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Abstract

For an irrational rotation, we use the symbolic dynamics on the sturmian coding to compute explicitly, according to the continued fraction approximation of the argument, the measure of the largest Rokhlin stack made with intervals, and the measure of the largest Rokhlin stack whose levels have one name for the coding. Each one of these measures is equal to one if and only if the argument has unbounded partial quotients.

Keywords: Combinatorics on words; Symbolic dynamics; Arithmetics

0. Introduction

In this paper, we use symbolic dynamics and combinatorics on words to solve some open problems in ergodic theory.

The rotations on the torus $\mathbb{T}_1$, defined by $R_\alpha x = x + \alpha \bmod 1$, are among the simplest dynamical systems to conceive, and to study for their properties in ergodic theory and topological dynamics. However, the various dynamical invariants which can be used for this study (measure-theoretic or topological entropy, symbolic complexity and more recently measure-theoretic complexity, see [9]), fail to distinguish between R_α and $R_{\alpha'}$, even when α and α' have very different arithmetic properties.

We propose here to consider topological invariants (though they use the unique measure-theoretic structure defined by this topology) which both come from the dynamics of the system, and, as we shall see, give indications on the arithmetics of α. They are linked with the notions of Rokhlin stacks and rank (for a general survey of these notions, the reader may consult [8]); more precisely, we shall use various notions of *covering numbers*, some of which having been defined in [17] (see also [15, 6]) and in [7].

In every measure-theoretic ergodic system, for every ε and arbitrarily high h, we can find (because of Rokhlin's Lemma, see, for example, [11]) disjoint sets $B, TB, \ldots, T^{h-1}B$ such that the union (called a *Rokhlin stack*) of these sets has measure greater than

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$1-\varepsilon$. Furthermore, a rotation is of *rank one* [4], which means that such stacks approximate every set: more precisely, in that case, their basis B can be taken of arbitrarily small diameter. However, this does not imply that B is a "good" set, of nonempty interior for example. Can we get the same property with B being an interval? The answer is positive if α has unbounded partial quotients ([16], see [8] for a proof), and as yet unknown in the opposite case.

A rotation can be naturally coded into a symbolic system, and this canonical coding corresponds to *sturmian* symbolic systems [12]; in the particular case where α is the golden ratio number, this is the system associated to the well-known Fibonacci sequence. This symbolic system can then be studied by using combinatorics on words, the main tool being the graph of words, or *Rauzy graph*, of the associated sequences, see, for example, [2]. These methods have already been used in [3] to give a new proof of the famous *three-length theorem* on the distribution of the sequence $\{k\alpha\}$ on the circle, which appeared simultaneously in [18–20]; the reader may consult the survey [1] on these questions. Under a symbolic form, we speak of *three-frequency theorem*, and these frequencies are computed explicitly in [3] from the Rauzy graphs.

Here, using the explicit values of the frequencies and a geometric argument, we compute precisely the measure of the largest possible Rokhlin stack made with intervals (see definition of F_I below), according to the continued fraction expansion of α, and show that it is smaller than 1 when α has bounded partial quotients (a lower bound for F_I for rotations appears in [10]). We also consider the arithmetic properties of the number F_I. We remark that a related problem is solved by similar methods in [5].

The rank one property translates on the sturmian coding into covering the considered sequences by disjoint iterates of one word, with arbitrarily small gaps and transcription errors ($F^\star = 1$ in the definition below); the invariant F_I is equal to a topological invariant of the symbolic system, denoted by F_C, which is then completely known. But we may ask a new question: can we cover the sequences by iterates of one word, with arbitrarily small gaps but without errors? The ergodic theory tells us that, because of rank one, this must be true for some codings [14], but it need not be true for the canonical sturmian coding. We show that the answer is yes for the sturmian coding if and only if α has unbounded partial quotients, and compute explicitly, according to the continued fraction expansion of α, the proportion of each sequence that we may cover thus (see definition of F above). The invariant F is not smaller than F_I, and is greater in some cases, such as the case of the golden ratio number.

1. Preliminaries

1.1. Covering numbers

For a measure-theoretic dynamical system (X, T, μ) (μ being a probability invariant by T), the (measure-theoretic) *covering number* $F^\star(T)$ ([17], see also [6, 15]) is defined by:

Definition 1. $F^\star(T)$ is the largest real number z such that for every measurable partition $P=\{P_1,\ldots,P_r\}$ of X, for every $\varepsilon>0$, for every integer h_0, there exist a subset B of X, an integer $h>h_0$ and a partition $P'=\{P'_1,\ldots,P'_r\}$ of X such that if $A=\bigcup_{j=0}^{h-1} T^jB$,
- $B, TB,\ldots,T^{h-1}B$ are disjoint,
- $\mu(A)>z$,
- $\sum_{i=1}^{r}\mu((P_i\,\Delta\,P'_i)\cap A)<\varepsilon$,
- each $P'_i\cap A$ is a union of sets T^jB, for some $0\leqslant j\leqslant h-1$.

$F^\star$ is an invariant for the notion of isomorphism for measure-theoretic dynamical systems (see [11] for a definition).

If we take for X the torus $\mathbb{T}_1$, and if T is the irrational rotation $Tx=x+\alpha \bmod 1$, the only probability μ preserved by T is the Lebesgue measure. The last two conditions in the definition of $F^\star$ are realized if the diameter of B is smaller than ε, and it is known that $F^\star=1$ [4], but we cannot guarantee that B is a nice set. Hence, we define a new invariant, the *covering number by intervals* F_I.

Let (X,T) be a topological dynamical system, defined on the torus $\mathbb{T}_1$ with the usual topology, and *uniquely ergodic*: there is a unique probability invariant by T, denoted by μ. We call *interval* an arc of the torus, open to the left and closed to the right.

Definition 2. $F_I(T)$ is the largest real number z such that, for every h_0, there exist $h\geqslant h_0$ and an interval B such that
- $B, TB,\ldots,T^{h-1}B$ are disjoint intervals,
- $\mu(\bigcup_{i=0}^{h-1} T^iB)\geqslant z$.

The result will not be changed if we take open or closed intervals, or if we ask only that the interiors of the T^jB are disjoint. The system is said to be of *rank one by intervals* whenever $F_I(T)=1$.

F_I depends a priori on both the topological and the measure-theoretic structure of the system; but the measure-theoretic structure is defined uniquely by the topology, and two uniquely ergodic systems defined on the torus have the same F_I when they are topologically conjugate. We say that F_I is an invariant of topological conjugacy in this class of systems.

Let Λ be a finite alphabet; we consider the one-sided sequences $(x_0,\ldots,x_n,\ldots)$, $x_n\in\Lambda$.

