Ain Shams Engineering Journal (2016) **xxx**, xxx–xxx

Ain Shams University
Ain Shams Engineering Journal
www.elsevier.com/locate/asej
www.sciencedirect.com

ENGINEERING PHYSICS AND MATHEMATICS

Numerical study of a class of variable order nonlinear fractional differential equation in terms of Bernstein polynomials

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Received 13 June 2014; revised 10 August 2015; accepted 5 July 2016

KEYWORDS

Bernstein polynomials;
Variable order fractional nonlinear differential equation;
Operational matrix;
Numerical solution;
Convergence analysis;
The absolute error

Abstract In this paper, we use Bernstein polynomials to seek the numerical solution of a class of nonlinear variable order fractional differential equation. The fractional derivative is described in the Caputo sense. Three different kinds of operational matrixes with Bernstein polynomials are derived and are utilized to transform the initial equation into the products of several dependent matrixes which can also be regarded as the system of nonlinear equations after dispersing the variable. By solving the system of equations, the numerical solutions are acquired. Numerical examples are provided to show that the method is computationally efficient and accurate.

1. Introduction

In recent years, fractional calculus has attracted many researchers successfully in different disciplines of science and engineering [1]. Recently, more and more researchers are finding that numerous important dynamical problems exhibit fractional order behavior which may vary with space and time. This fact illustrates that variable order calculus provides an effective mathematical framework for the description of complex dynamical problems. The concept of a variable order operator is a much more recent development, which is a new orientation in science. Different authors have proposed different definitions of variable order differential operators, each of these with a specific meaning to suit desired goals. The variable order operator definitions recently proposed in the science include the Riemann-Liouville definition, Caputo definition, Marchaud definition, Coimbra definition and Grünwald definition [2–7].

Since the kernel of the variable order operators is too complex for having a variable-exponent, the numerical solutions of

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Peer review under responsibility of Ain Shams University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.asej.2016.07.002
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variable order fractional differential equations are quite difficult to obtain, and have not attracted much attention. To the best of the authors' knowledge, there are few references appeared on the discussion of the numerical of variable order fractional differential equation. Coimbra [4] employed a consistent approximation with first-order accurate for the solution of variable order differential equations. Soon et al. [7] proposed a second-order Runge–Kutta method which is consisting of an explicit Euler predictor step followed by an implicit Euler corrector step to numerically integrate the variable order differential equation. Lin et al. [8] studied the stability and the convergence of an explicit finite-difference approximation for the variable-order fractional diffusion equation with a nonlinear source term. Zhuang et al. [9] obtained explicit and implicit Euler approximations for the fractional advection–diffusion nonlinear equation of variable-order. Aiming a variable-order anomalous subdiffusion equation, Chen et al. [10] employed two numerical schemes one fourth order spatial accuracy and with first order temporal accuracy, the other with fourth order spatial accuracy and second order temporal accuracy. However, as far as we know, no one had attempted to seek the numerical solution of the variable order fractional differential equations.

So in this paper, we introduce the Bernstein polynomials to seek the numerical solution of the variable order fractional equation. With the simple structure and perfect properties [11,12], Bernstein polynomials play an important role in the solution of integral equations and differential equations [11–17].

In this paper, our study focuses on a class of variable order fractional nonlinear differential equation as follows:

$$D^{\alpha(t)}(u^2(t)) + D^{\beta(t)}u(t) + u'(t) = f(t) \quad (0 < \alpha(t), \beta(t) \leqslant 1) \tag{1}$$

where $D^{\alpha(t)}(u^2(t))$ and $D^{\beta(t)}(u(t))$ are fractional derivative in Caputo sense. Among $u(t), f(t)$ are assumed to be casual functions on $[0,1]$, and $f(t)$ is known and $u(t)$ is unknown.

The reminder of the paper is organized as follows: Sections 2 and 3 are preparative, the definitions and properties of the variable order fractional order integrals and derivatives and Bernstein polynomials are given in Sections 2 and 3 respectively. In Section 4, three kinds of operational matrixes with Bernstein polynomials are derived, and we applied the operational matrixes to solve the equation as given at beginning. In Section 5, we present some numerical examples to illustrate the method and demonstrate efficiency of the method. We end the paper with a few concluding remarks in Section 6.

