Linear Transformations on Matrices:  
The Invariance of Generalized Permutation Matrices—III

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ABSTRACT

Let $F$ be a field, $F^*$ be its multiplicative group, and $\mathcal{K} = \{ H : H$ is a subgroup of $F^*$ and there do not exist $a, b \in F^*$ such that $Ha + b \subset H \}$. Let $D_n$ be the dihedral group of degree $n$, $H$ be a nontrivial group in $\mathcal{K}$, and $\Gamma_n(H) = \{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_i \in H \}$. For $\sigma \in D_n$ and $\alpha \in \Gamma_n(H)$, let $P(\sigma, \alpha)$ be the matrix whose $(i, j)$ entry is $\alpha_i \delta_{m(i)}$ (i.e., a generalized permutation matrix), and

$$P(D_n, H) = \{ P(\sigma, \alpha) : \sigma \in D_n, \alpha \in \Gamma_n(H) \}.$$  

Let $M_n(F)$ be the vector space of all $n \times n$ matrices over $F$ and

$$\mathcal{T} P(D_n, H) = \{ T : T$ is a linear transformation on $M_n(F)$

to itself and $T(P(D_n, H)) = P(D_n, H) \}.$$  

In this paper we classify all $T$ in $\mathcal{T} P(D_n, H)$ and determine the structure of the group $\mathcal{T} P(D_n, H)$ (Theorems 1 to 4). An expository version of the main results is given in Sec. 1, and an example is given at the end of the paper.

1. INTRODUCTION

Let $F$ be a field, $M_n(F)$ be the vector space of all $n$-square matrices with entries in $F$, and $\mathcal{U}$ be a subset of $M_n(F)$. It is of interest to determine the structure of linear map $T : M_n(F) \to M_n(F)$ such that $T(\mathcal{U}) \subseteq \mathcal{U}$. For example Dieudonné [1] showed that if $\mathcal{U} = \{ X \in M_n(F) : \det(X) = 0 \}$ then $T$ is of the

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form

\[ T(X) = UXV, \quad X \in M_n(F) \quad \text{or} \quad T(X) = U'XV, \quad X \in M_n(F), \]

(1.1)

where \( U, V \) are invertible matrices and \( \mathcal{T} \) is the transpose of \( X \); Marcus [3] showed that if \( \mathbb{U} \) is the unitary group, then \( T \) is also of the form (1.1), where \( U, V \) are unitary and \( F = \mathbb{C} \). Other results in this direction can be found in [4]. In [5], [6] we consider \( \mathbb{U} \) to be a set of generalized permutation matrices relative to some permutation group (set) and with entries in some nontrivial subgroup of \( F^* \), the multiplicative group of \( F \). More precisely, let \( S_n \) be the symmetric group acting on the set \{1, 2, ..., n\}, and if \( S \) is a subset of \( F \) define

\[ \Gamma_n(S) = \{ \alpha = (\alpha_1, \alpha_2, ..., \alpha_n) : \alpha_i \in S \}. \]

If \( \alpha \in \Gamma_n(F^*) \) and \( \sigma \in S_n \), then \( P(\sigma, \alpha) \) is the matrix whose \((i, i)\) entry is \( \alpha_i \delta_{\sigma(i)} \) (where \( \delta_{ij} = 1 \) if \( i = j \) and 0 elsewhere), and we call \( P(\sigma, \alpha) \) a generalized permutation matrix. If \( G \) is a nonempty subset of \( S_n \) and \( H \) is a subgroup of \( F^* \), we define

\[ P(G, H) = \{ P(\sigma, \alpha) : \alpha \in \Gamma_n(H) \text{ and } \sigma \in G \}, \]

\[ \mathcal{P}(G, H) = \{ T : T \text{ is a linear transformation on } M_n(F) \text{ to itself and } T(P(G, H)) = P(G, H) \}, \]

\[ \mathcal{S}(G, H) = \{ S : S \text{ is a linear transformation on } M_n(F) \text{ to itself and } S(P(G, H)) \subseteq P(G, H) \}. \]

Let

\[ \mathcal{K} = \{ H : H \text{ is a subgroup of } F^* \text{ and there do not exist } a, b \in F^* \text{ such that } Ha + b \subseteq H \}. \]

The set \( \mathcal{K} \) is not empty. For examples, \( F^* \) is in \( \mathcal{K} \) for every field \( F \), every nontrivial finite subgroup of \( F^* \) is in \( \mathcal{K} \), and every subgroup \( H \) of the unit circle \( \{ z : |z| = 1 \} \) of the complex plane and \( |H| > 2 \) is in \( \mathcal{K} \) [5]. But if \( F \) is a subfield of \( \mathbb{R} \), the real field, then \( F^* = \{ x \in F : x > 0 \} \) is not in \( \mathcal{K} \), since \( F^* + 1 \subseteq F^* \). Also, if \( F \) is a finite field of characteristic \( p > 2 \), then the trivial
group $H = \{1\}$ is not in $\mathcal{H}$, since $(p-1)H+2 = H$, but for $p=2$, the group $\{1\}$ is in $\mathcal{H}$. In [5] we classify the $T$ in $\mathcal{T}(G,H)$ and determine the structure of the group $\mathcal{T}(G,H)$ when $G$ is a regular subset or a doubly transitive subset (subgroup) of $S_n$ and $H$ a nontrivial group in $\mathcal{H}$. In [6] the $T$ in $\mathcal{T}(G,H)$ when $G$ is a regular subset of $S_n$ and $H$ is a nontrivial group in $\mathcal{H}$ has been characterized. In this paper we shall classify the $T$ in $\mathcal{T}(D_n,H)$ and determine the structure of $\mathcal{T}(D_n,H)$, where $D_n$ is the dihedral group of degree $n$ and $H$ is a nontrivial group in $\mathcal{H}$.

Recall that the dihedral group $D_n$ of degree $n$ is the subgroup of $S_n$ generated by the two elements $g,h$, where $g(i) = i + 1$, $i = 1,2,\ldots,n-1$; $g(n) = 1$, and where $h(1) = 1$; $h(i) = n - i + 2$, $i = 2,3,\ldots,n$. Let $g = g^{i-1}$, $i = 1,2,\ldots,n$. Then we may write $D_n = \{g,g,h : i = 1,2,\ldots,n\}$, and the diagonals are as follows shown in Fig. 1 ($n = 6$; solid lines denote the diagonals $g_i$, dotted lines the diagonals $g_i h$).

\[ g_1 \quad g_2 \quad g_3 \quad g_4 \quad g_5 \quad g_6 \]

\[ x \quad x \quad x \quad x \quad x \quad x \]

\[ x \quad x \quad x \quad x \quad x \quad x \]

\[ x \quad x \quad x \quad x \quad x \quad x \]

\[ x \quad x \quad x \quad x \quad x \quad x \]

\[ g_6 h \quad g_1 h \quad g_2 h \quad g_3 h \quad g_4 h \quad g_5 h \]

FIG. 1.

Roughly speaking, if $n = 3$ or $n > 5$ we prove that $T$ is in $\mathcal{T}(D_n,H)$ if and only if $T$ corresponds to some particular permutation among the diagonals $g_i g_i h$, $i = 1,2,\ldots,n$ and then each entry is multiplied by an element in $H$. More precisely, we denote by $S(\{x_1,x_2,\ldots,x_n\})$ the symmetric group acting on the set $\{x_1,x_2,\ldots,x_n\}$; by $(x_1 x_2 \cdots x_r)$ the cycle $\sigma$ such that $\sigma(x_1) = x_2, \sigma(x_2) = x_3,\ldots, \sigma(x_r) = x_1$; by $\circ$ the usual function composition; by $1$ the identity permutation; and we write $g_i h = g_i + i$, $i = 1,2,\ldots,n$. Then apart from multiplying each entry by an element in $H$, we have $T \in \mathcal{T}(D_n,H)$ if and only if $T$ transforms the ordered $2n$-tuple $(g_1, g_2, \ldots, g_{2n})$ to $(g_{\psi(1)}, g_{\psi(2)}, \ldots, g_{\psi(2n)})$ where for $n$ odd and $n > 3$, $\psi$ is in the subgroup $S_n \circ S(\{n+1,n+2,\ldots,2n\}) \circ \{1,(1n+1)(2n+2)\cdots (n2n)\}$ of $S_{2n}$, and for $n$ even and $n > 6$, $\psi$ is in the subgroup $\psi$ in the subgroup $\psi$. 
\[S\left(\{1,3,\ldots, n-1\}\right) \circ S\left(\{2,4,\ldots, n\}\right) \circ S\left(\{n+1,n+3,\ldots, 2n-1\}\right)\]

\[\circ S\left(\{n+2, n+4,\ldots, 2n\}\right)\]

\[\circ \left\{1,(12)(34)\cdots (n-1)n(n+1n+2)\cdots (2n-12n)\right\}\]

\[\circ \left\{1,(1n+1)(3n+3)\cdots (n-12n-1)\right\} \circ \left\{1,(2n+2)(4n+4)\cdots (n2n)\right\}\]

of \(S_{2n}\). For example, for \(n = 6\) if \(T \in \mathcal{T}\left(\mathcal{P}(D_6,H)\right)\), one possibility is as shown in Fig. 2 and Table 1 (solid lined diagonals are transformed to solid lined diagonals and dotted lined diagonals to dotted lined diagonals). Thus

\[
\begin{array}{cccccc}
X & X & X & X & X & X \\
X & X & X & X & X & X \\
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X & X & X & X & X & X \\
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X & X & X & X & X & X \\
\end{array} \quad \rightarrow \quad \begin{array}{cccccc}
X & X & X & X & X & X \\
X & X & X & X & X & X \\
X & X & X & X & X & X \\
X & X & X & X & X & X \\
X & X & X & X & X & X \\
X & X & X & X & X & X \\
\end{array}
\]

\[
\begin{array}{cccccc}
X & X & X & X & X & X \\
X & X & X & X & X & X \\
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X & X & X & X & X & X \\
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X & X & X & X & X & X \\
\end{array} \quad \rightarrow \quad \begin{array}{cccccc}
X & X & X & X & X & X \\
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X & X & X & X & X & X \\
X & X & X & X & X & X \\
X & X & X & X & X & X \\
X & X & X & X & X & X \\
\end{array}
\]