A *word* of length $l(w)=h$ is a finite sequence $w=w_1\ldots w_h$ of elements of Λ. The concatenation of two words v and w is denoted by vw. The word $w=w_1\ldots w_h$ *occurs* at place i in a sequence (x_n) or a word $x_0\ldots x_s$ if $x_i=w_1,\ldots,x_{i+h-1}=w_h$. Two occurrences of w, at places i and j, are *without overlap* if $j>i+h-1$.

We define the shift T on the space $\{0,1\}^{\mathbb{N}}$, by, if $x=(x_n)_{n\in\mathbb{N}}$, $(Tx)_n=x_{n+1}$. For a sequence u, we define the set X_u as the closure of $\{0,1\}^{\mathbb{N}}$ equipped with the discrete topology of the set $(T^nu)_{n\in\mathbb{N}}$. A *symbolic dynamical system* is the topological dynamical system (X_u,T) for a sequence u on an alphabet Λ. Every topological dynamical system we consider here is *minimal* (each orbit is dense) and uniquely ergodic.

In a symbolic dynamical system, for a word w, we call *cylinder* associated to w and denote by $[w]$ the set of x in X_u such that w occurs in x at place 0; the *height* of the cylinder $[w]$ is the length of w.

Let (X_u, T) be a symbolic dynamical system, minimal and uniquely ergodic. For such a system, the covering number of the associated measure-theoretic system has a symbolic expression [7]:

Definition 3. $F^{\star}$ is the largest real number z such that, for every $\varepsilon > 0$, for every integer h_0, there exists $h \geqslant h_0$, a word of length h, denoted by $w = w_1 \dots w_h$, and a sequence of indices $(i_n)_{n \in \mathbb{N}}$, such that

- $1/h \# \{1 \leqslant j \leqslant h; w_{i_n+j-1} \neq w_j\} < \varepsilon$,
- $i_{n+1} > i_n + h - 1$,
- $\#(\{i_n + m,\ n \in \mathbb{N},\ 0 \leqslant m \leqslant h-1\} \cap \{0, \dots, N-1\}) \geqslant zN$ for all N large enough.

We define now the *symbolic covering number* F [7] and the *covering number by cylinders* F_C for a symbolic system:

Definition 4. F is the largest real number z such that, for every $\varepsilon > 0$, for every integer h_0, there exists $h \geqslant h_0$, a word of length h, denoted by $w = w_1 \dots w_h$, and a sequence of indices $(i_n)_{n \in \mathbb{N}}$, such that

- w occurs in u at each place i_n,
- $i_{n+1} > i_n + h - 1$,
- $\#(\{i_n + m,\ n \in \mathbb{N},\ 0 \leqslant m \leqslant h-1\} \cap \{0, \dots, N-1\}) \geqslant zN$ for all N large enough.

Definition 5. F_C is the largest real number z such that, for every $\varepsilon > 0$, for every integer h_0, there exists $h \geqslant h_0$, a word of length h, denoted by $w = w_1 \dots w_h$, and a sequence of indices $(i_n)_{n \in \mathbb{N}}$, such that

- w occurs in u at each place i_n,
- w occurs in u only at places i_n,
- $i_{n+1} > i_n + h - 1$,
- $\#(\{i_n + m,\ n \in \mathbb{N},\ 0 \leqslant m \leqslant h-1\} \cap \{0, \dots, N-1\}) \geqslant zN$ for all N large enough.

Because of minimality and unique ergodicity, these definitions do not depend on the particular sequence u we used to define X_u, and we can immediately give for F and F_C definitions using the invariant probability μ:

Definition 6. F is the largest real number z such that, for every h_0, there exist $h \geqslant h_0$, a sequence w_j, $0 \leqslant j \leqslant h-1$, $1 \leqslant w_j \leqslant r$, and a subset B of $\bigcup_{j=0}^{h-1} T^{-j} P_{w_j}$ such that

- $B, TB, \dots, T^{h-1}B$ are disjoint,
- $\mu(\bigcup_{i=0}^{h-1} T^i B) \geqslant z$.

Definition 7. F_C is the largest real number z such that, for every h_0, there exist $h \geqslant h_0$ and a sequence w_j, $0 \leqslant j \leqslant h-1$, $1 \leqslant w_j \leqslant r$, such that the set $B = \bigcup_{j=0}^{h-1} T^{-j} P_{w_j}$ satisfies
- $B, TB, \ldots, T^{h-1}B$ are disjoint,
- $\mu(\bigcup_{i=0}^{h-1} T^i B) \geqslant z$.

We see that $F_C \leqslant F \leqslant F^\star$. We check also that F and F_C are invariants of topological conjugacy: if two minimal and uniquely ergodic topological dynamical systems are conjugate, they have the same F and the same F_C.

If B is a cylinder $[w]$, its measure $\mu([w])$ is also the *frequency* $f(w)$ defined as $\lim_{n\to+\infty} N_n(w)/(n+1)$, where $N_n(w)$ is the number of occurrences of w in $u_0 \ldots u_n$.

In the same way, if $[w]$ is a cylinder of height h, we call *frequency without overlap* $d(w)$ the maximal measure of a subset B of $[w]$ such that $B, TB, \ldots, T^{h-1}B$ are disjoint; $d(w)$ is also $\lim_{n\to+\infty} M_n(w)/(n+1)$, where $M_n(w)$ is the maximal number of occurrences without overlap of w that we can find in $u_0 \ldots u_n$.

To compute F for a symbolic system is the same as to compute, for arbitrarily long words, the greatest values of the quantity $\tau(w) = l(w)d(w)$; we call $\tau(w)$ the *tiling* of the word w. To compute F_C, we have to consider $\tau(w)$ for words whose all occurrences are without overlap.

In all this study, a set B such that $B, TB, \ldots, T^{h-1}B$ are disjoint is called a *basis of a Rokhlin stack of height* h; the Rokhlin stack is the set $\bigcup_{i=0}^{h-1} T^i B$.

1.2. Sturmian coding and Rauzy graph

In all what follows, $0 < \alpha < 1$ is an irrational number: let $\alpha = [0; a_1, \ldots, a_n, \ldots]$ be its simple continued fraction expansion, and $q_{n+1} = a_{n+1} q_n + q_{n-1}$, $p_{n+1} = a_{n+1} p_n + p_{n-1}$, $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, $q_0 = 1$.

(X, T, μ) is the dynamical system associated to the irrational rotation of angle α on $\mathbb{T}_1$. Throughout this paper, we consider either this rotation or its *sturmian coding*, that is the symbolic system defined in the following way: if $P_0 = [0, 1-\alpha[$ and $P_1 = [1-\alpha, 1[$, we associate to each point x the sequence $PN(x)$ defined by $PN(x)_n = i$ if $T^n x \in P_i$. We use the alphabet $A = \{0, 1\}$ and for example $u = PN(0)$; we still call T the shift on X_u and μ the unique invariant probability on the minimal and uniquely ergodic system (X_u, T). The coding by P gives a measure-theoretic isomorphism, and a semi-topological conjugacy (the isomorphism is continuous except on a countable number of points) between (X, T, μ) and (X_u, T, μ).

