $f_1 := L(s, \varphi_1^G), \ldots, f_m := L(s, \varphi_m^G)$. It holds:

$$\mathcal{B} = \mathbb{C}[f_1, \ldots, f_m].$$

Each coefficient $b_j$, $j = 0, \ldots, l - 1$, is a polynomial in $f_1, \ldots, f_m$:

$$b_j = P_j(f_1, \ldots, f_m) = \sum_{t_1 \geq 0, \ldots, t_m \geq 0} a(j)_{t_1 \ldots t_m} f_1^{t_1} \cdots f_m^{t_m}.$$

and (1) rewrites as

$$L(s, l\chi) + \sum_{j=0}^{l-1} \sum_{t_1 \geq 0, \ldots, t_m \geq 0} a(j)_{t_1 \ldots t_m} L(s, t_1\varphi_1^G + \cdots + t_m\varphi_m^G + j\chi) = 0. \quad (2)$$

By the linear independence of L-functions corresponding to different characters [2, Theorem 1, p. 179], and by (2) it follows that there exist $j \in \{0, \ldots, l - 1\}$ and $t_1 \geq 0, \ldots, t_m \geq 0$ such that

$$l\chi = t_1\varphi_1^G + \cdots + t_m\varphi_m^G + j\chi,$$

that is

$$(l - j)\chi = t_1\varphi_1^G + \cdots + t_m\varphi_m^G.$$

Since $\chi$ is an irreducible character there exist $u \in \{1, \ldots, m\}$ and $1 \leq k \leq l - j$ such that $k\chi = \varphi_u^G$, so $G$ is quasi-$M$-group.

(b) $\Rightarrow$ (a): Let $\chi \in \{\chi_1, \ldots, \chi_r\}$. Since $G$ is quasi-$M$-group, there exist a subgroup $H \subseteq G$, a 1-dimensional character $\varphi : H \to \mathbb{C}^\times$ and $k \geq 1$ such that

$$k\chi = \varphi^G.$$

Since $\chi$ is not trivial, the character $\varphi$ is not trivial. It holds

$$L(s, \chi)^k = L(s, k\chi) = L(s, \varphi^G) \in \mathcal{B},$$

so $L(s, \chi) \in \mathcal{B}'$. Hence

$$\mathcal{A} \subseteq \mathcal{B}'. \quad \square$$

It is not known whether there exist quasi-$M$-groups which are not $M$-groups. By a theorem of Taketa every $M$-group is solvable. It is not known whether every quasi-$M$-group is solvable.
Acknowledgments

I thank Peter Müller for Ref. [3] and for criticism which led to the correct formulation of Theorem 3.
I thank Michael Pohst, Florian Hess and Sebastian Pauli for the invitation to join the KANT-group at the Technical University Berlin, where this paper was finished.

References