**Fig. 2.**

<table>
<thead>
<tr>
<th>(T) transforms</th>
<th>the set of diagonals</th>
<th>to</th>
</tr>
</thead>
<tbody>
<tr>
<td>({g_1,g_3,g_5})</td>
<td>({g_2,g_4,g_6})</td>
<td></td>
</tr>
<tr>
<td>({g_2,g_4,g_6})</td>
<td>({g_1h,g_3h,g_5h})</td>
<td></td>
</tr>
<tr>
<td>({g_1h,g_3h,g_5h})</td>
<td>({g_2h,g_4h,g_6h})</td>
<td></td>
</tr>
<tr>
<td>({g_2h,g_4h,g_6h})</td>
<td>({g_1,g_3,g_5})</td>
<td></td>
</tr>
</tbody>
</table>

An example of this type is given at the end of the paper.
The main results will be stated in Sec. 5.
2. MORE DEFINITIONS AND NOTATION

Let 0 denote the additive identity of $F$, and 1 denote the multiplicative identity of $F$. The matrix with 1 at the $(i, j)$ position and 0 elsewhere will be denoted by $E_{i,j}$. If $\alpha \in \Gamma_n(F)$ is the sequence all of whose entries are equal to 1, we write $P(\alpha)$ for $P(\sigma, \alpha)$ and call $P(\sigma)$ the permutation matrix corresponding to $\sigma$. The $n$-square matrix all of whose entries are 0, the matrix all of whose entries are 1, and the identity matrix will be denoted by $O_n$, $I_n$, $I_n$ respectively (or $O, I, I$ if no ambiguity arises). If $A = (a_{ij})$ is an $n$-square matrix, let $A$ denote the matrix whose $(i, j)$ entry is $a_{i,n-j+1}$ for all $i, j = 1, 2, \ldots, n$. If $n = 2m$ is even, we denote by $A_o$ the matrix whose $(i, j)$ entry is $a_{ij}$ if $i + j$ is even and $0$ otherwise, and let $A_o = A - A_o$. If $A = (a_{ij})$ and $B = (b_{ij})$ are $n$-square matrices, then their Hadamard product $A \ast B = (c_{ij})$ is the matrix defined by $c_{ij} = a_{ij}b_{ij}$. If $A$ is an $n$-square matrix and $B$ is an $m$-square matrix, then $A \oplus B$ will denote their direct sum. If $X = (x_{ij}) \in M_n(F)$ and $\sigma \in S_n$, then $X_\sigma$ is the matrix whose $(i, j)$ entry is $x_{ij}$ if $\sigma(i) = j$ and 0 otherwise.

If $H$ is a subgroup of $F^*$, let $M_n(H)$ be the set of all $n$-square matrices with entries in $H$. It is easy to see that the set $M_n(H)$ with the operation Hadamard product form a group, which will be denoted by $M_n(H)$. Under the correspondence

$$A \mapsto (a_{11}, a_{12}, \ldots, a_{1n}, \ldots, a_{n1}, a_{n2}, \ldots, a_{nn}),$$

where $A = (a_{ij}) \in M_n(H)$, it is obvious that $M_n(H)$ is isomorphic to the direct product $H \times H \times \cdots \times H$ ($n^2$ times).

A chain of subgroups of a group $G'$, $G_k < G_{k-1} \cdots < G_1 < G_0 = G'$, is a composition series if each $G_i$ is a maximal normal subgroup of $G_{i-1}$, $i = 1, 2, \ldots, k$. If $G_1$ is a normal subgroup of $G_2$, we write $G_1 \triangleleft G_2$. If $S$ is a finite set, $|S|$ will be the order of $S$. If $S$ is a set of $\eta$ a mapping of $S$ into $S$, we denote the image of $s \in S$ under $\eta$ by $s^\eta$ or $\eta(s)$. Suppose $G', K$ are two groups and for every element $g' \in G'$ we are given an automorphism of $K$, $k \mapsto k^{g'}$ for all $k \in K$, such that

$$(k^{g'})^{g''} = k^{g'g''}, \quad g', g'' \in G'.$$

Then the symbols $\langle g', k \rangle$, $g' \in G'$, $k \in K$, form a group under the rule $\langle g', k_1 \rangle \cdot \langle g'', k_2 \rangle = \langle g'g'', k_1 k_2^{g'} \rangle$, which is called the semi-direct product of $K$ by $G'$ and will be denoted by $\langle G', K \rangle$. 
The linear transformations \( U, R \) on \( M_n(F) \) to itself are defined as follows:

\[
U(X) = XP(g^{-1}), \quad R(X) = X, \quad X \in M_n(F).
\]

For \( \sigma \in S_n \) let \( D(\sigma) = \{ (i, \sigma(i)) : i = 1, 2, \ldots, n \} \). For \( T \in \mathbb{P}(D_n, H) \) and \( \sigma \in D_n \) let \( T(\sigma) = \{ T(E_{\sigma(i)}): i = 1, 2, \ldots, n \} \).

If \( n (=2m) \) is even, let \( G \) be the subgroup of \( S_n \) generated by the transpositions \((i i + m), i = 1, 2, \ldots, m; K_n \) be the subgroup \( \{ \sigma \in S_n : \sigma \text{ maps even integers into even integers} \} \) of \( S_n \); and \( V \) and \( \Lambda(\lambda_1, \lambda_2, \ldots, \lambda_n) \), \( \lambda_1, \lambda_2, \ldots, \lambda_n \in G \), be the linear transformations on \( M_n(F) \) to itself defined by

\[
V(X) = X_0 + R(X, P(g^{-1})),
\]

\[
\Lambda(\lambda_1, \lambda_2, \ldots, \lambda_n)(X) = \sum_{i=1}^{n} P(\lambda_i)X_{i}P(g_{\lambda_i}g_i^{-1})
\]

and

\[
\Lambda = \{ \Lambda(\lambda_1, \lambda_2, \ldots, \lambda_n) : (\lambda_1, \lambda_2, \ldots, \lambda_n) \in G \times \cdots \times G \}.
\]

In Sec. 3, for any integer \( p \) we denote by \( \langle p \rangle \) the remainder of \( p \) in \( \{1, 2, \ldots, n\} \) after dividing by \( n \), i.e., we work modulo \( n \) using \( \{1, 2, \ldots, n\} \) as a system of distinct representatives.

3. THE MAPPINGS \( \varphi_i, \theta_i \)

It is well known that the subgroup \( D'_n = \{ g_i : i = 1, 2, \ldots, n \} \) of \( D_n \) is regular and \( D_n = D'_n \cup D'_nh \). Hence for each pair \( (i, j), 1 < i, j < n \), there exist exactly one \( k \) and one \( l, 1 < k, l < n \), such that \( g_k(i) = j \) and \( g_lh(i) = j \) or \( g_k(i) = g_lh(i) \). We then define

\[
\varphi_k(i) = l \quad \text{and} \quad \theta_l(i) = k.
\]

Since \( g \) is the full cycle \((12 \cdots n), g^k(i) = \langle k + i \rangle \) for \( i, k = 1, 2, \ldots, n \). Hence

\[
g_k(i) = g^{k-1}(i) = \langle k + i - 1 \rangle, \quad i, k = 1, 2, \ldots, n
\]

and

\[
g_lh(i) = g^{l-1}h(i) = \langle l - (i - 1) \rangle, \quad i, l = 1, 2, \ldots, n.
\]
Now $g_k(i) = g_lh(i)$ implies that $l = k + 2(i - 1)$ and $k = l - 2(i - 1)$. Hence
\[ \varphi_k(i) = \langle k + 2(i - 1), i, k = 1, 2, \ldots, n \]
and
\[ \theta_l(i) = \langle l - 2(i - 1), i, l = 1, 2, \ldots, n. \]
Consequently, if $i \neq j$, then $\varphi_k(i) = \varphi_k(j)$ or $\theta_k(i) = \theta_k(j)$ if and only if $\langle 2(i - j) \rangle = n$—i.e., if $n$ is odd, then $\varphi_k, \theta_k, k = 1, 2, \ldots, n$, are in $S_n$, and if $n$ is even, then $\varphi_k(i) = \varphi_k(i + n/2), \theta_k(i) = \theta_k(i + n/2)$ for $k = 1, 2, \ldots, n, i = 1, 2, \ldots, n/2$. Also $\varphi_k(i), \theta_k(i)$ are even if and only if $k$ is even for $i = 1, 2, \ldots, n$, where $n$ is even.

4. THE GROUPS $\langle S_n \times S_n, M_n(H) \rangle$ AND $\langle K_n \times K_n, M_n(H) \rangle$

Let $H$ be a subgroup of $F^*$. We first assume that $n$ is an odd positive integer. Then $\varphi_k, \theta_k \in S_n$ for $k = 1, 2, \ldots, n$. For $(\tau, \nu) \in S_n \times S_n$ and $A \in M_n(H)$ we define
\[ A^{(\tau, \nu)} = \sum_{i=1}^n P\left( \varphi_{\tau(i)}^{-1} \nu \varphi_i \right) A_{g_i} P\left( g_i \varphi_i^{-1} \nu^{-1} \varphi_{\tau(i)} g_i^{-1} \right), \quad (4.1) \]
i.e., for $i = 1, 2, \ldots, n$, $(\tau, \nu)$ permutes the entries within the $g_i$-diagonal of $A$ by $\varphi_{\tau(i)}^{-1} \nu \varphi_i$ and then transforms the entries in the $g_i$-diagonal to the $g_{\tau(i)}$-diagonal. For $A, B \in M_n(H)$, since $A_{g_i}$ and $B_{g_i}$ are $g_i$-diagonal matrices, it can be shown that $(A \ast B)^{\tau, \nu} = A^{(\tau, \nu)} \ast B^{(\tau, \nu)}$, and clearly $A^{(\tau, \nu)} = J$ if and only if $A = J$. Therefore $(\tau, \nu)$ is an automorphism of $M_n(H)$. Also, if $(\tau, \nu), (\sigma, \mu) \in S_n \times S_n$ and $A \in M_n(H)$, it can be shown that $(A^{(\tau, \nu)})^{(\sigma, \mu)} = A^{(\sigma, \mu)(\tau, \nu)}$. For $(\sigma, \mu), (\tau, \nu) \in S_n \times S_n$ and $A, B \in M_n(H)$ define
\[ \left\langle (\sigma, \mu), A \right\rangle \cdot \left\langle (\tau, \nu), B \right\rangle = \left\langle (\sigma, \mu)(\tau, \nu), A \ast B^{(\sigma, \mu)} \right\rangle \]
and denote by $\langle S_n \times S_n, M_n(H) \rangle$ the corresponding semi-direct product of $M_n(H)$ with $S_n \times S_n$.

