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On mep-relations in the wreath product of groups

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ABSTRACT

Let a, b be two long cycles in an alternating group A_n, satisfying relations $a = [a,_k b]$ and $b = [b,_k a]$. We show that every pair of elements of the form $x = (X, a)$, $y = (Y, b)$, where the sum of coefficients of X and Y is equal zero, satisfies relations $x = [x,_l y]$, $y = [y,_l x]$ in the wreath product $(S_n \wr Z_m)'$ for m coprime with n and for an l divisible by k. We show also that for $n = 5, 7, 13$ and for m coprime with n, $(S_n \wr Z_m)'$ is generated by such pairs.

1. Introduction

We use traditional notations and terminology (see for example [3]). If g, h are two elements of a group, then the commutator of g and h is an element $[g, h] = [g,_1 h] = g^{-1}h^{-1}gh$ and for an integer $k > 1$ $[g,_k h] = [[g,_{k-1} h], h]$ and $[g,_0 h] = g$. By $G' = [G, G]$ we denote the commutator subgroup of G (that is $G' = \langle [g, h]\colon\ g, h \in G \rangle$). If m is a positive integer, then Z_m is the ring of integers modulo m. As usual, S_n and A_n denote respectively the symmetric and the alternating group acting on $\{1, \ldots, n\}$. If P is a matrix then P^T is transpose of P. We call $\sigma \in S_n$ a long cycle if it is a cycle of length n.

In Kourovka Notebook [5], Brandl posed the following question (Problem 11.18): Is it true that if G is generated by elements a, b satisfying relations $a = [a, _k b]$ and $b = [b, _l a]$ for some positive integers k, l, then G is finite? He stated also that if every minimal simple group has generators satisfying above relations, then there exists a series of two-variable words which characterize soluble groups (see [1]). In [2] Heineken calls such pairs of elements mep-pairs (a mutually Engel periodic pair).

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Definition 1. We say that a pair of elements a, b is a mep-pair if there exist positive integers k, l, such that $a = [a, {}_k b]$ and $b = [b, {}_l a]$. If k and l are minimal, then we say that (k, l) is the mep-period of the mep-pair (a, b). We say that a group G is a mep-group if it is generated by a mep-pair.

An example of a mep-group is A_5, which is generated by a mep-pair $a = (1, 2, 3, 4, 5)$, $b = (1, 3, 5, 4, 2)$ satisfying relations $a = [a, {}_5 b]$, $b = [b, {}_5 a]$. Other examples of mep-pairs can be found in Section 3.

Heineken [2] showed that if a group G is generated by a mep-pair a, b, then G is perfect (that is $G = G'$) and elements a, b^{-1}, ab^{-1} are conjugate in G (see also Proposition 1 below). He studied mep-pairs in $\mathrm{Sl}_2(p)$ and he proved that for all primes p of the form $5 + 8t$ and some of the form $1 + 8t$, there exist mep-pairs generating $\mathrm{Sl}_2(p)$. He also showed that mep-pairs exist in $\mathrm{Sl}_2(q)$ where q is a prime such that $q^3 - q$ is divisible by 7 and he found mep-pairs for $q = p^3$, for the remaining primes p.

It can be deduced from [6] that for $n \geqslant 5$, every group $G = (S_n \wr Z_m)'$ is a perfect group and is two-generated. So, searching mep-groups among such groups is natural. At first, we used computer calculations for searching mep-pairs in groups $G = (S_5 \wr Z_m)'$ and we discovered that pairs of the form $x = ((0, 0, 0, 0, 0); (1, 2, 3, 4, 5))$, $y = ((0, 0, 0, 1, -1); (1, 3, 2, 5, 4))$ are mep-pairs in $G = (S_5 \wr Z_m)'$ for m not divisible by 5. At the end of the paper we present tables, which show these results. After analyzing the mep-pairs that we had obtained, we discovered that if x, y is a mep-pair of period (k, k) in $(S_5 \wr Z_{p^l})'$, then it is a mep-pair of period (pk, pk) in $(S_5 \wr Z_{p^{l+1}})'$, where p^l is a power of prime number p. This discovery and further analysis finally led us to Theorem 1, which says that every pair of the form (X, a), (Y, b), where $a, b \in A_n$ is a mep-pair of long cycles and X, Y are vectors with zero sum of coefficients modulo m. More information about mep-groups and groups with similar properties can be found in [7].

