Derivative polynomials and enumeration of permutations by number of interior and left peaks

Shi-Mei Ma

Department of Information and Computing Science, Northeastern University at Qinhuangdao, Hebei 066004, China

ARTICLE INFO

Article history:
Received 25 May 2011
Received in revised form 2 October 2011
Accepted 3 October 2011
Available online 22 October 2011

Keywords:
Derivative polynomials
Interior peaks
Left peaks

ABSTRACT

Derivative polynomials in two variables are defined by repeated differentiation of the tangent and secant functions. We establish the connections between the coefficients of these derivative polynomials and the number of interior and left peaks over the symmetric group. Properties of the generating functions for the number of interior and left peaks over the symmetric group, including recurrence relations, generating functions and real-rootedness, are studied.

1. Introduction

Let $\delta_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. A permutation $\pi = \pi(1) \pi(2) \cdots \pi(n) \in \delta_n$ is alternating if $\pi(1) > \pi(2) < \cdots < \pi(n)$. In other words, $\pi(i) < \pi(i+1)$ if $i$ is even and $\pi(i) > \pi(i+1)$ if $i$ is odd. Similarly $\pi$ is reverse alternating if $\pi(1) < \pi(2) > \cdots > \pi(n)$. Let $E_n$ denote the number of alternating permutations in $\delta_n$. For instance, $E_4 = 5$, corresponding to the permutations $2143$, $3142$, $3241$, $4132$, and $4231$. The number $E_n$ is called an Euler number because Euler considered the numbers $E_{2n+1}$. The bijection $\pi \mapsto \pi^c$ on $\delta_n$ defined by $\pi^c(i) = n+1-\pi(i)$ shows that $E_n$ is also the number of reverse alternating permutations in $\delta_n$. Alternating permutations have rich combinatorial structure and we refer the reader to the survey paper by Stanley [18] for recent progress on this subject.

In 1879, André [1] observed that

$$\sum_{n=0}^{\infty} \frac{E_n x^n}{n!} = \tan x + \sec x,$$

$$= 1 + x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 16\frac{x^5}{5!} + \cdots.$$

Since the tangent is an odd function and the secant is an even function, we have

$$\sum_{n=0}^{\infty} \frac{E_{2n+1} x^{2n+1}}{(2n+1)!} = \tan x \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{E_{2n} x^{2n}}{(2n)!} = \sec x.$$

For this reason the integers $E_{2n+1}$ are sometimes called the tangent numbers and the integers $E_{2n}$ are called the secant numbers.

© 2011 Elsevier B.V. All rights reserved.

doi:10.1016/j.disc.2011.10.003

This work is supported by the Fundamental Research Funds for the Central Universities (N100323013).

E-mail address: shimeima@yahoo.com.cn.
Let $D_x$ denote the differential operator $\frac{d}{dx}$. In 1995, Hoffman [12] considered two sequences of derivative polynomials defined respectively by
\begin{align}
D^n_\times(\tan(x)) = P_n(\tan(x)) \quad \text{and} \quad D^n_\times(\sec(x)) = \sec(x)Q_n(\tan(x))
\end{align}
for $n \geq 0$. From the chain rule it follows that the polynomials $P_n(u)$ satisfy $P_0(u) = u$ and $P_{n+1}(u) = (1 + u^2)P_n'(u)$, and similarly $Q_0(u) = 1$ and $Q_{n+1}(u) = (1 + u^2)Q_n'(u) + uQ_n(u)$. In particular, the numbers $P_n(0)$ and $Q_n(0)$ are respectively the tangent and secant numbers. Recently, several combinatorial formulas for these derivative polynomials have been extensively investigated (see [8,11,13,14] for instance). For example, let
\begin{align}
\tan^k(x) = \sum_{n\geq k} T(n,k) \frac{x^n}{n!}
\end{align}
and
\begin{align}
\sec(x) \tan^k(x) = \sum_{n\geq k} S(n,k) \frac{x^n}{n!}.
\end{align}
The numbers $T(n,k)$ and $S(n,k)$ are respectively called the tangent numbers of order $k$ (see [5, p. 428]) and the secant numbers of order $k$ ([see [4, p. 305]]). Clearly, the numbers $T(n,1)$ and $S(n,0)$ are the tangent and secant numbers, respectively. Cvijović [8, Theorem 1] obtained that
\begin{align}
P_n(x) = T(n,1) + \sum_{k=1}^{n+1} \frac{1}{k} T(n+1,k)x^k
\end{align}
and
\begin{align}
Q_n(x) = \sum_{k=0}^{n} S(n,k)x^k.
\end{align}
The organization of this paper is as follows. In Section 2, derivative polynomials in two variables are defined by repeated differentiation of the tangent and secant functions. In Section 3, we study the connections between interior peaks and left peak polynomials. In Section 4, we obtain several explicit formulas. In Section 5, we give both central and local limit theorems for the coefficients of certain polynomials.