2. Basic definitions and properties of the variable order fractional integrals and derivatives

In this section, before stating our main results, we firstly provide some basic definitions and properties of the variable order fractional order calculus [2–7].

Definition 2.1. Caputo's fractional derivative with order $\alpha(t)$, $(0 < \alpha(t) \leqslant 1)$ is

$$D^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{0+}^{t}(t-\tau)^{-\alpha(t)}u'(\tau)d\tau + \frac{(u(0+) - u(0-))t^{-\alpha(t)}}{\Gamma(1-\alpha(t))} \tag{2}$$

If we assume the starting time in a perfect situation, we can get the definition as follows:

$$D^{\alpha(t)}u(t) = \frac{1}{\Gamma(1-\alpha(t))}\int_{0+}^{t}(t-\tau)^{-\alpha(t)}u'(\tau)d\tau \tag{3}$$

Generally, we adopt Eq. (3) as the definition of fractional derivative in Caputo sense.

With the definition above, we can get the following formula:

$$D_*^{\alpha(t)}c = 0 \tag{4}$$

$$D_*^{\alpha(t)}x^{\beta} = \begin{cases} 0 & \beta = 0 \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha(t))}x^{\beta-\alpha(t)} & \beta = 1,2,3\cdots \end{cases} \tag{5}$$

3. Bernstein polynomials and their properties

3.1. The definition of Bernstein polynomials basis

The Bernstein Polynomials of degree n are defined by

$$B_{i,n}(x) = \binom{n}{i}x^i(1-x)^{n-i} \tag{6}$$

By using the binomial expansion of $(1-x)^{n-i}$, Eq. (6) can be expressed as

$$B_{i,n}(x) = \binom{n}{i}x^i(1-x)^{n-i} = \sum_{k=0}^{n-i}(-1)^k\binom{n}{i}\binom{n-i}{k}x^{i+k} \tag{7}$$

Now, we define

$$\boldsymbol{\Phi}(x) = [B_{0,n}(x), B_{1,n}(x), \cdots, B_{n,n}(x)]^T \tag{8}$$

where we can have

$$\boldsymbol{\Phi}(x) = \boldsymbol{A}\boldsymbol{T}_n(x) \tag{9}$$

where

$$\boldsymbol{A} = \begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & \cdots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0\binom{n}{1}\binom{n-1}{0} & \cdots & (-1)^{n-1}\binom{n}{1}\binom{n-1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^0\binom{n}{n} \end{bmatrix} \tag{10}$$

$$\boldsymbol{T}_n(x) = \left[1, x, x^2, \ldots, x^n\right]^T \tag{11}$$

Clearly

$$\boldsymbol{T}_n(x) = \boldsymbol{A}^{-1}\boldsymbol{\Phi}(x) \tag{12}$$

3.2. Function approximation

A function $f(x) \in L^2(0,1)$ can be expressed in terms of the Bernstein Polynomials basis. In practice, only the first $(n+1)$ terms of Bernstein Polynomials are considered. Hence

$$f(x) \simeq \sum_{i=0}^{n}c_iB_{i,n}(x) = c^T\boldsymbol{\Phi}(x) \tag{13}$$

where $c = [c_0, c_1, \ldots, c_n]^T$.

Then we have

$$c = Q^{-1}(f, \Phi(x)) \tag{14}$$

where Q is an $(n+1) \times (n+1)$ matrix, which is called the dual matrix of $\Phi(x)$.

$$\begin{aligned} Q &= \int_0^1 \Phi(x)\Phi^T(x)dx = \int_0^1 (AT_n(x))(AT_n(x))^T dx \\ &= A\left(\int_0^1 T_n(x)T_n^T(x)dx\right)A^T = AHA^T \end{aligned} \tag{15}$$

where H is a Hilbert matrix:

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{bmatrix} \tag{16}$$

3.3. Convergence analysis

Suppose that the function $f: [0,1] \to \mathbb{R}$ is $m+1$ times continuously differentiable, $f \in C^{m+1}[0,1]$, and $\mathbb{Y} = Span\{B_{0,n}, B_{1,n}, B_{2,n} \cdots, B_{n,n}\}$ is vector space. If $c^T\Phi(x)$ is the best approximation of f out of $\mathbb{Y}$, then the mean error bound is presented as follows:

$$\|f - c^T\Phi\|_2 \leqslant \frac{\sqrt{2}MS^{\frac{2m+3}{2}}}{(m+1)!\sqrt{2m+3}} \tag{17}$$

where $M = \max_{x\in[0,1]}|f^{(m+1)}(x)|, S = \max\{1 - x_0, x_0\}$.