2. Mep-pairs in $(S_n \wr Z_m)'$

Remark. If k is the least integer, such that for elements a, b of a group we have $a = [a, {}_k b]$, then for every positive integer n we have $[a, {}_n b] = [a, {}_r b]$ where $n \equiv r \bmod k$.

Proposition 1. *Let G be a mep-group generated by a mep-pair a, b. Then*

(a) *G is a perfect group, that is $G = G'$,*
(b) *elements a, b^{-1} and ab^{-1} are pairwise conjugate in G.*

Proof. Point (a) is clear, since a, b generate G and $a, b \in G'$. Since $a = [a, {}_k b] = [a, {}_{k-1} b]^{-1} b^{-1} [a, {}_{k-1} b] b$, we have $ab^{-1} = [a, {}_{k-1} b]^{-1} b^{-1} [a, {}_{k-1} b]$, so ab^{-1} and b^{-1} are conjugate. Similarly ba^{-1} and a^{-1} are conjugate and so ab^{-1} and a are conjugate. Hence by transitivity of conjugation, a and b^{-1} are also conjugate. □

The definition of a wreath product of groups can be found for example in [4]. We use the following natural representation of $S_n \wr Z_m$. Every element of $S_n \wr Z_m$ can be interpreted as a pair (X, σ), where $\sigma \in S_n$ and X is an $n \times 1$ matrix over Z_m (i.e. X is a column vector). The multiplication and inversion can be defined as follows:

$$(X, \sigma) \cdot (Y, \delta) = (X + A_\sigma Y, \sigma\delta), \qquad (X, \sigma)^{-1} = \left(-A_\sigma^{-1} X, \sigma^{-1}\right)$$

where $A_\sigma \in \mathrm{Gl}_n(Z_m)$ is a regular matrix representation of σ over Z_m. The neutral element of $S_n \wr Z_m$ is $(\bar{0}, id)$, where id is the neutral element of S_n, and $\bar{0}$ is a zero-vector.

If $x = (X, \sigma)$ and $y = (Y, \delta)$ then $[x, y] = (X, \sigma)^{-1}(Y, \delta)^{-1}(X, \sigma)(Y, \delta)$ and after calculations we get

$$[x, y] = \left(A_\sigma^{-1}\left(A_\delta^{-1} - I\right)X + A_\sigma^{-1} A_\delta^{-1}(A_\sigma - I)Y, [\sigma, \delta]\right). \tag{1}$$

Let a, b be long cycles in A_n, satisfying relations $[a,{}_kb]=a$ and $[b,{}_ka]=b$. Let A be the matrix representation of a and B be the matrix representation of b. If $x=(X,a)$, $y=(Y,b)$ then using formula (1) we have $[x,y]=(A^{-1}(B^{-1}-I)X+A^{-1}B^{-1}(A-I)Y,[a,b])$. We shall calculate iterated commutators $[x,{}_iy]$. Let $[x,{}_iy]=(V_iX+W_iY,[a,{}_ib])$. Then $[x,{}_{i+1}y]=(\overline{X},[a,{}_{i+1}b])$, where

$$\begin{aligned}\overline{X}&=[A,{}_iB]^{-1}\big(B^{-1}-I\big)(V_iX+W_iY)+[A,{}_iB]^{-1}B^{-1}\big([A,{}_iB]-I\big)Y\\&=[A,{}_iB]^{-1}\big(B^{-1}-I\big)V_iX+[A,{}_iB]^{-1}\big(\big(B^{-1}-I\big)W_i+B^{-1}\big([A,{}_iB]-I\big)\big)Y.\end{aligned}$$