2. Triangular arrays

Since $D_x(\tan(x)) = \sec^2(x)$ and $D_x(\sec(x)) = \tan(x) \sec(x)$, it is convenient to set $y = \tan(x)$ and $z = \sec(x)$. Hence we have $D_x(y) = z^2$ and $D_x(z) = yz$. It is natural to consider the triangular arrays $(W_{n,k})_{n \geq 1, 0 \leq k \leq \lfloor(n-1)/2\rfloor}$, $(W^l_{n,k})_{n \geq 1, 0 \leq k \leq n/2}$ and $(R_{n,k})_{n \geq 1, 0 \leq k \leq n}$ defined respectively by
\begin{align}
D^n_\times(y) = \sum_{k=0}^{\lfloor(n-1)/2\rfloor} W_{n,k}y^{n-2k-1}z^{2k+2}, \quad D^n_\times(z) = \sum_{k=0}^{n/2} W^l_{n,k}y^{n-2k}z^{2k+1}
\end{align}
and
\begin{align}
D^n_\times(y+z) = \sum_{k=0}^{n} R_{n,k}y^{n-k}z^{k+1}.
\end{align}
In the following discussion we will always assume that $n \geq 1$ and $0 \leq k \leq n$. By the linearity of $D_x$, we have
\begin{align}
R_{n,k} = \begin{cases} 
W_{n,k+1} & \text{if } k \text{ is odd,} \\
W^l_{n,k+1} & \text{if } k \text{ is even.}
\end{cases}
\end{align}

**Theorem 1.** For $n \geq 1$ and $0 \leq k \leq n$, we have
\begin{align}
R_{n+1,k} &= (k+1)R_{n,k} + (n-k+2)R_{n,k-2} , \\
W_{n,k} &= (2k+2)W_{n-1,k} + (n-2k)W_{n-1,k-1}
\end{align}
and
\begin{align}
W^l_{n,k} &= (2k+1)W^l_{n-1,k} + (n-2k+1)W^l_{n-1,k-1}.
\end{align}

**Proof.** Note that
\begin{align}
D_x^{n+1}(y+z) = D_x(D_x^n(y+z)) = \sum_{k=0}^{n} (k+1)R_{n,k}y^{n-k}z^{k+1} + \sum_{k=0}^{n} (n-k)R_{n,k}y^{n-k-1}z^{k+3}.
\end{align}
Thus we obtain (5). Similarly, we get (6) and (7). □
The numbers $W_{n,k}$ and $W_{n,k}^l$ arise often in combinatorics and other branches of mathematics (see [17, A008303, A008971]). In [15,16], Petersen studied the peak statistic over $\delta_n$. Let $\pi = \pi (1)\pi (2) \cdots \pi (n) \in \delta_n$. An interior peak in $\pi$ is an index $i \in \{2, 3, \ldots, n - 1\}$ such that $\pi (i - 1) > \pi (i) > \pi (i + 1)$. Let $\text{pk} (\pi)$ denote the number of interior peaks in $\pi$. An left peak in $\pi$ is an index $i \in \{n - 1\}$ such that $\pi (i - 1) > \pi (i) > \pi (i + 1)$, where we take $\pi (0) = 0$. Let $\text{lk} (\pi)$ denote the number of left peaks in $\pi$. For example, the permutation $\pi = 21435$ has $\text{pk} (\pi) = 1$ and $\text{lk} (\pi) = 2$.

