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ScienceDirect
Advances in Applied Mathematics 40 (2008) 377–400
ADVANCES IN Applied Mathematics
www.elsevier.com/locate/yaama

The BG-rank of a partition and its applications ☆

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Received 14 December 2006; accepted 9 April 2007

Available online 6 June 2007

Abstract

Let π denote a partition into parts $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots$. In a 2006 paper we defined BG-rank(π) as

$$\text{BG-rank}(\pi) = \sum_{j \geqslant 1} (-1)^{j+1} \frac{1-(-1)^{\lambda_j}}{2}.$$

This statistic was employed to generalize and refine the famous Ramanujan modulo 5 partition congruence. Let $p_j(n)$ denote the number of partitions of n with BG-rank $= j$. Here, we provide a combinatorial proof that

$$p_j(5n+4) \equiv 0 \pmod 5, \quad j \in \mathbb{Z},$$

by showing that the residue of the 5-core crank mod 5 divides the partitions enumerated by $p_j(5n+4)$ into five equal classes. This proof uses the orbit construction from our previous paper and a new identity for the BG-rank. Let $a_{t,j}(n)$ denote the number of t-cores of n with BG-rank $= j$. We find eta-quotient representations for

$$\sum_{n \geqslant 0} a_{t,\lfloor \frac{t+1}{4} \rfloor}(n) q^n \quad \text{and} \quad \sum_{n \geqslant 0} a_{t,-\lfloor \frac{t-1}{4} \rfloor}(n) q^n,$$

when t is an odd, positive integer. Finally, we derive explicit formulas for the coefficients $a_{5,j}(n)$, $j = 0, \pm 1$.
Published by Elsevier Inc.

MSC: primary 11P81, 11P83; secondary 05A17, 05A19

Keywords: Partitions; t-Cores; BG-rank; *eta*-Quotients; Lambert series; Theta series; Even–odd dissections

☆ Research of both authors was supported in part by NSA grant MSPF-06G-150.

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0196-8858/$ – see front matter Published by Elsevier Inc.
doi:10.1016/j.aam.2007.04.002

1. Introduction

A partition π is a nonincreasing sequence

$$\pi = (\lambda_1, \lambda_2, \lambda_3, \ldots)$$

of positive integers (parts) $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots$. The norm of π, denoted $|\pi|$, is defined as

$$|\pi| = \sum_{i \geqslant 1} \lambda_i.$$

If $|\pi| = n$, we say that π is a partition of n. The (Young) diagram of π is a convenient way to represent π graphically: the parts of π are shown as rows of unit squares (cells). Given the diagram of π we label a cell in the ith row and jth column by the least nonnegative integer $\equiv j - i \pmod t$. The resulting diagram is called a t-residue diagram [7]. We can also label cells in the infinite column 0 and the infinite row 0 in the same fashion and call the resulting diagram the extended t-residue diagram [5]. And so with each partition π and positive integer t we can associate the t-dimensional vector

$$\vec{r}(\pi, t) = \bigl(r_0(\pi, t), r_1(\pi, t), \ldots, r_{t-1}(\pi, t)\bigr)$$

with

$$r_i(\pi, t) = r_i, \quad 0 \leqslant i \leqslant t - 1,$$

being the number of cells colored i in the t-residue diagram of π. If some cell of π shares a vertex or edge with the rim of the diagram of π, we call this cell a rim cell of π. A connected collection of rim cells of π is called a rim hook if (diagram of π)\(rim hook) represents a legitimate partition. We say that a partition is a t-core, denoted $\pi_{t\text{-core}}$, if its diagram has no rim hooks of length t [7].

The Durfee square of π is the largest square that fits inside the diagram of π. Reflecting the diagram of π about its main diagonal, one gets the diagram of π' (the conjugate of π). More formally,

$$\pi' = \bigl(\lambda'_1, \lambda'_2, \lambda'_3, \ldots\bigr)$$

with λ'_i being the number of parts of π that are $\geqslant i$. In [2] we defined a new partition statistic

$$\text{BG-rank}(\pi) := \sum_{j \geqslant 1} (-1)^j \frac{(-1)^{\lambda_j} - 1}{2}. \tag{1.1}$$

It is easy to verify that

$$\text{BG-rank}(\pi) = r_0(\pi, 2) - r_1(\pi, 2) \tag{1.2}$$

and

$$\text{BG-rank}(\pi) \equiv |\pi| \pmod 2. \tag{1.3}$$

In [2] we proved the following (mod 5) congruences

$$p_j(5n) \equiv 0 \pmod 5 \quad \text{if } j \equiv 1, 2 \pmod 5, \tag{1.4}$$

$$p_j(5n+1) \equiv 0 \pmod 5 \quad \text{if } j \not\equiv 1, 2 \pmod 5, \tag{1.5}$$

$$p_j(5n+2) \equiv 0 \pmod 5 \quad \text{if } j \not\equiv 0, 3 \pmod 5, \tag{1.6}$$

$$p_j(5n+3) \equiv 0 \pmod 5 \quad \text{if } j \equiv 0, 3 \pmod 5, \tag{1.7}$$

$$p_j(5n+4) \equiv 0 \pmod 5 \quad \text{for all } j \in \mathbb{Z}. \tag{1.8}$$

Here $p_j(n)$ denotes the number of partitions of n with BG-rank $= j$. Clearly,

$$p(5n+4) = \sum_j p_j(5n+4)$$

with $p(n)$ denoting the number of unrestricted partitions of n. And so (1.8) implies the famous Ramanujan congruence [11]

$$p(5n+4) \equiv 0 \pmod 5.$$

In this paper, we build on the developments in [2] to provide a combinatorial proof of (1.8).

For t-odd it is surprising that the BG-rank($\pi_{t\text{-core}}$) assumes only finitely many values. In fact, we will show that if t is an odd, positive integer, then

$$-\left\lfloor \frac{t-1}{4} \right\rfloor \leqslant \text{BG-rank}(\pi_{t\text{-core}}) \leqslant \left\lfloor \frac{t+1}{4} \right\rfloor. \tag{1.9}$$

Here $\lfloor x \rfloor$ denotes the integer part of x.

We will establish the following identities. For odd $t > 1$

$$C_{t,(-1)^{\frac{t-1}{2}} \lfloor \frac{t-1}{4} \rfloor}(q) = q^{\frac{(t-1)(t-3)}{8}} F\bigl(t, q^2\bigr), \tag{1.10}$$

$$C_{t,(-1)^{\frac{t+1}{2}} \lfloor \frac{t+1}{4} \rfloor}(q) = q^{\frac{t^2-1}{8}} \frac{E^t(q^{4t})}{E(q^4)}, \tag{1.11}$$

where

$$C_{t,j}(q) = \sum_{n \geqslant 0} a_{t,j}(n) q^n,$$

$a_{t,j}(n)$ denotes the number of t-cores of n with BG-rank $= j$ and

$$E(q) = \prod_{j=1}^{\infty} \bigl(1 - q^j\bigr),$$

$$F(t,q) = \frac{E^{t-4}(q^{2t}) E^2(q^t) E^3(q^2)}{E^2(q)}.$$

We observe that (1.3) suggests that $C_{t,j}(q)$ is an even (odd) function of q if j is even (odd).

It is instructive to compare (1.10), (1.11) with the well-known identity [5] for unrestricted t-cores

$$\sum_{n\geqslant 0} a_t(n)q^n = \frac{E^t(q^t)}{E(q)}. \tag{1.12}$$

Here $a_t(n)$ denotes the number of t-cores of n.

The rest of this paper is organised as follows.

In Section 2 we discuss the Littlewood decomposition of π in terms of t-core and t-quotient of π. We describe the Garvan, Kim, Stanton bijection for t-cores and use a constant term technique to provide a simple proof of the Klyachko identity [8]

$$\sum_{\substack{\vec{n}\in\mathbb{Z}^t\\ \vec{n}\cdot\vec{1}_t=0}} q^{\frac{t}{2}\vec{n}\cdot\vec{n}+\vec{b}_t\cdot\vec{n}} = \frac{E^t(q^t)}{E(q)}. \tag{1.13}$$

Here $\vec{1}_t = (1,1,\ldots,1)\in\mathbb{Z}^t$, $\vec{b}_t = (0,1,2,\ldots,t-1)$.

In Section 3 we establish a fundamental identity connecting BG-rank and the Littlewood decomposition.

In Section 4 we discuss a combinatorial proof of (1.8).

Section 5 is devoted to the proof of the identities (1.10), (1.11).

Section 6 deals with 5-cores with prescribed BG-rank. There we derive the explicit formulas for the coefficients $a_{5,j}(n)$, $j=0,\pm 1$.