So matrices V_i and W_i are defined recursively as follows:

$$\begin{aligned}&V_1=A^{-1}\big(B^{-1}-I\big),\qquad W_1=A^{-1}B^{-1}(A-I),\\&V_{i+1}=[A,{}_iB]^{-1}\big(B^{-1}-I\big)V_i,\\&W_{i+1}=[A,{}_iB]^{-1}\big(\big(B^{-1}-I\big)W_i+B^{-1}\big([A,{}_iB]-I\big)\big)\quad\text{for } i>0.\end{aligned}$$

By symmetry if $[y,{}_ix]=(L_iX+T_iY,[b,{}_ia])$ then

$$\begin{aligned}&T_1=B^{-1}\big(A^{-1}-I\big),\qquad L_1=B^{-1}A^{-1}(B-I),\\&T_{i+1}=[B,{}_iA]^{-1}\big(A^{-1}-I\big)T_i,\\&L_{i+1}=[B,{}_iA]^{-1}\big(\big(A^{-1}-I\big)W_i+A^{-1}\big([B,{}_iA]-I\big)\big)\quad\text{for } i>0.\end{aligned}$$

If C is a permutation matrix, then $(1,\dots,1)C=(1,\dots,1)$, so if $P\in\{V_i,W_i,T_i,L_i\}$, then $(1,\dots,1)P=0$, and hence the sum of coefficients in every column of a matrix P is equal zero. Since $[a,{}_kb]=a$, $[b,{}_ka]=b$ then we have $[A,{}_kB]=A$ and $[B,{}_kA]=B$.

Proposition 2. *Let k be a least integer such that $[a,{}_kb]=a$ and $[b,{}_ka]=b$. If $x=(X,a)$, $y=(Y,b)\in S_n\wr Z_m$ is a mep-pair satisfying relations $[x,{}_ly]=x$ and $[y,{}_lx]=y$ then $k|l$.*

Proof. If $x=[x,{}_ly]$ and $y=[y,{}_lx]$ then since $x=(X,a)$, $y=(Y,b)$ we have $a=[a,{}_lb]$ and $b=[b,{}_la]$. Using the above Remark we get $k|l$. □

Remark. If x, y is a mep-pair satisfying relations $[x,{}_ly]=x$ and $[y,{}_lx]=y$ then $l=kt$ for an integer t and

$$V_{kt}X+W_{kt}Y=X,\qquad L_{kt}X+T_{kt}Y=Y. \tag{2}$$

Let $V=V_k$, $T=T_k$, $W=W_k$, $L=L_k$. We know from our above calculations that

$$V=[A,{}_{k-1}B]^{-1}\big(B^{-1}-I\big)\dots[A,B]^{-1}\big(B^{-1}-I\big)A^{-1}\big(B^{-1}-I\big). \tag{3}$$

Lemma 1. *For every positive integer t,*

(i) $V_{kt}=V^t$, $T_{kt}=T^t$,
(ii) $W_{kt}=(V^{t-1}+V^{t-2}+\dots+V+I)W$, $L_{kt}=(T^{t-1}+T^{t-2}+\dots+T+I)L$.