Let
\[
W_n(x) = \sum_{\pi \in \delta_n} x^{\text{pk} (\pi)} \quad \text{and} \quad W_n^l(x) = \sum_{\pi \in \delta_n} x^{\text{lk} (\pi)}.
\]

Note that $\deg W_n(x) = [(n - 1)/2]$ and $\deg W_n^l(x) = [n/2]$. Using (6) and (7), it is easy to verify that $W_{n,k}$ is the number of permutations in $\delta_n$ with $k$ interior peaks, and $W_{n,k}^l$ is the number of permutations in $\delta_n$ with $k$ left peaks. Hence
\[
W_n(x) = \sum_{k \geq 0} W_{n,k} x^k \quad \text{and} \quad W_n^l(x) = \sum_{k \geq 0} W_{n,k}^l x^k.
\]

Therefore, by (6), the polynomials $W_n(x)$ satisfy
\[
W_{n+1}(x) = (nx - x + 2)W_n(x) + 2x(1 - x)D_x W_n(x),
\]
with initial values $W_1(x) = 1$, $W_2(x) = 2$ and $W_3(x) = 4 + 2x$. By (7), the polynomials $W_n^l(x)$ satisfy
\[
W_{n+1}^l(x) = (nx + 1)W_n^l(x) + 2x(1 - x)D_x W_n^l(x),
\]
with initial values $W_1^l(x) = 1$, $W_2^l(x) = 1 + x$ and $W_3^l(x) = 1 + 5x$.

We now present a connection between the polynomials $W_n(x)$ and $W_n^l(x)$.

**Theorem 2.** For $n \geq 1$, we have
\[
W_n(u) = (u - 1)^{\frac{n+1}{2}} u^{-\frac{1}{2}} P_n((u - 1)^{-\frac{1}{2}})
\]
and
\[
W_n^l(u) = (u - 1)^{\frac{n}{2}} Q_n((u - 1)^{-\frac{1}{2}}).
\]

**Proof.** Set $y = \tan(x)$ and $z = \sec(x)$. Note that $z^2 = y^2 + 1$. Comparing (1) with (2), we have
\[
P_n(y) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} W_{n,k} y^{n-2k-1} (1 + y^2)^k = (y^{n-1} + y^{n+1}) W_n(1 + y^2),
\]
and
\[
Q_n(y) = \sum_{k=0}^{\lfloor n/2 \rfloor} W_{n,k}^l y^{n-2k} (1 + y^2)^k = y^n W_n^l(1 + y^2).
\]

Let $u = 1 + y^{-2}$. Then the desired results follows immediately. \( \square \)

Let
\[
R_n(x) = \sum_{k=0}^n R_{n,k} x^k.
\]
By (5), we have
\[
R_{n+1}(x) = (1 + nx^2)R_n(x) + x(1 - x^2)R_n^l(x) \quad \text{for} \quad n \geq 1.
\]
with initial values $R_1(x) = 1 + x$, $R_2(x) = 1 + 2x + x^2$ and $R_3(x) = 1 + 4x + 5x^2 + 2x^3$. Set $R_0(x) = 1$. For $0 \leq n \leq 6$, the coefficients of $R_n(x)$ can be arranged as follows with $R_{n,k}$ in row $n$ and column $k$:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>2</th>
<th>1</th>
<th>4</th>
<th>5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>18</td>
<td>16</td>
<td>5</td>
<td>16</td>
<td>58</td>
<td>88</td>
<td>61</td>
</tr>
<tr>
<td>1</td>
<td>32</td>
<td>179</td>
<td>416</td>
<td>479</td>
<td>272</td>
<td>61</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Using (4), we get
\[
R_n(x) = xW_n(x^2) + W_n^l(x^2).
\]

Therefore, $R_{n,0} = 1$, $R_{n,1} = 2^{n-1}$, $\sum_{k=0}^n R_{n,k} = 2n!$ and $R_{n,n} = R_{n-1,n-2} = E_n$ for $n \geq 2$. 

3. Identities

Recently, much attention has been paid to the properties of the polynomials \(W_n(x)\) and \(W_n^l(x)\). For a permutation \(\pi \in \delta_n\), we define a descent to be a position \(i\) such that \(\pi(i) > \pi(i + 1)\). Denote by \(\text{des}(\pi)\) the number of descents of \(\pi\). The generating function for descents is

\[
A_n(x) = \sum_{\pi \in \delta_n} x^{\text{des}(\pi)},
\]

which is well known as the Eulerian polynomial. The exponential generating function for \(A_n(x)\) is

\[
A(x, z) = 1 + \sum_{n \geq 1} A_n(x) \frac{z^n}{n!} = \frac{(1 - x)e^{(1-x)z}}{1 - xe^{(1-x)}}.
\]  

(10)

By the theory of enriched \(P\)-partitions, Stembridge [19, Remark 4.8] showed that

\[
W_n \left( \frac{4x}{(1 + x)^2} \right) = \frac{2^{n-1}}{(1 + x)^{n-1}} A_n(x).
\]  

