On a kind of restricted edge connectivity of graphs

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Abstract

Let $G=(V,E)$ be a connected graph and $S \subset E$. $S$ is said to be an $m$-restricted edge cut ($m$-RC) if $G - S$ is disconnected and each component contains at least $m$ vertices. The $m$-restricted edge connectivity $\lambda^{(m)}(G)$ is the minimum size of all $m$-RCs in $G$. Based on the fact that $\lambda^{(3)}(G) \leq \xi_3(G)$, where $\xi_m(G) = \min \{\omega(X); X \subset V, |X| = m \text{ and } G[X] \text{ is connected}\}$ (\omega(X) denotes the number of edges with one end vertex in $X$ and the other in $V \setminus X$), we call a graph $G$ super-$\lambda^{(3)}$ if $\lambda^{(m)}(G) = \xi_m(G) (1 \leq m \leq 3)$. We proved that regular graphs with order more than 5 have at least one 3-RC, and show that vertex-and edge-transitive graphs other than cycles are super-$\lambda^{(3)}$. We also characterize super-$\lambda^{(3)}$ circulant graphs. As a consequence, we give the counting formula for the number of $i$-cutsets $N_i$ of these graphs (including the Star graphs, the Hypercubes and the Harary graphs) for $i$, $2k - 2 \leq i < \xi_3(G)$, where $k$ is the regular degree of $G$. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $G(V,E)$ be a connected (undirected) graph. An edge cutset of $G$ is a set of edges whose removal disconnects $G$. The edge connectivity $\lambda$ is the minimum size of edge cutset of $G$. The number of edge cutsets of size $i$, denoted by $N_i(G)$, was called the high-order edge connectivity measure in [4]. Assume that all vertices are perfectly reliable and all edges failed independently, with the same probability $p$, then,
the probability $P(G)$ of $G$ being disconnectedness is given by

$$P(G) = \sum_{i=0}^{q} N_i(G) p^i (1 - p)^{q-i},$$

here $q$ is the number of edges. In general, it is difficult to determine $N_i$ of a graph. To minimum $N_i$, Bauer et al. [2] defined the so-called super-$\lambda$ graphs.

**Definition 1.1.** A connected graph $G$ is said to be super-$\lambda$ if every edge cutset of size $\lambda$ isolates a vertex.

Clearly, if $G$ is super-$\lambda$, then its edge connectivity attains its minimum degree. As a natural generalization of classical (edge) connectivity, Harary [6] proposed the concept of conditional edge connectivity. The $P$-edge connectivity $\lambda(G,P)$ of graph $G$ has been defined as the minimum cardinality $|S|$ of a set $S$ of edges such that $G - S$ is disconnected and every component of $G - S$ has the given graph property $P$. In particular, Esfahanian and Hakimi [5] considered a special kind of conditional edge connectivity $\lambda(G,P)$, where $P$ is defined as follows: A graph $H$ satisfies property $P$ if it contains more than one vertex. Here, we will consider a more general question.

**Definition 1.2.** A set $S$ of edges of a connected graph $G$ is called a $m$-restricted edge cutset ($m$-RC) if $G - S$ is disconnected and each component contains at least $m$ vertices. The $m$-restricted edge connectivity $\lambda^{(m)}(G)$ is the minimum size of all $m$-RCs in $G$.

Clearly, $\lambda^{(1)}(G) = \lambda(G)$, $\lambda^{(2)}(G)$ is just the kind of conditional edge connectivity mentioned above. In [4], the authors characterized the graphs which have 2-RC and give a upper bound for $\lambda^{(2)}(G)$ as follows:

**Proposition 1.3.** If $G$ is a connected graph with at least four vertices and it is not a star graph $K_{1,m}$. Then, $\lambda^{(2)}(G)$ is well defined and $\lambda(G) \leq \lambda^{(2)}(G) \leq \bar{\zeta}(G)$, where $\bar{\zeta}(G) = \min \{d(e) = d(x) + d(y) - 2 : e = (x,y) \in E(G)\}$ and is called the minimum edge degree.