In Section 7 we give a generalization of the BG-rank and state a number of results.

2. Two bijections

In this section we will follow closely the discussion in [4,5] to recall some basic facts about t-cores and t-quotients. A region r in the extended t-residue diagram of π is the set of cells (i,j) satisfying $t(r-1)\leqslant j-i<tr$. A cell of π is called exposed if it is at the end of a row. One can construct t bi-infinite words $W_0, W_1, \ldots, W_{t-1}$ of two letters N, E as

$$\text{the } r\text{th letter of } W_i = \begin{cases} E, & \text{if there is an exposed cell labelled } i \text{ in the region } r,\\ N, & \text{otherwise.}\end{cases}$$

It is easy to see that the word set $\{W_0, W_1, \ldots, W_{t-1}\}$ fixes π uniquely.

Let P be the set of all partitions and $P_{t\text{-core}}$ be the set of all t-cores. There is a well-known bijection

$$\phi_1 : P \to P_{t\text{-core}} \times P \times P \times P \times \cdots \times P$$

which goes back to Littlewood [9]

$$\phi_1(\pi) = (\pi_{t\text{-core}}, \hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1})$$

such that

$$|\pi| = |\pi_{t\text{-core}}| + t\sum_{i=0}^{t-1}|\hat{\pi}_i|.$$

Multipartition $(\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1})$ is called the t-quotient of π. We remark that (1.12) is the immediate corollary of the Littlewood bijection. We describe ϕ_1 in full detail a bit later.

The second bijection

$$\phi_2 : P_{t\text{-core}} \to \left\{\vec{n}\colon\ \vec{n} \in \mathbb{Z}^t,\ \vec{n}\cdot\vec{1}_t = 0\right\}$$

was introduced in [5]. It is for t-cores only

$$\phi_2(\pi_{t\text{-core}}) = \vec{n} = (n_0, n_1, \ldots, n_{t-1})$$

where for $0 \leqslant i \leqslant t-2$

$$n_i = r_i(\pi_{t\text{-core}}, t) - r_{i+1}(\pi_{t\text{-core}}, t) \tag{2.1}$$

and

$$n_{t-1} = r_{t-1}(\pi_{t\text{-core}}, t) - r_0(\pi_{t\text{-core}}, t). \tag{2.2}$$

Clearly,

$$\sum_{i=0}^{t-1} n_i = \vec{n}\cdot\vec{1}_t = 0.$$

Moreover,

$$|\pi_{t\text{-core}}| = \frac{t}{2}\vec{n}\cdot\vec{n} + \vec{b}_t\cdot\vec{n}, \tag{2.3}$$

as shown in [5]. And so

$$\sum_{n\geqslant 0} a_t(n)q^n = \sum_{\substack{\vec{n}\in\mathbb{Z}^t\\ \vec{n}\cdot\vec{1}_t=0}} q^{\frac{t}{2}\vec{n}\cdot\vec{n}+\vec{b}_t\cdot\vec{n}}. \tag{2.4}$$

Note that (1.12), (2.4) imply the Klyachko identity (1.13). The reader may wonder if (2.1), (2.2) can be used to define $\phi_2(\pi) = \vec{n}$ for any partition π. This, of course, can be done. However, in general ϕ_2 is not a 1–1 function and so ϕ_2^{-1} cannot be defined. Indeed, if $\pi_1 \neq \pi_2$, but $\pi_{t\text{-core}}$ is a t-core of both π_1 and π_2 then

$$\phi_2(\pi_1) = \phi_2(\pi_2) = \phi_2(\pi_{t\text{-core}}).$$

When a partition is a t-core, ϕ_2 can be inverted. To do this we recall that the partition is a t-core iff for $0 \leqslant i \leqslant t-1$

$$\begin{array}{rcccccc} \text{Region}: & \cdots\cdots\cdots & n_i-1 & n_i & n_i+1 & n_i+2 & \cdots\cdots\cdots \\ W_i: & \cdots\cdots\cdots & E & E & N & N & \cdots\cdots\cdots \end{array}$$

as explained in [5]. For example, the word image of $\phi_2^{-1}((2,-1,-1))$ is

$$\begin{array}{rccccccc} \text{Region}: & \cdots\cdots & -1 & 0 & 1 & 2 & 3 & \cdots\cdots \\ W_0: & \cdots\cdots & E & E & E & E & N & \cdots\cdots \\ W_1: & \cdots\cdots & E & N & N & N & N & \cdots\cdots \\ W_2: & \cdots\cdots & E & N & N & N & N & \cdots\cdots. \end{array}$$

This means that

$$\phi_2^{-1}\big((2,-1,-1)\big) = (4,2). \tag{2.5}$$

More generally, if

$$\phi_1(\pi) = (\pi_{t\text{-core}}, \hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1})$$

with

$$\hat{\pi}_i = \big(\lambda_1^{(i)}, \lambda_2^{(i)}, \ldots, \lambda_{m_i}^{(i)}\big), \quad 0 \leqslant i \leqslant t-1,$$

then cells colored i are not exposed only in the regions

$$n_i + j - \lambda_j^{(i)}, \quad 1 \leqslant j \leqslant m_i,$$

and

$$n_i + m_i + k, \quad k \geqslant 1.$$

For example, if $\hat{\pi}_i = (\lambda_1, \lambda_2, \lambda_3)$ then

$$\begin{array}{rccccccccc} \text{Region}: & \cdots\cdots & n_i+1-\lambda_1 & \cdots\cdots & n_i+2-\lambda_2 & \cdots\cdots & n_i+3-\lambda_3 & \cdots\cdots & n_i+4 & \cdots\cdots \\ W_i: & \cdots\cdots & E\ N\ E & \cdots\cdots & E\ N\ E & \cdots\cdots & E\ N\ E & \cdots\cdots & E\ N & \cdots\cdots. \end{array}$$

Clearly, one can easily determine $\vec{n}$ and $(\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1})$ from the word set $\{W_0, W_1, \ldots, W_{t-1}\}$. And so

$$\phi_1(\pi) = \big(\phi_2^{-1}(\vec{n}), \hat{\pi}_0, \ldots, \hat{\pi}_{t-1}\big).$$

We illustrate the above with the following example. If $t=3$ and $\pi = (7,5,4,3,2)$ then

$$\begin{array}{rcccccccccc} \text{Region}: & \cdots\cdots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots\cdots \\ W_0: & \cdots\cdots & E & E & E & N & E & E & N & N & \cdots\cdots \\ W_1: & \cdots\cdots & E & N & N & E & N & N & N & N & \cdots\cdots \\ W_2: & \cdots\cdots & E & N & E & N & N & N & N & N & \cdots\cdots. \end{array}$$

We have

$$\begin{aligned} n_0 &= 2, & \hat{\pi}_0 &= (2), \\ n_1 &= -1, & \hat{\pi}_1 &= (1,1), \\ n_2 &= -1, & \hat{\pi}_2 &= (1). \end{aligned}$$

Using (2.5), we obtain

$$\phi_1\big((7,5,4,3,2)\big) = \big((4,2),(2),(1,1),(1)\big).$$

To proceed further we recall some standard q-hypergeometric notations [6]:

$$(a_1, a_2, a_3, \dots; q)_N = (a_1; q)_N (a_2; q)_N (a_3; q)_N \dots$$

where

$$(a;q)_N = (a)_N = \begin{cases} \prod_{j=0}^{N-1}(1 - aq^j), & N > 0, \\ 1, & N = 0, \\ \prod_{j=1}^{-N}(1 - aq^{-j})^{-1}, & N < 0. \end{cases}$$

We shall also require the Jacobi triple product identity [6, (II.28)]

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \left(q^2, -zq, -\frac{q}{z}; q^2\right)_\infty. \tag{2.6}$$

We are now ready to prove the Klyachko identity (1.13). We will employ a so-called constant term technique. To this end we rewrite the left-hand side of (1.13) as

$$\text{LHS of (1.13)} = \big[z^0\big] \sum_{\vec{n} \in \mathbb{Z}^t} q^{\frac{t}{2}\vec{n}\cdot\vec{n} + \vec{b}_t\cdot\vec{n}} z^{\vec{n}\cdot\vec{1}_t} = \big[z^0\big] \prod_{i=0}^{t-1} \sum_{n_i=-\infty}^{\infty} q^{\frac{t}{2}n_i^2 + in_i} z^{n_i}$$

where $[z^i]f(z)$ is the coefficient of z^i in the expansion of $f(z)$ in powers of z. With the aid of (2.6) we derive

$$\begin{aligned} \text{LHS of (1.13)} &= \big[z^0\big] \prod_{i=0}^{t-1} \left(q^t, -q^{i+\frac{t}{2}} z, -\frac{q^{\frac{t}{2}}}{q^i z}; q^t\right)_\infty \\ &= \big[z^0\big] \frac{E^t(q^t)}{E(q)} \left(q, -q^{\frac{t}{2}} z, -\frac{q}{q^{\frac{t}{2}} z}; q\right)_\infty \\ &= \big[z^0\big] \left(\frac{E^t(q^t)}{E(q)} \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2} + \frac{t-1}{2} n} z^n\right) \\ &= \frac{E^t(q^t)}{E(q)}, \end{aligned}$$

as desired. The above proof is just a warm-up exercise to prepare the reader for a more sophisticated proof of (1.10) discussed in Section 5.