(11)

In order to simplify the identities (11) and (12), we will present another connection between \(W_n(x)\) and \(W_n^l(x)\). Let \(C_n\) denote the set of signed permutations of \(\pm[n]\) such that \(\omega(-i) = -\omega(i)\) for all \(i\), where \(\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}\). Let

\[
\tilde{C}_n(x) = \sum_{\omega \in C_n} x^{\text{des}(\omega)} \quad \text{and} \quad \tilde{C}_n(x) = \sum_{\omega \in C_n} x^{\text{ades}(\omega)},
\]

where

\[
\text{des}(\omega) = |\{i \in [0, n-1] : \omega(i) > \omega(i + 1)\}|
\]

and

\[
\text{ades}(\omega) = |\{i \in [0, n] : \omega(i) > \omega(i + 1)\}|
\]

with \(\omega(0) = \omega(n + 1) = 0\). For example, if \(\omega = (-2, -4, 6, -8, 1, 3, 7, 5)\), then \(\text{des}(\omega) = |\{0, 1, 3, 7\}| = 4\) and \(\text{ades}(\omega) = |\{0, 1, 3, 7, 8\}| = 5\). Recently, Dilks et al. [10] studied the combinatorial expansions for the affine Eulerian polynomials \(\tilde{C}_n(x)\). From [10, Cor. 5.6 and Cor. 5.7], we get

\[
2xW_n \left( \frac{4x}{(1 + x)^2} \right) = \frac{\tilde{C}_n(x)}{(1 + x)^{n-1}}
\]

and

\[
(1 + x)W_n^l \left( \frac{4x}{(1 + x)^2} \right) = \frac{C_n(x)}{(1 + x)^{n-1}}.
\]

(12)

For \(n \geq 1\), set

\[
T_n(x) = C_n(x^2) + \frac{1}{x}\tilde{C}_n(x^2).
\]

From their generating functions (see [10, Sec. 6.2]), we get

\[
\tilde{C}(x, z) = 1 + \sum_{n \geq 1} \frac{\tilde{C}_n(x)}{n!} z^n = \frac{1 - x}{1 - xe^{2z(1-x)}},
\]

\[
C(x, z) = 1 + (1 + x)z + \sum_{n \geq 2} C_n(x) \frac{z^n}{n!} = \frac{(1 - x)e^{(1-x)z}}{1 - xe^{2z(1-x)}}.
\]

Therefore,

\[
T(x, z) = 1 + \sum_{n \geq 1} T_n(x) \frac{z^n}{n!} = C(x^2, z) + \frac{1}{x}(\tilde{C}(x^2, z) - 1) = \frac{e^{(1-x^2)} - x}{1 - xe^{2x(1-x)}}.
\]  

(13)

Comparing (10) with (13), we get

\[
x + T(x, z) = (1 + x)A(x, z(1 + x)).
\]  

(14)

Using (14), we immediately obtain the following result.
Theorem 3. For $n \geq 1$, we have
\[ T_n(x) = (1 + x)^{n+1} A_n(x). \]  

Motivated by the connections between $C_n(x)$ and $\widetilde{C}_n(x)$, in the following sections we will study the properties of the polynomials $R_n(x)$.

4. Explicit formulas

The goal of this section is to find an explicit formula for the polynomials $R_n(x)$. Let
\[ P(x, z) = \sum_{n \geq 0} R_n(x) \frac{z^n}{n!}. \]

Multiplying both sides of (8) by $x^n/n!$ and summing over all values of $n$, we get that $P(x, z)$ satisfies the following partial differential equation:
\[ x(x^2 - 1) \frac{\partial P(x, z)}{\partial x} + (1 - x^2) \frac{\partial P(x, z)}{\partial z} = P(x, z) + x. \]  

It is well known [17, A008303, A008971] that
\[ W(x, z) = \sum_{n \geq 1} W_n(x) \frac{z^n}{n!} = \frac{\sinh(z \sqrt{1 - x})}{\sqrt{1 - x} \cosh(z \sqrt{1 - x}) - \sinh(z \sqrt{1 - x})} \]
and
\[ W'(x, z) = 1 + \sum_{n \geq 1} W_n'(x) \frac{z^n}{n!} = \frac{\sqrt{1 - x}}{\sqrt{1 - x} \cosh(z \sqrt{1 - x}) - \sinh(z \sqrt{1 - x})}. \]