Next assume that $n = 2m$ is a positive even integer, and define $1 < q_j^{-1} \nu \varphi_{\tau(j)}(i) \leq m$ if and only if $1 < j < m$ for all $1 < j < n$ and $\tau, \nu \in K_n$. Now by the definition of $q_{i}$, the numbers $q_{i}(i), i = 1, 2, \ldots, n$, are even if and only if $j$ is even; thus $q_{j}^{-1} \nu \varphi_{\tau(j)}$, $j = 1, 2, \ldots, n$, are well defined and are in $S_n$. Now for $(\tau, \nu) \in K_n \times K_n$ and $A \in M_n(H)$ define
\[ A^{(\tau, \nu)} = \sum_{i=1}^n P\left( \varphi_{\tau(i)}^{-1} \nu \varphi_i \right) A_{g_i} P\left( g_i \varphi_i^{-1} \nu^{-1} \varphi_{\tau(i)} g_i^{-1} \right), \quad (4.2) \]
Then, as in the case where \( n \) is odd, it can be shown that \((\tau, \nu)\) is an automorphism of \( M_n(H) \). Also, since \((q_{\sigma(i)}^{-1} \mu q_{\tau(i)} q_{\nu(i)}^{-1} q_{\tau(i)}^{-1} \mu q_{\nu(i)})=q_{\sigma(i)}^{-1} \mu q_{\nu(i)}, i=1,2,\ldots,n,\) for \((\sigma, \mu), (\tau, \nu) \in K_n \times K_n\) it follows that \((A^{(\tau, \nu)}(\sigma, \mu))=A^{(\sigma, \mu)}(\tau, \nu),\) \( A \in M_n(H)\). Hence for \( A, B \in M_n(H) \) and \((\sigma, \mu), (\tau, \nu) \in K_n \times K_n\) we define

\[
\langle (\sigma, \mu), A \rangle \cdot \langle (\tau, \nu), B \rangle = \langle (\sigma, \mu)(\tau, \nu), A \ast B^{(\sigma, \mu)} \rangle
\]

and denote by \( \langle K_n \times K_n, M_n(H) \rangle \) the semi-direct product of \( M_n(H) \) with \( K_n \times K_n\).

In the following we shall define

\[
1 \leq q_{\sigma(i)}^{-1} \mu q_{\nu(i)}(f), q_{\sigma(i)}^{-1} \mu q_{\nu(i)}(f) \leq m \quad \text{if and only if} \quad 1 \leq f \leq m
\]

for \( \sigma, \mu \) both in \( K_n \) or \( K_n g \) unless otherwise stated.

5. MAIN RESULTS

Let \( D_n \) be the dihedral group of degree \( n \), and \( H \) be a nontrivial group in \( \mathcal{K} \). If \( T \in TP(D_n, H) \) and \( n=3 \) or \( n \geq 5 \), then for \( 1 \leq i, j \leq n \) there exist \( 1 \leq p, q \leq n \) and \( \alpha_i \in H \) such that

\[
T(E_{ij}) = \alpha_i E_{pq}
\]

and for distinct \( (i, j) \) we have distinct \( (p, q) \), i.e., the matrix representation of \( T \) is a generalized permutation matrix with respect to the usual basis \( \{E_{ij}; i, j=1,2,\ldots,n\} \) (Lemmas 3 and 6). Furthermore we prove the following results.

**Theorem 1.** If \( n \) is odd, \( n \geq 3 \), and \( H \) is a nontrivial group in \( \mathcal{K} \), then \( T \in TP(D_n, H) \) if and only if there exist \( \sigma, \mu \in S_n \) and \( \alpha_k \in H, k, l=1,2,\ldots,n, \) such that

\[
T(E_{kg}(k)) = \alpha_{kg}(k) E_{g^{-1} g_i} \mu g_i(k), g_i(0) g_i(1) \mu g_i(k), \quad i, k=1,2,\ldots,n \quad (5.1)
\]

or

\[
T(E_{kg}(k)) = \alpha_{kg}(k) E_{g_i(0) g_i(1) \mu g_i(k), g_i(0) g_i(1) \mu g_i(k),} \quad i, k=1,2,\ldots,n. \quad (5.2)
\]
THEOREM 2. Suppose $n$ is even, $n \geq 6$, and $H$ is a nontrivial group in $\mathcal{K}$. Then $T \in \mathcal{F}(D_n, H)$ if and only if either

(i) there exist $\sigma, \mu \in K_n$ or $K_n g_i$, $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in G \times G \times \cdots \times G$ and $\alpha_{kl} \in H$, $k, l = 1, 2, \ldots, n$, such that

$$T(E_{\lambda_g, (k)}) = \alpha_{kg, (k)} E_{\lambda_{\omega(k)} \theta_{g(k)} \sigma(k), \mu_{g(k)} \eta_{g(k)}}(k), \quad i, k = 1, 2, \ldots, n; \quad (5.3)$$

(ii) there exist $\tau, \nu \in K_n$ or $K_n g$, $(\kappa_1, \kappa_2, \ldots, \kappa_n) \in G \times G \times \cdots \times G$ and $\alpha_{kl} \in H$, $k, l = 1, 2, \ldots, n$, such that

$$T(E_{\kappa_g, (k)}) = \alpha_{kg, (k)} E_{\kappa_{\omega(k)} \theta_{g(k)} \sigma(k), \mu_{g(k)} \eta_{g(k)}}(k), \quad i, k = 1, 2, \ldots, n, \quad (5.4)$$

or

(iii) there exist $\omega, \omega', \pi, \pi' \in K_n$ or $K_n g$, $\alpha_{kl} \in H$, $k, l = 1, 2, \ldots, n$; for $i \in \Omega_1$ there exists $\lambda^{(i)} \in G$; and for $i \in \Omega_2$ there exists $\kappa^{(i)} \in G$, such that for $i \in \Omega_1$,

$$T(E_{\lambda_g, (k)}) = \alpha_{kg, (k)} E_{\lambda_{\omega(k)} \theta_{g(k)} \sigma(k), \mu_{g(k)} \eta_{g(k)}}(k), \quad k = 1, 2, \ldots, n; \quad (5.5a)$$

and for $i \in \Omega_2$,

$$T(E_{\kappa_g, (k)}) = \alpha_{kg, (k)} E_{\kappa_{\omega(k)} \theta_{g(k)} \sigma(k), \mu_{g(k)} \eta_{g(k)}}(k), \quad k = 1, 2, \ldots, n \quad (5.5b)$$

where $\Omega_1 = \{2i - 1: i = 1, 2, \ldots, m\}$ or $\{2i: i = 1, 2, \ldots, m\}$, and $\Omega_2 = \{1, 2, \ldots, n\} - \Omega_1$.

Generally speaking, Theorems 1 and 2 state that for $n = 3$ or $n > 5$, if $T$ is in $\mathcal{F}(D_n, H)$, then $T$ is a composition of three linear transformations, i.e., $T = T_3 \circ T_2 \circ T_1$, where $T_1$ permutes the entries within each diagonal, $T_2$ permutes the diagonals, and $T_3$ multiplies each entry by an element in $H$. For example, in (5.4), $T_1$ permutes the entries at the positions $(k, g_i(k))$, $k = 1, 2, \ldots, n$, by $\eta = \kappa^{(i)} \theta^{(i)} \sigma^{(i)} \omega^{(i)}$, i.e., $T_1$ transforms the entry at the position $(k, g_i(k))$ to the position $(\eta(k), g_i(k))$; $T_2$ transforms the entries in the diagonal $g_i$ to the diagonal $g_{\tau^{(i)}(k)} h_i$, i.e., $T_2$ transforms the entry at the position $(\eta(k), g_i(k))$ to $(\eta(k), g_{\tau^{(i)}(k)} h_i(k))$; and $T_3$ multiplies the entry at $(\eta(k), g_{\tau^{(i)}(k)} h_i(k))$ by $\alpha_{kg, (k)}$ in $H$. Recall the notation in Sec. 1. For $n = 3$ or $n > 5$ and $T \in \mathcal{F}(D_n, H)$, $T_2 \circ T_1$ is, in fact, equivalent to a permutation among the diagonals $g_i$, $g_i h_i$, $i = 1, 2, \ldots, n$, i.e., $T$ transforms the ordered $2n$-tuple $(g_{\xi_1}, g_{\xi_2}, \ldots, g_{\xi_{2n}})$ to $(g_{\psi_1}, g_{\psi_2}, \ldots, g_{\psi_{2n}})$. More specifically, for $n$ odd and $n \geq 3$, $T$ is (5.1) if and only if $\psi$ is in the group $S_n \circ S((n + 1, n + 2, \ldots, 2n))$, and $T$ is (5.2) if and only if $\psi$ is in $S_n \circ S((n + 1, n + 2, \ldots, 2n))$.
For $n$ even and $n > 6$, if we write
\[ G_1 = S(\{1,3,\ldots,n-1\}) \circ S(\{2,4,\ldots,n\}) \]
\[ \circ S(\{n+1,n+3,\ldots,2n-1\}) \circ S(\{n+2,n+4,\ldots,2n\}), \]
\[ \alpha = (12)(34) \cdots (n-1n)(n+1n+2) \cdots (2n-12), \]
\[ \beta = (1n+1)(3n+3) \cdots (n-12n-1), \]
\[ \gamma = (2n+2)(4n+4) \cdots (n2n), \]
then we have the correspondences in Table 2.