3. The Littlewood decomposition and BG-rank

The main goal of this section is to establish the following identities for BG-rank. If t is even and $(n_0, \dots, n_{t-1}) = \phi_2(\pi)$, then

$$\text{BG-rank}(\pi) = \sum_{i=0}^{\frac{t-2}{2}} n_{2i}. \tag{3.1}$$

If t is odd then

$$\text{BG-rank}(\pi_{t\text{-core}}) = bg(\vec{n}), \tag{3.2}$$

where $\vec{n} = \phi_2(\pi_{t\text{-core}})$ and

$$bg(\vec{n}) := \frac{1 - \sum_{j=0}^{t-1}(-1)^{j+n_j}}{4}. \tag{3.3}$$

Moreover, if t is odd and $\phi_1(\pi) = (\pi_t\text{-core}, \hat{\pi}_0, \dots, \hat{\pi}_{t-1})$ then

$$\text{BG-rank}(\pi) = \text{BG-rank}(\pi_{t\text{-core}}) + \sum_{j=0}^{t-1}(-1)^{j+n_j}\,\text{BG-rank}(\hat{\pi}_j). \tag{3.4}$$

The proof of (3.1) is straightforward. It is sufficient to observe that if some cell is colored i in the t-residue diagram of π, then it is colored $(1-(-1)^i)/2$ in the 2-residue diagram of π. And so we obtain with the aid of (1.2)

$$\begin{aligned}\text{BG-rank}(\pi) &= (r_0 + r_2 + r_4 + \cdots + r_{t-2}) - (r_1 + r_3 + r_5 + \cdots + r_{t-1})\\ &= (r_0 - r_1) + (r_2 - r_3) + \cdots + (r_{t-2} - r_{t-1})\\ &= n_0 + n_2 + \cdots + n_{t-2},\end{aligned}$$

as desired. Next, let $D(\pi) = D$ denote the size of the Durfee square of π. To prove (3.2) we begin by rewriting (1.1) as

$$\text{BG-rank}(\pi) = \frac{1}{2}\left(\text{par}(\nu) + \sum_{j=1}^{\nu}(-1)^{\lambda_j - j}\right). \tag{3.5}$$

Here $\pi = (\lambda_1, \lambda_2, \dots, \lambda_\nu)$ and $\text{par}(x)$ is defined as

$$\text{par}(x) := \frac{1-(-1)^x}{2}.$$

Next, let π_1, π_2 denote the partitions constructed from the first $D = D(\pi_{t\text{-core}})$ rows, columns of $\pi_{t\text{-core}}$, respectively. Let π_3 denote a partition whose diagram is the Durfee square of $\pi_{t\text{-core}}$. It is plain that

$$\begin{aligned}\text{BG-rank}(\pi_{t\text{-core}}) &= \text{BG-rank}(\pi_1) + \text{BG-rank}(\pi_2) - \text{BG-rank}(\pi_3)\\ &= \text{BG-rank}(\pi_1) + \text{BG-rank}(\pi_2) - \text{par}(D).\end{aligned} \tag{3.6}$$

We shall also require the following sets

$$P_+ := \{i \in \mathbb{Z}\colon\ 0 \leqslant i \leqslant t-1,\ n_i > 0\},$$
$$P_- := \{i \in \mathbb{Z}\colon\ 0 \leqslant i \leqslant t-1,\ n_i < 0\}.$$

Here n_i's are the components of $\phi_2(\pi_{t\text{-core}})$. Note that if $i \in P_+$, then i is exposed in all positive regions $\leqslant n_i$ of π_1. This observation together with (3.5) implies that

$$\begin{aligned}\text{BG-rank}(\pi_1) &= \frac{1}{2}\left(\text{par}(D) + \sum_{i\in P_+}\sum_{k=1}^{n_i}(-1)^{t(k-1)+i}\right)\\ &= \frac{1}{2}\left(\text{par}(D) + \sum_{i\in P_+}(-1)^i\,\text{par}(n_i)\right).\end{aligned} \tag{3.7}$$

In [5], the authors showed that under conjugation $\phi_2(\pi_{t\text{-core}})$ transforms as

$$(n_0, n_1, n_2, \ldots, n_{t-1}) \to (-n_{t-1}, -n_{t-2}, -n_{t-3}, \ldots, -n_0).$$

Also it is easy to see that

$$\text{BG-rank}(\pi_2) = \text{BG-rank}\bigl(\pi_2'\bigr).$$

It follows that

$$\text{BG-rank}(\pi_2) = \frac{1}{2}\left(\text{par}(D) + \sum_{i\in P_-}(-1)^i\,\text{par}(n_i)\right). \tag{3.8}$$

Combining (3.6)–(3.8) and taking into account that $\text{par}(0) = 0$ we get

$$\text{BG-rank}(\pi_{t\text{-core}}) = \frac{1}{2}\sum_{i\in P_-\cup P_+}(-1)^i\,\text{par}(n_i) = \frac{1}{2}\sum_{i=0}^{t-1}(-1)^i\,\text{par}(n_i) = \frac{1-\sum_{i=0}^{t-1}(-1)^{i+n_i}}{4},$$

as desired. Note that formula (3.2) implies that BG-rank of odd-t-cores is bounded, as stated in (1.9). Next, let $\tilde{\pi}_{0,i}, \tilde{\pi}_{2,i}, \tilde{\pi}_{3,i}, \ldots$ denote the partitions constructed from $\phi_1(\pi) = (\pi_{t\text{-core}}, \hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{t-1})$, for odd t as follows

$$\tilde{\pi}_{0,i} = \phi_1^{-1}\bigl(\pi_{t\text{-core}}, \hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{i-1}, (0), \hat{\pi}_{i+1}, \ldots, \hat{\pi}_{t-1}\bigr),$$
$$\tilde{\pi}_{1,i} = \phi_1^{-1}\bigl(\pi_t\text{-core}, \hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{i-1}, (\lambda_1), \hat{\pi}_{i+1}, \ldots, \hat{\pi}_{t-1}\bigr),$$
$$\tilde{\pi}_{2,i} = \phi_1^{-1}\bigl(\pi_{t\text{-core}}, \hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_{i-1}, (\lambda_1, \lambda_2), \hat{\pi}_{i+1}, \ldots, \hat{\pi}_{t-1}\bigr),$$
$$\cdots\cdots\cdots.$$

Here $\hat{\pi}_i = (\lambda_1, \lambda_2, \ldots, \lambda_\nu)$. Note that the W_i word of $\tilde{\pi}_{0,i}$ is

$$\begin{array}{rccccc} \text{Region}: & \cdots\cdots\cdots & n_i & n_i+1 & \cdots\cdots\cdots \\ W_i: & \cdots\cdots\cdots & E & N & \cdots\cdots\cdots. \end{array}$$

To convert $\tilde{\pi}_{0,i}$ into $\tilde{\pi}_{1,i}$ we attach a rim hook of length $t\lambda_1$ to $\tilde{\pi}_{0,i}$ so that W_i becomes

$$\begin{array}{rccccc} \text{Region}: & \cdots\cdots\cdots & n_i+1-\lambda_1 & \cdots\cdots\cdots & n_i+2, & \cdots\cdots\cdots \\ W_i: & \cdots\cdots\cdots & E\ N\ E & \cdots\cdots\cdots & E\ N\ N & \cdots\cdots\cdots. \end{array}$$

It is not hard to verify that the color of the head (north-eastern) cell of the added rim-hook in the 2-residue diagram of $\tilde{\pi}_{1,i}$ is given by $\mathrm{par}(tn_i + i) = \mathrm{par}(n_i + i)$. Observe that zeros and ones alternate along the added hook rim. This means that BG-rank does not change if λ_1 is even. If λ_1 is odd then the change is determined by the color of the added head cell, i.e.

$$\begin{aligned} \text{BG-rank}(\tilde{\pi}_{1,i}) &= \text{BG-rank}(\tilde{\pi}_{0,i}) + \mathrm{par}(\lambda_1)\bigl(1 - 2\,\mathrm{par}(n_i + i)\bigr) \\ &= \text{BG-rank}(\tilde{\pi}_{0,i}) + \mathrm{par}(\lambda_1)(-1)^{n_i+i}. \end{aligned}$$

