On the Completeness of the Spherical Vector Wave Functions

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The set of the regular and radiating spherical vector wave functions (SVWF) is shown to be complete in $L_2(S)$ where $S$ is a Lyapunov surface. The completeness fails when $k^2$ is an eigenvalue of the Dirichlet problem for the Helmholtz equation $(\nabla^2 + k^2)F = 0$ in the region $D$, bounded by $S$. On the other hand, the set of radiating SVWF is shown to be complete for all values of $k^2$. It is also proved that any vector function, which is continuous in $D + S$ and satisfies the Helmholtz equation in $D$, can be approximated uniformly in $D$, and in the mean square sense on $S$ by a sequence of linear combinations of the regular SVWF (assuming the set is complete). Similar results are obtained for the exterior problem with the set of radiating SVWF. These results are extended to the set composed of the regular and radiating SVWF on two nonintersecting Lyapunov surfaces, one of which encloses the other.

1. INTRODUCTION

The characteristic solutions to the scalar Helmholtz equation $(\nabla^2 + k^2)u = 0$ obtained by the separation of variables in suitable coordinate systems are well established [11]. These functions are used in solving boundary value problems by series expansions on regions $D$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ having boundaries $S$ which conform with the specific coordinate system. Furthermore, such series which have coefficients determined from the boundary values converge uniformly in closed subsets of $D$. It is of interest to know what happens when the the boundary of $D$ does not conform with a coordinate system. This problem was first considered for the spherical wave functions (SWF) by Vekua [16].

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Vekua established the completeness of the sets of regular and radiating (outgoing) SWF on a Lyapunov surface $S$ [15]. However, he showed that when $k^2$ is an eigenvalue of the Dirichlet problem in $D$, (the region bounded by $S$), the set of regular SWF is not complete, and that completeness can be preserved by the addition of suitable functions. He also demonstrated that the solutions to the Helmholtz equation in $D$, which are continuous in $D + S$ can be approximated by a series of orthogonalized regular SWF which converge uniformly in $D$.

Hizal [7] extended Vekua's [16] results to doubly connected regions bounded by two Lyapunov surfaces, by considering the set formed by the union of regular and radiating SWF on such composite surfaces.

A more general analysis for the solutions of the scalar Helmholtz equation in $\mathbb{R}^3$ was presented by Müller and Kersten [14]. They demonstrated that the sets of functions obtained by separation of variables in various coordinate systems are complete in $L^2(S)$ and that they may be used in solving Dirichlet and Neumann problems.

Millar [9] showed the completeness of the set of outgoing cylindrical wave functions and the set of their normal derivatives on $C$, which is a closed smooth curve in $\mathbb{R}^2$. In a more recent paper [10], he made corrections, as well as improvements in the proof for the normal derivatives. Colton [3, 4] and Millar [10] also established the completeness in $L^2(C)$ of sets of functions formed from solutions of the Helmholtz equation, each function being a linear combination of a solution (obtained by separation of variables) and its normal derivative on $C$. One set is constructed from regular Bessel functions for use in interior problems and the other set contains the Hankel function which satisfies the Sommerfeld radiation condition and is utilized in exterior problems. Millar [10] also dealt with the completeness of the union of these two sets on the boundary of a doubly connected annular region.

The solutions of the vector Helmholtz equation $(\nabla^2 + k^2)F = 0$ with zero divergence $\nabla \cdot F = 0$ in $\mathbb{R}^3$, which are equivalent to

$$(\nabla \times \nabla \times - k^2)F = 0 \tag{1}$$

have been considered by Calderon [2] and Müller [13]. Calderon dealt with the completeness in $L_2(S_\text{tan})$ of the set formed from the tangential components of multipole fields, where $L_2(S_\text{tan})$ is the space of square integrable tangential vector functions on a surface $S$ which may be composed of disjoint closed smooth surfaces. He showed that there exists a series in terms of multipole fields which uniformly approximates a radiating solution to (1) in the region exterior to $S$.

Müller [13] considered both the regular and radiating spherical vector
wave functions (SVWF) with zero divergence, i.e., the solutions to (1) in the spherical coordinate system. He proved that the tangential components of the regular SVWF form a basis in $L^2(S_{\text{tan}})$ where $S$ is a Lyapunov surface. Furthermore, similar to Calderon's result for the exterior boundary value problem [2], he showed that any radiating solution to (1) in the region exterior to $S$ can be approximated uniformly by a series of SVWF which has coefficients determined from the tangential component of the boundary value of $F$.