Before proceeding, we give some notations and definitions. Let $G = (V,E)$ be a graph and $X,Y \subseteq V$. $E(X,Y)$ denotes the set of edges with one end vertex in $X$ and the other in $Y$, and $G[X]$ the induced subgraph by $X$. In particular, $E[X] = E[X,V \setminus X]$. Let $\omega(X) = |E[X]|$. Then, the following inequality is well-known [7].

$$\omega(X \cup Y) + \omega(X \cap Y) \leq \omega(X) + \omega(Y).$$

Let $n = |V|$ and $1 \leq m \leq n$. Define

$$\bar{\zeta}_m(G) = \min_{\substack{|X| = m \\text{G[X] is connected}}} \omega(X).$$
Clearly, $\xi_1(G) = \delta(G)$ is the minimum degree of $G$ and $\xi_2(G) = \bar{\tau}(G)$ is the minimum edge degree. It is easy to see that $\lambda(G) = \lambda(G) \leq \lambda(G) \leq \lambda(G)$. Thus, we define $G$ to be super-$\lambda$ if it is $\lambda(G)$ and $\lambda(G)$ is the super-

Let $A$ be a group, $S \subseteq A \setminus \{1\}$ with $S = S^{-1}$. The Cayley graph $C(A, S)$ of $A$ with respect to $S$ is a graph with vertex set $A$, and for any $x$ and $y$ in $A$, there is an edge connecting $x$ and $y$ if and only if $x^{-1} y \in S$. Circulant graphs are Cayley graphs of the cyclic groups. Let $\Gamma_n$ be the cyclic group of integers modulo $n$, and $a, a_2, \ldots, a_k$ be $k$ integers with $1 \leq a < a_2 < \cdots < a_k \leq n/2$ in $\Gamma_n$, for explicity, we use $C_n(a_1, a_2, \ldots, a_k)$ to denote the circulant graph $C(\Gamma_n, \{a_1, \pm a_2, \ldots, \pm a_k\})$. For details on Cayley graphs, see the excellent work [9] for reference.

A graph $G$ is called vertex-transitive if the automorphism group $\text{Aut}(G)$ acts transitively on $V(G)$, and $G$ is called edge-transitive if $\text{Aut}(G)$ acts transitively on $E(G)$. It is known that Cayley graphs are vertex-transitive but not necessarily edge-transitive. It is well known that a circulant graph $C_n(a_1, a_2, \ldots, a_k)$ is connected if and only if $\gcd(n, a_1, a_2, \ldots, a_k) = 1$, and the edge connectivity of every connected vertex-transitive graph attains its minimum degree. In [4], Boesch and Wang gave a necessary and sufficient conditions for a circulant graph to be super-$\lambda$ and they also determined $\lambda_i(2k = i \leq 4k - 3)$ for Harary graphs $C_n(1, 2, \ldots, k), k < n/2$.

**Theorem 1.4.** (i) Every connected circulant is super-$\lambda$ unless it is $C_n(a)$ or $C_2(2, 4, \ldots, j-1, j)$ for $j > 1$ odd.

(ii) Let $H = C_n(1, 2, \ldots, k), k < n/2$, then $\lambda_i(H) = n^{\frac{n^2 - 2k}{i - 2k}}$.

Based on Proposition 1.3, in [7], the authors defined the so called optimal super-$\lambda$ graphs as follows: A super-$\lambda$ graph $G$ is said to be optimal super-$\lambda$ if $\lambda(G) = \xi(G)$. They also characterized optimal super-$\lambda$ graphs.

**Theorem 1.5.** Let $G = C_n(a_1, a_2, \ldots, a_k)$ be a connected circulant graph with $k \geq 2$, then $G$ is optimal super-$\lambda$ if and only if one of the three conditions holds:

(i) $a_k < n/2$;

(ii) $a_k = n/2$ and $\gcd(n, a_1, \ldots, a_{k-1}) = 1$; or

(iii) $a_k = n/2$, $\gcd(n, a_1, \ldots, a_{k-1}) = 2$ and $n \geq 8k - 8$.

In [7], the authors also determined $\lambda_i(G)$ for $m \leq i \leq \xi(G) - 1$, where $m$ is the regular degree of $G$. In fact, we have the following more general result.