Next, we convert $\tilde{\pi}_{1,i}$ into $\tilde{\pi}_{2,i}$ by adding the new hook rim of length $t\lambda_2$ to $\tilde{\pi}_{1,i}$ so that W_i becomes

$$\begin{array}{rccccccc} \text{Region}: & \cdots\cdots & n_i+1-\lambda_1 & \cdots\cdots & n_i+2-\lambda_2 & \cdots\cdots & n_i+3 & \cdots\cdots \\ W_i: & \cdots\cdots & E\ N\ E & \cdots\cdots & E\ N\ E & \cdots\cdots & E\ N & \cdots\cdots. \end{array}$$

The color of the new head cell is given by

$$\mathrm{par}\bigl(t(n_i+1)+i\bigr) = \mathrm{par}(n_i+1+i),$$

and so

$$\begin{aligned} \text{BG-rank}(\tilde{\pi}_{2,i}) &= \text{BG-rank}(\tilde{\pi}_{1,i}) + \mathrm{par}(\lambda_2)\bigl(1 - 2\,\mathrm{par}(n_i + 1 + i)\bigr) \\ &= \text{BG-rank}(\tilde{\pi}_{0,i}) + (-1)^{n_i+i}\bigl(\mathrm{par}(\lambda_1) - \mathrm{par}(\lambda_2)\bigr). \end{aligned}$$

Proceeding as above we arrive at

$$\begin{aligned} \text{BG-rank}(\pi) &= \text{BG-rank}(\tilde{\pi}_{0,i}) + (-1)^{n_i+i}\sum_{j=1}^{\nu}(-1)^{j+1}\,\mathrm{par}(\lambda_j) \\ &= \text{BG-rank}(\tilde{\pi}_{0,i}) + (-1)^{n_i+i}\text{BG-rank}(\hat{\pi}_i). \end{aligned} \tag{3.9}$$

Formula (3.4) follows easily from (3.9). Let us now define $\vec{B}_t, \vec{\tilde{B}}_t \in \mathbb{Z}^t$ as

$$\vec{B}_t = \begin{cases} \sum_{i=0}^{\frac{t-1}{2}} \vec{e}_{2i}, & \text{if } t \equiv 1 \pmod 4, \\ \sum_{i=0}^{\frac{t-3}{2}} \vec{e}_{1+2i}, & \text{if } t \equiv -1 \pmod 4) \end{cases}$$

and

$$\vec{\tilde{B}}_t = \vec{B}_t + \sum_{i=0}^{t-1} \vec{e}_i = \begin{cases} \sum_{i=0}^{\frac{t-3}{2}} \vec{e}_{1+2i}, & \text{if } t \equiv 1 \pmod 4, \\ \sum_{i=0}^{\frac{t-1}{2}} \vec{e}_{2i}, & \text{if } t \equiv -1 \pmod 4. \end{cases}$$

Here $\vec{e}_i$'s are standard unit vectors in $\mathbb{Z}^t$ defined as $e_0 = (1, 0, \ldots, 0), \ldots, \vec{e}_{t-1} = (0, \ldots, 0, 1)$.

We conclude this section with the following important observation. If odd $t > 1$, $k = 0, 1, \ldots, (t-1)/2$ and $\vec{n} \in \mathbb{Z}^t$, $\vec{n} \cdot \vec{1}_t = 0$, then

$$bg(\vec{n}) = (-1)^{\frac{t-1}{2}} \left(\left\lfloor \frac{t}{4} \right\rfloor - k \right) \tag{3.10}$$

iff $\vec{n} \equiv \vec{B}_t + \vec{e}_{i_0} + \vec{e}_{i_1} + \cdots + \vec{e}_{i_{2k}} \pmod 2$ for some $0 \leqslant i_0 < i_1 < i_2 < \cdots < i_{2k} \leqslant t-1$. In particular, if $\vec{n} \in \mathbb{Z}^t$, $\vec{n} \cdot \vec{1}_t = 0$, then

$$bg(\vec{n}) = (-1)^{\frac{t+1}{2}} \left\lfloor \frac{t+1}{4} \right\rfloor \tag{3.11}$$

iff $\vec{n} \equiv \vec{\tilde{B}}_t \pmod 2$. We leave the proof as an exercise for the interested reader.

4. Combinatorial proof of $p_j(5n+4) \equiv 0 \pmod 5$

Throughout this section we assume that

$$|\pi| \equiv 4 \pmod 5$$

and

$$|\pi_{5\text{-core}}| \equiv 4 \pmod 5.$$

To prove (1.8) we shall require a few definitions. Following [5], we define the 5-core crank as

$$c_5(\pi) := 2\big(r_0(\pi, 5) - r_4(\pi, 5)\big) + \big(r_1(\pi, 5) - r_3(\pi, 5)\big) + 1 \pmod 5. \tag{4.1}$$

Note that if $|\pi_{5\text{-core}}| \equiv 4 \pmod 5$, then obviously

$$n_0 + n_1 + n_2 + n_3 + n_4 = 0, \tag{4.2}$$

$$n_1 + 2n_2 + 3n_3 + 4n_4 \equiv 4 \pmod 5. \tag{4.3}$$

Here, $\vec{n} = (n_0, n_1, n_2, n_3, n_4) = \phi_2(\pi_{5\text{-core}})$. Let us introduce a new vector $\vec{\alpha}(\vec{n}) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$, defined as

$$\alpha_0 = \frac{n_0 - 3n_1 - 2n_2 - n_3 + 1}{5}, \tag{4.4}$$

$$\alpha_1 = \frac{-3n_0 - n_1 - 4n_2 - 2n_3 + 2}{5}, \tag{4.5}$$

$$\alpha_2 = \frac{-3n_0 - n_1 + n_2 - 2n_3 + 2}{5}, \tag{4.6}$$

$$\alpha_3 = \frac{n_0 + 2n_1 + 3n_2 + 4n_3 + 1}{5}, \tag{4.7}$$

$$\alpha_4 = \frac{4n_0 + 3n_1 + 2n_2 + n_3 - 1}{5}. \tag{4.8}$$

Using (4.2), (4.3) it is easy to verify that $\vec{\alpha}(\vec{n}) \in \mathbb{Z}^5$ and that

$$(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 1. \tag{4.9}$$

Inverting (4.4)–(4.8) we find that

$$n_0 = \alpha_0 + \alpha_4, \tag{4.10}$$

$$n_1 = -\alpha_0 + \alpha_1 + \alpha_4, \tag{4.11}$$

$$n_2 = -\alpha_1 + \alpha_2, \tag{4.12}$$

$$n_3 = -\alpha_2 + \alpha_3 - \alpha_4, \tag{4.13}$$

$$n_4 = -\alpha_3 - \alpha_4. \tag{4.14}$$

Note that in terms of these new variables we have

$$c_5(\pi) \equiv \sum_{i=0}^{4} i\alpha_i \pmod 5, \tag{4.15}$$

$$|\pi| = 5Q(\vec{\alpha}) - 1 + 5\sum_{i=0}^{4} |\hat{\pi}_i|, \tag{4.16}$$

and

$$\begin{aligned} \text{BG-rank}(\pi) = {} & \frac{1 - (-1)^{\alpha_0+\alpha_1} - (-1)^{\alpha_1+\alpha_2} - \cdots - (-1)^{\alpha_4+\alpha_0}}{4} \\ & + (-1)^{\alpha_0+\alpha_4}\text{BG-rank}(\hat{\pi}_0) \\ & + (-1)^{\alpha_2+\alpha_3}\text{BG-rank}(\hat{\pi}_1) \\ & + (-1)^{\alpha_1+\alpha_2}\text{BG-rank}(\hat{\pi}_2) \\ & + (-1)^{\alpha_0+\alpha_1}\text{BG-rank}(\hat{\pi}_3) \\ & + (-1)^{\alpha_3+\alpha_4}\text{BG-rank}(\hat{\pi}_4). \end{aligned} \tag{4.17}$$