In this paper, using the methods presented by Vekua [16], as in [1], we show that the set of regular SVWF are complete on any Lyapunov surface $S$ as long as $k^2$ is not an eigenvalue of the interior homogeneous Dirichlet problem. We demonstrate that it is possible to preserve the completeness by adding suitable vector functions to the set. When the set is complete, it is shown that vector functions satisfying the Helmholtz equation in $D$, and continuous in $D_1 + S$ can be represented by linear combinations of regular SVWF which converge in the mean square sense on $S$ and uniformly in closed subsets of $D_1$.

Similar results are obtained for the set of radiating SVWF. The major difference in this case is that the completeness is preserved for all values of $k^2$. This set is shown to be applicable for the exterior boundary value problem.

The above results are extended to a doubly connected region $D$, which is bounded by two Lyapunov surfaces $S_1$ and $S_2$. Completeness in $L_2(S_1 + S_2)$ of the set formed by both the regular and radiating SVWF is established and related approximation results are presented.

2. INTERIOR PROBLEM

The regular SVWF

$$\{\mathbf{M}_{nmr}^l(kr), \mathbf{N}_{nmr}^l(kr), \mathbf{L}_{nmr}^l(kr)\},$$

$$n = 0, 1, 2, \ldots, m = 0, \ldots, n, \ell = 0, \ell$$

are the solutions of the vector Helmholtz equation

$$\nabla^2 \mathbf{F} + k^2 \mathbf{F} = 0 \quad (k = \text{constant})$$

which are finite at the origin. These functions can be expressed as [12]

$$\mathbf{M}_{nmr}(kr) = \nabla \times \left[ rf_{mn}(\phi) P_n^m(\cos \theta) j_n(kr) \right]$$

(4a)
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\[ N_{nm}(kr) = \frac{1}{k} \nabla \times M_{nm}(kr) \quad \text{and} \]

\[ M_{nm}(kr) = \frac{1}{k} \nabla \times N_{nm}(kr) \quad (4b) \]

\[ L_{nm}(kr) = \frac{1}{k} \nabla [f_m(\phi) P_n^m(\cos \theta) j_n(kr)] \quad (4c) \]

where

\( P_n^m(\cos \theta) \) — Associated Legendre function

\( j_n(kr) \) — Spherical Bessel function

\[ i = (-1)^{1/2} \]

\[ f_m(\phi) = \cos m\phi, \quad t = e. \]

\[ = \sin m\phi, \quad t = o. \]

Let \( S \) be the boundary of a finite region \( D_i \) in \( \mathbb{R}^3 \), which is connected and contains the origin. Assume \( S \) to be a Lyapunov surface [15] and the exterior \( D_e \) of \( D_i \) to be connected. Let \( D_o \) be a spherical region centered at the origin and containing \( D_i \) in its interior. Denote the spectrum of eigenvalues of the homogeneous Dirichlet problem for the scalar Helmholtz equation in \( D_i \) by \( \sigma(D_i) \). Let \( L_2(S) \) be the space of square integrable vector functions on \( S \).

**Lemma.** The spectrum of eigenvalues for the interior Dirichlet problem of the scalar and vector Helmholtz equations are identical.

**Proof.** Let \( \omega \) be an eigenfunction of the scalar Helmholtz equation \((\nabla^2 + k^2)\omega = 0\) in \( D_i \). Then one can construct an eigenfunction \( \Phi \) of the vector Helmholtz equation \((\nabla^2 + k^2)\Phi = \mathbf{0}\) by letting one of the cartesian components of \( \Phi \) to be \( \omega \) and the remaining components to be zero. Conversely, if \( \Phi \) is an eigenfunction of the vector Helmholtz equation, then each cartesian component is an eigenfunction of the scalar Helmholtz equation. Thus, the proof is completed.

**Theorem 1.** The set of regular SVWF given in (2) is complete in \( L_2(S) \) if and only if \( k^2 \notin \sigma(D_i) \). But if \( k^2 \in \sigma(D_i) \) then the completeness fails and the set has a deficiency equal to \( 3M \) (where \( M \) is the multiplicity of the eigenvalue \( k^2 \) for the scalar problem).