Proof. We consider the Taylor polynomials

$$\begin{aligned} f_1(x) &= f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2} + \cdots \\ &\quad + f^{(m)}(x_0)\frac{(x - x_0)^m}{m!} \end{aligned}$$

which we know

$$|f(x) - f_1(x)| = \left|f^{(m+1)}(\varepsilon)\right|\frac{(x - x_0)^{m+1}}{(m+1)!} \quad \exists \varepsilon \in (0,1)$$

Since $c^T\Phi(x)$ is the best approximation of f, so we have

$$\begin{aligned} \|f - c^T\Phi\|_2^2 &\leqslant \|f - f_1\|_2^2 = \int_0^1 (f(x) - f_1(x))^2 dx \\ &= \int_0^1 \left(\left|f^{(m+1)}(\varepsilon)\right|\frac{(x-x_0)^{m+1}}{(m+1)!}\right)^2 dx \\ &\leqslant \frac{M^2}{[(m+1)!]^2}\int_0^1 (x - x_0)^{2m+2}dx \\ &\leqslant \frac{2M^2S^{2m+3}}{[(m+1)!]^2(2m+3)} \end{aligned}$$

And taking square roots we have the above bound.

4. The operational matrix in terms of Bernstein polynomials

4.1. The operational matrix of the section as $u'(t)$ in terms of Bernstein polynomials

If we approximate the function $u(t)$ with Bernstein polynomials, it can be written as Eq. (13), namely $u(t) \simeq c^T\Phi(t)$.

The differentiation of vector $\Phi(t)$ can be expressed as

$$\Phi'(t) = D\Phi(t) \tag{18}$$

where D is the $(n+1) \times (n+1)$ operational matrix of derivatives for Bernstein polynomials. Form Eq. (11) we have

$$\Phi'(t) = A\begin{bmatrix} 0 \\ 1 \\ \vdots \\ nt^{n-1} \end{bmatrix} \tag{19}$$

Define the $(n+1) \times (n)$ matrix $V_{(n+1)\times n}$ and vector $T_n^*(t)$ as

$$V_{(n+1)\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{bmatrix}, \quad T_n^*(x) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{n-1} \end{bmatrix}_{(n\times 1)} \tag{20}$$

Eq. (19) may then be restated as

$$\Phi'(t) = AV_{(n+1)\times n}T_n^*(t) \tag{21}$$

We now expand vector $T_n^*(t)$ in terms of $\Phi(t)$. From Eq. (12), we get

$$T_n^*(t) = \mathbf{B}^*\Phi(t) \tag{22}$$

where

$$B^* = \begin{bmatrix} A_{[1]}^{-1} \\ A_{[2]}^{-1} \\ \vdots \\ A_{[n]}^{-1} \end{bmatrix} \tag{23}$$

$A_{[k]}^{-1}$ is kth row of A^{-1}, $k = 1, 2, \cdots, n$.

Then we have

$$\Phi'(t) = AV_{(n+1)\times n}B^*\Phi(t) \tag{24}$$

Therefore we get the operational matrix of the section as $u'(t)$ as follows:

$$u'(t) = [c^T\Phi(t)]' = c^T\Phi'(t) = c^TAV_{(n+1)\times n}B^*\Phi(t) \tag{25}$$

4.2. The operational matrix of the section as $D^{\alpha(t)}(u^2(t))$ in terms of Bernstein polynomials

According to Eqs. (5) and (13), we have

$$\begin{aligned} & D^{\alpha(t)}(u^2(t)) \\ &= D^{\alpha(t)}(c^T\Phi(t)\Phi^T(t)c) \\ &= D^{\alpha(t)}(c^TAT_n^*(t)(AT_n^*(t))^Tc) \\ &= D^{\alpha(t)}(c^TAT_n^*(t)T_n^{*T}(t)A^Tc) \\ &= c^TAD^{\alpha(t)}(T_n^*(t)T_n^{*T}(t))A^Tc \\ &= c^TAD^{\alpha(t)}\left(\begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix}[1 \ t \ \ldots \ t^n]\right)A^Tc \end{aligned}$$