**Theorem 1.6.** Let $G$ be a $k$-regular optimal super-$\lambda$ graph with order $n$, then for each $i, k \leq i \leq \xi(G) - 1$,

$$\lambda_i(G) = n \left( \frac{1}{i} nk - k \right).$$

In the next section, we show that regular graphs with order at least 6 have 3-RCs and $\lambda_2(G) \leq \lambda_3(G) \leq \xi_3(G)$. Further, we study the fundamental properties of atoms
with respect to $\lambda^{(3)}(G)$. In Section 3, we show that vertex- and edge-transitive graphs other than cycles are super-$\lambda^{(3)}$. We also characterize super-$\lambda^{(3)}$ circulant graphs. As a consequence, we determine $N_i$ for super-$\lambda^{(3)}$ circulant graphs (including Harary graphs) for $i$, $\xi_2(G) \leq i \leq \xi_3(G) - 1$.

2. Atoms of 3-restrictive connectivity

Before discussing the 3-restrictive connectivity, we have the following theorem.

**Theorem 2.1.** A connected regular graph $G$ with $|V(G)| \geq 6$ has 3-RCs, and therefore $\lambda^{(3)}(G) \leq \xi_3(G)$.

**Proof.** We call a path $P$ of length 2 an open 2-path if $x_1$ and $x_3$ are not adjacent. Otherwise, we call it a closed 2-path. If $d(G) = 2$, then $G \cong C_n$, $n \geq 6$. Clearly, it has 3-RCs and $\lambda^{(3)}(G) = \xi_3(G) = 2$. Thus, we assume $d(G) \geq 3$ in the following.

If $girth(G) = 3$, then there is a closed 2-path $P = x_1x_2x_3$. Let $X = \{x_1, x_2, x_3\}$. We claim that $E[X]$ is a 3-RC. If not, there is a component $C$ of $G \setminus E[X]$ isomorphic to $K_1$ or $K_2$. If $C \cong K_4$, then $G \cong K_4$ as $G$ is regular and $d(G) \geq 3$. Similarly, $G \cong K_4$ if $C \cong K_2$. Contradicting $|V(G)| \geq 6$.

If $girth(G) > 3$, then there is an open 2-path $P = x_1x_2x_3$. Let $X = \{x_1, x_2, x_3\}$. We claim that $E[X]$ is a 3-RC. If not, there is a component $C$ of $G \setminus E[X]$ isomorphic to $K_1$ or $K_2$. By the fact that $G$ is regular and $d(G) \geq 3$, we can deduce that $G$ contains triangles. Contradicting $girth(G) > 3$. \square

In what follows we always suppose that $G$ has 3-RCs, thus $\lambda^{(3)}(G)$ is well-defined. $G$ is said to be super-$\lambda^{(3)}$, if for $i$, $1 \leq i \leq 3$, $\lambda^{(i)}(G) = \xi_i(G)$. To study the 3-restricted edge connectivity of graphs, we first define the so called atoms of 3-restricted edge connectivity, and discuss its fundamental properties.

Let $G = (V, E)$ be a connected graph, $F$ a non-empty subset of $V$. $F$ is called a fragment with respect to 3-restricted edge connectivity (or simply, a 3-RF) if $E[F]$ is a 3-RC with $\omega(F) = \lambda^{(3)}(G)$. A 3-RF with least cardinality is called an atom with respect to 3-restricted edge connectivity (or simply, a 3-RA). $\omega^{(3)}(G)$ denotes the cardinality of a 3-RA. Clearly, $\omega^{(3)}(G) \geq 3$, and a super-$\lambda^{(2)}$ graph is super-$\lambda^{(3)}$ if and only if $\omega^{(3)}(G) = 3$.

We use $C_m[K_2]$ and $C_m \times K_2$ to denote the lexicographic product and the cartesian product of a cycle $C_m$ and $K_2$, respectively. The following result is important in considering 3-restricted edge connectivity of transitive graphs.

**Theorem 2.2.** Let $G$ be a super-$\lambda^{(2)}$ vertex-transitive graph with $\omega^{(3)}(G) \geq 4$. If $G \not\cong C_m[K_2]$ or $C_m \times K_2$, then the intersection of distinct 3-RAs of $G$ is empty.

**Proof.** By contradiction. Let $G$ be a super-$\lambda^{(2)}$ vertex-transitive graph, $A_1$ and $A_2$ be two distinct 3-RAs with $A_1 \cap A_2 \neq \emptyset$. Then we have the following claims.
Claim 1. \(G[A_1 \cup A_2]\) and \(G[V \setminus (A_1 \cap A_2)]\) are connected.

In fact, by definition of RA, \(G[A_1], G[A_2], G[V \setminus A_1]\) and \(G[V \setminus A_2]\) are connected. The results then follow from the facts that \(A_1 \cap A_2 \neq \emptyset\) and \((V \setminus A_1) \cap (V \setminus A_2) \neq \emptyset\).