<table>
<thead>
<tr>
<th>$T$ is</th>
<th>if and only if $\psi$ is in</th>
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<tbody>
<tr>
<td>(5.3) with $\sigma, \mu \in K_n$</td>
<td>$G_1$</td>
</tr>
<tr>
<td>(5.3) with $\sigma, \mu \in K_n g$</td>
<td>$G_1 \circ \alpha$</td>
</tr>
<tr>
<td>(5.4) with $\tau, \nu \in K_n$</td>
<td>$G_1 \circ \alpha \circ \beta \circ \gamma$</td>
</tr>
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<td>$G_1 \circ \alpha \circ \beta \circ \gamma$</td>
</tr>
<tr>
<td>(5.5) with $\omega, \omega', \pi, \pi' \in K_n$ and $\Omega_1 = {1,3,\ldots,n-1}$</td>
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</tr>
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<td>$G_1 \circ \alpha \circ \beta$</td>
</tr>
</tbody>
</table>

Regarding the structure of the group $\mathcal{G}P(D_n, H)$, we have

**Theorem 3.** Let $n$ be odd, $n > 3$, and $H$ be a nontrivial group in $\mathcal{K}$. If $\langle (\alpha, \mu), A \rangle \in \langle S_n, XS_n, M_n(H) \rangle$ we define

\[ X^{\langle (\alpha, \mu), A \rangle} = \Lambda \sum_{\substack{i=1}}^{n} P(q_{\alpha(i)}^{-1}q_{\mu(i)}^{-1}q_{\alpha(i)} q_{\mu(i)}^{-1} q_{\alpha(i)} q_{\mu(i)}^{-1}), \quad X \in M_n(F), \]
then \( \mathcal{T}P(D_n, H) \) is equal to the group

\[
\langle S_n \times S_n, M_n(H) \rangle \circ \{ I, R \}.
\]

As an abstract group, there exists a subgroup \( \mathcal{H}_1 P(D_n, H) \) of index 2 in \( \mathcal{H}P(D_n, H) \), and \( \mathcal{H}_1 P(D_n, H) \) is isomorphic to the group

\[
\left\langle S_n \times S_n, H \times H \times \cdots \times H \right\rangle. \quad n^2 \text{ times}
\]

If \( |H| \) is finite, the order of \( \mathcal{H}P(D_n, H) \) is \( 2(n!)^2 |H|^{n^2} \).

**Theorem 4.** Suppose \( n \) is even, \( n \geq 6 \), and \( H \) is a nontrivial group in \( \mathcal{H} \). If for \( \langle (\sigma, \mu), A \rangle \in \langle K_n \times K_n, M_n(H) \rangle \) we define

\[
X^{\langle (\sigma, \mu), A \rangle} = A \ast \sum_{i=1}^{n} P(\varphi_{A(i)}^{-1} \mu \varphi_i) X g_i P(g_i \varphi_i^{-1} \mu^{-1} \varphi_{A(i)}^{-1} g_{A(i)}^{-1}), \quad X \in M_n(F),
\]

then \( \mathcal{H}P(D_n, H) \) is equal to the group

\[
\Lambda \circ \left\langle K_n \times K_n, M_n(H) \right\rangle \circ \{ I, U \} \circ \{ I, R \} \circ \{ I, V \}.
\]

As an abstract group, \( \mathcal{H}P(D_n, H) \) has subgroups \( \mathcal{H}_i P(D_n, H) \), \( i = 0, 1, 2, 3 \), such that

\[
\mathcal{H}_1 P(D_n, H) \triangle \mathcal{H}_2 P(D_n, H) \triangle \mathcal{H}_3 P(D_n, H) \triangle \mathcal{H}P(D_n, H)
\]

is a composition series, \( \mathcal{H}_1 P(D_n, H) \triangle \mathcal{H}P(D_n, H) \), \( \mathcal{H}_0 P(D_n, H) \) is a subgroup of index \( 2^{n^2/2} \) in \( \mathcal{H}_1 P(D_n, H) \), and

\[
\mathcal{H}_0 P(D_n, H) \cong \left\langle K_n \times K_n, H \times H \times \cdots \times H \right\rangle, \quad n^2 \text{ times}
\]

\[
\frac{\mathcal{H}P(D_n, H)}{\mathcal{H}_1 P(D_n, H)} \cong D_4,
\]

where \( \cong \) is the group isomorphism. If \( |H| \) is finite, the order of \( \mathcal{H}P(D_n, H) \) is \( 2^{mn} + 3(m!)^4 |H|^{n^2} \), where \( m = n/2 \).

To complete the list we have
THEOREM 5. If \(|H| > 2\) and \(H \in \mathcal{H}\), then Theorems 2 and 4 are true when \(n = 4\). If \(H = \{1, -1\}\), then \(\mathcal{P}(D_4, H)\) consists of the group of linear transformations generated by those stated in Theorem 2 (when \(n = 4\)) together with the linear transformation \(S\) defined as follows:

\[
S(E_{11}) = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad S(E_{13}) = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
S(E_{31}) = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad S(E_{33}) = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[S(E_{ij}) = E_{ij} \text{ if } (i, j) \notin \{(1, 1), (1, 3), (3, 1), (3, 3)\}.
\]

6. ASSUMED RESULTS

We will make use of the following results whose proof can be found in [5]. Suppose \(K\) is a nonempty subset of \(S_n\), and \(H\) is a nontrivial group in \(\mathcal{H}\). A subset \(E = \{A_1, A_2, \ldots, A_n\}\) of \(M_n(F)\) is called a \(K-H\) unitary set if \(E\) is a linearly independent set and for \(\alpha = (a_1, a_2, \ldots, a_n) \in \Gamma_n(H)\), \(\sum_{i=1}^{n} a_i A_i \in P(K, H)\).

**Proposition 1.** Suppose \(|H| > 2\), and \(\{A_1, A_2, \ldots, A_n\} \subseteq M_n(F)\) is a \(K-H\) unitary set. Then there exist \(a_1, a_2, \ldots, a_n \in H\), \(\tau \in S_n\), \(\sigma \in K\) such that

\[A_i = a_i E_{\sigma(i) \tau(i)}, \quad i = 1, 2, \ldots, n.\]

**Proposition 2.** If \(|H| = 2\) and \(\{A_1, A_2, \ldots, A_n\} \subseteq M_n(F)\) is a \(K-H\) unitary set, then there exist permutation matrices \(P\) and \(Q\), an integer \(r (0 < r < n)\) and \(\epsilon_\tau, \epsilon_j \in H\) such that \(n - r\) is even and if \(P\{A_1, A_2, \ldots, A_n\}Q = \{E_1, E_2, \ldots, E_n\}\), then

\[E_1 = [\epsilon_1] \oplus O_{n-1},\]

\[E_2 = O_1 \oplus [\epsilon_2] \oplus O_{n-2},\]

\[\vdots\]

\[E_r = O_{r-1} \oplus [\epsilon_r] \oplus O_{n-r},\]
\[ E_{r+1} = O_r \oplus \frac{1}{2} \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{13} & \xi_{14} \end{bmatrix} \oplus O_{n-r+2}, \]

\[ E_{r+2} = O_r \oplus \frac{1}{2} \begin{bmatrix} \pm \xi_{11} & \pm \xi_{12} \\ \mp \xi_{13} & \pm \xi_{14} \end{bmatrix} \oplus O_{n-r-2}, \]

\[ \vdots \]

\[ E_{n-1} = O_{n-2} \oplus \frac{1}{2} \begin{bmatrix} \xi_{t1} & \xi_{t2} \\ \xi_{t3} & \xi_{t4} \end{bmatrix}, \quad t = \frac{1}{2}(n-r), \]

\[ E_n = O_{n-2} \oplus \frac{1}{2} \begin{bmatrix} \pm \xi_{t1} & \pm \xi_{t2} \\ \mp \xi_{t3} & \pm \xi_{t4} \end{bmatrix}. \]

**Proposition 3.** If \( K \) is a transitive subset of \( S_n \) and \( H \) is a nontrivial subgroup of \( F^* \), then \( \mathcal{P}(K,H) \) is a subgroup of the group of all nonsingular \( n^2 \times n^2 \) matrices over \( F \).

**Lemma 1.** Suppose \( K \) is a transitive subset of \( S_n \) and \( H \) is a nontrivial subgroup of \( F^* \). If \( T \in \mathcal{P}(K,H) \) and \( a \in K \), then \( T(a^{-1}) \) is a \( K-H \) unitary set.

7. **The Structure of the Group** \( \mathcal{P}(D_n,H) \), \( n \) Odd and \( n \geq 3 \)

In this section we assume that \( H \) is a nontrivial group in \( \mathfrak{X} \) and \( n \) is an odd positive integer, \( n \geq 3 \).

**Lemma 2.** For each pair \( g, g_k h \) in \( D_n \), \( 1 \leq j,k \leq n \), we have \( |D(g_j) \cap D(g_k h)| = 1 \). In fact

\[ D(g_j) \cap D(g_k h) = \{ (q_j^{-1}(k), g_j q_j^{-1}(k)) \} = \{ (\theta_k^{-1}(i), g_k h \theta_k^{-1}(j)) \}. \]

**Proof.** Since \( \varphi_j \in S_n \), there exists exactly one \( i, 1 \leq i \leq n \), such that \( \varphi_j(i) = k \). By the definition of \( \varphi_j \), \( g_j(i) = g_k h(i) \) and \( \theta_k(i) = j \). \( \square \)
Lemma 3. If \( n \) odd, \( n > 3 \), and \( T \in \mathcal{S}(D_n, H) \), then for \( 1 \leq i, j \leq n \) there exist integers \( 1 \leq p, q \leq n \) and \( \alpha_{ij} \in H \) such that

\[
T(E_{ij}) = \alpha_{ij}E_{pq}.
\]

Proof. If \( |H| > 2 \), then the result follows from Proposition 1 and Lemma 1, since there exists \( g_k \) such that \( g_k(i) = i \), and we consider the \( D_n \)-\( H \) unitary set \( T(g_k) \). We suppose that \( |H| = 2 \); then Proposition 2 and Lemma 1 apply. If \( r = n \) [i.e., no matrices of the second type appear in \( T(g_k) \)], the result follows. Hence we assume for some \( i' \neq i \) (only writing the appropriate 2-square matrices)

\[
T(E_{ig_k(i)}) = \frac{1}{2} \begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \pm \epsilon_3 & \mp \epsilon_4 \end{bmatrix},
\]

Now since \( g_k(i) = g_k(i)h(i) \), we have

\[
T(E_{ig_k(i)h(i)}) = \frac{1}{2} \begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \pm \epsilon_3 & \mp \epsilon_4 \end{bmatrix}.
\]

Repeating the argument for \( T(g_k(i)h) \), by Proposition 2 there must exist \( l \neq i \) such that

\[
T(E_{lg_k(i)h(l)}) = \frac{1}{2} \begin{bmatrix} \pm \epsilon_1 & \mp \epsilon_2 \\ \mp \epsilon_3 & \pm \epsilon_4 \end{bmatrix} = T(F_{ig_k(i)}) + T(F_{ig_k(i)}) = \pm T(F_{ig_k(i)}).
\]

By Lemma 2, \( (i', g_k(i')) \neq (l, g_k(i)h(l)) \). Hence \( T \) is singular, which is, by Proposition 3, a contradiction.