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$$= \boldsymbol{c}^T \boldsymbol{A} \boldsymbol{D}^{\alpha(t)} \begin{bmatrix} 1 & t & \dots & t^n \\ t & t^2 & \dots & t^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ t^n & t^{2n} & \dots & t^{2n} \end{bmatrix} \boldsymbol{A}^T \boldsymbol{c}$$

$$= \boldsymbol{c}^T \boldsymbol{A} \begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))} t^{1-\alpha(t)} & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{n-\alpha(t)} \\ \frac{\Gamma(2)}{\Gamma(2-\alpha(t))} t^{1-\alpha(t)} & \frac{\Gamma(3)}{\Gamma(3-\alpha(t))} t^{2-\alpha(t)} & \dots & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))} t^{n+1-\alpha(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{n-\alpha(t)} & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))} t^{n+1-\alpha(t)} & \dots & \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha(t))} t^{2n-\alpha(t)} \end{bmatrix} \boldsymbol{A}^T \boldsymbol{c}$$

$$= \boldsymbol{c}^T \boldsymbol{AMA}^T \boldsymbol{c}$$

Let

$$\boldsymbol{M} = \begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))} t^{1-\alpha(t)} & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{n-\alpha(t)} \\ \frac{\Gamma(2)}{\Gamma(2-\alpha(t))} t^{1-\alpha(t)} & \frac{\Gamma(3)}{\Gamma(3-\alpha(t))} t^{2-\alpha(t)} & \dots & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))} t^{n+1-\alpha(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))} t^{n-\alpha(t)} & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))} t^{n+1-\alpha(t)} & \dots & \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha(t))} t^{2n-\alpha(t)} \end{bmatrix} \tag{27}$$

$\boldsymbol{M}$ is called the operational matrix of the section as $D^{\alpha(t)}(u^2(t))$ in terms of Bernstein polynomials. So we have

$$D^{\alpha(t)}(u^2(t)) = \boldsymbol{c}^T \boldsymbol{AMA}^T \boldsymbol{c} \tag{28}$$

4.3. The operational matrix of the section as $\boldsymbol{D}^{\beta(t)}\boldsymbol{u}(\boldsymbol{t})$ in terms of Bernstein polynomials

Similar to the process above, we have

$$\begin{aligned} D^{\beta(t)} u(t) &= D^{\beta(t)} \boldsymbol{c}^T \boldsymbol{\Phi}(t) \\ &= \boldsymbol{c}^T D^{\beta(t)} \boldsymbol{\Phi}(t) \\ &= \boldsymbol{c}^T D^{\beta(t)} \boldsymbol{A} \boldsymbol{T}_n(t) \\ &= \boldsymbol{c}^T \boldsymbol{A} D^{\beta(t)} \begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} = \boldsymbol{c}^T \boldsymbol{A} \begin{bmatrix} 0 \\ \frac{\Gamma(2)}{\Gamma(2-\beta(t))} x^{-\beta(t)} \\ \frac{\Gamma(3)}{\Gamma(3-\beta(t))} x^{2-\beta(t)} \\ \vdots \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\beta(t))} x^{n-\beta(t)} \end{bmatrix} \\ &= \boldsymbol{c}^T \boldsymbol{A} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\beta(t))} t^{-\beta(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\beta(t))} t^{-\beta(t)} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ \vdots \\ t^n \end{bmatrix} \\ &= \boldsymbol{c}^T \boldsymbol{ANA}^{-1} \boldsymbol{\Phi}(t) \end{aligned} \tag{29}$$

We define

$$\boldsymbol{N} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{\Gamma(2)}{\Gamma(2-\beta(t))} t^{-\beta(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\Gamma(n+1)}{\Gamma(n+1-\beta(t))} t^{-\beta(t)} \end{bmatrix} \tag{30}$$

So the initial equation is transformed to the form as follows:

$$\boldsymbol{c}^T \boldsymbol{AMA}^T \boldsymbol{c} + \boldsymbol{c}^T \boldsymbol{ANA}^{-1} \boldsymbol{\Phi}(t) + \boldsymbol{c}^T \boldsymbol{A} \boldsymbol{V}_{(n+1)\times n} \boldsymbol{B}^* \boldsymbol{\Phi}(t) = f(t) \tag{31}$$

Dispersing Eq. (31) with the variable t, by using Mathematica 9.0, we can obtain $\boldsymbol{c}$. So the numerical solution of the original problem is obtained ultimately.