In the sequel, we identify the elements of X and X_u: for a subset B, we can simultaneously ask whether it is an interval, as a subset of X, and whether it is a cylinder, as a subset of X_u.

If $L_h(u)$ is the set of words of length h occurring in u, then $\# L_h(u) = h + 1$, that is why we call the coding *sturmian*. For each word $w = w_0 \ldots w_{h-1}$ of $L_h(u)$, the cylinder $[w]$ is the nonempty interval $\bigcap_{i=0}^{h-1} T^{-i} P_{w_i}$, and these $h+1$ intervals are exactly the intervals delimited on the circle by the points $0, T^{-1}0, \ldots, T^{-h}0$.

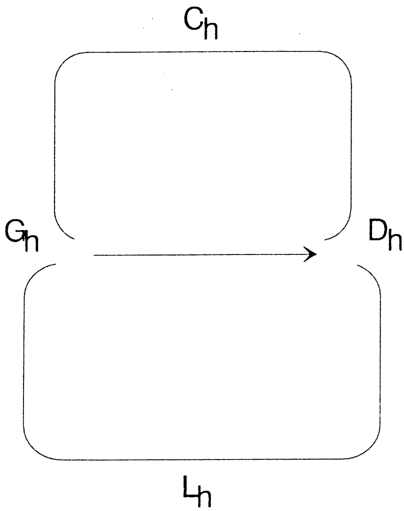

Fig. 1. Sturmian graph.

We define the *Rauzy graph* Γ_h in the following way: the vertices are the points of $L_h(u)$, with an edge from w to w' if w and w' occur successively in u, that is if $w=av$ and $w'=vb$ for letters a and b and a word v of $L_{h-1}(u)$; we label this edge by $avb \in L_{h+1}$, and the set of edges is L_{h+1}.

The Rauzy graphs of sturmian systems and their evolution with h are described in [2, 3]; we state here what we shall use in our study.

The graph Γ_h contains one vertex D_h which is *right special*: it has two outgoing edges D_h0 and D_h1; and one vertex G_h is *left special*: it has two incoming edges $0G_h$ and $1G_h$; D_h and G_h may be the same vertex; every vertex except G_h has one incoming edge, every vertex except D_h has one outgoing edge.

We say that the *central branch* contains the vertex G_h, its successors as far as D_h (included) and (if they exist) the edges between them. The other vertices and edges form two branches, beginning with one of the two outgoing edges of D_h, ending with one of the two incoming edges of G_h, and containing the edges and (if they exist) vertices between them. The lengths of these last two branches are always different, and we call them respectively *short branch* and *long branch*. The *short circuit* C_h (resp. *long circuit* L_h) begins with G_h and is made with the central branch followed by the short (resp. long) branch (Fig. 1).

In the sequel, when we speak of *words* of a branch, this will always mean *vertices* of that branch. The *length* of a branch is the number of its vertices, and the *frequency* of a branch is the common frequency of its vertices.

The vertices of Γ_{h+1} are the edges of Γ_h; if $D_h \neq G_h$, the vertices of a branch of Γ_{h+1} are the edges of the same branch of Γ_h (there is a *split* of an edge). If $G_h = D_h$, the central branch of Γ_h is reduced to one vertex, and there is a *burst*: for a *reversing* burst (RB), the vertices of the central branch of Γ_{h+1} are the edges of the long branch of Γ_h, while for a *non-reversing* burst (NRB) the vertices of the central branch of Γ_{h+1} are the edges of the short branch of Γ_h; in both cases, the short branch of Γ_{h+1} is reduced to one edge.

If we enumerate successively the words of length h occurring in $u = PN(0)$ at place i for $i = 0, \dots, n, \dots$, we get an infinite path γ_h in the graph Γ_h, beginning with G_h and made with a succession of short and long circuits. If for h there is a split, γ_h is deduced from γ_{h+1} by replacing C_{h+1} by C_h, L_{h+1} by L_h; if there is a non-reversing burst, γ_h is deduced from γ_{h+1} by replacing C_{h+1} by C_h, L_{h+1} by C_hL_h; if there is a reversing burst, γ_h is deduced from γ_{h+1} by replacing C_{h+1} by L_h, L_{h+1} by L_hC_h.

2. Covering number by intervals

2.1. Geometric computation of F_I

Let B be an interval and b its length.

Lemma 1. *The property "$B, TB, \dots, T^{h-1}B$ are disjoint", depends only on b.*

Proof. This property is invariant by translation. □

We now ask the question: what is the maximal possible length of B such that $B, TB, \dots, T^{h-1}B$ are disjoint?

Lemma 2. *The maximal possible length for an interval which is a basis of a Rokhlin stack of height h is the smallest frequency of a cylinder of length $h-1$.*

Proof. Because of Lemma 1, we take $B = [0, b[$. Then $B \cap TB = \emptyset$ if the point $0 + b$ is situated before $T0$ on the oriented circle, and the point $T0 + b$ is situated before 0 on the oriented circle, that is:

$$B \cap TB = \emptyset \quad \text{if and only if} \quad b \leqslant |T0 - 0|.$$

In the same way, for T^2B disjoint from B and TB:

$$T^2B \cap B = \emptyset \Leftrightarrow b \leqslant |T^20 - 0|,$$

$$T^2B \cap TB = \emptyset \Leftrightarrow b \leqslant |T^20 - T0|.$$

Eventually, we have

$$T^iB \cap T^jB = \emptyset \Leftrightarrow b \leqslant |T^i0 - T^j0|,$$

for $i \neq j$, $i, j \in [0, h-1]$.

Hence, the greatest possible b is

$$b = \min_{\substack{i \neq j \\ i,j \in [0,h-1]}} |T^i0 - T^j0| = \min_{\substack{i \neq j \\ i,j \in [0,h-1]}} |T^{-i}0 - T^{-j}0|.$$

These form h intervals, which are the h cylinders of height $h-1$, hence the result.

2.2. Evolution of the Rauzy graph

The following analysis is inspired from [3]. We use different notations and give additional data (the lengths of the circuits) which are not explicit in [3].

The reversing bursts take place for [3]

$$h = q_n + q_{n-1} - 2,$$

and the lengths of the circuits are q_n and q_{n-1}. It can be shown, by recursion, that the three frequencies are

$f_L = |p_{n-1} - q_{n-1}\alpha|$ for the words which are vertices of the long branch,

$f_C = |p_n - q_n\alpha|$ for the words of the short branch,

$f^+ = |p_n - p_{n-1} - (q_n - q_{n-1})\alpha|$ for the words of the central branch,

with $f_L > f_C$, $f^+ = f_C + f_L$.