Claim 2. \(|A_1 \cap A_2| < 3\).

If not, \(|A_1 \cap A_2| \geq 3\). Then by definition, if \(G[A_1 \cap A_2]\) is connected, we have
\[
\omega(A_1 \cap A_2) \geq \lambda^{(3)}(G).
\]
Otherwise, if \(G[A_1 \cap A_2]\) is not connected, we claim that
\[
\omega(A_1 \cap A_2) > \lambda^{(3)}(G).
\]
In fact, if the number of components of \(G[A_1 \cap A_2]\) is at least 3, then since \(G\) is super-\(\lambda^{(2)}\), we have \(\lambda^{(1)}(G) = k\), where \(k\) is the regular degree of \(G\) (Note that \(G\) is regular since it is vertex-transitive), and so \(\omega(A_1 \cap A_2) \geq 3k > \lambda^{(3)}(G)\). If \(G[A_1 \cap A_2]\) has exactly two components, say, \(B_1\) and \(B_2\), then one of \(B_1\) and \(B_2\) has more than one vertex, thus \(\omega(A_1 \cap A_2) \geq k + \lambda_2(G) = k + \xi_2(G) > \xi_3(G) \geq \lambda_3(G)\). Similarly, if \(G[V \setminus (A_1 \cup A_2)]\) is connected, then
\[
\omega(A_1 \cup A_2) \geq \lambda_3(G).
\]
Otherwise, if \(G[V \setminus (A_1 \cup A_2)]\) is disconnected, then
\[
\omega(A_1 \cup A_2) > \lambda_3(G).
\]
But from the following inequality:
\[
\omega(A_1 \cap A_2) + \omega(A_1 \cup A_2) \leq \omega(A_1) + \omega(A_2) = 2\lambda^{(3)}(G),
\]
we conclude that both \(G[A_1 \cap A_2]\) and \(G[V \setminus (A_1 \cup A_2)]\) are connected, and \(\omega(A_1 \cap A_2) = \omega(A_1 \cup A_2) = \lambda^{(3)}(G)\). Thus \(A_1 \cap A_2\) is a 3-RF. But \(|A_1 \cap A_2| < |A_1|\), this is impossible.

Claim 3. \(|A_1 \cap A_2| > 1\).

If not, then \(|A_1 \cap A_2| = 1\). Consider \(A_1 \setminus A_2\). Set \(B_1 = V \setminus A_2\). Then \(A_1 \setminus A_2 = A_1 \cap B_1\), and
\[
|A_1 \cap B_1| = |A_1| - |A_1 \cap A_2| = |A_1| - 1 \geq 3,
\]
\[
|V - (A_1 \cap B_1)| \geq \frac{|V(G)|}{2} \geq 3,
\]
\[
|V - (A_1 \cup B_1)| = |A_2 \setminus A_1| \geq 3.
\]
By a similar argument to that of Claim 2, we can derive a contradiction.
Claim 4. \(|A_1 \cap A_2| = 2\) and \(|A_1| = |A_2| = 4\).

In fact, if \(|A_1| > 4\), consider \(A_1 \setminus A_2\). Similarly as in the proof of Claim 3, we can obtain a contradiction. Thus \(|A_1| = 4\). Similarly, \(|A_2| = 4\).

Clearly, if \(\text{girth}(G) = 3\), then, \(\xi_3(G) = 3k - 6\), and if \(\text{girth}(G) > 3\), then, \(\xi_3(G) = 3k - 4\).

Claim 5. If \(\text{girth}(G) > 3\), then \(G \cong C_m \times K_2\).

In fact, from the following inequality:

\[
4k - 12 \leq \omega(A_1) = \lambda^{(3)}(G) \leq \xi_3(G) - 1 = 3k - 5,
\]