5. Numerical examples

In order to demonstrate the efficiency and the practicability of the proposed method, we present an example and find its numerical solutions through the method described in Section 4.

Example 1.

$$\begin{aligned} &D^{\frac{t}{4}}(u^2(t)) + D^{\frac{t}{3}} u(t) + u'(t) = f(t) \\ &u(0) = 0 \quad t \in [0, 1] \end{aligned}$$

where

$$f(t) = 2t + \frac{18 t^{2-\frac{t}{3}}}{(18 - 9t + t^2)\Gamma\left(1 - \frac{t}{3}\right)} + \frac{6144 t^{4-\frac{t}{4}}}{(-16+t)(-12+t)(-8+t)(-4+t)\Gamma\left(1 - \frac{t}{4}\right)}.$$

The exact solution of the problem is $u(t) = t^2$.

We solved the problem by adopting of the technique described in Section 4 with using of Mathematica 9.0.

Taking $n = 2$, dispersing $t_i = \frac{k_i}{2} - \frac{1}{4}$ $(k_i = 1, 2)$, we get $\boldsymbol{c} = \begin{bmatrix} 0 & -1.25 \times 10^{-16} & 1 \end{bmatrix}^T$, so the numerical solution is $u(t) = c^T \boldsymbol{\Phi}(t)$, where $\boldsymbol{\Phi}(t) = \begin{bmatrix} (1-t)^2 & 2(1-t)t & t^2 \end{bmatrix}^T$. In other words, the algebraic expression is $u(t) = -1.25 \times 10^{-16}(1-t)t + t^2$. The absolute error between the numerical solution and exact solution is displayed as in Fig. 1.

Taking $n = 3$, dispersing $t_i = \frac{k_i}{3} - \frac{1}{6}(k_i = 1, 2, 3)$, we get $\boldsymbol{c} = [0 \quad 0 \quad 1 \quad 1]^T$, so the numerical solution is $u(t) = \boldsymbol{c}^T \boldsymbol{\Phi}(t)$ $\quad \boldsymbol{\Phi}(t) = \begin{bmatrix} (1-t)^3 & 3(1-t)^2 t & 3(1-t)t^2 & t^3 \end{bmatrix}$. In other words, the algebraic expression is $u(t) = (1-t)t^2 + t^3$. The absolute error between the numerical solution and exact solution is displayed as in Fig. 2.

Taking $n = 4$, dispersing $t_i = \frac{k_i}{4} - \frac{1}{8}(k_i = 1, 2, 3, 4)$, we get $\boldsymbol{c}_1 = [0 \quad 0 \quad 1 \quad 2 \quad 1]^T$, so the solution is $u(t) = \boldsymbol{c}^T \boldsymbol{\Phi}(t)$, where $\boldsymbol{\Phi}(t) = \left[(1-t)^4, (1-t)^3 t, (1-t)^2 t^2, (1-t)t^3, t^4\right]$. In other words, the algebraic expression is $u(t) = (1-t)^2 t^2 + 2(1-t)t^3 + t^4$.

The absolute error between the numerical solution and exact solution is displayed as in Fig. 3.

Example 2.

$$\begin{aligned} &D^{\frac{\cos t}{3}}(u^2(t)) + D^{\frac{t}{4}} u(t) + u'(t) = f(t) \\ &u(0) = 0 \quad t \in [0, 1] \end{aligned}$$

where

$$f(x,t) = 1 + 2t + \frac{4 t^{1-\frac{t}{4}}(8+7t)}{(-8+t)(-4+t)\Gamma\left(1-\frac{t}{4}\right)} + \frac{18 t^{2-\frac{\cos t}{3}}\left[108(1+t)^2 - 3(7+6t)\cos t + \cos^2 t\right]}{(\cos t - 12)(\cos t - 9)(\cos t - 6)(\cos t - 3)\Gamma\left(1 - \frac{\cos t}{3}\right)}$$

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The exact solution of the above problem is $u(t) = t + t^2$.