Proof. We will only show that $V_{kt}=V^t$ and $W_{kt}=(V^{t-1}+\dots+V+I)W$, because proofs of two other equations are similar. We know that $[x,{}_ky]=(V_kX+W_kY,[a,{}_kb])=(V_kX+W_kY,a)$. Let us denote $X_1=V_kX+W_kY=VX+WY$. The kt commutator is equal to $[x,{}_{kt}y]=(V_{kt}X+W_{kt}Y,[a,{}_{kt}b])$, but on the other hand the commutator $[x,_{kt} y]$ is the $k(t-1)$ commutator of elements (X_1, a) and (Y, b) and since V_i and W_i does not depend on X, Y but only on a, b we get $[x,_{kt} y] = (V_{k(t-1)}X_1 + W_{k(t-1)}Y, [a,_{k(t-1)} b])$, and using the formula for X_1, $[x,_{kt} y] = (V_{k(t-1)}(VX + WY) + W_{k(t-1)}Y, a)$. Hence $V_{kt} = V_{k(t-1)}V$ and $W_{kt} = V_{k(t-1)}W + W_{k(t-1)}$. So using the induction on t we get $V_{kt} = V_{k(t-1)}V = V^{t-1}V = V^t$ and $W_{kt} = V_{k(t-1)}W + W_{k(t-1)} = V^{t-1}W + W_{k(t-1)} = (V^{t-1} + V^{t-2} + \cdots + V + I)W$. $\square$

Theorem 1. *Let a, b be long cycles that form a mep-pair in A_n and let m be an integer coprime with n. If X, Y are $n \times 1$ matrices over Z_m with coefficients summing to 0, then $x = (X, a)$, $y = (Y, b)$ is a mep-pair in the group $(S_n \wr Z_m)'$.*

Proof. We have to show that there exists t, such that X, Y satisfy Eqs. (2). We show that there exists t satisfying the first equation, because the appropriate t for the second equation can be found similarly. In a view of Lemma 1 the first equation has the form

$$V^t X + \left(V^{t-1} + \cdots + I\right)WY = X.$$

Let $R \subseteq Z_m^n$ consist of all vectors, whose sums of coefficients are equal to zero. Then R^T is the set of all column vectors X, such that $X^T \in R$. The numbers m and n are coprime, so $e = (1, \ldots, 1)$ does not belong to R and we get a direct sum:

$$Z_m^n = \mathbb{Z}e \oplus R.$$

Both summands are invariant under the action of A and B, so by (3) they are also invariant under the action of V.

If b is a long cycle, then b^{-1} is a long cycle and we shall deduce that the restriction $\overline{B^{-1} - I}$ of $B^{-1} - I$ on R is a bijection. Since R is finite, we only need to show that it is injective. If for $r = (r_1, \ldots, r_n) \in R$ we have $(B^{-1} - I)r^T = 0$, then $B^{-1}r^T = r^T$ and as b^{-1} is a long cycle, this can happen only if $r_1 = r_2 = \cdots = r_n = u$. As $r \in R$, this implies that $nu = 0$ and as n and m are coprime we get $u = 0$.

Hence, $\overline{V}$ the restriction of V on R also is a bijection (by (3) it is a product of bijective maps on R). It acts on a finite set, so has a finite order, s say. Hence, $\overline{V}^s = I$ and we get:

$$\begin{aligned}\overline{V}^{ms-1} + \cdots + \overline{V} + I &= \left(\overline{V}^{s(m-1)} + \cdots + \overline{V}^s + I\right)\left(\overline{V}^{s-1} + \cdots + \overline{V} + I\right)\\ &= m\left(\overline{V}^{s-1} + \cdots + \overline{V} + I\right) = 0.\end{aligned}$$

So for $t = ms$ and every $U \in R^T$ we have $(V^{t-1} + \cdots + V + I)U = 0$. Hence, for $X \in R^T$ we have $(V^t - I)X = (V - I)(V^{t-1} + \cdots + I)X = 0$, so $V^t X = X$. A sum of every column of a matrix W is equal to 0, so $(V^{t-1} + \cdots + I)W = 0$. Finally, for $X, Y \in R^T$ the required equation $V^t X + (V^{t-1} + \cdots + I)WY = X$ holds. $\square$

3. Mep-pairs generating $(S_n \wr Z_m)'$

Let R_m be the set of all $n \times 1$ matrices over Z_m with zero sum of coefficients, that is:

$$R_m = \left\{[x_1, \ldots, x_n]^T\colon\ x_i \in Z_m,\ \sum_{i=1}^{n} x_i = 0\right\} = \left\{X\colon\ [1, \ldots, 1]X = 0\right\}.$$

For example $Z = A_\sigma^{-1}(A_\delta^{-1} - I)X + A_\sigma^{-1}A_\delta^{-1}(A_\sigma - I)Y$ is an element of R_m, because $[1, \ldots, 1]Z = 0$.