**Proof.** Since completeness and closedness imply each other [6], it is
sufficient to prove that a nonzero vector function \( \mathbf{U}(r) \in L_2(S) \) that satisfies the following equations simultaneously.

\[
\begin{align*}
\langle \mathbf{M}_{nmt}^1, \mathbf{U} \rangle & \equiv \int_S \mathbf{U}^*(r_s) \cdot \mathbf{M}_{nmt}^1(k \mathbf{r}_s) \, dS = 0 \\
\langle \mathbf{N}_{nmt}^1, \mathbf{U} \rangle & = 0 \\
\langle \mathbf{L}_{nmt}^1, \mathbf{U} \rangle & = 0 \\
n & = 0, 1, 2, \ldots \text{ and } m = 0, \ldots, n, \quad t = e, o
\end{align*}
\]  

exists if and only if \( k^2 \in \sigma(D_1) \). The star * means complex conjugate.

Let us multiply Eqs. (5a), (5b), and (5c) by the radiating SVWF (Section 3): \( C_{nmt} \mathbf{M}_{nmt}^2(k \mathbf{r}), C_{nmt} \mathbf{N}_{nmt}^2(k \mathbf{r}), \text{and } C_{nmt} n(n+1) \mathbf{L}_{nmt}^2(k \mathbf{r}) \), respectively, where \( r \) denotes a point in \( \mathbb{R}^3 \setminus D_0 \), and

\[
C_{nmt} = i k \frac{(2n+1)}{2 \pi n (n+1)} \frac{(n-m)!}{(n+m)!} \varepsilon_m
\]  

After summing the equations for all \( n, m, \) and \( t \) and interchanging the summation and integration signs, the resulting series converges to the dyadic Green's function for unbounded space [12, p. 1875], and we obtain

\[
\mathbf{T}(r) \equiv \int_S \mathbf{U}^*(r_s) \cdot \mathbf{F} e^{ikR} \, dS = \mathbf{0}
\]  

where \( \mathbf{F} \) is the identity dyadic and \( R = |r - r_s| \).

\( \mathbf{T}(r) \) as defined in (7) is a solution of the vector Helmoltz equation in \( D_e \) and is identically zero for \( r \in \mathbb{R}^3 \setminus D_0 \). Since the solutions of the Helmholtz equation are analytical, one can conclude that \( \mathbf{T}(r) = \mathbf{0} \) in \( D_e \), and all its derivatives are zero as well. Therefore, we can write

\[
\mathbf{T}(r_s)_+ = \left( \frac{\partial}{\partial n_s} \mathbf{T}(r_s) \right)_+ = \mathbf{0}.
\]

The plus sign indicates the values on the surface when approached from \( D_e \). By a straightforward extension to the vector case of the properties of single layer potentials with square integrable densities [8], one obtains [14, p. 61]

\[
\int_S \mathbf{U}^*(r_s) \frac{e^{ikR}}{2 \pi R} \, dS - \mathbf{U}^*(r) = \mathbf{0}, \quad r \in S
\]
in $L_2(S)$. Equation (9) can also be written as

$$KU^* = U^* \quad (10)$$

where $K$ is an integral operator with a weakly singular kernel. Iteration yields

$$K^nU^* = U^*. \quad (11)$$

For sufficiently large $n$, $K^n$ is an integral operator with a continuous kernel [14]. Therefore, $U^*$ can be replaced with a continuous vector function $U^*$ where

$$U^* = U^*_t \quad (12)$$

in $L_2(S)$. Then using Eq. (8) and the continuity of single layer potentials with continuous kernels and the discontinuity of their normal derivatives [18], one obtains

$$T(r_s)_- = 0 \quad (13)$$

$$\left( \frac{\partial}{\partial n_s} T(r_s)_- \right)_- = 2U^*_t(r_s) \quad (14)$$

Since we have assumed that $k^2 \notin \sigma(D_i)$, (13) requires that

$$T(r) = 0, \quad r \in D_i \quad (15)$$

which leads to

$$\left( \frac{\partial}{\partial n_s} T(r_s) \right)_- = 0. \quad (16)$$

Therefore, one can conclude from (12), (14), and (16) that

$$U^* = 0$$

in $L_2(S)$. This completes the first part of the proof.