Using (9), we get
\[ P(x, z) = 1 + \sum_{n \geq 1} [x W_n(x^2) + W_n'(x^2)] \frac{z^n}{n!} = x W(x^2, z) + W(x^2, z). \]

Thus
\[ P(x, z) = \frac{x \sinh(z \sqrt{1 - x^2}) + \sqrt{1 - x^2}}{\sqrt{1 - x^2} \cosh(z \sqrt{1 - x^2}) - \sinh(z \sqrt{1 - x^2})}. \]  

and this formula gives a solution to the partial differential equation (16). It should be noted that the generating function of the polynomials $R_n(x)$ has also been studied by Charalambos [6, p. 542]. Recall that the hyperbolic cosine and the inverse of the hyperbolic cosine are defined respectively by
\[ \cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \text{arccosh } (x) = \ln(x + \sqrt{x^2 - 1}) \quad \text{for } x \geq 1. \]

Let
\[ R(x, t) = \sum_{n \geq 0} R_{n+1}(x) \frac{t^n}{n!}. \]

Clearly,
\[ \frac{\partial P(x, t)}{\partial t} = R(x, t). \]

By the method of characteristics (see [20] for instance), Charalambos [6, p. 542] obtained a two-variable generating function
\[ R(x, t) = \frac{1 - x^2}{x(\cosh z - 1)}, \]  

(18)
with \( z = -t \sqrt{1 - x^2} + \arccosh (1/x) \). Comparing (17) with (18), the latter is simpler. To obtain an explicit formula for \( R_n(x) \), we will use (18). Let \( \{x_i\}_{i=1} \) be a sequence of variables. The partial Bell polynomials \( B_{n,k} =: B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \) are defined by the generating function

\[
\sum_{n \geq k} B_{n,k} \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{i=1}^{\infty} x_i \frac{t^i}{i!} \right)^k,
\]

with \( B_{0,0} = 1 \) and \( B_{n,0} = 0 \) for \( n > 0 \) (see [7, p. 133]).

**Proposition 4.** Let \( B_{n,k} \) be the partial Bell polynomials. When \( x_i = (1 - x^2)^{(i-1)/2} \) for each \( i \geq 1 \), we have

\[
R_{n+1}(x) = \sum_{k=1}^{n} (-1)^{n-k} k!(1 + x)^{k+1} B_{n,k}.
\]

**Proof.** Let \( z = -t \sqrt{1 - x^2} + \arccosh (1/x) \). Note that

\[
x \left( \frac{e^z + e^{-z}}{2} - 1 \right) = \frac{e^{-t \sqrt{1 - x^2}} + e^{t \sqrt{1 - x^2}}}{2} + \sqrt{1 - x^2} e^{-t \sqrt{1 - x^2}} - e^{t \sqrt{1 - x^2}} - x
\]

\[
= 1 - x + \sum_{i \geq 1} (-1)^i y_i \frac{t^i}{i!},
\]

where \( y_i = (1 - x^2)^{(i+1)/2} \). Thus

\[
R(x, t) = \frac{1 - x^2}{1 - x + \sum_{i \geq 1} (-1)^i y_i \frac{t^i}{i!}}
\]

\[
= \frac{1}{1 + x + \sum_{i \geq 1} (-1)^i x_i \frac{t^i}{i!}},
\]

where \( x_i = (1 - x^2)^{(i-1)/2} \). Then the desired result follows immediately from the formula for the geometric series and (19). \( \square \)

Here we provide an example for illustration of Proposition 4.

**Example 5.** Consider the case \( n = 4 \), we have

\[
B_{4,1} = x_4, \quad B_{4,2} = 4x_1x_3 + 3x_2^2, \quad B_{4,3} = 6x_1^2x_2, \quad B_{4,4} = x_4^4
\]

(see [7, p. 307] for instance). When \( x_i = (1 - x^2)^{(i-1)/2} \) for each \( i \geq 1 \), we get

\[
B_{4,1} = 1 - x^2, \quad B_{4,2} = 7 - 4x^2, \quad B_{4,3} = 6, \quad B_{4,4} = 1.
\]

Thus

\[
\sum_{k=1}^{4} (-1)^{4-k} k!(1 + x)^{k+1} B_{4,k} = 1 + 16x + 58x^2 + 88x^3 + 61x^4 + 16x^5.
\]