We will identify the subgroup $\{(\bar{0}, \sigma)\colon\ \sigma \in A_n\}$ with A_n, and the normal subgroup $\{(Z, id)\colon\ Z \in R_m\}$ with R_m.

Let for $i = 1, 2, \ldots, n-1$, a_i denote an element (X_i, id), where X_i has 1 on i-th position, -1 on n-th position and 0 elsewhere. Elements $a_1, a_2, \ldots, a_{n-1}$ generate R_m, because for every $s = (S, id) \in R_m$ we have $s = a_1^{s_1} a_2^{s_2} \cdots a_{n-1}^{s_{n-1}}$, where $s_1, s_2, \ldots, s_n$ are coefficients of S.

Proposition 3. *An element (X, α) is in the commutator subgroup of $S_n \wr Z_m$ for $n \geqslant 2$ if and only if $\alpha \in A_n$ and $X \in R_m$. Moreover, we have $(S_n \wr Z_m)' \cong R_m \rtimes A_n$.*

Proof. Let $x = (X, \sigma)$, $y = (Y, \delta) \in S_n \wr Z_m$. Then by (1) and from $Z \in R_m$ we have $[x, y] = (Z, [\sigma, \delta]) = (Z, id) \cdot (\bar{0}, [\sigma, \delta]) \in R_m A_n$.

To establish the converse, it is enough to prove that A_n and R_m are subgroups of $(S_n \wr Z_m)'$. Clearly $A_n < (S_n \wr Z_m)'$. Let $g = (X, id)$ where $X = [0, \ldots, 0, 1]^T$ and $h_i = (\bar{0}, (i, n))$ for $i = 1, \ldots, n-1$. Then $a_i = [g, h_i] \in (S_n \wr Z_m)'$ and $R_m = \langle a_1, \ldots, a_{n-1} \rangle < (S_n \wr Z_m)'$. □

Proposition 4. *Let A be a subset of A_n ($n \geqslant 4$) such that $\langle A \rangle = A_n$ and $\overline{A} = \{(X, \sigma) \mid \sigma \in A\} \subset (S_n \wr Z_m)'$. Then for every $i \in \{1, \ldots, n-1\}$, we have $\langle \overline{A} \cup \{a_i\} \rangle = (S_n \wr Z_m)'$.*

Proof. Let i, j, k are different integers from the set $\{1, \ldots, n-1\}$. Since $\langle A \rangle = A_n$ there exists $u = (U, (i, j, k)) \in \langle \overline{A} \rangle$. Hence $u a_i u^{-1} = a_j$ and $a_1, \ldots, a_{n-1} \in \langle \overline{A} \cup \{a_i\} \rangle$. So $R_m \subseteq \langle \overline{A} \cup \{a_i\} \rangle$. From assumptions we know that for every $\sigma \in A$ there exists $X \in R_m$, such that (X, σ) belongs to $\overline{A}$. So $(\bar{0}, \sigma) = (X, \sigma) \cdot (X, id)^{-1}$ is in $\langle \overline{A} \cup \{a_i\} \rangle$. It means that A_n is also a subgroup of $\langle \overline{A} \cup \{a_i\} \rangle$. Hence, by Proposition 3, $\langle \overline{A} \cup \{a_i\} \rangle = (S_n \wr Z_m)'$. □

Theorem 2. *Let a, b be elements generating A_n, where $n > 3$ is an odd integer. If there exists positive integer k such that $a^k b$ is a cycle of length less than n then there exists X, such that $x = (\bar{0}, a)$, $y = (X, b)$ generate $(S_n \wr Z_m)'$.*