If $k^2 \notin \sigma(D_i)$, then the interior homogeneous Dirichlet problem for the Helmholtz equation does not have a unique solution. Let $X_1(r), X_2(r), \ldots, X_{3M}(r)$ be the complete set of corresponding vector eigenfunctions. Note that in three dimensions there are three times as many eigenfunctions for the vector problem than the scalar problem because each cartesian component of the vector function satisfies the Helmholtz equation and one can construct three linearly independent vector functions from one scalar function. It can be shown based on the scalar case [17, pp. 325, 326]
that these eigenfunctions are continuous and continuously differentiable up to order 2 in the closed region $D_i + S$. Since $X_m(r) = 0$, $m = 1, 2, ..., 3M$ on $S$, then by Green's formula

$$X_m(r) = \int_S K_m(r_s) \cdot \hat{I} \frac{e^{ikr}}{2\pi R} \, dS$$  \tag{17}$$

where

$$K_m(r_s) = \frac{\partial}{\partial n_s} X_m(r_s) = (\hat{n}_s \cdot \nabla) X_m(r_s), \quad m = 1, 2, ..., 3M$$

$\hat{n}_s =$ surface unit normal vector pointing away from $D_i$.

Using the linear independence of $X_m, m = 1, 2, ..., 3M$, one can show that $K_m(r), m = 1, 2, ..., 3M$ are linearly independent. Since $X_m(r), m = 1, 2, ..., 3M$, are zero on $S$ and since (17) satisfies the radiation condition at infinity (Section 3), then as a result of the uniqueness property of the exterior radiation problem we have [19]

$$X_m(r) = 0 \quad \text{in} \quad D_c, \quad m = 1, 2, ..., 3M \tag{18}$$

Thus, the functions $K_m(r_s), m = 1, 2, ..., 3M,$ satisfy (7). It can be proved that any function satisfying (5) simultaneously is a linear combination of $K_1, ..., K_{3M}$ as follows: Let $h$ be such a function, then

$$u(r) = \int_S h(r_s) \cdot \hat{I} \frac{e^{ikr}}{2\pi R} \, dS$$

and $u = 0$ in $D_c$. Since $X_m, m = 1, 2, ..., 3M,$ is the complete set of eigenfunctions, $u$ must be a linear combination of these, i.e.,

$$u = \sum_{m=1}^{3M} a_m X_m. \tag{19}$$

Using similar arguments as before for the single layer potentials with square integrable densities, one obtains

$$h = \sum_{m=1}^{3M} a_m \frac{\partial}{\partial n_s} X_m \tag{20}$$

in $L_2(S)$. This shows that for $k^2 \in \sigma(D_i)$, the set of regular SVWF is not complete in $L_2(S)$ unless the functions $K_m^*(r), m = 1, 2, ..., 3M,$ are added. Thus, the proof is completed.

Before proceeding with an important theorem on the representation of functions using the SVWF, let us orthonormalize the set of regular SVWF
with respect to the surface S using the Schmidt process and denote the new set as

\[ \{ V_n^1(kr) \}, \quad n = 1, 2, \ldots \] (21)

with the property

\[ \langle V_n^1, V_m^1 \rangle = 0, \quad n \neq m \]
\[ = 1, \quad n = m. \] (22)

**Theorem 2.** If \( k^2 \notin \sigma(D_i) \), then every \( H \) satisfying the Helmholtz equation in \( D_i \) and taking the continuous value \( H_s \) on \( S \) can be expanded as

\[ H(r) = \sum_{n=1}^{\infty} \langle H_s, V_n^1 \rangle V_n(r) \] (23)

where the series converges uniformly to \( H \) in closed subsets of \( D_i \) and in the mean square sense to \( H_s \) on \( S \).

**Proof.** Theorem 1 and the definition of the completeness of a system of functions lead to the result that any square integrable vector function on a Lyapunov surface can be approximated in the mean square sense arbitrarily closely by a linear combination of the regular SVWF if and only if \( k^2 \notin \sigma(D_i) \). In other words, let

\[ H_N(r) = \sum_{n=1}^{N} \langle H_s, V_n^1 \rangle V_n(r) \] (24)

and we then have

\[ \lim_{N \to \infty} \int_{S} |H_s - H_N|^2 \, dS = 0. \] (25)