Now by Lemma 1, for each \( g' \in D_n \), \( T(g') \) is a \( D_n \)-\( H \) unitary set; hence \( T(g') = \{ T(E_{ig''(i)}): i = 1, 2, \ldots, n \} \) for some \( g'' \in D_n \), and we write \( T(D(g')) = D(g'') \).
LEMMA 4. If \( n \) is odd, \( n > 3 \), and \( T \in \mathcal{J}P(D_n, H) \), then either

\[
T(D(g_i)) = D(g_{\sigma(i)}), \quad T(D(g_h)) = D(g_{\mu(i)}h), \quad i = 1, 2, \ldots, n, \tag{7.1}
\]

for some \( \sigma, \mu \in S_n \) or

\[
T(D(g_i)) = D(g_{\tau(i)}h), \quad T(D(g_h)) = D(g_{\nu(i)}), \quad i = 1, 2, \ldots, n, \tag{7.2}
\]

for some \( \tau, \nu \in S_n \).

Proof. Suppose \( T(D(g_i)) = D(g_k) \) and \( T(D(g_j)) = D(g_lh) \) for some \( 1 \leq i, j, k, l \leq n \). Since \( |D(g_i) \cap D(g_j)| = n \) or 0 and \( |D(g_k) \cap D(g_lh)| = 1 \), it follows that \( T \) is singular, a contradiction. 

Now by Lemma 2,

\[
|D(g_i) \cap D(g_j)| = |D(g_{\sigma(i)}) \cap D(g_{\mu(i)}h)| = 1,
\]

\[
|D(g_i) \cap D(g_j)| = |D(g_{\tau(i)}h) \cap D(g_{\nu(i)})| = 1.
\]

Hence each of (7.1) and (7.2) completely describes the linear transformation \( T \). On the other hand, it is easy to see that for any choices of \( \sigma, \mu, \tau, \nu \) in \( S_n \), the \( T \)'s are in \( \mathcal{J}P(D_n, H) \). Let

\[
\mathcal{J}_1 P(D_n, H) = \{ T \in \mathcal{J}P(D_n, H) : T \text{ satisfies (7.1) with } \sigma, \mu \in S_n \},
\]

\[
\mathcal{J}_2 P(D_n, H) = \{ T \in \mathcal{J}P(D_n, H) : T \text{ satisfies (7.2) with } \tau, \nu \in S_n \}.
\]

For \( T \in \mathcal{J}_1 P(D_n, H) \), since

\[
D(g_i) \cap D(g_j) = \{(p_i^{-1}(j), g_j p_i^{-1}(j))\},
\]

\[
D(g_{\sigma(i)} \cap D(g_{\mu(i)}h) = \{(p_{\sigma(i)}^{-1}(j), g_{\mu(i)} p_{\sigma(i)}^{-1}(j))\},
\]

it follows that

\[
T(E_{p_i^{-1}(j), g_j p_i^{-1}(j)}) = a_{p_i^{-1}(j), g_j p_i^{-1}(j)} E_{p_{\sigma(i)}^{-1}(j), g_{\mu(i)} p_{\sigma(i)}^{-1}(j)}.
\]
or, if we set \( k = q_i^{-1}(j) \),

\[
T'(E_{k_i}(k)) = \alpha_{k_i}(k)E_{q_i^{-1}(i)}\mu q_i(k), \quad i, k = 1, 2, \ldots, n.
\]  

(7.3)

Similarly, if \( T' \in \mathcal{G}_2 P(D_n, H) \), it follows from

\[
D(g) \cap D(g h) = \left\{ (q_i^{-1}(j), g q_i^{-1}(j)) \right\},
\]

\[
D(g_{\tau(i)} h) \cap D(g_{\tau(i)} f) = \left\{ (\theta_{\tau(i)}^{-1} f(j), g_{\tau(i)} h \theta_{\tau(i)}^{-1} f(j)) \right\}
\]

that

\[
T'(E_{q_i^{-1}(j)} g q_i^{-1}(j)) = \alpha_{q_i^{-1}(j)} g q_i^{-1}(j)E_{q_i^{-1}(j)}\mu q_i(k), \quad i, k = 1, 2, \ldots, n.
\]

This proves Theorem 1.

Now for \( T \in \mathcal{G}_1 P(D_n, H) \), since \( X_{g_i} = \sum_{i=1}^n x_{k_i}(k)E_{k_i}(k) \), it follows from

(7.3) that

\[
T(X_{g_i}) = \sum_{k=1}^n \alpha_{k_i}(k) x_{k_i}(k)E_{q_i^{-1}(i)}\mu q_i(k), \quad i = 1, 2, \ldots, n.
\]

Set \( q_{\sigma(i)}^{-1} \mu q_i = \omega^1 \) and \( \omega^{-1}(k) = l \). Then \( k = \omega(l) \), and

\[
T(X_{g_i}) = \sum_{l=1}^n \alpha_{\omega(l)}(l) g_{\omega(l)}(l) x_{\omega(l)}(l) g_{\omega(l)}(l)E_{g_{\omega(l)}(l)}(l), \quad i = 1, 2, \ldots, n.
\]

Since for \( \sigma \in S_n \),

\[
\sum_{i=1}^n a_{\sigma(i)} E_{\sigma(i)} = \text{diag}(a_{1\sigma(1)}, a_{2\sigma(2)}, \ldots, a_{n\sigma(n)}) P(\sigma^{-1})
\]

and

\[
\text{diag}(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}) = P(\sigma^{-1}) \text{diag}(a_1, \ldots, a_n) P(\sigma),
\]
where diag$(a_1, a_2, \ldots, a_n)$ is the matrix with entry $a_i$ at the $(i, i)$ position and zero elsewhere, it follows that

$$T(X_g) = P(\omega^{-1}) \text{diag}(x_{1g,(1)}a_{1g,(1)}, \ldots, x_{ng,(n)}a_{ng,(n)})P(\omega g^{-1}_\sigma(i))$$

$$= P(q_{\sigma(i)}^{-1}g^{-1}_{\sigma(i)})X_g \ast A'_g P( g_{\sigma(i)}q_{\sigma(i)}^{-1}q_{\sigma(i)}^{-1}g_{\sigma(i)})$$

where $A' = (\alpha_{jk}) \in M_n(H)$. Since $X = \sum_{i=1}^n X_{g_i}$,

$$T(X) = A \ast \sum_{i=1}^n P(q_{\sigma(i)}^{-1}g_{\sigma(i)}q_{\sigma(i)}^{-1}g_{\sigma(i)})X_g \in M_n(F)$$

where

$$A = \sum_{i=1}^n P(q_{\sigma(i)}^{-1}q_{\sigma(i)}^{-1}g_{\sigma(i)}q_{\sigma(i)}^{-1}g_{\sigma(i)}^{-1}) \in M_n(H).$$

Let $S$ be another element in $\mathcal{P}(D_n, H)$ which is associated with a pair $\sigma', \mu' \in S_n$ and $B \in M_n(H)$, i.e.,

$$S(X) = B \ast \sum_{i=1}^n P(q_{\sigma(i)}^{-1}g_{\sigma(i)}q_{\sigma(i)}^{-1}g_{\sigma(i)})X_g \in M_n(F).$$

Then a few step computation shows that

$$ST(X) = B \ast A^{(\sigma', \mu')} \ast \sum_{i=1}^n P(q_{\sigma'\sigma(i)}^{-1}q_{\sigma'\sigma(i)}^{-1}g_{\sigma'\sigma(i)}^{-1}g_{\sigma'\sigma(i)}^{-1})X_g \in M_n(F)$$

where $A^{(\sigma', \mu')}$ is defined by (4.1), i.e., $ST$ is associated with the $\sigma', \mu' \in S_n$ and $B \ast A^{(\sigma', \mu')} \in M_n(H)$. Also, if $T \in \mathcal{P}(D_n, H)$ is associated with $\sigma = \mu = 1$ (the identity element of $S_n$) and $A = I$, then $T$ is the identity linear transformation on $M_n(F)$. This shows that $\mathcal{P}(D_n, H)$ is isomorphic to the group $\langle S_n \times S_n, M_n(H) \rangle$.

Recall that $R(X) = X$, $X \in M_n(F)$. Clearly $R$ is in $\mathcal{P}(D_n, H)$ and satisfies

$$R(D(g_i)) = D(g_{\sigma(i)}h), R(D(g_{\sigma(i)}) h) = D(g_{\sigma(i)}), \quad i = 1, 2, \ldots, n,$$
where \( \tau(i) = v(i) = n - i + 1 \), \( i = 1, 2, \ldots, n \). If \( T \in \mathcal{T}_1 P(D_n, H) \) and is associated with \( \sigma, \mu \in S_n \), then

\[
TR(D(g_i)) = D(g_{\sigma v(i)} h), \quad T(D(g_i h)) = D(g_{\sigma v(i)}), \quad i = 1, 2, \ldots, n,
\]

i.e., \( TR \in \mathcal{T}_2 P(D_n, H) \). On the other hand, if \( S \in \mathcal{T}_2 P(D_n, H) \) and is associated with \( \tau', v' \in S_n \), then since \( R \) is nonsingular, \( R^{-1} \) exists and

\[
R^{-1}(D(g_i)) = D(g_{\tau'^{-1}(i)} h), \quad R^{-1}(D(g_i h)) = D(g_{\tau'^{-1}(i)}), \quad i = 1, 2, \ldots, n,
\]

we obtain

\[
SR^{-1}(D(g_i)) = D(g_{\tau'^{-1}(i)}), \quad SR^{-1}(D(g_i h)) = D(g_{\tau'^{-1}(i)}), \quad i = 1, 2, \ldots, n,
\]

i.e., \( SR^{-1} \) is in \( \mathcal{T}_1 P(D_n, H) \). Hence \( S \) is in \( \mathcal{T}_2 P(D_n, H) \) if and only if \( S = TR \), where \( T \) is in \( \mathcal{T}_1 P(D_n, H) \), and Theorem 3 follows.

8. STRUCTURE OF \( \mathcal{T}_P(D_n, H) \): \( n \) EVEN AND \( n \geq 6 \)

Let \( H \) be a nontrivial group in \( \mathcal{H} \), and \( n \) be a positive even integer, \( n = 2m \). In Lemma 5, we assume \( 1 < \varphi_k^{-1}(i), \theta_k^{-1}(i) < m \) for all appropriate \( i \) and \( k \).