Proof. By Proposition 4, it is enough to prove that there exists i such, that $a_i \in \langle x, y \rangle$. Let $a^k b = (i_1, \ldots, i_l)$. Since n is odd, we have got $l \leqslant n-2$. Moreover l is greater then 2, because for $l = 1$, $a^k b$ would be equal id, which is impossible, and for $l = 2$, $a^k b$ would be odd, which also is impossible. We will denote by A (resp. B) a matrix representation of a (resp. b). If $x = (\bar{0}, a)$ and $y = (X, b)$ then $x^k y = (A^k X, a^k b)$. Since A^k is invertible, $A^k X$ can take any value Y in R_m. Since $a^k b$ is a cycle of length l we have $(x^k y)^l = ((I + C + \cdots + C^{l-1})Y, id)$, where $C = A^k B$. It is easy to see that if c_{ij} are entries of the matrix $I + C + \cdots + C^{l-1}$ then $c_{ii} = l$ if $i \notin \{i_1, \ldots, i_l\}$ and $c_{ij} = 1$ if $i, j \in \{i_1, \ldots, i_l\}$ and remaining coordinates are zero. So if $Y = [y_1, \ldots, y_n]^T$, $Z^T = (I + C + \cdots + C^{l-1})Y$ and $Z = [z_1, \ldots, z_n]$ then $z_i = l y_i$ for $i \notin \{i_1, \ldots, i_l\}$ and $z_j = y_{i_1} + \cdots + y_{i_l}$ for $j \in \{i_1, \ldots, i_l\}$. If we choose exactly one $i \notin \{i_1, \ldots, i_l\}$ and we put $y_i = -1$ and exactly one $y_j = 1$ for $j \in \{i_1, \ldots, i_l\}$ then Z has exactly one coefficient $-l$ and exactly $l \geqslant 3$ coefficients 1 and rest (at least one) of coefficients are zero. Hence there exist p, r, s, t such that $z_p = 1$, $z_r = 1$, $z_s = 1$ and $z_t = 0$. Since $\langle a, b \rangle = A_n$, there exists U such that $u = (U, (p, r)(s, t)) \in \langle x, y \rangle$. Hence $u(Z, id)u^{-1} \in \langle x, y \rangle$ and $u(Z, id)u^{-1} = (W, id)$, where $w_p = z_p = 1$, $w_r = z_r = 1$, $w_s = z_t = 0$, $w_t = z_s = 1$ and the rest coefficients of W are the same as coefficients of Z. Then $(Z, id)(W, id)^{-1} = (P, id)$, where P has 1 on s-th position, -1 on t-th position and zero elsewhere. If $t \neq n$, then conjugation by the element of the form $(U, (i, j)(t, n))$ move -1 to the last position. □

Corollary 1. *Let a, b be a mep-pair of long cycles, generating A_n, and let m be a positive integer coprime with n. If a and b satisfy the assumption of Theorem 2, then there exists X such that $(\bar{0}, a)$, (X, b) is a mep-pair generating $(S_n \wr Z_m)'$ (that is $(S_n \wr Z_m)'$ is a mep-group).*

Proof. It follows immediately from Theorems 1 and 2. □

Examples. Using Theorem 2 and Corollary 1 we give examples that show that $(S_n \wr Z_m)'$ are mep-groups for $n = 5, 7, 13$ and m coprime with n.

1. A_5 is a mep-group generated by a mep-pair

$$a = (1, 2, 3, 4, 5), \qquad b = (1, 3, 5, 4, 2),$$

satisfying $a = [a, {}_5b]$, $b = [b, {}_5a]$. Since $ab = (143)$, $(S_5 \wr Z_m)'$ are mep-groups.