Since \( k^2 \notin \sigma(D_i) \) we can express \( H \) in \( D_i \) as

\[ H(r) = \int_{S} H_s(r_s) \frac{\partial}{\partial n_s} G(r, r_s) \, dS \] (26)

where \( G(r, r_s) \) is the Green's function for the Helmholtz equation in \( D_i \) which vanishes on \( S \). Similarly, each regular SVWF can be expressed in the form (26). Hence, we have the equality

\[ H(r) - H_N(r) = \int_{S} [H_s(r_s) - H_N(r_s)] \frac{\partial}{\partial n_s} G(r, r_s) \, dS. \] (27)
Since \( r \) and \( r_s \) represent points in \( D_i \) and on \( S \), respectively, 
\(|(\partial/\partial n_s) G(r, r_s)|^2 \) is integrable on \( S \). Then by the Cauchy–Schwarz inequality we obtain for each cartesian component of (27) \((j = 1, 2, 3 \) representing the cartesian components) 
\[
|H_j(r) - H_{N_j}(r)| \leq \left[ \int_S \left| H_{N_j}(r_s) - H_{N_j}(r_s) \right|^2 \, dS \right]^{1/2} \cdot \left[ \int_S \left| \frac{\partial}{\partial n_s} G(r, r_s) \right|^2 \, dS \right]^{1/2}. \tag{28}
\]
Thus, by (25) and (28) the uniform convergence of \( H_N \) to \( H \) in all closed subsets of \( D_i \) is established, i.e., 
\[
\lim_{N \to \infty} |H(r) - H_{N}(r)| = 0 \tag{29}
\]
where \(|H| = (|H_1|^2 + |H_2|^2 + |H_3|^2)^{1/2}\).

3. EXTERIOR PROBLEM

In this section we consider the completeness of the set of radiating SVWF on a Lyapunov surface \( S \) and their approximation properties related to the solutions of the Helmholtz equation in the region exterior to \( S \).

The radiating SVWF 
\[
\{M^2_{nmt}(kr), N^2_{nmt}(kr), L^2_{nmt}(kr)\}, \quad n = 0, 1, 2, \ldots, m = 0, \ldots, n, t = e, o \tag{30}
\]
can be obtained from (4) by replacing \( j_n(kr) \) with \( h_n^{(1)}(kr) \), the spherical Hankel functions of the first kind. They are the solutions of the Helmholtz equation which satisfy the radiation condition \[5\]
\[
- \frac{r}{r} \times \nabla \times F + \frac{r}{r} \nabla \cdot F - ikF = o \left( \frac{1}{r} \right) \tag{31}
\]
uniformly for all directions of \( r/r \). Note that the vector wave functions satisfying (31) have cartesian components, each of which satisfies Sommerfeld’s radiation condition
\[
\frac{\partial F_j}{\partial r} - ikF_j = o \left( \frac{1}{r} \right), \quad j = 1, 2, 3 \tag{32}
\]
uniformly for all directions of \( r/r \) \[5\].
THEOREM 3. The set of radiating SVWF given in (30) is complete in $L_2(S)$.

The proof is very similar to that of Theorem 1. However, in this case, one makes use of the uniqueness of the exterior radiation problem [19], thus the completeness is valid for any value of $k^2$.

Orthonormalizing the set of radiating SVWF with respect to the surface $S$, we obtain the new set \( \{ V_n^2 \}, n = 1, 2, \ldots \).

THEOREM 4. Every $H$ satisfying the Helmholtz equation in $D_\varepsilon$ and taking the continuous value $H_s$ on $S$ can be expanded as

\[
H(r) = \sum_{n=1}^{\infty} \langle H_s, V_n^2 \rangle V_n^2(r)
\]

where the series converges to $H$ in closed subsets of $D_\varepsilon$ and in the mean square sense to $H_s$ on $S$.

The proof is similar to that of Theorem 2.

4. EXTENSIONS TO A DOUBLY CONNECTED REGION

In this section we will extend the results of Section 2 to a more complex geometry involving two surfaces using both the regular and radiating SVWF. Let $S_1$ and $S_2$ be two Lyapunov surfaces with $S_2$ enclosing $S_1$. Let the finite doubly connected region between $S_1$ and $S_2$ be denoted by $D_1$, the connected regions interior to $S_1$ and $S_2$ by $D_{i1}$ and $D_{i2}$, respectively, and exterior to $S_2$ by $D_{e2}$. We assume that the origin is in $D_{i1}$. Let $D_{o1}$ and $D_{o2}$ be two spherical regions centered at the origin such that $D_{o1}$ is a subset of $D_{i1}$ and $D_{o2}$ contains $D_{i2}$. The spectrum of eigenvalues of the homogeneous interior Dirichlet problem for the Helmholtz equation in $D_1$ will be denoted by $\sigma(D_1)$. Let $L_2(S_1 + S_2)$ be the space of square integrable vector functions on the surface formed by $S_1$ and $S_2$.