Recall that

\[
B_{n,k}(1, 1, 1, \ldots) = S(n, k),
\]

where \( S(n, k) \) is a Stirling number of the second kind (see [7, p. 135]). Hence when \( x = 0 \), the formula (20) reduces to

\[
1 = \sum_{k=0}^{n} (-1)^{n-k} k! S(n, k).
\]

Moreover, when \( x = 1 \), the formula (20) reduces to

\[
(n + 1)! = \sum_{k=1}^{n} (-1)^{n-k} k! 2^k B_{n,k}(1, 1, 0, \ldots, 0).
\]
From (20), we observe that \( R_n(x) \) is divisible by \((1 + x)^2 \) for \( n \geq 2 \). It is well known that the Eulerian polynomial \( A_n(x) \) has only real nonpositive simple zeros (see [3, Theorem 1.33]). In the next section, we will show that the polynomial \( R_n(x) \) also has only real zeros.

5. Central and local limit theorems

Let \( \text{sgn} \) denote the sign function defined on \( \mathbb{R} \), i.e.,
\[
\text{sgn} \, x = \begin{cases} 
+1 & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
-1 & \text{if } x < 0.
\end{cases}
\]

Let \( R \) denote the set of real polynomials with only real zeros. Furthermore, denote by \( R_i \) the set of such polynomials all whose zeros are in the interval \( I \). Suppose that \( f, F \in R \). Let \( \{r_i\} \) and \( \{s_j\} \) be all zeros of \( f \) and \( F \) in nonincreasing order respectively. We say that \( f \) separates \( F \), denoted by \( f \preceq F \), if \( \deg f \leq \deg F \leq \deg f + 1 \) and
\[
s_1 \geq r_1 \geq s_2 \geq r_2 \geq s_3 \geq r_3 \geq \cdots.
\]

It is well known that if \( f \in R \), then \( f' \in R \) and \( f' \preceq f \).

**Theorem 6.** For \( n \geq 1 \), we have \( R_n(x) \in R[-1, 0) \) and \( R_n(x) \preceq R_{n+1}(x) \). More precisely, \( R_n(x) \) has \( \lceil n/2 \rceil - 1 \) simple zeros, and the zero \( x = -1 \) with the multiplicity \( \lceil n/2 \rceil + 1 \).

**Proof.** From (8), we have \( \deg R_{n+1}(x) = \deg R_n(x) + 1 \) and \( R_n(0) = 1 \). Note that \( R_1(x) = 1 + x \), \( R_2(x) = (1 + x)^2 \), \( R_3(x) = (1 + x)^2(1 + 2x) \) and \( R_4(x) = (1 + x)^3(1 + 5x) \). So it suffices to consider the case \( n \geq 4 \). We proceed by induction on \( n \). Let \( m = \lceil n/2 \rceil + 1 \). Suppose that
\[
R_n(x) = \prod_{i=1}^{k} (x - r_i)(x + 1)^m,
\]
where \( 0 > r_1 > r_2 > \cdots > r_k > -1 \).

Let \( g(x) = \prod_{i=1}^{k} (x - r_i) \) and \( G(x) = R_{n+1}(x)/(x + 1)^m \). Then \( \deg G(x) = \deg g(x) + 1 \). By (8), we have
\[
G(x) = (1 + nx^2)g(x) + x(1 - x^2) \sum_{j=1}^{k} g(x) \frac{x}{x - r_j} + mx(1 - x)g(x).
\]

Note that \( \text{sgn} \, G(r_i) = (-1)^i \) for \( 1 \leq i \leq k \) and \( G(-1) = (1 + n - 2m)g(-1) \). Hence \( G(x) \) has precisely one zero in each of \( k \) intervals \((r_k, r_{k-1}), \ldots, (r_2, r_1), (r_1, 0)\). Note that
\[
1 + n - 2m = n - 2\lceil n/2 \rceil - 1 = \begin{cases} 
0 & \text{if } n \text{ is odd,} \\
-1 & \text{if } n \text{ is even.}
\end{cases}
\]

Thus \(-1\) is a simple zero of \( G(x) \) if \( n \) is odd. If \( n \) is even, then \( \text{sgn} \, G(-1) = (-1)^{k+1} \) and \( G(x) \) has precisely one zero in the interval \((-1, r_1)\). Hence \( R_{n+1}(x) \in R[-1, 0) \) and \( R_n(x) \preceq R_{n+1}(x) \). This completes the proof.