Taking $n = 2$, dispersing $t_i = \frac{k_i}{2} - \frac{1}{4}(k_i = 1, 2)$, we get $\boldsymbol{c} = [0 \quad 0.5 \quad 2]^T$, so the numerical solution is $u(t) = \boldsymbol{c}^T\boldsymbol{\Phi}(t)$, where $\boldsymbol{\Phi}(t) = \left[(1-t)^2 \quad 2(1-t)t \quad t^2\right]^T$. In other words, the algebraic expression is $u(t) = (1-t)t + 2t^2$. The absolute error between the numerical solution and exact solution is displayed as in Fig. 4.

Taking $n = 3$, dispersing $t_i = \frac{k_i}{3} - \frac{1}{6}$ $(k_i = 1, 2, 3)$, we get $c = \left[0 \quad \frac{1}{3} \quad 1 \quad 2\right]^T$, so the numerical solution is $u(t) = c^T\boldsymbol{\Phi}(t)$, and $\boldsymbol{\Phi}(t) = \left[(1-t)^3 \quad 3(1-t)^2t \quad 3(1-t)t^2 \quad t^3\right]$. The algebraic expression is $u(t) = (1-t)t^2 + 3(1-t)t^2 + 2t^3$. The absolute error between the numerical solution and exact solution is displayed as in Fig. 5.

Taking $n = 4$, dispersing $t_i = \frac{k_i}{4} - \frac{1}{8}$ $(k_i = 1, 2, 3, 4)$, we get $\boldsymbol{c}_1 = \left[0 \quad \frac{1}{4} \quad \frac{2}{3} \quad \frac{5}{4} \quad 2\right]^T$, so the solution is $u(t) = \boldsymbol{c}^T\boldsymbol{\Phi}(t)$, $\boldsymbol{\Phi}(t) = \left[(1-t)^4, 4(1-t)^3t, 6(1-t)^2t^2, 4(1-t)t^3, t^4\right]$. In other words, the algebraic expression is $u(t) = (1-t)^3 + 4(1-t)^2t^2 + 5(1-t)t^3 + 2t^4$. The absolute error between the numerical solution and exact solution is displayed as in Fig. 6.

Example 3.

$$D^{\frac{e^t}{3}}(u^2(t)) + D^{\frac{t}{2}}u(t) + u'(t) = f(t)$$
$$u(0) = 0 \quad t \in [0, 1]$$

where

$$f(x,t) = 3t^2 + \frac{720t^{6-\frac{e^t}{3}}}{\Gamma\left(7 - \frac{e^t}{3}\right)} - \frac{48t^{3-\frac{t}{2}}}{(t-6)(t-4)(t-2)\Gamma\left(1 - \frac{t}{2}\right)}$$

The exact solution of the problem is $u(t) = t^3$

Taking $n = 3$, dispersing $t_i = \frac{k_i}{3} - \frac{1}{6}(k_i = 1, 2, 3)$, we can get the coefficient as $\boldsymbol{c} = \left[0 \quad 4.09 \times 10^{-17} \quad 7.77 \times 10^{-17} \quad 1\right]^T$, so the numerical solution is $u(t) = \boldsymbol{c}^T\boldsymbol{\Phi}(t)$, where $\boldsymbol{\Phi}(t) = \left[(1-t)^3 \quad 3(1-t)^2t \quad 3(1-t)t^2 \quad t^3\right]$. In other words, the algebraic expression is $u(t) = -1.23 \times 10^{-16}(1-t)^2t + 2.33 \times 10^{-16}(1-t)t^2 + t^3$. The absolute error between the numerical solution and exact solution is displayed as in Fig. 7.

Taking $n = 4$, dispersing $t_i = \frac{k_i}{4} - \frac{1}{8}$ $(k_i = 1, 2, 3, 4)$, we get $\boldsymbol{c}_1 = \left[0 \quad 0 \quad 0 \quad \frac{1}{4} \quad 1\right]^T$, so the solution is $u(t) = \boldsymbol{c}^T\boldsymbol{\Phi}(t)$, $\boldsymbol{\Phi}(t) = \left[(1-t)^4, 4(1-t)^3t, 6(1-t)^2t^2, 4(1-t)t^3, t^4\right]$. In other words, the algebraic expression is $u(t) = (1-t)t^3 + t^4$. The absolute error between the numerical solution and exact solution is displayed as in Fig. 8.

From Examples 1–3, we can draw a conclusion that no matter what type of the derivative is, the numerical solutions are in quite agreement with the exact solutions.