Just after the reversing burst, $h = q_n + q_{n-1} - 1$, with circuits of length q_n and $q_n + q_{n-1}$. At this stage there are only two frequencies:

$$f^+ = |p_{n-1} - q_{n-1}\alpha|,$$

$$f_L = |p_n - q_n\alpha|.$$

The new frequency $f_C = |p_n + p_{n-1} - (q_n + q_{n-1})\alpha|$ appears for $h = q_n + q_{n-1}$; for this value h is also equal to the length of the long circuit. Then the lengths and frequencies remain the same until the following burst. If $a_{n+1} = 1$, the following burst is reversing and takes place for $h = q_{n+1} + q_n - 2$. If $a_{n+1} > 1$, the following bursts take place for $h = kq_n + q_{n-1} - 2$, $2 \leqslant k \leqslant a_{n+1}$, and are non-reversing.

If $a_{n+1} \geqslant 2$, $2 \leqslant k \leqslant a_{n+1}$, $(k-1)q_n + q_{n-1} - 1 \leqslant h \leqslant kq_n + q_{n-1} - 2$, the length of the long circuit is $|L| = (k-1)q_n + q_{n-1}$, the length of the short circuit is $|C| = q_n$; $|h| > |L|$ and $|L| > (a_{n+1} + 1)|C|$. The corresponding frequencies are

$$f^+ = |(k-2)p_n + p_{n-1} - \alpha[(k-2)q_n + q_{n-1}]|, \quad (1)$$

$$f_C = |(k-1)p_n + p_{n-1} - \alpha[(k-1)q_n + q_{n-1}]|, \quad (2)$$

$$f_L = |p_n - q_n\alpha|. \quad (3)$$

We remark however that there is no word on the short branch for $h = (k-1)q_n + q_{n-1} - 1$.

We have still the same f_L, with $f_C > f_L$, and $f^+ = f_C + f_L$, until $h = a_{n+1}q_n + q_{n-1} - 2 = q_{n+1} - 2$, the last non-reversing burst; this f_L will remain the smallest frequency for $h = a_{n+1}q_n + q_{n-1} - 1$, because there is no frequency for the short branch at that stage.

We have the following frequencies for $a_{n+1}q_n+q_{n-1}-1\leqslant h\leqslant(a_{n+1}+1)q_n+q_{n-1}-2$ (except for $h=a_{n+1}q_n+q_{n-1}-1$ where there is no word on the short branch):

$$\begin{aligned} f_L &= |p_n - q_n\alpha| \quad \text{for the words of the long branch,}\\ f_C &= |p_{n+1} - q_{n+1}\alpha| \quad \text{for the words of the short branch,}\\ f^+ &= |p_{n+1} - p_n - (q_{n+1} - q_n)\alpha| \quad \text{for the words of the central branch,} \end{aligned}$$

and now $f_L > f_C$ (the next burst will be reversing), and $f^+ = f_C + f_L$.

It will be useful to compare h with the lengths of the circuits: if $(k-1)q_n + q_{n-1} - 1 \leqslant h \leqslant kq_n + q_{n-1} - 2$, we have always $h \leqslant |L| + |C|$; among these values of h, we have $h \leqslant |L|$ if $h \leqslant (k-1)q_n + q_{n-1}$, and $h > |L|$ otherwise; among these values of h, we have $(k-1)|C| < h \leqslant k|C|$ if $h \leqslant kq_n$, and $k|C| < h \leqslant (k+1)|C|$ otherwise.

2.3. Arithmetic computation of F_I

The previous analysis shows that if $q_n < h \leqslant q_{n+1}$, the smallest frequency of a cylinder of height $h-1$ remains $|p_n - q_n\alpha|$. Hence, we have the following value for F_I, which appears as a lower bound in [10]:

Lemma 3. *For the rotation of angle α, whose convergents are p_n/q_n,*

$$F_I = \limsup_{n\to\infty} (q_{n+1}|p_n - q_n\alpha|).$$

Proposition 1. *We define*

$$v_n = [0; a_n, a_{n-1}, \ldots, a_1],$$

$$t_n = [0; a_{n+1}, a_{n+2}, \ldots].$$

Then, for the rotation of angle $\alpha = [0; a_1, a_2, \ldots]$,

$$F_I = \limsup_{n\to\infty} \frac{1}{1 + t_n v_n}.$$

Proof. We call

$$|p_n - q_n\alpha| = f_n \tag{4}$$

with $\alpha = [0; a_1, a_2, \ldots, a_n, \ldots]$. If we use the previous analysis for $h = q_n + q_{n-1} - 2$, and write that the sum of all frequencies is equal to one (knowing the frequencies, the lengths of the circuits and hence the length of each branch as the central branch is reduced to one vertex), we get the classic relation

$$q_n f_{n-1} + q_{n-1} f_n = 1 \tag{5}$$

and we can write

$$q_n f_{n-1} = \frac{1}{1 + \frac{q_{n-1}}{q_n}\frac{f_n}{f_{n-1}}}. \tag{6}$$

We then use Jager's notations (see, for example, [13]), and put $v_n = (q_{n-1})/q_n$, $t_n = f_n/(f_{n-1})$; the recursion formulas $q_{n+1} = a_{n+1}q_n + q_{n-1}$ and $f_{n-1} = a_{n+1}f_n + f_{n+1}$ give the values for t_n and v_n which we claim in our proposition. □

2.4. Properties of F_I

Proposition 2. *$F_I = 1$ if and only if α has unbounded partial quotients.*

Proof. If there exists a sequence n_k such that $a_{n_k} \to \infty$, we have $v_{n_k} \leqslant 1/a_{n_k} \to 0$, $t_n \leqslant 1$ for all n.

Conversely, if for all n, $a_n \leqslant M$, there exist K and L such that $K \leqslant t_n v_n \leqslant L$ and

$$F_I < 1. \qquad \square$$

Corollary 1. *Let $\theta = \frac{1}{2}(1+\sqrt{5})$ be the golden ratio number; for any $\alpha \in [0,1[\cap \mathbb{Z}\theta + \mathbb{Z}$,*

$$F_I = \frac{\theta + 2}{5} = \frac{5+\sqrt{5}}{10}.$$

Proof. For such an α, the a_n are ultimately equal to one; $v_n \to 1/\theta$ if $n \to \infty$, $t_n = 1/\theta$ for n large enough, hence

$$F_I = \frac{1}{1 + \frac{1}{\theta}\frac{1}{\theta}}$$

and the result. □

Corollary 2. *For every α,*

$$F_I \geqslant \frac{5+\sqrt{5}}{10}.$$

For every $\alpha \notin \mathbb{Z}\theta + \mathbb{Z}$,

$$F_I \geqslant \tfrac{3}{4}.$$

Proof. If infinitely often $a_n \geqslant 3$, $F_I \geqslant \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$. If infinitely often $a_n \geqslant 2$ and $a_{n+1} \geqslant 2$, $F_I \geqslant \frac{1}{1+\frac{1}{2}\frac{1}{2}} = \frac{4}{5}$.