we derive that \(k \leq 7\), and \(7\) if and only if \(A_1 \cong K_4\). Since the sum of the vertex degrees of \(G[A_1]\) is equal to \(4k - \omega(A_1)\), we conclude that \(\omega(A_1)\) is even. Thus \(\omega(A_1) \neq 4k - 11, 4k - 9\) or \(4k - 7\). Therefore \(k \neq 2, 4\) or \(6\). If \(k = 7\), then \(G[A_1] \cong K_4\), which means that \(G\) has triangles, a contradiction. Similarly, if \(k = 6\), then \(G[A_1]\) has 5 edges, and so \(G[A_1]\) is isomorphic to the graph \(K_4 \setminus e\), which also has triangles. Thus \(k = 3\). Now, it is easy to see that \(G[A_1]\) has 4 edges. Since it is triangle-free, we conclude that \(A_1 \cong C_4\), the cycle of length 4. Since \(|A_1 \cap A_2| = 2\) and \(k = 3\), \(G[A_1 \cap A_2] \cong K_2\). Assume that \(G[A_1]\) is the cycle \(Q_1 = x_1y_1z_1w_1\), and \(G[A_2]\) is the cycle \(Q_2 = x_2y_2z_2w_2\), where \(y_1 = x_2\), and \(z_1 = w_2\) (see Fig. 1).

Since \(y_1\) is in exactly 2 cycles of length 4, by vertex-transitivity of \(G\), \(y_2\) and \(z_2\) must also be in exactly 2 cycles of length 4. As \(k = 3\), we see that \(y_2\) and \(z_2\) are in the same cycles of length 4. Let \(Q_3 = x_3y_3z_3w_3\) be the cycle of length 4 containing \(y_2\) and \(z_2\) and different from \(Q_2\), where \(x_3 = y_2\) and \(w_3 = z_2\). Continuing this process, we get a sequence of cycles \(Q_i = x_iy_iz_iw_i\) (\(i \geq 1\)) with \(x_i = y_{i-1}\) and \(w_i = z_{i-1}\) such that the intersection of the two consecutive ones is \(K_2\). As \(G\) is finite, there exists an integer \(m\) such that \(y_{m+1} = x_1\), and \(w_{m+1} = w_1\). Then \(G \cong C_m \times K_2\).

Claim 6. If \(\text{girth}(G) = 3\), then \(G \cong C_m[K_2]\).

In fact, since \(4k - 12 \leq \omega(A_1) = \lambda^{(3)}(G) \leq \xi_3(G) - 1 = 3k - 7\), we have \(k \leq 5\). Similarly as above, we see that \(k\) is odd. Thus \(k = 3\) or \(5\). If \(k = 3\), then \(G[A_1] \cong G[A_2] \cong K_4 \setminus e\), but as \(|A_1 \cap A_2| = 2\), this is impossible. Thus \(k = 5\). As \(|E(G[A_1])| = 6\),
we have $G[A_1] \cong G[A_2] \cong K_4$. Note that $|A_1 \cap A_2| = 2$, we have $G[A_1 \cap A_2] \cong K_2$. By a similar argument as above, we deduce that $G \cong C_m[K_2]$.

In all cases, we obtain contradictions, thus $A_1 \cap A_2 = \emptyset$.  

3. Edge-transitive graphs and circulant graphs

In [3], Boesch and Tindell proved the following:

**Theorem 3.1.** The only connected edge transitive graphs which are not super-$\lambda$ are cycles $C_n$.

The result is generalized as follows in [8]:

**Theorem 3.2.** The only connected edge transitive graphs which are not super-$\lambda^{(2)}$ are the cycles and the stars.

Here, we give the following

**Theorem 3.3.** The only connected edge and vertex transitive graphs which are not super-$\lambda^{(3)}$ are cycles.

To prove Theorem 3.3, we cite a result from [9] which is useful in the following discussions.

An imprimitive block for a permutation group $\Phi$ of a set $T$ is a proper, nontrivial subset $A$ of $T$ such that if $\phi \in \Phi$, then either $\phi(A) = A$ or $\phi(A) \cap A = \emptyset$. The following proposition indicates why imprimitivity is so useful.

**Proposition 3.4.** Let $G = (V, E)$ be a connected graph and $Y$ be the subgraph induced by an imprimitive block $A$ of $G$. Then

(i) If $G$ is vertex-transitive, then so is $Y$.

(ii) If $G$ is edge-transitive and $A$ is a proper subset of $V$, then $A$ is an independent subset of $G$.

(iii) If $G = C(H, S)$ and $A$ contains the identity of $H$, then $A$ is a subgroup of $H$.