From Figs. 1–8, we can see that the absolute error is very small and only a small number of Bernstein polynomials are needed to get a satisfactory result. From the above example, we can draw a conclusion that the approach proposed in this paper can be effectively used in seeking the numerical solution of the variable order fractional nonlinear integral-differential

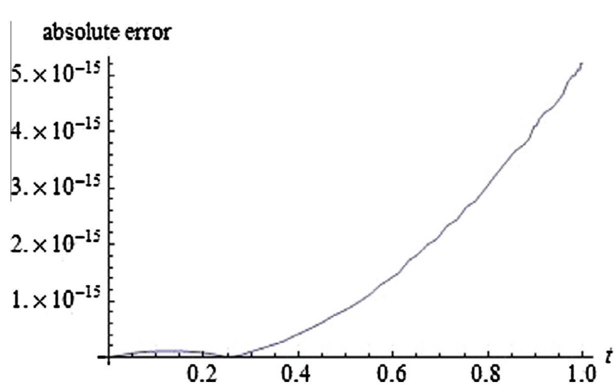

Figure 1 The absolute error when $n = 2$ for Example 1.

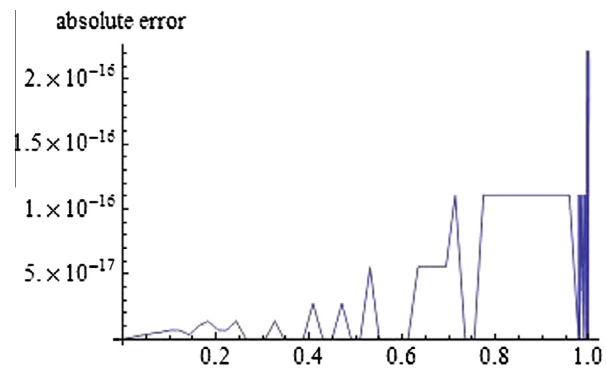

Figure 2 The absolute error when $n = 3$ for Example 1.

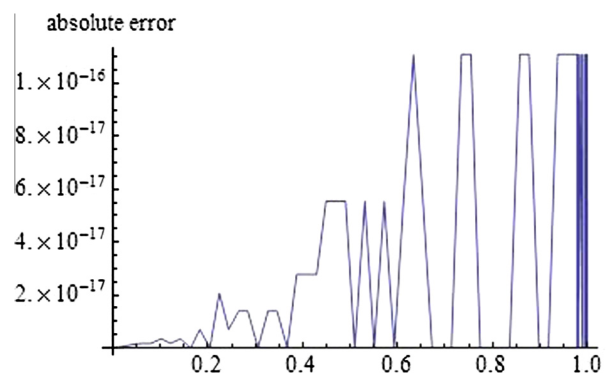

Figure 3 The absolute error when $n = 4$ for Example 1.

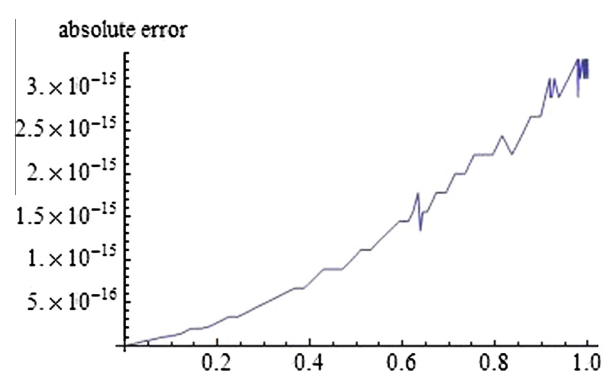

Figure 4 The absolute error when $n = 2$ for Example 2.

Please cite this article in press as: Chen Y-m et al., Numerical study of a class of variable order nonlinear fractional differential equation in terms of Bernstein polynomials, Ain Shams Eng J (2016), http://dx.doi.org/10.1016/j.asej.2016.07.002

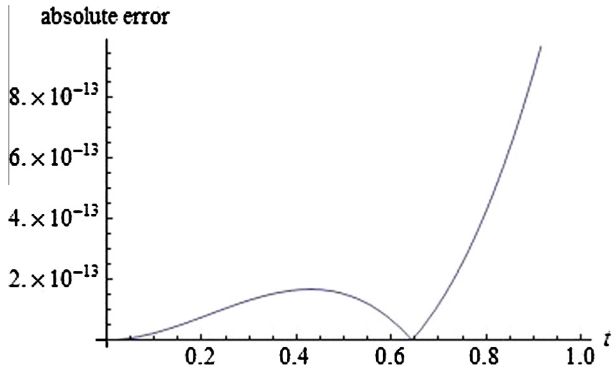

Figure 5 The absolute error when $n = 3$ for Example 2.