There remains the case where $a_n = 1$ ultimately, except for isolated values of n for which $a_n = 2$. If $a_n = 2$ infinitely often, we get $F_I \geqslant \frac{1}{1+\frac{4}{10}\frac{3}{4}} = \frac{10}{13}$; for $a_n = 1$ ultimately, see the last corollary. □

The first assertion of Corollary 2 is stated in [10]. We remark that there is no value of α giving an F_I between $\frac{1}{10}(5+\sqrt{5})$ and $\frac{3}{4}$.

If α and β have the same partial quotients a_n for all n large enough, they have the same F_I; thus F_I is not a complete invariant of topological conjugacy, even if we restrict ourselves to the class of bounded partial quotients. But α and β may also give the same F_I without having any a_n in common: we check that for the periodic expansion $\alpha = [0; r, s, r, s, \ldots, r, s, \ldots]$, the associated F_I depends only on the product rs.

If the expansion of α is ultimately periodic, or equivalently if α is an algebraic number of degree 2, F_I is still an algebraic number of degree 2. If the period is 2, the upper limit giving F_I is a true limit, though this is not the case in general.

2.5. Computation of F_C

Lemma 4. *For every natural integer h, there exists an interval B, of maximal length for the property "B is a basis of a Rokhlin stack of height h", which is a cylinder of height h.*

Proof. We know that the length of B is the smallest frequency of an $(h-1)$-cylinder. Hence, we can take for B an $(h-1)$-cylinder. The word in Γ_{h-1} corresponding to this cylinder is not right special (otherwise, it would be on the central branch of Γ_{h-1} and its frequency would not be the smallest). Hence, we can take the unique right extension of this $h-1$-word, which is an h-word corresponding to an h-cylinder. □

Proposition 3. *For rotations and the sturmian coding*

$$F_C = F_I.$$

Proof. Immediate because of the previous lemma and Definition 7. □

3. Symbolic covering number

In all this part, we intend to compute F for the rotation of angle α equipped with the partition P defined in Section 1.2, or, equivalently, the symbolic covering number of the sturmian coding. We already know, because of Proposition 3, that $F \geqslant F_I$.

We have to estimate the frequency without overlap of a word w: Section 2.2 gives the frequency of words, but also, by comparing their length with the length of the circuits, which of their occurrences are without overlap. We shall study first the extremal cases, that is the values of h for which the frequencies or the integral parts of the ratios between h and the lengths of the circuits change values. Then we shall show that the other cases do not bring any new situation. h being fixed, we denote by C the short circuit C_h and L the long circuit L_h. In this section, t_n, v_n and f_n are the quantities defined in the statement and the proof of Proposition 1.

In all what follows, for fixed h: if w is on the short (resp. long) branch of the graph Γ_h, it can occur only on the short (resp. long) circuit and its frequency without overlap is not greater than the frequency without overlap of a word of the central branch. On the contrary, a word of the central branch occurs both on the long and the short circuit; hence both the frequency and the frequency without overlap are maximal for these words. So we shall always take a word w *of the central branch* of the graph;

an occurrence of w can take place either on the short circuit, C, of Γ_h, or on its long circuit, L.

3.1. Study of the cases $a_{n+1}\geqslant 2$, $h=kq_n+q_{n-1}-2$, $2\leqslant k\leqslant a_{n+1}$

For this value of h, there is a non-reversing burst. The analysis of Section 2.2 shows that $|L|<h<|L|+|C|$ and $k|C|<h<(k+1)|C|$.

We have to select disjoint occurrences of w in an optimal way. For that, we need to know the succession of L and C in the infinite path γ_h (see Section 1.2). Taking into account the formulas at the end of Section 1.2, and the sequence of bursts after h, we see that γ_h is made with blocks of the form $LC^{a_{n+1}-k+1}$ or $LC^{a_{n+1}-k+2}$.

Lemma 5. *If on the N first blocks of the path γ_h, there are N_0 blocks of the form $LC^{a_{n+1}-k+2}$,*

$$\frac{N_0}{N}\to t_{n+1} \quad \text{when } N\to +\infty.$$

Proof. To the first N blocks correspond N' circuits. But

$$\frac{N}{N'}\to f_L \quad \text{and} \quad \frac{(a_{n+1}-k+2)N_0+(a_{n+1}-k+1)(N-N_0)}{N'}\to f_C,$$

hence $N_0/N\to f_C/f_L-a_{n+1}+k-1$. Hence, the result because of the expression of f_C and f_L and the recursion relation on f_n (see the proof of Proposition 1). □

In fact, it can be shown that the succession of blocks is itself a sturmian sequence, associated to the irrational rotation of angle t_{n+1}; we chose to deduce Lemma 5 from already known parameters.

Henceforth, we put $p_0=t_{n+1}$, $p_1=1-t_{n+1}$ and call "p_0-blocks" the blocks $LC^{a_{n+1}-k+2}$ and "p_1-blocks" the blocks $LC^{a_{n+1}-k+1}$.

Lemma 6. *For $a_{n+1}\geqslant 2$, $h=kq_n+q_{n-1}-2$, $2\leqslant k\leqslant a_{n+1}$ and w a word of the central branch of Γ_h there exists $0\leqslant e_n\leqslant 2$, such that*

$$\tau(w)=\left(1-\frac{2}{kq_n+q_{n-1}}\right)(kq_n+q_{n-1})\left(\frac{a_{n+1}}{k+1}-i+e_n\right)f_n \tag{7}$$

$$=\left(1-\frac{2}{kq_n+q_{n-1}}\right)(k+v_n)\left(\frac{a_{n+1}}{k+1}-i+e_n\right)q_nf_n. \tag{8}$$

Proof. We put

$$a_{n+1}=q(k+1)+i,\quad 0\leqslant i\leqslant k. \tag{9}$$

In each block, we select occurrences without overlap of w. If we have selected an occurrence in a circuit C, we cannot select an occurrence in the following k circuits C, nor in the following circuit L if there are less than k circuits C before it. If we have selected an occurrence in a circuit L, we cannot select an occurrence in the following circuit (necessarily C). In each block, we number the circuits C from 1 to $(a_{n+1}-k+1)$ or $(a_{n+1}-k+2)$. In each case, we can select an occurrence of w, either in the circuit C number one, or in the circuit C number two.