Here $\phi_1(\pi) = (\pi_{5\text{-core}}, \hat{\pi}_0, \ldots, \hat{\pi}_4)$ and $Q(\vec{\alpha}) := \vec{\alpha} \cdot \vec{\alpha} - (\alpha_0\alpha_1 + \alpha_1\alpha_2 + \cdots + \alpha_4\alpha_0)$. It is convenient to combine $\phi_1, \phi_2, \vec{\alpha}$ into a new invertible function Φ, defined as

$$\Phi(\pi) = \bigl(\vec{\alpha}\bigl(\phi_2(\pi_{5\text{-core}})\bigr), \vec{\hat{\pi}}\bigr),$$

where $\vec{\hat{\pi}} := (\hat{\pi}_0, \ldots, \hat{\pi}_4)$. Following [2] we define

$$\begin{aligned}
\widehat{C}_1(\vec{\alpha}) &= (\alpha_4, \alpha_0, \alpha_1, \alpha_2, \alpha_3),\\
\widehat{C}_2(\vec{\hat{\pi}}) &= (\hat{\pi}_4, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_0, \hat{\pi}_1),\\
\widehat{O}(\pi) &= \Phi^{-1}\bigl(\widehat{C}_1(\vec{\alpha}), \widehat{C}_2(\vec{\hat{\pi}})\bigr).
\end{aligned}$$

We observe that operator $\widehat{O}$ has the following properties

$$\begin{aligned}
\bigl|\widehat{O}(\pi)\bigr| &= |\pi|,\\
\widehat{O}^5(\pi) &= \pi,\\
\text{BG-rank}\bigl(\widehat{O}(\pi)\bigr) &= \text{BG-rank}(\pi),\\
c_5\bigl(\widehat{O}(\pi)\bigr) &\equiv 1 + c_5(\pi) \pmod 5.
\end{aligned} \tag{4.18}$$

Clearly, $\widehat{O}$ preserves the norm and the BG-rank of the partition. And so we can assemble all partitions of $5n+4$ with BG-rank $= j$ into disjoint orbits:

$$\pi, \quad \widehat{O}(\pi), \quad \widehat{O}^2(\pi), \quad \widehat{O}^3(\pi), \quad \widehat{O}^4(\pi).$$

Here, π is some partition of $5n+4$ with BG-rank $= j$. Formula (4.18) suggests that all five members of the same orbit are distinct. Clearly,

$$p_j(5n+4) = 5 \cdot (\text{number of orbits}).$$

Hence, $p_j(5n+4) \equiv 0 \pmod 5$, as desired. In fact, we have the following

Theorem 4.1. *Let j be any fixed integer. The residue of the* 5*-core crank* mod 5 *divides the partitions enumerated by $p_j(5n+4)$ into five equal classes.*

We note that this theorem generalizes Theorem 4.1 [2, p. 717].

5. Identities for odd t-cores with extreme BG-rank values

The main object of this section is to provide a proof of formulas (1.10) and (1.11). Throughout this section t is presumed to be a positive odd integer. We will prove (1.11) first. To this end we employ the observation (3.10) together with (2.3) to rewrite it as

$$\sum_{\substack{\vec{n} \in \mathbb{Z}^t,\, \vec{n} \cdot \vec{1}_t = 0\\ \vec{n} \equiv \vec{\tilde{B}}_t \pmod 2}} q^{\widetilde{Q}(\vec{n})} = q^{\frac{t^2-1}{8}} \frac{E^t(q^{4t})}{E(q^4)}, \tag{5.1}$$

where

$$\widetilde{Q}(\vec{n}) := \frac{t}{2}\vec{n}\cdot\vec{n} + \vec{b}_t\cdot\vec{n}. \tag{5.2}$$

Next we introduce new summation variables $\vec{\tilde{n}} = (\tilde{n}_0, \ldots, \tilde{n}_{t-1}) \in \mathbb{Z}^t$ as follows

$$\vec{n} = 2\vec{\tilde{n}} + \sum_{i=0}^{\lfloor\frac{t-3}{4}\rfloor} (\vec{e}_{\frac{t-3}{2}-2i} - \vec{e}_{\frac{t+1}{2}+2i}). \tag{5.3}$$

Obviously, $\vec{\tilde{n}}$ is subject to the constraint

$$\vec{\tilde{n}}\cdot\vec{1}_t = 0. \tag{5.4}$$

Note that in terms of new variables we have

$$\widetilde{Q}(\vec{n}) = \widetilde{Q}(\vec{n}) + (t-1)\vec{1}_t\cdot\vec{\tilde{n}} = \frac{t^2-1}{8} + 4\left\{\frac{t}{2}\vec{\tilde{n}}\cdot\vec{\tilde{n}} + \sigma_1 + \sigma_2 + \sigma_3\right\}, \tag{5.5}$$

where

$$\sigma_1 = \sum_{i=0}^{\lfloor\frac{t-3}{4}\rfloor} (t-1-i)\tilde{n}_{\frac{t-3}{2}-2i},$$

$$\sigma_2 = \sum_{i=0}^{\lfloor\frac{t-3}{4}\rfloor} i\tilde{n}_{2i+\frac{t+1}{2}},$$

$$\sigma_3 = \sum_{i=-\lfloor\frac{t-1}{4}\rfloor}^{\lfloor\frac{t-1}{4}\rfloor} \left(\frac{t-1}{2} + i\right)\tilde{n}_{\frac{t-1}{2}+2i}.$$

At this point it is natural to perform further changes:

$$\tilde{n}_{\frac{t-3}{2}-2i} \to \tilde{n}_{t-1-i}, \quad 0 \leqslant i \leqslant \left\lfloor\frac{t-3}{4}\right\rfloor,$$

$$\tilde{n}_{\frac{t+1}{2}+2i} \to \tilde{n}_i, \quad 0 \leqslant i \leqslant \left\lfloor\frac{t-3}{4}\right\rfloor,$$

$$\tilde{n}_{\frac{t-1}{2}+2i} \to \tilde{n}_{\frac{t-1}{2}+i}, \quad -\left\lfloor\frac{t-1}{4}\right\rfloor \leqslant i \leqslant \left\lfloor\frac{t-1}{4}\right\rfloor.$$

This way we obtain

$$\widetilde{Q}(\vec{n}) = \frac{t^2-1}{8} + 4\widetilde{Q}(\vec{\tilde{n}}),$$
$$\vec{\tilde{n}} \in \mathbb{Z}^t, \quad \vec{\tilde{n}}\cdot\vec{1}_t = 0.$$

And so with the aid of the Klyachko identity (1.13) we find that

$$C_{t,(-1)^{\frac{t+1}{4}}\lfloor\frac{t+1}{4}\rfloor}(q)=\sum_{\substack{\vec{\tilde{n}}\in\mathbb{Z}^t\\ \vec{\tilde{n}}\cdot\vec{1}_t=0}} q^{\frac{t^2-1}{8}+4\widetilde{Q}(\vec{\tilde{n}})}=q^{\frac{t^2-1}{8}}\frac{E^t(q^{4t})}{E(q^4)}, \tag{5.6}$$

as desired. To prove (1.10) we shall require the following lemma.

Lemma 5.1. *For a positive odd t*

$$\psi^2(q^2)=q^{\frac{t-1}{2}}\psi^2(q^{2t})+\frac{E^3(q^{4t})}{f(-q^t,-q^{3t})}\sum_{i=0}^{\frac{t-3}{2}}q^i\frac{f(q^{t-1-2i},-q^{1+2i})}{f(-q^{4i+2},-q^{4t-2-4i})} \tag{5.7}$$

holds.

In the above we employed the Ramanujan notations

$$\psi(q):=\frac{E^2(q^2)}{E(q)}=\sum_{n\geqslant 0}q^{\binom{n+1}{2}}, \tag{5.8}$$

$$f(a,b):=(ab,-a,-b;ab)_\infty. \tag{5.9}$$

Using (2.6) we can easily show that

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}}. \tag{5.10}$$

Setting $a=q^{t-1-2i}$, $b=-q^{1+2i}$, $0\leqslant i\leqslant\frac{t-3}{2}$, in (5.10) and dissecting we obtain

$$f(q^{t-1-2i},-q^{1+2i})=f(-q^{2+t+4i},-q^{3t-2-4i})+q^{t-1-2i}f(-q^{2-t+4i},-q^{5t-2-4i}). \tag{5.11}$$

To prove the above lemma we start with the Ramanujan ${}_1\psi_1$-summation formula [6, II.29]

$$\sum_{n=-\infty}^{\infty}\frac{(a)_n}{(b)_n}z^n=\frac{(az,\frac{q}{az},q,\frac{b}{a};q)_\infty}{(z,\frac{b}{az},b,\frac{q}{a};q)_\infty},\quad \left|\frac{b}{a}\right|<|z|<1. \tag{5.12}$$

We set $b=aq$ to obtain

$$\sum_{n=-\infty}^{\infty}\frac{z^n}{1-aq^n}=\frac{(az,\frac{q}{az},q,q;q)_\infty}{(z,\frac{q}{z},a,\frac{q}{a};q)_\infty}=\frac{E^3(q)f(-az,-\frac{q}{az})}{f(-z,-\frac{q}{z})f(-a,-\frac{q}{a})},\quad |q|<|z|<1. \tag{5.13}$$