**Lemma 5.** For each pair \( g_i, g_k h \) in \( D_n \) \( (j, k = 1, 2, \ldots, n) \), \( |D(g_i) \cap D(g_k h)| = 0 \) if \( 2 \nmid j-k \) and \( |D(g_i) \cap D(g_k h)| = 2 \) if \( 2|j-k \). In fact, if \( 2|(j-k) \),

\[
D(g_i) \cap D(g_k h) = \{ (\varphi_i^{-1}(k), g_i \varphi_i^{-1}(k)), (\varphi_i^{-1}(k) + m, g_i (\varphi_i^{-1}(k) + m)) \}
\]

\[
= \{ (\theta_k^{-1}(j), g_k h \theta_k^{-1}(j)), (\theta_k^{-1}(j) + m, g_k h (\theta_k^{-1}(j) + m)) \}.
\]

**Proof.** \( \varphi_i(i) \) is even if and only if \( j \) is even for all \( i = 1, 2, \ldots, n \), \( \varphi_i(i) = \varphi_i(i + m) \) for \( i = 1, 2, \ldots, m \), and \( \varphi_i(i) \neq \varphi_i(k) \) if \( 1 < i, k < m, i \neq k \). Therefore we see that if \( j, k \) are both even or both odd, then there exist \( i, 1 < i < m \), such that \( g_i(i) = g_k h(i), g_i(i + m) = g_k h(i + m) \); and if one of \( j, k \) is odd and the other is even, then \( g_i(i) \neq g_k h(i) \) for all \( i = 1, 2, \ldots, n \).
LEMMA 6. Let \( n \) be even and \( n > 6 \). If \( T \in \mathcal{P}(D_n, H) \) and \( 1 \leq i, j \leq n \), then there exist \( 1 \leq p, q \leq n \) and \( \alpha_{ij} \in H \) such that

\[
T(E_{ij}) = \alpha_{ij} E_{pq}.
\]

Proof. As in the case that \( n \) is odd (Lemma 3), the result follows from Proposition 1 and Lemma 1 if \( |H| > 2 \), since for \( 1 \leq i, j \leq n \) there exist \( g_k \in D_n \) such that \( g_k(i) = j \). Consider the \( D_n-H \) unitary set \( T(g_k) \). Suppose \( |H| = 2 \); then again Proposition 2 and Lemma 1 apply. If the \( r \) in Proposition 2 is equal to \( n \) [i.e., no matrices of the second type appear in \( T(g_k) \)], the result follows. Hence we assume that there exist \( 1 \leq i, i' < n \), \( i \neq i' \) such that (only writing the appropriate 2-square matrices)

\[
T(E_{gg(i),i}) = \frac{1}{2} \begin{bmatrix}
\alpha_{ru} & \alpha_{rv} \\
\alpha_{su} & \alpha_{sv}
\end{bmatrix}, \quad T(E_{gg(i),i'}) = \frac{1}{2} \begin{bmatrix}
\pm \alpha_{ru} & \mp \alpha_{rv} \\
\mp \alpha_{su} & \pm \alpha_{sv}
\end{bmatrix},
\]

where \( 1 \leq r, s, u, v < n \), \( r \neq s \), \( u \neq v \), and \( \alpha_{ru}, \alpha_{rv}, \alpha_{su}, \alpha_{sv} \in H \). Since \( g_{gg(i),i}h(i) = g_k(i) \), there exist \( 1 \leq i'', n < i'' \neq i \) such that

\[
T(E_{gg(i),h(i'')}) = \frac{1}{2} \begin{bmatrix}
\pm \alpha_{ru} & \mp \alpha_{rv}
\mp \alpha_{su} & \pm \alpha_{sv}
\end{bmatrix} = \pm T(E_{gg(i),i}).
\]

If \( i'' \neq i' \) or \( g_{gg(i),i}h(i'') \neq g_k(i') \) then \( T \) is singular, a contradiction. Hence \( i'' = i' \), \( g_{gg(i),i}h(i'') = g_k(i') \), and therefore \( i'' = i'' = i + m \). We may assume that there exists an integer \( 1 \leq t \leq m \) such that

\[
T(E_{gg(i),t}) = \frac{1}{2} \begin{bmatrix}
\alpha_{nu} & \alpha_{nv} \\
\alpha_{su} & \alpha_{sv}
\end{bmatrix}, \quad T(E_{t+m,gg(i),t+m}) = \frac{1}{2} \begin{bmatrix}
\pm \alpha_{nu} & \mp \alpha_{nv} \\
\mp \alpha_{su} & \pm \alpha_{sv}
\end{bmatrix}, \quad 1 \leq l \leq t,
\]

where \( u_i \neq v_i, \eta_i \neq s_i \); \( u_i \neq u_p, v_i \neq v_p, \eta_i \neq r_p, s_i \neq s_p \) if \( l \neq p \); and \( T(E_{gg(i),l}), l \neq 1, 2, \ldots, t, 1+m, 2+m, \ldots, t+m \), are of the first type. Then

\[
\sum_{l=1}^{n} T(E_{gg(i),l}) = P(\alpha, \sigma),
\]

\[
\sum_{l \neq 1+m} T(E_{gg(i),l}) - T(E_{1+m,gg(i),1+m}) = P(\beta, \tau),
\]
where $\alpha, \beta \in \Gamma_n(H)$, $\sigma, \tau \in D_n$, and $\sigma(r_i) \neq \tau(r_i)$, $\sigma(s_i) \neq \tau(s_i)$, $\sigma(l) = \tau(l)$ for all $l \neq r_i, s_i$. By Lemma 5, $|D(\sigma) \cap D(\tau)| = 0$ or 2, which requires $n \leq 4$, a contradiction, so the result follows.

In the following we assume that $n \geq 6$.

**Lemma 7.** Suppose $n = 2m$, where $m \geq 3$ and $T \in \overline{\mathbb{P}}(D_n, H)$.

(i) If $T(D(g_i)) = D(g_j)$, $T(D(g_k)) = D(g_k)$, then $2|i - k$ if and only if $2|j - l$.

(ii) If $T(D(g_i)) = D(g_j)$, $T(D(g_k)) = D(g_k)$, then $2|i - k$ if and only if $2|j - l$.

(iii) If $T(D(g_i)) = D(g_j)$, $T(D(g_k)) = D(g_k)$, then $2|i - k$ if and only if $2|j - l$.

(iv) If $T(D(g_i)) = D(g_j)$, $T(D(g_k)) = D(g_k)$, then $2|i - k$ if and only if $2|j - l$.

(v) If $T(D(g_i)) = D(g_j)$, $T(D(g_k)) = D(g_k)$, then $2|i - k$ if and only if $2|j - l$.

(vi) If $T(D(g_i)) = D(g_j)$, $T(D(g_k)) = D(g_k)$, then $2|i - k$ if and only if $2|j - l$.

(vii) If $T(D(g_i)) = D(g_j)$, $T(D(g_k)) = D(g_k)$, then $2|i - k$ if and only if $2|j - l$.

(viii) If $T(D(g_i)) = D(g_j)$, $T(D(g_k)) = D(g_k)$, then $2|i - k$ if and only if $2|j - l$.

(ix) If $T(D(g_i)) = D(g_j)$, $T(D(g_k)) = D(g_k)$, then $2|i - k$ if and only if $2|j - l$.

**Proof.**

(i) First assume that $2|i - k$. Then either both $i, k$ are even or both are odd. If both $i, k$ are even, then $D(g_i) \cap D(g_k) \neq \emptyset$ and $D(g_i) \cap D(g_{2h}) \neq \emptyset$. Hence there exist $(r, g_{2h}(r)) \in D(g_i)$, $(s, g_{2h}(s)) \in D(g_k)$, $1 \leq r, s \leq n$, and

$$T(E_{rg_{2h}(r)}) = \alpha_{rg_{2h}(r)} E_{rg_{2h}(r)}, \quad \text{for some } 1 \leq t \leq n,$$

$$T(E_{sg_{2h}(s)}) = \alpha_{sg_{2h}(s)} E_{sg_{2h}(s)}, \quad \text{for some } 1 \leq u \leq n.$$

If $2|i - l$, then $g_j \neq g_l$; hence $T(D(g_l)) = D(g_l)$ for some $1 \leq v \leq n$. But for each $1 \leq q \leq n$ either $D(g_q) \cap D(g_j) = \emptyset$ or $D(g_q) \cap D(g_j) = \emptyset$, a contradiction. If $i, k$ are both odd, then $D(g_i) \cap D(g_k) \neq \emptyset$, $D(g_i) \cap D(g_{2h}) \neq \emptyset$ and the result can be obtained in the same way. On the other hand if $2|i - l$ we can proceed as above to prove that $2|i - k$.

(ii) and (iv) can be proved as in (i).
(iii) Suppose $2 | j - k$. If $2 \nmid j - l$, then $D(g_j) \cap D(g_kh) \neq \varnothing$ and $D(g_j) \cap D(g_kh) = \varnothing$, which is impossible. Similarly $2 | j - l$ implies that $2 | i - k$.

Since $T^{-1}$ exists, (v) follows from (iv). (vi) can be proved as in (iii).

(vii) Suppose $2 | i - k$ and $i, k$ are odd. If $2 | j - l$, then $D(g_j) \cap D(g_kh) \neq \varnothing$. But $D(g_j) \cap D(g_k) = \varnothing$, a contradiction; hence $2 \nmid j - l$. We first assume that $j$ is odd and $l$ is even. Let $q$ be odd and $1 \leq q < n$. If $T(D(g_qh)) = D(g_h)$ for some $1 \leq r < n$, then $D(g_qh) \cap D(g_k) \neq \varnothing$, $D(g_j) \cap D(g_i) = \varnothing$ if $r$ is odd; $D(g_qh) \cap D(g_k) \neq \varnothing$, $D(g_h) \cap D(g_j) = \varnothing$ if $r$ is even; and both are impossible. Hence consider $T(D(g_qh)) = D(g_q)$ for some $1 \leq r < n$. If $r$ is odd, then $D(g_qh) \cap D(g_k) \neq \varnothing$ and $D(g_q) \cap D(g_qh) = \varnothing$, which is impossible. If $r$ is even, then $D(g_qh) \cap D(g_k) \neq \varnothing$ and $D(g_q) \cap D(g_j) = \varnothing$, which is again impossible. If $j$ is even and $l$ is odd, we can proceed as above to obtain the result.

(viii) can be proved as in (vii).

(ix) Since $T$ is nonsingular, $j \neq l$. If $2 | i - k$, then $D(g_j) \cap D(g_kh) \neq \varnothing$ and $D(g_j) \cap D(g_kh) = \varnothing$, a contradiction.

(x) can be proved as in (ix).