2. A_7 is a mep-group generated by a mep-pair

$$a = (1, 2, 3, 4, 5, 6, 7), \qquad b = (1, 3, 6, 2, 4, 7, 5),$$

satisfying $a = [a, {}_{49}b]$, $b = [b, {}_{49}a]$. Since $a^4b = (172)$, $(S_7 \wr Z_m)'$ are mep-groups.

3. A_{13} is a mep-group generated by a mep-pair

$$a = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13), \qquad b = (1, 3, 8, 11, 7, 12, 5, 4, 10, 2, 13, 9, 6),$$

satisfying $a = [a, {}_{2708}b]$, $b = [b, {}_{2708}a]$.

Since $a^4b = (1, 7, 3, 12, 9, 10, 6, 5, 8, 2, 4)$, $(S_{13} \wr Z_m)'$ are mep-groups.

4. Tables

Let $a = (1, 2, 3, 4, 5)$, $b = (1, 3, 5, 4, 2)$, $m \in \mathbb{N} \setminus \{0, 1\}$. We define $x = (\bar{0}, a)$, $y = (X, b) \in (S_5 \wr Z_m)'$, such that x, y is a mep-pair of a mep-period (k, k), so $[x, {}_k y] = x$, $[y, {}_k x] = y$.

Let $G = \langle x, y\rangle \subseteq (S_5 \wr Z_m)'$. We think that the case when m is divisible by 3 is special. Here are some examples obtained by computer calculations.

m	X^T	k	$\|G\|$	$G = (S_5 \wr Z_m)'$
3	[0, 0, 0, 1, 2]	130	4860	yes
3	[1, 1, 2, 0, 2]	5	60	no
6	[0, 0, 0, 1, 5]	390	77 760	yes
6	[1, 1, 2, 0, 2]	15	960	no
9	[0, 0, 0, 1, 8]	390	393 660	yes
9	[1, 1, 2, 0, 5]	15	4860	no

Next table shows the results of our computer search for mep-pairs in groups $(S_5 \wr Z_m)'$. Elements a, b are as previously, $x = ([0, 0, 0, 0, 0]^T, a)$, $y = ([0, 0, 0, 1, -1]^T, b)$, $z = ([z_1, z_2, z_3, z_4, z_5]^T, b)$, $k_{\max}$ is a maximal mep-period, $k_{\min}$ is a minimal mep-period of a mep-pair x, z in $(A_5 \wr Z_m)'$ and ? means unknown value.

m	$k_{\max}$	z max	$k_{\min}$	z min
2	15	y	= max	
3	130	y	= max	
4	30	y	= max	
6	390	y	= max	
7	1710	y	15	$([1, 3, 2, 3, 5]^T, b)$
8	60	y	= max	
9	390	y	= max	
11	190	y	10	$([1, 10, 7, 1, 3]^T, b)$
12	390	y	= max	
13	840	y	420	$([0, 1, 3, 0, 9]^T, b)$
14	1710	y	15	$([1, 3, 2, 3, 5]^T, b)$
16	120	y	= max	
17	27 840	y	= max	
18	390	y	= max	
19	34 290	y	90	$([5, 1, 7, 8, 17]^T, b)$
21	22 230	y	?	
22	570	y	30	$([1, 10, 7, 1, 3]^T, b)$

m	$k_{\max}$	z max	$k_{\min}$	z min
23	279 840	y	?	
24	780	y	= max	
26	840	y	420	$([0, 1, 3, 0, 9]^T, b)$
27	1170	y	= max	
28	1710	y	30	$([1, 3, 2, 10, 12]^T, b)$
29	60 970	y	?	
31	230 880	y	?	
32	240	y	= max	
33	2470	y	?	
34	27 840	y	?	
36	390	y	?	
37	253 260	y	?	
38	34 290	y	90	$([5, 1, 7, 8, 17]^T, b)$
39	10 920	y	?	
41	34 460	y	?	
42	22 230	y	?	
43	4620	y	?	

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