If we replace $q\to q^4$, $z=q$, $a=q^2$ in (5.13) we find that

$$\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{2+4n}}=\psi^2(q^2). \tag{5.14}$$

Next we split the sum on the left of (5.14) as

$$\psi^2(q^2) = \sum_{\substack{i=0\\ i\neq\frac{t-1}{2}}}^{t-1} \sum_{m_i=-\infty}^{\infty} q^i \frac{q^{tm_i}}{1-q^{2+4i}q^{4tm_i}} + \sum_{m=-\infty}^{\infty} q^{\frac{t-1}{2}} \frac{q^{tm}}{1-q^{2t}q^{4tm}}. \tag{5.15}$$

Using (5.14) with $q \to q^t$ it is easy to recognize the last sum in (5.15) as $q^{(t-1)/2}\psi^2(q^{2t})$. And so we have

$$\psi^2(q^2) = q^{\frac{t-1}{2}}\psi^2(q^{2t}) + \frac{E^3(q^{4t})}{f(-q^t,-q^{3t})} \sum_{\substack{i=0\\ i\neq\frac{t-1}{2}}}^{t-1} q^i \frac{f(-q^{2+4i+t},-q^{3t-2-4i})}{f(-q^{2+4i},-q^{4t-2-4i})}, \tag{5.16}$$

where we have made a multiple use of (5.13). Finally, folding the last sum in half and using (5.11) we arrive at

$$\begin{aligned}\psi^2(q^2) &= q^{\frac{t-1}{2}}\psi^2(q^{2t}) + \sum_{i=0}^{\frac{t-3}{2}} \frac{E^3(q^{4t})q^i}{f(-q^t,-q^{3t})f(-q^{2+4i},-q^{4t-2-2i})}\\ &\quad\times\left\{f\left(-q^{2+4i+t},-q^{3t-2-4i}\right)+q^{t-1-2i}f\left(-q^{5t-2-4i},-q^{2-t+4i}\right)\right\}\\ &= q^{\frac{t-1}{2}}\psi^2(q^{2t}) + \frac{E^3(q^{4t})}{f(-q^t,-q^{3t})}\sum_{i=0}^{\frac{t-3}{2}} q^i \frac{f(q^{t-1-2i},-q^{1+2i})}{f(-q^{2+4i},-q^{4t-2-4i})}.\end{aligned} \tag{5.17}$$

This concludes the proof of Lemma 5.1.

We now move on to prove (1.10). Again, using the observation (3.10), we can rewrite it as

$$\sum_{j=0}^{t-1} \sum_{\substack{\vec{n}\in\mathbb{Z}^t,\,\vec{n}\cdot\vec{1}_t=0\\ \vec{n}\equiv\vec{B}_t+\vec{e}_j \pmod 2}} q^{\widetilde{Q}(\vec{n})} = q^{\frac{(t-1)(t-3)}{8}} F\left(t,q^2\right). \tag{5.18}$$

Remarkably, (5.18) is just the constant term in z of the following more general identity

$$\sum_{j=0}^{t-1} \sum_{\substack{\vec{n}\in\mathbb{Z}^t\\ \vec{n}\equiv\vec{B}_t+\vec{e}_j \pmod 2}} q^{\widetilde{Q}(\vec{n})} z^{\frac{\vec{n}\cdot\vec{1}_t}{2}} = q^{\frac{(t-1)(t-3)}{8}} F\left(t,q^2\right) \sum_{n=-\infty}^{\infty} q^{2n^2+(t-1)n} z^n. \tag{5.19}$$

To prove (5.19) we observe that its right-hand side satisfies the *first* order functional equation

$$\widehat{D}_{t,q}\left(f(z)\right) = f(z), \tag{5.20}$$

where

$$\widehat{D}_{t,q}\left(f(z)\right) := zq^{t+1} f\left(zq^4\right).$$

After a bit of labor one can verify that for $0 \leqslant i \leqslant t-1$

$$\widehat{D}_{t,q}\left(\sum_{\substack{\vec{n}\in\mathbb{Z}^t\\ \vec{n}\equiv\vec{B}_t+\vec{e}_i \pmod 2}} q^{\widetilde{Q}(\vec{n})} z^{\frac{\vec{n}\cdot\vec{1}_t}{2}}\right) = \sum_{\substack{\vec{n}\in\mathbb{Z}^t\\ \vec{n}\equiv\vec{B}_t+\vec{e}_{i+2} \pmod 2}} q^{\widetilde{Q}(\vec{n})} z^{\frac{\vec{n}\cdot\vec{1}_t}{2}}, \tag{5.21}$$

where $\vec{e}_t := \vec{e}_0$ and $\vec{e}_{t+1} := \vec{e}_1$. Clearly, (5.21) implies that the left-hand side of (5.19) satisfies (5.20), as well. It remains to verify (5.19) at one nontrivial point. To this end we set

$$z = \begin{cases} 1, & \text{if } t \equiv -1 \pmod 4,\\ q^2, & \text{if } t \equiv 1 \pmod 4) \end{cases}$$

in (5.19), and then replace $q^2 \to q$ to get with the help of (2.6)

$$q^{\frac{t-1}{2}}\psi\left(q^{2t}\right)\prod_{j=0}^{\frac{t-3}{2}} f^2\left(q^{1+2j}, q^{2t-1-2j}\right)\left\{1+\sum_{i=1}^{\frac{t-1}{2}} q^{-i}\frac{f(q^t,q^t)f(q^{2i},q^{2t-2i})}{\psi(q^{2t})f(q^{t+2i},q^{t-2i})}\right\} = \psi\left(q^2\right)F(t,q). \tag{5.22}$$

To proceed further we need to verify two product identities

$$\psi\left(q^2\right)\prod_{j=0}^{\frac{t-3}{2}} f^2\left(q^{1+2j}, q^{2t-1-2j}\right) = \psi\left(q^{2t}\right)F(t,q)$$

and

$$\psi\left(q^{2t}\right)\frac{f(q^t,q^t)f(q^{2i},q^{2t-2i})}{f(q^{t+2i},q^{t-2i})} = E^3\left(q^{4t}\right)\frac{f(q^{2i},-q^{t-2i})}{f(-q^t,-q^{3t})f(-q^{2t+4i},-q^{2t-4i})}, \quad i \in \mathbb{N}.$$

Next, we multiply both sides of (5.22) by $\frac{\psi(q^2)}{F(t,q)}$ and simplify to arrive at

$$q^{\frac{t-1}{2}}\psi^2\left(q^{2t}\right) + \frac{E^3(q^{4t})}{f(-q^t,-q^{3t})}\sum_{i=1}^{\frac{t-1}{2}} q^{\frac{t-1}{2}-i}\frac{f(q^{2i},-q^{t-2i})}{f(-q^{2t+4i},-q^{2t-4i})} = \psi^2\left(q^2\right), \tag{5.23}$$

which is essentially the identity in Lemma 5.1. This concludes our proof of (5.19). It follows that (5.18), (1.10) hold true.

6. 5-cores with prescribed BG-rank

Formula (1.9) suggests that BG-rank($\pi_{5\text{-core}}$) can assume just three values: $0, \pm 1$. This means that

$$a_5(n) = a_{5,-1}(n) + a_{5,0}(n) + a_{5,1}(n). \tag{6.1}$$

The generating function of version (6.1) is

$$\frac{E^5(q^5)}{E(q)} = C_{5,-1}(q) + C_{5,0}(q) + C_{5,1}(q). \tag{6.2}$$

In the last section we proved (1.10), (1.11). These identities with $t = 5$ state that

$$C_{5,-1}(q) = q^3 \frac{E^5(q^{20})}{E(q^4)}, \tag{6.3}$$

$$C_{5,1}(q) \quad = qF\big(5, q^2\big). \tag{6.4}$$

By (1.3) we observe that $C_{t,j}(q)$ is either an odd or an even function of q with parity determined by the parity of j. Therefore, $C_{5,0}(q)$ is an even function of q, and $C_{5,\pm 1}(q)$ are odd functions of q. Consequently, we see that

$$\operatorname{ep}\left(\frac{E^5(q^5)}{E(q)}\right) = C_{5,0}(q), \tag{6.5}$$

where

$$\operatorname{ep}\big(f(x)\big) := \frac{f(x) + f(-x)}{2}.$$

In this section we will show that $C_{5,0}(q)$ can be expressed as a sum of two infinite products

$$C_{5,0}(q) = R\big(q^2\big), \tag{6.6}$$

where

$$R(q) := \frac{E^4(q^{10})E(q^5)E^2(q^4)}{E^2(q^{20})E(q)} + q\frac{E^2(q^{20})E^3(q^5)E^6(q^2)}{E^2(q^{10})E^2(q^4)E^3(q)}. \tag{6.7}$$