Now if $T \in \mathcal{G}(D_n, H)$ and $g' \in D_n$, then $T(D(g')) = D(g'')$ for some $g'' \in D_n$. Hence either

(I) $T(D(g_i)) = D(g_{\sigma(i)})$, $T(D(g_kh)) = D(g_{\mu(i)}h)$, $i = 1, 2, \ldots, n$, for some $\sigma, \mu \in S_n$;

(II) $T(D(g_i)) = D(g_{\tau(i)}h)$, $T(D(g_kh)) = D(g_{\sigma(i)}h)$, $i = 1, 2, \ldots, n$, for some $\tau, \nu \in S_n$;

(III) there exist partitions $\{\Omega_1, \Omega_2\}$, $\{\Omega_3, \Omega_4\}$ of $\{1, 2, \ldots, n\}$ such that

$$T(D(g_i)) = D(g_{\omega(i)}) \quad \text{for} \ i \in \Omega_1, \quad T(D(g_i)) = D(g_{\rho(i)}h) \quad \text{for} \ i \in \Omega_2$$

and

$$T(D(g_kh)) = D(g_{\omega'(i)}h) \quad \text{for} \ i \in \Omega_3, \quad T(D(g_kh)) = D(g_{\pi(i)}h) \quad \text{for} \ i \in \Omega_4,$$

where $\omega, \omega', \rho, \pi \in S_n$.

We shall consider the three cases separately in the following.

Case (I)

In view of Lemma 7(i) and (ii), $2 | i - k$ if and only if $2 | \sigma(i) - \sigma(k)$ and $2 | i - k$ if and only if $2 | \mu(i) - \mu(k)$, i.e., $\sigma$ maps all even integers into even integers or all even into odd, and $\mu$ has the same form. By Lemma 7(iii), $\sigma$ maps all even into even if and only if $\mu$ maps all even into even. Now let $K_n = \{\sigma \in S_n : \sigma \text{ maps all even integers into even integers}\}$. It is easy to see
that $K_n$ is a subgroup of $S_n$, $K_n$ is isomorphic to $S_m \times S_m$, and the elements in $K_n$ map all even integers into odd integers. Hence if $T \in \mathcal{F}(D_n, H)$ and is of type (I), then $\sigma, \mu \in K_n$ or $\sigma, \mu \in K_n g$.

**Proposition 4.** $T \in \mathcal{F}(D_n, H)$ and is of type (I) if and only if there exist $\sigma, \mu \in K_n$ or $\sigma, \mu \in K_n g$, $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in G \times \cdots \times G$ and $\alpha_{kl} \in H$, $k, l = 1, 2, \ldots, n$, such that

$$
T\left(E_{\alpha_{kl}(k)}\right) = \alpha_{\alpha_{kl}(k)} E_{\lambda_\sigma(i) \lambda_\sigma(i) \mu_{\sigma(i)} \mu_{\sigma(i)} \mu_{\sigma(i)}} \alpha_{\alpha_{kl}(k)}, \quad 1 \leq i, k \leq n.
$$

**Proof.** We first define $1 < q_i^{-1}(j) < m$ for all appropriate $i$ and $j$. By the above remark, $T(D(g)) = D(g_{i(j)}, i = 1, 2, \ldots, n$, and $T(D(g_{j})) = D(g_{j(h)}, j = 1, 2, \ldots, n$, where $\sigma, \mu$ are both in $K_n$ or both in $K_n g$. By Lemma 5, if $2|i - j$ and $1 \leq i, j \leq n$, then

$$
D(g) \cap D(g_j h) = \left\{(q_i^{-1}(j), g_{q_i^{-1}(j)}), (q_i^{-1}(j) + m, g_{q_i^{-1}(j) + m})\right\},
$$

$$
D(g_{i(\sigma)}) \cap D(g_{j(\mu)}) = \left\{(q_{\alpha(i)}^{-1}(j), g_{\alpha(i)}(q_{\alpha(i)}^{-1}(j)), (q_{\alpha(i)}^{-1}(j) + m, g_{\alpha(i)}(q_{\alpha(i)}^{-1}(j) + m))\right\}.
$$

Hence by Lemma 6 either

$$
T\left(E_{q_i^{-1}(j)} g_{q_i^{-1}(j)} \right) = \alpha_{q_i^{-1}(j)} E_{q_{\sigma(i)}^{-1}(j)} g_{q_{\sigma(i)}^{-1}(j)},
$$

$$
T\left(E_{q_i^{-1}(j)} + m, g_{i(q_i^{-1}(j) + m)} \right) = \alpha_{q_i^{-1}(j) + m} E_{q_{\sigma(i)}^{-1}(j) + m} g_{i(q_{\sigma(i)}^{-1}(j) + m)}
$$
or

$$
T\left(E_{q_i^{-1}(j)} + m, g_{i(q_i^{-1}(j) + m)} \right) = \alpha_{q_i^{-1}(j) + m} E_{q_{\sigma(i)}^{-1}(j) + m} g_{i(q_{\sigma(i)}^{-1}(j) + m)} g_{i(q_{\sigma(i)}^{-1}(j) + m)}
$$

That is, there exist $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in G \times \cdots \times G$ such that

$$
T\left(E_{q_i^{-1}(j)} g_{q_i^{-1}(j)} \right) = \alpha_{q_i^{-1}(j)} E_{q_i^{-1}(j)} g_{q_i^{-1}(j)},
$$

$$
T\left(E_{q_i^{-1}(j)} + m, g_{i(q_i^{-1}(j) + m)} \right) = \alpha_{q_i^{-1}(j) + m} E_{q_i^{-1}(j) + m} g_{i(q_i^{-1}(j) + m)}
$$

$$\times E_{q_{\sigma(i)}^{-1}(j) + m} g_{i(q_{\sigma(i)}^{-1}(j) + m)} g_{i(q_{\sigma(i)}^{-1}(j) + m)}.$$
Set \( q_i^{-1}(j) = k \). Then for \( k = 1, 2, \ldots, m \),

\[
T\left(E_{kg_i(k)}\right) = \alpha_{kg_i(k)}E_{\lambda_i(i(q_{o(i)})^{-1}q_{i(k)})g_{o(i)}\lambda_i(i(q_{o(i)})^{-1}q_{i(k)})},
\]

\[
T\left(E_{k+m,g_i(k+m)}\right) = \alpha_{k+m,g_i(k+m)}E_{\lambda_i(i(q_{o(i)})^{-1}q_{i(k)}+m)g_{o(i)}\lambda_i(i(q_{o(i)})^{-1}q_{i(k)}+m)}.
\]

If we define \( 1 \leq q_{o(i)}^{-1}q_{i(k)} < m \) if and only if \( 1 \leq k \leq m \), the result follows.

Conversely, for any choice of \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \in G \times \cdots \times G \), \( \alpha_{kl} \in H \) and \( \sigma, \mu \in K_n \) or \( K_n g \), it is easy to see that the \( T \) is in \( \mathcal{T} P(D_n, H) \) and is of type (I).

Now let \( \mathcal{T}_1 P(D_n, H) \) be the set of all linear transformations in \( \mathcal{T} P(D_n, H) \) and of type (I) with \( \sigma, \mu \in K_n \). Since \( K_n \) is a group, \( \mathcal{T}_1 P(D_n, H) \) is a subgroup of \( \mathcal{T} P(D_n, H) \). Suppose \( T \in \mathcal{T}_1 P(D_n, H) \). Then in view of Proposition 4 and since \( X_i = \sum_{k=1}^{n} X_{g_i(k)} E_{kg_i(k)} \), it follows by a few step computation as in Sec. 7 that

\[
T(X) = \left( A' \right) \sum_{i=1}^{n} P\left( \lambda_{o(i)} q_{o(i)}^{-1} q_{i(i)} \right) P\left( g_{o(i)} q_{o(i)}^{-1} q_{i(i)} \right) X_{g_i}, \quad X \in M_n(F),
\]

where \( A' = (\alpha_{kl}) \in M_n(H) \). Since \( X = \sum_{i=1}^{n} X_{g_i} \), it follows that

\[
T(X) = A \sum_{i=1}^{n} P\left( \lambda_{o(i)} q_{o(i)}^{-1} q_{i(i)} \right) X_{g_i} P\left( g_{o(i)} q_{o(i)}^{-1} q_{i(i)} \right) X_{g_i}, \quad X \in M_n(F),
\]

where

\[
A = \sum_{i=1}^{n} P\left( \lambda_{o(i)} q_{o(i)}^{-1} q_{i(i)} \right) A' P\left( g_{o(i)} q_{o(i)}^{-1} q_{i(i)} \right) \in M_n(H).
\]

Let \( \mathcal{T}_0 P(D_n, H) \) be the set of all linear transformations in \( \mathcal{T}_1 P(D_n, H) \) with \( \lambda_1 = \lambda_2 = \cdots = \lambda_n = 1 \), the identity element of \( S_n \), i.e., \( T \in \mathcal{T}_0 P(D_n, H) \) if and only if

\[
T(X) = A \sum_{i=1}^{n} P\left( q_{o(i)}^{-1} q_{i(i)} \right) X_{g_i} P\left( g_{o(i)} q_{o(i)}^{-1} q_{i(i)} \right), \quad X \in M_n(F),
\]

with \( \sigma, \mu \in K_n \) and \( A \in M_n(H) \). Let \( T' \in \mathcal{T}_0 P(D_n, H) \) and be associated with \( \sigma', \mu' \in K_n \) and \( B \in M_n(H) \), i.e.,

\[
T'(X) = B \sum_{i=1}^{n} P\left( q_{o(i)}^{-1} q_{i(i)} \right) X_{g_i} P\left( g_{o(i)} q_{o(i)}^{-1} q_{i(i)} \right), \quad X \in M_n(F).
\]
Then
\[
T' T(X) = B \sum_{i=1}^{n} P \left( \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right) A g_i P \left( g_i \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right)^{-1} g_{\sigma(i)}^{-1} \times \sum_{i=1}^{n} P \left( \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right) P \left( \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right) X g_i P \left( g_i \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right)^{-1} g_{\sigma(i)}^{-1} \right) \times P \left( g_i \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right)^{-1} g_{\sigma(i)}^{-1} \right).
\]

\[
= B A^{(\sigma', \mu')} \sum_{i=1}^{n} P \left( \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right) X g_i P \left( g_i \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right)^{-1} g_{\sigma(i)}^{-1} \right).
\]

if we define \( A^{(\sigma', \tau)} \) by (4.2). Hence \( T' T \) associates with the matrix \( B A^{(\sigma', \tau)} \) in \( M_n(H) \) and \( \sigma', \mu', \mu \in \mathbb{K}_n \). Also if \( T \) is associated with the identity matrix \( A = I \) in \( M_n(H) \) and \( \sigma = \mu = 1 \) (the identity element in \( \mathbb{K}_n \)), then clearly \( T \) is the identity linear transformation on \( M_n(F) \). Hence \( \mathcal{S}_0 P(D_n, H) \) is isomorphic to the group \( \langle K_n \times K_n, M_n(H) \rangle \).