**Proof of Theorem 3.3.** By contradiction. Suppose $G$ is a connected edge- and vertex-transitive graph with regular degree $k \geq 3$ which is not super-$\lambda^{(3)}$. Then by Theorem 3.2 we conclude that $\lambda^{(3)}(G) < \xi_3(G)$. It follows that the cardinality of its RAs is at least four. Let $A$ be a RA of $G$. By Theorem 2.2, we see that $A$ is an imprimitive block of $\text{Aut}(G)$, and it follows from Proposition 3.4 that $A$ is an independent set of $G$. Thus $\omega(A) = 4k > \xi_3(G)$, which is impossible.  

**Corollary 3.5.** Star graphs $S_n$ and hypercubes $Q_n$ [1] are super-$\lambda^{(3)}$ for $n \geq 3$. 

Theorem 3.6. Let $G$ be a $k$-regular optimal super-$\lambda^{(3)}$ graph with $n$ vertices and $m$ edges and $i$ be an integer satisfying $2k - 2 \leq i \leq \bar{\lambda}_3(G) - 1$. Then

$$N_i(G) = \begin{cases} 
    m + n \left( \frac{n - k}{k - 2} \right), & i = 2k - 2, \\
    m \left( \frac{m - 2k + 1}{2} \right) + \left( \frac{n(n - 1)}{2} - m \right) \left( \frac{m - 2k}{i - 2k} \right) + n \left( \frac{m - k}{i - k} \right) - k \left( \frac{m - 2k + 1}{i - 2k + 1} \right), & 2k - 1 \leq i \leq \bar{\lambda}_3(G) - 1.
\end{cases}$$

Proof. We only show the case where $2k - 1 \leq i \leq \bar{\lambda}_3(G) - 1$, the other case can be shown similarly. In this case, the first and second terms are the number of edge cuts creating exactly two isolated vertices which are and/or are not adjacent in $G$, respectively, the third term is those which create exactly one isolated vertex, and the last term is those which create an isolated edge. The result follows.

Now, we turn our attention to circulant graphs $C_n(a_1, a_2, \ldots, a_k)$ with $n \geq 6$, $k \geq 2$ and $0 < a_1 < a_2 < \cdots < a_k$. We first consider the case where $a_k < n/2$. Recall that, in this case, $C_n(a_1, a_2, \ldots, a_k)$ is super-$\lambda^{(2)}$.

Theorem 3.7. Let $G = C_n(a_1, a_2, \ldots, a_k)$ with $n \geq 6$, $k \geq 2$ and $a_k < n/2$. Then $G$ is not super-$\lambda^{(3)}$ if and only if one of the following conditions is satisfied.

(i) girth $(G) = 3$ and there exists a $(k-1)$-elements subset $T$ of $S$ such that $4 \leq \langle T \rangle < \min\{n - 1, 3k - 4\}$, where $\langle T \rangle$ denotes the subgroup of $Z_n$ generated by $T$.

(ii) girth $(G) > 3$ and there exists a $(k-1)$-elements subset $T$ of $S$ such that $4 \leq \langle T \rangle < \min\{n - 1, 3(k - 1)\}$.

Proof. If condition (i) is satisfied, let $A = \langle T \rangle$, then it is easy to see that the subgraph $G[A]$ generated by $A$ is $C(A, T)$. Thus, if we let $h = |T|$, then $G[A]$ is $2h$-regular. Then

$$\omega(A) = 2(k - h)|A| \leq 2(3k - 4)(k - h) = 6k - 8 < \bar{\lambda}_3(G).$$

On the other hand, suppose without loss of generality, that $S \setminus T = \{a_k\}$ and let $l$ be the least integer such that $lak \in A$. Then,

$$V(G) = Z_n = A \cup (A + a_k) \cup \cdots \cup (A + (l - 1)a_k).$$

Clearly, $G[A + ia_k] \cong G[A]$, and $G[E[A]] \supseteq (A + a_k) \cup \cdots \cup (A + (l - 1)a_k)$. It follows that the number of vertices in a connected component of $G[E[A]]$ is at least $|A|$. Thus, $E[A]$ is a 3-RC with cardinality less than $\bar{\lambda}_3(G)$. So $G$ is not super-$\lambda^{(3)}$. Similarly, if condition (ii) is satisfied, $G$ is not super-$\lambda^{(3)}$ either.