It is easy to rewrite (6.7) in a manifestly positive way as

$$R(q) = f\big(q, q^4\big) f\big(q^2, q^3\big)\big\{\varphi\big(q^5\big)\psi\big(q^2\big) + q\varphi(q)\psi\big(q^{10}\big)\big\},$$

where

$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{E^5(q^2)}{E^2(q^4)E^2(q)},$$

and $\psi(q)$ is defined in (5.8). Formula (6.6) enabled us to discover and prove the new Lambert series identity

$$R(q) = \sum_{i=0}^{1} \sum_{n=-\infty}^{\infty} (-1)^i q^{5n+i} \frac{1 + q^{1+2i+10n}}{(1 - q^{1+2i+10n})^2}. \tag{6.8}$$

In what follows we will require three identities:

$$\left[ux, \frac{u}{x}, vy, \frac{v}{y}; q\right]_\infty = \left[uy, \frac{u}{y}, vx, \frac{v}{x}; q\right]_\infty + \frac{v}{x}\left[xy, \frac{x}{y}, uv, \frac{u}{v}; q\right]_\infty \tag{6.9}$$

[6, Ex. 5.21],

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af\left(\frac{b}{c}, \frac{c}{b}(abcd)\right)f\left(\frac{b}{d}, \frac{d}{b}(abcd)\right), \tag{6.10}$$

provided $ab = cd$ [1], and

$$\frac{E^5(q^5)}{E(q)} = \sum_{i=1}^{2}\sum_{n=-\infty}^{\infty}(-1)^{i+1}\frac{q^{5n+i-1}}{(1-q^{5n+i})^2} \tag{6.11}$$

[6, Ex. 5.7], [5, p. 8]. Here

$$[a;q]_\infty = \left(a, \frac{q}{a}; q\right)_\infty,$$

$$[a_1, a_2, \dots, a_n; q]_\infty = \prod_{i=1}^{n}[a_i; q]_\infty.$$

Next, we wish to establish the validity of

$$F(5,q) = \frac{E(q^{10})E^2(q^5)E^3(q^2)}{E^2(q)} = \frac{E^5(q^5)}{E(q)} + q\frac{E^5(q^{10})}{E(q^2)}. \tag{6.12}$$

To this end we multiply both sides of (6.12) by

$$\frac{[q, q^3; q^{10}]_\infty^2[q^2, q^4; q^{10}]_\infty}{E^4(q^{10})}$$

to obtain after simplification that

$$\left[q^2, q^2, q^4, q^6; q^{10}\right]_\infty = \left[q, q^3, q^5, q^5; q^{10}\right]_\infty + q\left[q, q, q^3, q^3; q^{10}\right]_\infty. \tag{6.13}$$

But the last equation is nothing else but (6.9) with q replaced by q^{10} and $u = q^2$, $v = q^5$, $x = 1$, $y = q$. We now combine

$$\mathrm{ep}\left(q\frac{E^5(q^5)}{E(q)}\right) = qC_{5,-1}(q) + qC_{5,1}(q),$$

with (6.3), (6.5), and (6.12) to obtain

$$\mathrm{ep}\left(q\frac{E^5(q^5)}{E(q)}\right) = 2q^4\frac{E^5(q^{20})}{E(q^4)} + q^2\frac{E^5(q^{10})}{E(q^2)}. \tag{6.14}$$

This can be stated as the following eigenvalue problem

$$T_2\left(q\frac{E^5(q^5)}{E(q)}\right)=q\frac{E^5(q^5)}{E(q)}, \tag{6.15}$$

where for prime p the Hecke operator T_p is defined by its action as

$$T_p\left(\sum_{n\geqslant 0}a_nq^n\right)=\sum_{n\geqslant 0}a_{pn}q^n+p\left(\frac{p}{5}\right)\sum_{n\geqslant 0}a_nq^{pn},$$

with $(\frac{a}{b})$ being the Legendre symbol. We remark that (6.15) is the $p=2$ case of the more general formula

$$T_p\left(q\frac{E^5(q^5)}{E(q)}\right)=\left(p+\left(\frac{p}{5}\right)\right)\left(q\frac{E^5(q^5)}{E(q)}\right), \tag{6.16}$$

which can be deduced from (6.11). We shall not supply the details. Instead, we note that (6.16) together with (6.3)–(6.5) implies that

$$T_{\tilde{p}}\big(qC_{5,j}(q)\big)=\left(\tilde{p}+\left(\frac{\tilde{p}}{5}\right)\right)\big(qC_{5,j}(q)\big),\quad j=0\pm 1. \tag{6.17}$$

Here, $\tilde{p}$ is an odd prime.

To prove (6.6) we use (6.12) to deduce that

$$\mathrm{ep}\left(\frac{E^5(q^5)}{E(q)}\right)=\mathrm{ep}\big(F(5,q)\big)=E\big(q^{10}\big)E^3\big(q^2\big)\cdot\mathrm{ep}\left(\frac{E^2(q^5)}{E^2(q)}\right). \tag{6.18}$$

To proceed further we employ (6.10) with $a=q, b=q^9, c=q^3, d=q^7$ to get

$$\begin{aligned}\frac{E(q^5)}{E(q)}&=\frac{E(q^4)}{E(q^{20})E^2(q^2)}f\big(q,q^9\big)f\big(q^3,q^7\big)\\&=\frac{E(q^4)}{E(q^{20})E^2(q^2)}\big\{f\big(q^4,q^{16}\big)f\big(q^8,q^{12}\big)+qf\big(q^6,q^{14}\big)f\big(q^2,q^{18}\big)\big\}\\&=\frac{E^2(q^{20})E(q^8)}{E(q^{40})E^2(q^2)}+q\frac{E(q^{40})E(q^{10})E^3(q^4)}{E(q^{20})E(q^8)E^3(q^2)}.\end{aligned} \tag{6.19}$$

It is clear that

$$\mathrm{ep}\left(\frac{E^2(q^5)}{E^2(q)}\right)=\frac{E^4(q^{20})E^2(q^8)}{E^2(q^{40})E^4(q^2)}+q^2\frac{E^2(q^{40})E^2(q^{10})E^6(q^4)}{E^2(q^{20})E^2(q^8)E^6(q^2)}. \tag{6.20}$$

Combining (6.18) and (6.20) we find that

$$\mathrm{ep}\left(\frac{E^5(q^5)}{E(q)}\right)=R\big(q^2\big). \tag{6.21}$$

The last formula together with (6.5) implies (6.6). Next, we rewrite (6.11) as

$$\frac{E^5(q^5)}{E(q)} = \sum_{i=1}^{2} \sum_{n=-\infty}^{\infty} (-1)^{i+1} \frac{q^{5n+i-1}(1+2q^{5n+i}+q^{10n+2i})}{(1-q^{10n+2i})^2}.$$

Clearly,

$$\begin{aligned} \mathrm{ep}\left(\frac{E^5(q^5)}{E(q)}\right) &= \sum_{i=1}^{2} \sum_{\substack{n=-\infty \\ n\equiv i-1 \pmod 2}}^{\infty} (-1)^{i+1} \frac{q^{5n+i-1}(1+q^{10n+2i})}{(1-q^{10n+2i})^2} \\ &= \sum_{i=0}^{1} \sum_{n=-\infty}^{\infty} (-1)^{i} \frac{q^{10n+i}(1+q^{20n+4i+2})}{(1-q^{20n+4i+2})^2}. \end{aligned} \tag{6.22}$$

Formula (6.8) with $q \to q^2$ follows easily from (6.21) and (6.22). Before we move on we wish to summarize some of the above observations in the formula below

$$\begin{aligned} \frac{E^5(q^5)}{E(q)} &= \left\{ \frac{E^4(q^{20})E(q^{10})E^2(q^8)}{E^2(q^{40})E(q^2)} + q^2 \frac{E^2(q^{40})E^3(q^{10})E^6(q^4)}{E^2(q^{20})E^2(q^8)E^3(q^2)} \right\} \\ &\quad + q\left\{ \frac{E^5(q^{10})}{E(q^2)} + 2q^2 \frac{E^5(q^{20})}{E(q^4)} \right\}. \end{aligned} \tag{6.23}$$

In [5], the authors used (6.11) to find explicit formulas for the coefficients

$$a_5(n) = \frac{2^{d+1}+(-1)^d}{3} \cdot 5^c \cdot \prod_{i=1}^{s} \frac{p_i^{a_i+1}-1}{p_i-1} \prod_{j=1}^{t} \frac{q_j^{b_j+1}+(-1)^{b_j}}{q_j+1}. \tag{6.24}$$