Consider the set of the regular and radiating SVWF:

\[
\{ \mathbf{M}_{nm}, \mathbf{N}_{nm}, \mathbf{M}_{nm}^2, \mathbf{N}_{nm}^2, \mathbf{V}_{nm}^2 \},
\]

\[
n = 0, 1, 2, \ldots, m = 0, \ldots, n, t = e, o.
\]

THEOREM 5. The set of SVWF given in (34) is complete in $L_2(S_1 + S_2)$ if and only if $k^2 \notin \sigma(D_1)$. But if $k^2 \in \sigma(D_1)$, then the completeness fails and the set has a deficiency equal to $3M$ (where $M$ is the multiplicity of the eigenvalue $k^2$ for the scalar problem).
Proof. It is sufficient to show that a nonzero vector function \( U \in L_2(S_1 + S_2) \) that satisfies the following equations simultaneously

\[
\langle M_{nm}, U \rangle = 0 \quad (35a)
\]

\[
\langle N_{nm}, U \rangle = 0 \quad (35b)
\]

\[
\langle L_{nm}, U \rangle = 0 \quad (35c)
\]

\[
\langle M^2_{nm}, U \rangle = 0 \quad (35d)
\]

\[
\langle N^2_{nm}, U \rangle = 0 \quad (35e)
\]

\[
\langle L^2_{nm}, U \rangle = 0 \quad (35f)
\]

exists if and only if \( k^2 \in \sigma(D_1) \). Let us multiply Eqs. (35a, b, c, d, e, f) by \( C_{nm} M^2_{nm}(kr_1) \), \( C_{nm} N^2_{nm}(kr_2) \), \( C_{nm} n(n+1) L^2_{nm}(kr_2) \), \( C_{nm} M^1_{nm}(kr_1) \), \( C_{nm} N^1_{nm}(kr_1) \), \( C_{nm} n(n+1) L^1_{nm}(kr_1) \), respectively, where \( r_1 \in D_{o1} \) and \( r_2 \in R^3 \backslash D_{o2} \) and \( C_{nm} \) is given in (6). After summing Eqs. (35a, b, d) for all \( n, m, \) and \( t \) values, we obtain

\[
T(r_1) = \int_{S_1 + S_2} U^*(r_3) \cdot \frac{e^{i k |r_1 - r_3|}}{2 \pi |r_1 - r_3|} dS = 0 \quad (36)
\]

and similarly from Eqs. (35d, e, f)

\[
T(r_2) = 0. \quad (37)
\]

Using analytic continuation arguments as was done for Eq. (7), we obtain

\[
T(r) = 0 \quad \text{for} \quad r \in D_{o1} \text{ and } r \in D_{o2}. \quad (38)
\]

Following the same procedure as in the proof of Theorem 1, we conclude that for \( k^2 \notin \sigma(D_1) \)

\[
U^* = 0 \quad (39)
\]

in \( L_2(S_1 + S_2) \), which completes the first part of the proof.

If \( k^2 \in \sigma(D_1) \) then the completeness of (34) fails. Let \( X_m(r) \), \( m = 1, 2, ..., 3M \), be the complete set of corresponding vector eigenfunctions. Again, by similar arguments as for Theorem 1, we conclude that the com-
pleteness of (34) can be preserved by the addition of the functions $K_m^r(r)$, $m = 1, 2, ..., 3M$, where

$$K_m^r(r) = \frac{\partial}{\partial n_s} X_m$$

on $S_1 + S_2$. (40)

Thus, the proof is completed.

Let us orthonormalize the set given in (34) with respect to the surface $S = S_1 + S_2$ and denote the new set as

$$\{ V_n(kr) \}, \quad n = 1, 2, 3, ... \quad (41)$$

with the property

$$\langle V_n, V_m \rangle = \int_{S_1 + S_2} V_n \cdot V_m dS = 0, \quad n \neq m \quad (42)$$

$$= 1, \quad n = m.$$

**THEOREM 6.** If $k \notin \sigma(D_1)$, then every $H$ satisfying the Helmholtz equation in $D_1$ and taking the continuous value $H_s$ on $S$ can be expanded as

$$H(r) = \sum_{n=1}^{\infty} \langle H_s, V_n \rangle V_n \quad (43)$$

where the series converges uniformly to $H$ in closed subsets of $D_1$ and in the mean square sense to $H_s$ on $S$.

The proof is similar to that of Theorem 2.

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**REFERENCES**