Conversely, if $G$ is not super-$\lambda^{(3)}$, let $A$ be the 3-RA containing the zero element 0 of $Z_n$, then $A$ is a subgroup of $Z_n$ by Theorem 2.2 and Proposition 3.4. Let $T = A \cap S$. Then, since $G[A]$ is connected and $|A| \geq 4, T \neq \emptyset$. Clearly, $G[A] = C(A, T)$, thus it is
2h-regular, where \( h = |T| \). It follows that

\[
\omega(A) = 2(k - h)|A| < \xi_3(G).
\]  

(1)

We claim that \( k \geq 3 \), since otherwise, \( k = 2 \) and \( h = 1 \), hence \( \omega(A) \geq 8 \geq \xi_3(G) \), this is impossible. We consider two cases.

**Case 1:** \( girth(G) = 3 \). Then, since \( G \) is not super-\( \lambda(3) \), by (1) we conclude that \( |A| \leq 3h - 1 \). On the other hand, it is clear that \( |A| \geq 2h + 1 \). Let \( |A| = 2h + c \), \( c \geq 1 \).

Since \( \omega(A) = 2(k - h)(2h + c) \) attains its minimum \( 4(2k + c - 4) \geq 6k - 6 = \xi_3(G) \) at \( h = k - 1 \) over \( 1 \leq h \leq k - 2 \), it follows that \( h = k - 1 \), \( |A| \leq 3k - 4 \) and \( A = \langle T \rangle \).

Thus, condition (i) is satisfied.

**Case 2:** \( girth(G) > 3 \). Similar as in Case 1, we have \( |A| \leq 3h \) and \( h = k - 1 \). Thus condition (ii) is satisfied. \( \Box \)

**Theorem 3.8.** Let \( G = C_n(a_1, a_2, \ldots, a_k) \) with \( n \geq 6 \), \( k \geq 2 \), \( a_k = n/2 \). Suppose that \( G \not\cong C(n/2) \times K_2 \) and \( G \not\cong C(n/2) \downarrow K_2 \). Then \( G \) is not super-\( \lambda(3) \) if and only if one of the following conditions is satisfied.

(i) \( girth(G) = 3 \), \( g.c.d.(n, a_1, \ldots, a_{k-1}) = 1 \) and there exists a \((k - 1)\)-elements subset \( T \) of \( S \) with \( n/2 \in T \) such that \( 4 \leq |\langle T \rangle| \leq \min\{n - 1, 3k - 5\} \).

(ii) \( girth(G) > 3 \), \( g.c.d.(n, a_1, \ldots, a_{k-1}) = 1 \) and there exists a \((k - 1)\)-elements subset \( T \) of \( S \) with \( n/2 \in T \) such that \( 4 \leq |\langle T \rangle| \leq \min\{n - 1, 3k - 4\} \).

(iii) \( girth(G) = 3 \), \( g.c.d.(n, a_1, \ldots, a_{k-1}) = 2 \) and \( n/2 < 6k - 9 \) or there exists a \((k - 1)\)-elements subset \( T \) of \( S \) with \( n/2 \in T \) such that \( 4 \leq |\langle T \rangle| \leq \min\{n - 1, 3k - 5\} \).

(iv) \( girth(G) > 3 \), \( g.c.d.(n, a_1, \ldots, a_{k-1}) = 2 \) and \( n/2 < 6k - 7 \) or there exists a \((k - 1)\)-elements subset \( T \) of \( S \) with \( n/2 \in T \) such that either \( 4 \leq |\langle T \rangle| \leq \min\{n - 1, 3k - 4\} \).

**Proof.** For the if part of the theorem, we only consider the case where condition (i) is satisfied, the other cases can be similarly proved. Let \( A = \langle T \rangle \) and \( |T| = h \). Then \( G[A] = C(A, T) \) is a \((2h - 1)\)-regular graph. So

\[
\omega(A) = 2(k - h)|A| \leq 2(k - h)((3k - 5)) \leq 6k - 10 < \xi_3(G).
\]

The conclusion then follows.

For the only if part of the theorem, the 3-RA \( A \) containing the zero element 0 of \( Z_n \) is a subgroup of \( Z_n \). Let \( T = A \cap S \) and \( h = |T| \). Then \( T \neq \emptyset \) as \( G[A] \) is connected. Clearly, \(|A| \geq 4 \) and \( G[A] = C(A, T) \). We consider two cases.