Case 1: In a block we select an occurrence of w in the circuit number 1. We can a priori select an occurrence in the following circuits:

$$\underset{1}{\underline{\quad C \quad}} \cdots \underset{1+(k+1)}{\underline{\quad C \quad}} \cdots \underset{1+(q-1)(k+1)}{\underline{\quad C \quad}} \cdots \underset{1+(q-1)(k+1)+(k+1)}{\underline{\quad C \text{ or } L \text{ or none} \quad}}$$

Hence for $i \in [0,k]$ we can select occurrences:

$i=0$	q times in the C	not in the following L	if p_1-block
	q times in the C	not in the following L	if p_0-block
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$i=k-3$	q times in the C	not in the following L	if p_1-block
	q times in the C	not in the following L	if p_0-block
$i=k-2$	q times in the C	not in the following L	if p_1-block
	q times in the C	in the following L	if p_0-block
$i=k-1$	q times in the C	in the following L	if p_1-block
	$q+1$ times in the C	not in the following L	if p_0-block
$i=k$	$q+1$ times in the C	not in the following L	if p_1-block
	$q+1$ times in the C	not in the following L	if p_0-block

(10)

Case 2: In a block, we select one occurrence of w in the initial circuit L and in the circuit C number 2; we select occurrences:

$i=0$	1 time in the initial L	q times in the C	not in the following L	if p_1
	1 time in the initial L	q times in the C	not in the following L	if p_0
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$i=k-2$	1 time in the initial L	q times in the C	not in the following L	if p_1
	1 time in the initial L	q times in the C	not in the following L	if p_0
$i=k-1$	1 time in the initial L	q times in the C	not in the following L	if p_1
	1 time in the initial L	q times in the C	in the following L	if p_0
$i=k$	1 time in the initial L	q times in the C	in the following L	if p_1
	1 time in the initial L	$q+1$ times in the C	not in the following L	if p_0

(11)

Hence, in a block we could take q, $q+1$ or $q+2$ occurrences of the word. We consider the first N blocks, where N is chosen arbitrarily large; on average, we took $q+e_n$ occurences of w for each block, for some $0 \leqslant e_n \leqslant 2$. The number of occurrences without overlap we have selected is $N(q+e_n)$, and hence, from the proof of Lemma 5, we get $d(w)=(q+e_n)f_L$, $l(w)=kq_n+q_{n-1}-2$ and we can compute the tiling of the word w:

$$\tau(w)=(kq_n+q_{n-1}-2)f_L(q+e_n),$$

and from (3), (4), (5) and (9) we get (7) and (8).

In Section 2.3 we had, with the notations of Proposition 1

$$F_I=\limsup_{n\to\infty} q_{n+1}|p_n-q_n\alpha| \\ =\limsup_{n\to\infty}(a_{n+1}+v_n)q_nf_n.$$

We see now that if $e_n-i\leqslant 0$, then

$$\tau(w)\leqslant(k+v_n)\left(\frac{a_{n+1}}{k+1}\right)q_nf_n\leqslant(k+1)\left(\frac{a_{n+1}}{k+1}\right)q_nf_n=a_{n+1}q_nf_n\leqslant F_I.$$

Hence in that case, $\tau(w)\leqslant F_I$, which is not useful for computing F as we know already $F\geqslant F_I$.

We have still to study the case $e_n-i>0$, and, as $e_n\leqslant 2$, there remain three possibilities, $i\in[0,2]$. Because of (10) and (11) we get:

- $0=i<k-2$: We can take $q+1$ occurrences of w in a block provided we take an occurrence in the initial L; but then we cannot take more than $q-1$ occurrences in the following block, which leaves an average of q. Hence $e_n-i=0$:
- $0=i=k-2$: $e_n-i\geqslant 0$;
- $0=i=k-1$: $k=1$, impossible as $k>1$;
- $0=i=k$: $k=0$, impossible as $k\neq 0$;
- $1=i<k-2$: $e_n-i\leqslant 0$;
- $1=i=k-2$: $e_n-i\leqslant 0$;
- $1=i=k-1$: $e_n-i\leqslant 0$;
- $2=i=k$: $e_n-i\leqslant 0$.

Hence the only useful case is $0=i=k-2$.

3.1.1. *Study of the sub-case* $0=i=k-2$

We take $h=2q_n+q_{n-1}-2$; for this value, a non-reversing bursting takes place. With $k=2$ and (9) we get: $a_{n+1}=3q$, $a_{n+1}>1$ and there are blocks $LC^{a_{n+1}-1}$ with frequency p_1 and $LC^{a_{n+1}}$ with frequency p_0 (in the sense of Lemma 5).

Lemma 7. *For* $h=2q_n+q_{n-1}-2$, *w a word of the central branch of* Γ_h *and* $a_{n+1}=3q$, $q\in\mathbb{N}$

$$\tau(w)=\left(1-\frac{2}{2q_n+q_{n-1}}\right)(2q_n+q_{n-1})\left(\frac{a_{n+1}}{3}+\frac{1}{2}\right)f_n$$

$$=\left(1-\frac{2}{2q_n+q_{n-1}}\right)\left(q+\frac{1}{2}\right)\frac{1+v_{n+1}(2-3q)}{1+t_{n+1}v_{n+1}} \qquad (12)$$

if $a_{n+2} = 1$ *and*

$$\tau(w) = \left(1 - \frac{2}{2q_n + q_{n-1}}\right)(2q_n + q_{n-1})\left(\frac{a_{n+1}}{3} + t_{n+1}\right) f_n$$
$$= \left(1 - \frac{2}{2q_n + q_{n-1}}\right)(q + t_{n+1})\frac{1 + v_{n+1}(2 - 3q)}{1 + t_{n+1}v_{n+1}} \tag{13}$$

if $a_{n+2} > 1$.

Proof. We have blocks of type $LC^{a_{n+1}}$ or $LC^{a_{n+1}-1}$. The ulterior evolution gives

$$\begin{aligned} C, L &\xrightarrow{NRB} C, CL \\ &\ \ \vdots \quad \vdots \\ &\rightarrow C, C^{a_{n+1}-1}L \\ &\xrightarrow{RB} C^{a_{n+1}-1}L, C^{a_{n+1}-1}LC. \end{aligned} \tag{14}$$

According to the following a_{n+2} we distinguish two cases.

Case 1: $a_{n+2} = 1$. In that case we continue (14) in the following way: $\xrightarrow{RB} C^{a_{n+1}-1}LC$ $C^{a_{n+1}-1}LC^{a_{n+1}}L$.
We see that after a block $LC^{a_{n+1}-1}$ there is always a block $LC^{a_{n+1}}$.

We choose the occurrences of w that we keep; we call "block of type L" a block where we choose an occurrence of w in the initial L, "block of type C" a block where we choose an occurrence of w in the initial C. After having made our choice, we call, among all the blocks,

A	the proportion of blocks which are	$LC^{a_{n+1}}$	and of type L	
B	the proportion of blocks which are	$LC^{a_{n+1}-1}$	and of type L	(15)
D	the proportion of blocks which are	$LC^{a_{n+1}}$	and of type C	
E	the proportion of blocks which are	$LC^{a_{n+1}-1}$	and of type C.	