Here

$$n+1 = 2^d 5^c \prod_{i=1}^{s} p_i^{a_i} \prod_{j=1}^{t} q_j^{b_j} \tag{6.25}$$

is the prime factorization of $n+1$ and $p_i \equiv \pm 1 \pmod 5$, $1 \leqslant i \leqslant s$, and $q_j \equiv \pm 2 \pmod 5$, $1 \leqslant j \leqslant t$, are odd primes. Formulas (6.3)–(6.5) and (6.12) suggest the following relations. For $n \in \mathbb{N}$ and $r = 0, 1, 2, 3$ one has

$$a_{5,0}(n) = \begin{cases} a_5(n), & \text{if } n \equiv 0 \pmod 2, \\ 0, & \text{otherwise}, \end{cases} \tag{6.26}$$

$$a_{5,-1}(4n+r) = \begin{cases} a_5(n), & \text{if } r = 3, \\ 0, & \text{otherwise}, \end{cases} \tag{6.27}$$

$$a_{5,1}(4n+r) = \begin{cases} a_5(2n), & \text{if } r = 1, \\ a_5(n) + a_5(2n+1), & \text{if } r = 3, \\ 0, & \text{if } r = 0, 2. \end{cases} \tag{6.28}$$

These relations together with (6.24) enabled us to derive explicit formulas for $a_{5,j}(n)$ with $-1 \leqslant j \leqslant 1$. In particular, if the prime factorization of $n+1$ is given by (6.25), then

$$a_{5,1}(4n+3) = 2^{d+1}5^c \prod_{i=1}^{s} \frac{p_i^{a_i+1}-1}{p_i-1} \prod_{j=1}^{t} \frac{q_j^{b_j+1}+(-1)^{b_j}}{q_j+1}. \tag{6.29}$$

We would like to conclude this section with the following discussion. It is easy to check that (6.17) implies that

$$a_{5,j}(pn+p-1) + p\left(\frac{p}{5}\right)a_{5,j}\left(\frac{n+1}{p}-1\right) = \left(p+\left(\frac{p}{5}\right)\right)a_{5,j}(n), \quad j=0,\pm 1, \tag{6.30}$$

where p is odd prime, $n \in \mathbb{N}$ and $a_{5,j}(x) = 0$ if $x \notin \mathbb{Z}$. Setting $p = 5$ we find that

$$a_{5,j}(5n+4) = 5a_{5,j}(n), \quad j = 0, \pm 1. \tag{6.31}$$

This is a refinement of the well-known result

$$a_5(5n+4) = 5a_5(n), \tag{6.32}$$

proven in [5]. We can prove (6.31) by adapting the combinatorial proof in [5].

Let us define

$$\vec{n} = (n_0, n_1, n_2, n_3, n_4) = \phi_2(\pi_{5\text{-core}})$$

for some $\pi_{5\text{-core}}$ with BG-rank$(\pi_{5\text{-core}}) = j$ and $|\pi_{5\text{-core}}| = n$. Consider map $\vec{n} \to \vec{\tilde{n}} = (\tilde{n}_0, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4)$ with

$$\begin{aligned}
\tilde{n}_0 &= n_1 + 2n_2 + 2n_4 + 1, \\
\tilde{n}_1 &= -n_1 - n_2 + n_3 + n_4 + 1, \\
\tilde{n}_2 &= 2n_1 + n_2 + 2n_3, \\
\tilde{n}_3 &= -2n_2 - 2n_3 - n_4 - 1, \\
\tilde{n}_4 &= -2n_1 - n_3 - 2n_4 - 1.
\end{aligned}$$

Obviously $\vec{\tilde{n}} \in \mathbb{Z}^5$ and $\vec{\tilde{n}} \cdot \vec{1}_5 = 0$ and so we can define $\tilde{\pi}_{5\text{-core}} = \phi_2^{-1}(\vec{\tilde{n}})$. It is easy to check that

$$|\tilde{\pi}_{5\text{-core}}| = 5n + 4,$$

and that

$$\text{BG-rank}(\tilde{\pi}_{5\text{-core}}) = j,$$

and

$$c_5(\tilde{\pi}_{5\text{-core}}) \equiv 4 \pmod 5.$$

Recall that the orbit $\{\tilde{\pi}_{5\text{-core}}, \widehat{O}(\tilde{\pi}_{5\text{-core}}), \ldots, \widehat{O}^4(\tilde{\pi}_{5\text{-core}})\}$ contains just one member with $c_5 \equiv 4 \pmod 5$. And so each 5-core of n with BG-rank j is in 1–1 correspondence with an appropriate 5-member orbit of t-cores of $5n+4$ with BG-rank j. This observation yields a combinatorial proof of (6.31).

7. Outlook

Given our combinatorial proof of

$$p_j(5n+4) \equiv 0 \pmod 5, \quad j \in \mathbb{Z},$$

one may wonder about a combinatorial proof of the other mod 5 congruences (1.4)–(1.7). We strongly suspect that such proof will be dramatically different from the one discussed in Section 4. In addition, one would like to have combinatorial insights into (6.30) for $p \neq 5$.

In this paper we found "positive" *eta*-quotient representations for $C_{5,j}(q)$, $-1 \leqslant j \leqslant 1$. In the general case (odd t, $-\lfloor (t-1)/4 \rfloor \leqslant j \leqslant \lfloor (t+1)/4 \rfloor$), we established such representation only for $C_{t,\pm\lfloor (t\pm1)/4 \rfloor}(q)$. Clearly, one wants to find "positive" *eta*-quotient representations for other admissible values of BG-rank. (See [3] for a fascinating discussion of the $t=7$ case.)

Finally, we observe that (1.2) is the $s=2$ case of the following more general definition

$$\text{gbg-rank}(\pi, s) = \sum_{j=0}^{s-1} r_j(\pi, s)\omega_s^j$$

with

$$\omega_s = e^{i\frac{2\pi}{s}}.$$

Many identities, proven here, can be generalized further. For example, we can prove that if $(s,t)=1$ then

$$\text{gbg-rank}(\pi_{t\text{-core}}, s) = \frac{\sum_{i=0}^{t-1} \omega_s^{i+1}(\omega_s^{tn_i} - 1)}{(1-\omega_s^t)(1-\omega_s)} \tag{7.1}$$

and for $1 \leqslant i \leqslant s-1$ that

$$\sum_{\text{gbg-rank}(\pi_{t\text{-core}}, s) = g(i)} q^{|\pi_{t\text{-core}}|} = q^{a(i)} F_i\big(q^s\big). \tag{7.2}$$

Here,

$$(n_0, n_1, \ldots, n_{t-1}) = \phi_2(\pi_{t\text{-core}}),$$

$$a(i) = \frac{(t^2-1)(s^2-1)}{24} - \frac{(t-1)(s-i)i}{2},$$

$$g(i) = \frac{1}{(1-\omega_s)(1-\frac{1}{\omega_s})} - \omega_s^{\frac{t-1}{2}} \frac{1 + \frac{t-1}{\omega_s^i}}{(1-\omega_s^t)(1-\frac{1}{\omega_s})},$$

$$F_i(q) = E\left(q^s\right)E\left(q^{st}\right)^{t-2}\frac{[q^{it};q^{st}]_\infty}{[q^i;q^s]_\infty}.$$

Setting $s = 2$ in (7.1), (7.2) we obtain (3.2), (1.10), respectively.

In addition we can show that

$$\sum_{\text{gbg-rank}(\pi_{t\text{-core}},s)=g(0)} q^{|\pi_{t\text{-core}}|} = q^{a(0)}\frac{E(q^{s^2t})^t}{E(q^{s^2})}. \tag{7.3}$$

Setting $s = 2$ in (7.3) we get (1.11).

In [10] Olsson and Stanton defined so-called (s,t)-good partitions. Surprisingly, t-cores with gbg-rank $= g(0)$ coincide with (t,s)-good partitions.

Let $\nu(t,s)$ denote a number of distinct values that gbg-rank$(\pi_{t\text{-core}}, s)$ may assume. Then it can be shown that

$$\nu(s,t) \leqslant \frac{\binom{t+s}{t}}{t+s},$$

provided that $(s,t) = 1$. Moreover, if s is prime or if s is a composite number and $t < 2p$ then

$$\nu(s,t) = \frac{\binom{t+s}{t}}{t+s}.$$

Here, p is the smallest prime divisor of s and $(s,t) = 1$.

Details of these and related results will be left to a later paper.

Acknowledgments

We would like to thank Robin Chapman, Ole Warnaar, Herbert Wilf and Hamza Yesilyurt for their kind interest and stimulating discussions.

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