For \( n \geq 1 \), set
\[
R_n(x) = (1 + x)^{\lceil n/2 \rceil + 1} G_n(x).
\]

Then the polynomial \( G_n(x) \) has only positive integer coefficients.

**Open Problem 7.** Is the similarity between (15) and (22) just a coincidence? Can either equation be given a combinatorial interpretation?

Let \( \{a(n, k)\}_{0 \leq k \leq n} \) be a sequence of positive real numbers. It has no internal zeros if there exist no indices \( i < j < k \) with \( a(n, i)a(n, k) \neq 0 \) but \( a(n, j) = 0 \). Let \( A_n = \sum_{k=0}^{n} a(n, k) \). We say the sequence \( \{a(n, k)\} \) satisfies a central limit theorem with mean \( \mu_n \) and variance \( \sigma_n^2 \) provided
\[
\lim_{n \to +\infty} \sup_{x \in \mathbb{R}} \left| \frac{\sum_{k=0}^{\mu_n+x\sigma_n} a(n, k)}{A_n} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt \right| = 0.
\]

The sequence satisfies a local limit theorem on \( B \in \mathbb{R} \) if
\[
\lim_{n \to +\infty} \sup_{x \in B} \left| \frac{\sigma_n a(n, \mu_n + x\sigma_n)}{A_n} - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0.
\]
Recall the following Bender’s theorem.

**Theorem 8** ([2]). Let \( \{P_n\}_{n \geq 1} \) be a sequence of polynomials with only real zeros. The sequence of the coefficients of \( P_n \) satisfies a central limit theorem with

\[
\mu_n = \frac{P'_n(1)}{P_n(1)} \quad \text{and} \quad \sigma_n^2 = \frac{P''_n(1)}{P_n(1)} + \frac{P'_n(1)}{P_n(1)} - \left( \frac{P'_n(1)}{P_n(1)} \right)^2,
\]

provided that \( \lim_{n \to \infty} \sigma_n^2 = +\infty \). If the sequence of coefficients of each \( P_n(x) \) has no internal zeros, then the sequence of coefficients satisfies a local limit theorem.

Combining Theorems 6 and 8, we obtain the following result.

**Theorem 9.** The sequence \( \{R_{n,k}\}_{0 \leq k \leq n} \) satisfies both a central limit theorem and a local limit theorem with \( \mu_n = (2n - 1)/3 \) and \( \sigma_n^2 = (8n + 8)/45 \), where \( n \geq 4 \).

**Proof.** By differentiating (8), we obtain the recurrence

\[
x_{n+1} = (4n)n! + (n - 1)x_n
\]

for \( x_n = R_n'(1) \), and this has the solution \( x_n = (4n - 2)n!/3 \) for \( n \geq 2 \). By Theorem 8, we have \( \mu_n = (2n - 1)/3 \). Another differentiation leads to the recurrence

\[
y_{n+1} = \frac{4}{3}n!(4n^2 - 5n + 3) + (n - 3)y_n
\]

for \( y_n = R_n''(1) \). Set \( y_n = (an^2 + bn + c)n! \) and solve for \( a, b, c \) to get \( y_n = n!(40n^2 - 84n + 56)/45 \) for \( n \geq 4 \). Hence \( \sigma_n^2 = (8n + 8)/45 \), where \( n \geq 4 \). Thus \( \lim_{n \to \infty} \sigma_n^2 = +\infty \) as desired. \( \square \)

Let \( P(x) = \sum_{i=0}^{n} a_i x^i \) be a polynomial. Let \( m \) be an index such that \( a_m = \max_{0 \leq i \leq n} a_i \). Darroch [9] showed that if \( P(x) \in \mathbb{R}[\neg \infty, 0] \), then

\[
\left[ \frac{P'(1)}{P_n(1)} \right] \leq m \leq \left[ \frac{P''(1)}{P_n(1)} \right].
\]

So the following result is immediate.

**Corollary 10.** Let \( i \) be an index such that \( R_{n,i} = \max_{0 \leq k \leq n} R_{n,k} \). If \( (2n - 1)/3 \) is an integer, then \( i = (2n - 1)/3 \); Otherwise, \( i = \lceil (2n - 1)/3 \rceil \) or \( i = \lfloor (2n - 1)/3 \rfloor \).

**Acknowledgment**

The author would like to thank the referee for many detailed suggestions leading to substantial improvement of this paper.

**References**