Remembering that after $LC^{a_{n+1}-1}$ there is always an $LC^{a_{n+1}}$ and using the above table, we choose the optimal situation: after A we take D or E; after B, D; after D, A or B; and after E, D; thus we take, on average on the first N blocks for an arbitrarily large N, see Lemma 5 and its proof, $(q+1)A + (q+1)B + qD + qE$ occurrences of the word w. The previous relations imply $A+B+E = D+E$, $A+D = p_0$ and $B+E = p_1$, which give in the best case $A = \frac{1}{2}(p_0 - p_1)$, $B = p_1$, $D = \frac{1}{2}(p_0 + p_1)$ and $E = 0$. Hence $q + e_n = q(A+B+D+E) + A + B$ and

$$e_n = \frac{p_0 + p_1}{2} = \frac{1}{2}.$$

Replacing e_n in (7) and (8) by $\frac{1}{2}$ and k by 2, using (5) and $a_{n+1} = 3q$ we get

$$\tau(w) = \left(1 - \frac{2}{2q_n + q_{n-1}}\right)(2q_n + q_{n-1})\left(\frac{a_{n+1}}{3} + \frac{1}{2}\right) f_n$$

$$= \left(1 - \frac{2}{2q_n + q_{n-1}}\right)(2 + v_n)\left(q + \frac{1}{2}\right) q_n f_n,$$

hence (12), as we check that $q_n f_n = v_{n+1}/(1 + t_{n+1} v_{n+1})$ and $1/v_{n+1} = 3q + v_n$.

Case 2: $a_{n+2} > 1$. In that case we continue (14) in the following way: $\xrightarrow{NRB} C^{a_{n+1}-1}L$, $C^{a_{n+1}-1}LC^{a_{n+1}-1}LC$.

We see that after an $LC^{a_{n+1}}$ there is always an $LC^{a_{n+1}-1}$, hence with the notations of (15) and in the optimal situation: after A we take E; after B, D or E; after D, B; and after E, D or E. The previous relations imply that $A + B + E = D + E$, $A + D = p_0$, $B + E = p_1$; the relations are the same as in the previous case, but the optimum is not the same because of the constraint that A, B, D, E cannot be negative, which gives in the best case $D = p_0$, $E = p_1 - p_0$, $D = B$ and $A = 0$. Hence

$$e_n = A + B = p_0 = t_{n+1}.$$

Replacing e_n in (7) and (8) by t_{n+1} and k by 2, using (5) and $a_{n+1} = 3q$ we get

$$\tau(w) = \left(1 - \frac{2}{2q_n + q_{n-1}}\right)(2q_n + q_{n-1})\left(\frac{a_{n+1}}{3} + t_{n+1}\right) f_n$$

$$= \left(1 - \frac{2}{2q_n + q_{n-1}}\right)(2 + v_n)(q + t_{n+1}) q_n f_n,$$

hence (13) because $q_n f_n = v_{n+1}/(1 + t_{n+1} v_{n+1})$ and $1/v_{n+1} = 3q + v_n$.

3.2. *Study of the cases* $h = kq_n + q_{n-1} - 2$, $k = a_{n+1} + 1$

For this value of h, a reversing burst takes place,

$$h = (a_{n+1} + 1)q_n + q_{n-1} - 2 = q_{n+1} + q_n - 2.$$

We have $|L| = q_{n+1}$, $|C| = q_n$ and $h > |L|$, $h > (a_{n+1} + 1)|C|$, but $h < |L| + |C|$.

Lemma 8. *For* $h = q_{n+1} + q_n - 2$ *and* w *a word of the central branch of* Γ_h

$$\tau(w) = \left(1 - \frac{2}{q_{n+1} + q_n}\right)\frac{q_{n+1} + q_n}{2}(f_{n+1} + f_n)$$

$$= \left(1 - \frac{2}{q_{n+1} + q_n}\right)\frac{(1 + t_{n+1})(1 + v_{n+1})}{2(1 + t_{n+1} v_{n+1})}. \qquad (16)$$

Proof. Then $CL \xrightarrow{RB} L, LC$. If $a_{n+2} = 1$, $L, LC \to LC, LCL$, if $a_{n+2} > 1$, $L, LC \to L, L^2C$. In both cases the short circuit is isolated, and we can keep an occurrence of w if and only if we take neither the previous one nor the following one; so we can take one occurrence in two; $d(w) = \frac{1}{2} f^+$.

We remember that $f^+ = f_C + f_L = (|p_n - q_n\alpha| + |p_{n+1} - q_{n+1}\alpha|)$ and $l(w) = q_{n+1} + q_n - 2$, hence the result. □

3.3. Study of the cases $a_{n+1} \geqslant 2$, $h = (k-1)q_n + q_{n-1}$, $2 \leqslant k \leqslant a_{n+1}$

For this value of h there is a split, the next burst will be non-reversing, and $h = |L|$; we have $(k-1)|C| < h < k|C|$; the frequencies are given by (1)–(3).

We reason like in Section 3.1, but with two modifications:
- as $h = |L|$, we can always take an occurrence of w in the first circuit C of each block,
- as $h < k|C|$, we take q such that $a_{n+1} = qk + i$, $0 \leqslant i \leqslant k-1$.

As previously, we have only to study the case $i = 0 = k-2$.

Lemma 9. *For $h = q_n + q_{n-1}$, w a word of the central branch of Γ_h and $a_{n+1} = 2q$, $q \in \mathbb{N}$,*

$$\tau(w) = (q_n + q_{n-1})\left(\frac{a_{n+1}}{2} + t_{n+1}\right) f_n \tag{17}$$

$$= (q + t_{n+1})\frac{1 + v_{n+1}(1-2q)}{1 + t_{n+1}v_{n+1}}. \tag{18}$$

Proof. We have $k = 2$, $a_{n+1} = 2q$ and blocks $LC^{a_{n+1}-1}$ with frequency p_1 and $LC^{a_{n+1}}$ with frequency p_0. For $i = 0 = k-2$ we can take q occurences of w in circuits C of a block, and one occurrence in the following circuit L if and only if we are in a block $LC^{a_{n+1}}$. By the same technique as in Section 3.1.1 we find an analogous formula for $d(w)$, with e_n replaced by $e'_n = p_0 = t_{n+1}$, which gives

$$\tau(w) = (q_n + q_{n-1})\left(\frac{a_{n+1}}{2} + t_{n+1}\right) f_n$$

$$= (1 + v_n)(q + t_{n+1})q_n f_n,$$

hence (18) because $q_n f_n = v_{n+1}/(1 + t_{n+1}v_{n+1})$ and $1/v_{n+1} = 2q + v_n$. □

However, we check that, if $q > 1$, $\tau(w) \leqslant 1/(1 + t_{n+1}v_{n+1})$ and that if $q = 1$, $\tau(w) \leqslant (1 + t_{n+1})(1 + v_{n+1})/2(1 + t_{n+1}v_{n+1})$. So these cases do not bring anything new in the final formula.