**Case 1:** \( g.c.d.(n, a_1, \ldots, a_{k-1}) = 1 \).

If \( girth(G) = 3 \), then \( \xi_3(G) = 6k - 9 \). We claim that \( n/2 \in T \). For otherwise, \( G[A] \) is a \( 2h \)-regular graph and \( h < k - 1 \) (if \( h = k - 1 \), then as \( g.c.d.(n, a_1, \ldots, a_{k-1}) = 1 \), we have \( \langle A \rangle = Z_n \), this is impossible). Clearly, \(|A| \geq 2h + 1 \). Thus \( \omega(A) = (2k - 1 - 2h)|A| \geq (2k - 1 - 2h)(2h + 1) \geq 6k - 9 = \xi_3(G) \), this is impossible. Therefore, \( n/2 \in T \), and \( G[A] \) is of regular degree \( 2h - 1 \). It follows that \( \omega(A) = 2(k - h)|A| \) and \( |A| \geq 2h \).

Since \( G \) is not super-\( \lambda(3) \), we have \(|A| \leq 3h - 2 \) and \( h = k - 1 \). Condition (i) is satisfied.
If \( \text{girth}(G)>3 \), we claim that \( n/2 \in T \). For otherwise, \( n/2 \notin T \), then \( G[A] \) is a \( 2h \)-regular graph. Then \( \omega = (2k−1−2h)|A| \geq (2k−1−2h)(2h+1) \). Similarly as above, we have \( h \leq k−2 \). Since \( G \) is not super-\( \lambda(3) \), we can deduce that \( |A|=2h+1 \) and \( h=k−2 \). But then \( G[A]=C(A,T) \) is a complete graph with more than three vertices, contradicting the hypothesis \( \text{girth}(G)>3 \). Thus, \( n/2 \in T \), and \( G[A] \) is a \( (2h−1) \)-regular graph. Then \( \omega(A)=2(k−h)|A|, \ h \leq k−1 \) and \( |A| \geq 2h \). As \( G \) is not super-\( \lambda(3) \) and \( \text{girth}(G)>3 \), we have \( |A| \leq 3h−1 \), \( h=k−1 \). Condition (ii) is satisfied.

Case 2. \( \text{g.c.d.}(n,a_1,\ldots,a_{k−1})=2 \).

If \( \text{girth}(G)=3 \), then \( \xi_3(G)=6k−9 \). Now suppose that \( n/2 \geq 6k−9 \), then \( G \) is super-\( \lambda(3) \). We claim that \( n/2 \notin T \). For otherwise, \( G[A] \) is \( 2h \)-regular. It follows that

\[
\omega(A)=2(k−1−2h)|A| \text{ and } |A| \geq 2h+1.
\]

If \( h=k−1 \), then, since \( |A|=n/2 \), we have \( \omega(A)=\frac{2h}{2} \geq 6k−9 \), a contradiction. If \( h \leq k−2 \), then \( \omega(A)=(2k−1−2h)|A| \geq 3(2(k−2)+1)=6k−9 \), again a contradiction. Thus \( n/2 \in T \) and \( G[A] \) is regular of degree \( 2h−1 \). Therefore \( \omega(A)=2(k−h)|A| \). As \( G \) is not super-\( \lambda(3) \), we have \( |A| \leq 3h−2 \) and \( h=k−1 \). Condition (iii) is satisfied.

If \( \text{girth}(G)>3 \), then \( \xi_3(G)=6k−7 \). Now assume that \( n/2 \geq 6k−7 \), then \( G \) is super-\( \lambda(3) \). We claim that \( n/2 \notin T \). For otherwise, \( \omega(A)=(2k−1−2h)|A| \). It is easy to see that \( \omega(A)<\xi_3(G) \) only when \( |A|=2h+1 \) and \( h=k−2 \). But then \( |A|=2k−3 \). Thus every non-zero element of \( A \) is in \( S \cup (−S) \), this means that \( G[A]=C(A,T) \) is a complete graph, contradicting the hypothesis \( \text{girth}(G)>3 \). Thus, \( n/2 \in T \) and \( \omega(A)=2(k−h)|A| \). Since \( G \) is not super-\( \lambda(3) \) and \( \text{girth}(G)>3 \), it follows that \( |A| \leq 3h−1 \) and \( h=k−1 \). Condition (iv) is satisfied. \( \square \)

**Corollary 3.9.** Harary graphs \( G=C_n(1,2,\ldots,k) \) with \( 2 \leq k<n/2 \) are super-\( \lambda(3) \).

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**References**