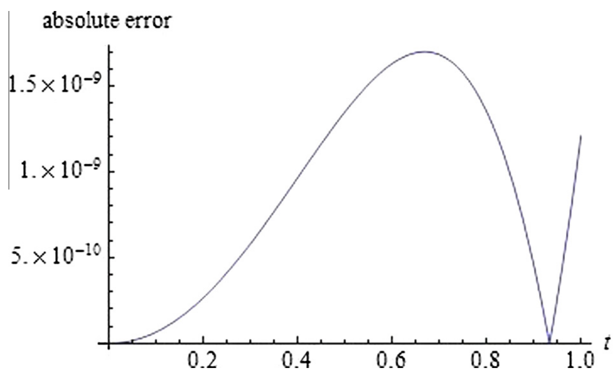

Figure 6 The absolute error when $n = 4$ for Example 2.

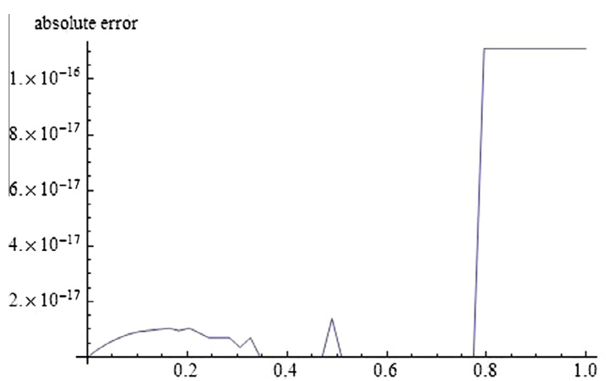

Figure 7 The absolute error when $n = 3$ for Example 3.

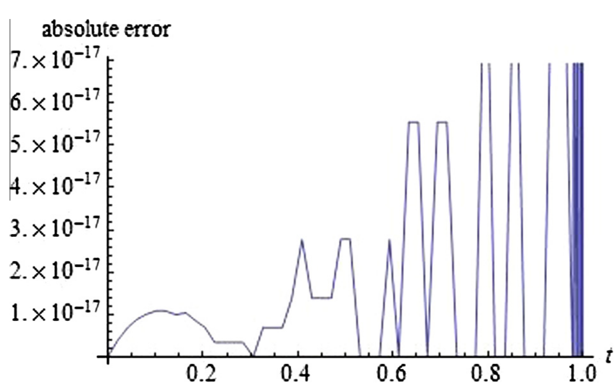

Figure 8 The absolute error when $n = 4$ for Example 3.

equation. At the same time we also prove the feasibility of the method. Also we can find that the numerical solutions are in good agreement with the exact solution. At last, it is worth mentioning that the proposed method is more convenient in computation than other methods such as the method in Ref. [8–10].

6. Conclusion

In this work, three kinds of fractional operational matrixes which contain the variable x or t in terms of Bernstein polynomials are derived and are utilized to seek the numerical solution of the variable order fractional nonlinear equations. With the operational matrixes, we transformed the initial equation into the products of some matrixes which can also be viewed as the system of algebraic nonlinear equations after dispersing the variable. Solving the nonlinear equations, the numerical solutions can be obtained.

As is known to all, it is difficult to solve the fractional nonlinear differential equations. The method proposed in this article is simple in theory and easy in computation, so this method has deserving applications in solving the various kinds of fractional differential equations.

Acknowledgment

(1) This work was supported by the Natural Foundation of Hebei Province (A2012203407).
(2) This work was supported by Qinhuangdao research and development program of science and technology, Adaptive Boundary Element Method of precision rolling process simulation (201201B019).
(3) This work was supported by Qinhuangdao Technology Bureau 2013 research and development projects of science and technology (201302A023).

The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

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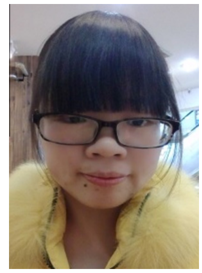

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