3.4. Study of the cases $a_{n+1} \geqslant 2$, $h = kq_n$, $2 \leqslant k \leqslant a_{n+1}$

For this value of h there is a split, the next burst will be non-reversing, and h is a multiple of $|C|$; we have $h > |L|$, $h = k|C|$; the frequencies are given by (1)–(3); we make the computations for a word of the central branch. The reasoning is as in Section 3.1, except that as $h = k|C|$, we take q such that $a_{n+1} = qk + i$, $0 \leqslant i \leqslant k-1$.

As previously, we have only to consider the case $i = 0 = k-2$.

We have $k = 2$, $a_{n+1} = 2q$ and blocks $LC^{a_{n+1}-1}$ with frequency p_1 and $LC^{a_{n+1}}$ with frequency p_0. For $i = 0 = k-2$, if we take the occurrence of w in the first C of a block, we can take q occurrences w in a circuit C, and its occurrence in the following circuit L if and only if we are in a block $LC^{a_{n+1}}$. If we take the occurrence of w in the second C of a block, we can take q occurrences of w in a circuit C and not its occurrence in the following circuit L if we are in a block $LC^{a_{n+1}}$, $q-1$ occurrences of w in a circuit C and its occurrence in the following circuit L if we are in a block $LC^{a_{n+1}-1}$. The technique of Section 3.1.1 gives an analogous formula for $d(w)$, with e_n replaced by $e_n''=\frac{1}{2}p_0=\frac{1}{2}t_{n+1}$, which gives

$$\tau(w)=2q_n\left(\frac{a_{n+1}}{2}+\frac{t_{n+1}}{2}\right)f_n$$

$$=\frac{q_nf_n}{t_n}=\frac{1}{1+t_nv_n}\leqslant F_I,$$

these cases bring nothing new.

3.5. Study of the other cases

The other extremal cases to study are the case $h=|L|$ and the case where h is a multiple of $|C|$, but when the next burst is reversing.

The first one corresponds to $h=a_{n+1}q_n+q_{n-1}$; we check that the frequency without overlap of a word of the central branch is not greater than the one of a word of the long branch, and that $\tau(w)$ is not greater than F_I (it will be close to F_I if n is large enough).

The second one corresponds to $h=(a_{n+1}+1)q_n$, and gives an estimate not greater than for $h=(a_{n+1}+1)q_n+q_{n-1}-2$.

The previous analyses show that if $h_1<h_2$ give the same frequencies and the same relations with the lengths of the circuits (namely, for $h=h_1$ and $h=h_2$, the same values for the upper integral parts of $h/|L_h|$ and $h/|C_h|$), the greatest frequency without overlap of a word of length h_i is the same for $i=1$ and 2. The tiling $\tau(w)$ will then be the best for $h=h_2$. Hence, the general case of any h between $q_n+q_{n-1}-2$ and $q_{n+1}+q_n-2$ always give tilings not greater than in one of the particular cases studied above.

3.6. Final formula and properties of F

Proposition 4. *If*

$$v_n=[0;a_n,a_{n-1},\ldots,a_1],$$

$$t_n=[0;a_{n+1},a_{n+2},\ldots],$$

then, for the rotation of angle $\alpha=[0;a_1,a_2,\ldots]$ *and the sturmian coding,*

$$F=\max\left\{\limsup_{n\to+\infty}\frac{1}{1+t_nv_n},\right. \tag{19}$$

$$\limsup_{n\to+\infty}\frac{(1+t_n)(1+v_n)}{2(1+t_nv_n)}, \tag{20}$$

$$\limsup_{\substack{n\to+\infty\\ a_n=3\\ a_{n+1}=1}} \left(\frac{3}{2}\right)\frac{1-v_n}{1+t_nv_n}, \tag{21}$$

$$\left.\limsup_{\substack{n\to+\infty\\ a_n\equiv 0 \bmod 3\\ a_{n+1}=2}} \left(\frac{a_n}{3}+t_n\right)\frac{1+v_n(2-a_n)}{1+t_nv_n}\right\}. \tag{22}$$

Proof. The above discussion gives these formulas, after replacing n by $n-1$ for aesthetic reasons. The comparison of formulas, see below, implies that (21) does not apply if $a_n>3$ (because then (20) prevails) and that (22) does not apply if $a_{n+1}>2$ (because then (19) prevails).

Proposition 5. *$F=1$ if and only if α has unbounded partial quotients.*

Proof. Every quantity in formulae (19)–(22) is bounded away from 0 and 1 if α has bounded partial quotients.

If α and β have the same a_n for all n large enough, they have the same F; F is not a complete invariant for measure-theoretic isomorphism, even if we restrict ourselves to the class of bounded partial quotients. □

Comparison of formulas: For a given value of n,

- (19) prevails over (20) if and only if $(1+t_n)(1+v_n)\leqslant 2$; (20) prevails if $(a_n,a_{n+1})=(1,1)$, $(1,2)$ ou $(2,1)$, and (19) prevails if $a_n\geqslant 2$, $a_{n+1}\geqslant 2$ and at least one of them is not smaller than 3;
- if $a_n=3$ and $a_{n+1}=1$, (21) always prevails over (19);
- if $a_n=3$ and $a_{n+1}=1$, (21) prevails over (20) if and only if $v_{n-1}+2t_{n+1}(1+v_{n-1})\geqslant 2$;
- if $a_n=3q$ for an integer q and $a_{n+1}=2$, (19) prevails over (22) if and only if $v_{n-1}\leqslant(q-2t_n)/(q+t_n-1)$.

Corollary 3. *For $\alpha=\theta-1$*

$$F=\frac{4\theta+3}{10}=\frac{5+2\sqrt{5}}{10},$$

which represents the proportion of the Fibonacci sequence which can be covered by occurrences without overlap of one word (see Definition 4).

Other examples: F is given by formula (20) for the inverse of the golden ratio number $\alpha=\theta-1$, but also for example for the periodic expansion $[0;1,6,1,6,\ldots]$.

We may have $F=F_I$ also for α with bounded partial quotients; this is the case, for example, as soon as all the a_n are not smaller than 3. For the periodic expansions $\alpha=[0;4,6,4,6,\ldots]$ and $\beta=[0;3,8,3,8,\ldots]$, α and β give the same F_I and the same F without having any a_n in common.

Unfortunately, we have not been able to find an explicit example where F is given by formula (21).

F is given by formula (22), for example, for the periodic expansion $\alpha = [0; 2, 3, 2, 3, \ldots]$. If $\beta = [0; 1, 6, 1, 6, \ldots]$, α and β give the same F_I but different F.

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