Now for \((\lambda_1, \lambda_2, \ldots, \lambda_n) \in G \times \cdots \times G \) it is clearly the case that \( \Lambda_i \Lambda_1, \Lambda_2, \ldots \Lambda_n \) is in \( \mathcal{S}_1 P(D_n, H) \) associated with \( \sigma = \mu = 1 \), \( A = I \), and \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \in G \times \cdots \times G \). Furthermore \( S \) is in \( \mathcal{S}_0 P(D_n, H) \) associated with \((\lambda_1, \lambda_2, \ldots, \lambda_n) \in G \times \cdots \times G \), \( \sigma, \mu \in \mathbb{K}_n \) if and only if \( S = \Lambda(\lambda_1, \lambda_2, \ldots, \lambda_n) T \), where \( T \) is in \( \mathcal{S}_0 P(D_n, H) \) associated with \( \sigma, \mu \in \mathbb{K}_n \) for

\[
A \sum_{i=1}^{n} P \left( \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right) X g_i P \left( g_i \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right)^{-1} g_{\sigma(i)}^{-1} \right) = \Lambda(\lambda_1, \ldots, \lambda_n) \left( A'' \sum_{i=1}^{n} P \left( \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right) X g_i P \left( g_i \varphi_{\sigma(i)}^{-1} \mu' \varphi_i \right)^{-1} g_{\sigma(i)}^{-1} \right),
\]

where \( A'' = \sum_{i=1}^{n} P(\lambda_i) A g_i P( g_i \varphi_{\sigma(i)}^{-1} ) \). Hence \( \mathcal{S}_1 P(D_n, H) = \Lambda \circ \mathcal{S}_0 P(D_n, H) \). Also, since \( G \) is the group generated by the transpositions \( (1 m + 1), \ldots, (m - 12 m - 1), (m - 2 m) \), we have \( |G| = 2^m \). Hence \( |G \times G \times \cdots \times G| = 2^{mn} = 2^{n^2/2} \), and \( \mathcal{S}_0 P(D_n, H) \) is a subgroup of index \( 2^{n^2/2} \) in \( \mathcal{S}_1 P(D_n, H) \).

Recall that \( U(X) = XP(g^{-1}) \) for every \( X \in M_n(F) \). Since \( U(X) \)

\[
(U(X))_{i g_j(i)} = x_{i g_j(i)} = x_{i g_j(i)},
\]

\[
(U(X))_{j g_i(h)} = x_{j g_i(h)} = x_{i g_j(i)}.
\]
for $i, j = 1, 2, \ldots, n$, it follows that

$$U(D(g_i)) = D(g_{U(i)}), \quad U(D(g_ih)) = D(g_{U(i)}h), \quad j = 1, 2, \ldots, n,$$

i.e., $U$ is of type (I) and is associated with $g$ in $K_n^g$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1$, $A = J$. Let $\mathcal{S}_2 P(D_n, H) = \{ T \in \mathcal{S}P(D_n, H) : T$ is of type (I) \}. Since $K_n \cup K_n^g$ is a group, it follows that $\mathcal{S}_2 P(D_n, H)$ is a group. Also it is easy to see that $\mathcal{S}_2 P(D_n, H) = \mathcal{S}_1 P(D_n, H) \circ \{ I, U \}$.

**Case (II)**

As in (I), by Lemma 7(iv), (v) and (vi), if $T \in \mathcal{S}P(D_n, H)$ and $T$ is of type (II), then $\tau, \nu \in K_n$ or $K_n^g$. We first state a proposition whose proof is similar to that of Proposition 4.

**Proposition 5.** $T \in \mathcal{S}P(D_n, H)$ and $T$ is of type (II) if and only if there exist $\tau, \nu \in K_n$ or $K_n^g$, $(\kappa_1, \ldots, \kappa_n) \in G \times \cdots \times G$ and $\alpha_{kl} \in H$, $k, l = 1, 2, \ldots, n$, such that

$$T(E_{kg_i}(k)) = \alpha_{kg_i(k)}E_{\kappa_{i(0)}\alpha_{i(0)}}(k), \quad i, k = 1, 2, \ldots, n.$$

Recall that $R$ is a linear transformation which satisfies $R(D(g_i)) = D(g_{\tau(i)}), R(D(g_ih)) = R(g_{\nu(i)}), \quad i = 1, 2, \ldots, n$, where $\tau(i) = \nu(i) = n - i + 1, i = 1, 2, \ldots, n$. Hence $\tau, \nu$ are in $K_n^g$, and $R$ is of type (II) in $\mathcal{S}P(D_n, H)$ associated with $\kappa_1 = \kappa_2 = \cdots = \kappa_n = 1$. Now let $S$ be of type (II) in $\mathcal{S}P(D_n, H)$, i.e., there exist $\tau, \nu \in K_n$ or $K_n^g$ such that

$$S(D(g_i)) = D(g_{\tau(i)}h), \quad S(D(g_ih)) = D(g_{\nu(i)}), \quad i = 1, 2, \ldots, n. \quad (8.1)$$

Then since $R^{-1}$ exists in $\mathcal{S}P(D_n, H)$,

$$SR^{-1}(D(g_i)) = D(g_{\nu^{-1}(i)}), \quad SR^{-1}(D(g_ih)) = D(g_{\tau(i)}h), \quad i = 1, 2, \ldots, n.$$
Then \( \mathcal{P}(D_n, H) \) and it is easily seen that \( \mathcal{P}(D_n, H) \) is a group.

**Case (III)**

By Lemma 7(vii) we have either \( \Omega_1 = \{2i : i = 1, 2, \ldots, m\} \) or \( \Omega_1 = \{2i - 1 : i = 1, 2, \ldots, m\} \), and by Lemma 7(viii) either \( \Omega_3 = \{2i : i = 1, 2, \ldots, m\} \) or \( \Omega_3 = \{2i - 1 : i = 1, 2, \ldots, m\} \). By Lemma 7(ix) and (x) we have \( \Omega_1 = \Omega_3 \) and \( \Omega_2 = \Omega_4 \). By Lemma 7(iii), either both \( \omega, \omega' \) are in \( K_n \) or both are in \( K_{ng} \), and by Lemma 7(vi) either both \( \pi, \pi' \) are in \( K_n \) or both are in \( K_{ng} \). By Lemma 7(vii), (viii), (ix) and (x) \( \omega, \omega' \in K_n \) if and only if \( \pi, \pi' \in K_n \). Hence either all \( \omega, \omega', \pi, \pi' \) are in \( K_n \) or all are in \( K_{ng} \).

**Proposition 6.** \( T \in \mathcal{P}(D_n, H) \) and \( T \) is of type (III) if and only if there exist \( \omega, \omega', \pi, \pi' \in K_n \) or in \( K_{ng} \), \( \alpha_{kl} \in H \), \( k, l = 1, 2, \ldots, n \), and for each \( i \in \Omega_1 \) there exists \( \lambda_{\omega(i)} \) in \( G \), for each \( i \in \Omega_2 \) there exists \( \kappa_{\pi(i)} \) in \( G \) such that for \( i \in \Omega_1 \),

\[
T(E_{g_\omega(k)}(k)) = \alpha_{g_\omega(k)}E_{\lambda_{\omega(i)}}(g_{\omega'(i)})\phi_{\omega(i)}(k), \quad k = 1, 2, \ldots, n
\]

and for \( i \in \Omega_2 \),

\[
T(E_{g_\omega(k)}(k)) = \alpha_{g_\omega(k)}E_{\phi_{\omega(i)}}(g_{\omega'(i)})\phi_{\omega(i)}(k), \quad k = 1, 2, \ldots, n,
\]

where \( \Omega_1 = \{2i : i = 1, 2, \ldots, m\} \) or \( \Omega_1 = \{2i - 1 : i = 1, 2, \ldots, m\} \), and \( \Omega_2 = \{1, 2, \ldots, n\} - \Omega_1 \).

**Proof.** By the above remark there exist \( \omega, \omega', \pi, \pi' \in K_n \) or \( K_{ng} \) and a partition \( \{\Omega_1, \Omega_2\} \) of \( \{1, 2, \ldots, n\} \), where \( \Omega_1 \) is equal to either \( \{2i : i = 1, 2, \ldots, m\} \) or \( \{2i - 1 : i = 1, 2, \ldots, m\} \), such that

\[
T(D(g_i)) = D(g_{\omega(i)}), \quad i \in \Omega_1; \quad T(D(g_i)) = D(g_{\omega'(i)}h), \quad i \in \Omega_2;
\]

\[
T(D(g_ih)) = D(g_{\omega'(i)}), \quad j \in \Omega_1; \quad T(D(g_ih)) = D(g_{\omega'(i)}), \quad j \in \Omega_2.
\]

By Lemma 5, if \( i, j \in \Omega_1 \) or \( \Omega_2 \),

\[
D(g_i) \cap D(g_j) = \{(\varphi_i^{-1}(i)g_\varphi_i^{-1}(i)), (\varphi_i^{-1}(i) + mg_\varphi_i^{-1}(i) + m)\};
\]

if \( i, j \in \Omega_1 \), then

\[
D(g_{\omega(i)}) \cap D(g_{\omega'(i)}h) = \{(\varphi_{\omega(i)}^{-1}\omega(j)g_{\varphi_{\omega(i)}^{-1}\omega'(j)}), (\varphi_{\omega(i)}^{-1}\omega'(j) + mg_{\varphi_{\omega(i)}^{-1}\omega'(j)} + m)\},
\]
and if \( i,j \in \Omega_2 \), then

\[
D(g_{\pi(i)}h) \cap D(g_{\pi(j)}) = \left\{ \left( \theta_{\pi(i)}^{-1} \pi'(j) g_{\pi(i)}h \theta_{\pi(i)}^{-1} \pi'(i) \right), \right.
\]

\[
\left( \theta_{\pi(i)}^{-1} \pi'(j) + m \theta_{\pi(i)}h \left( \theta_{\pi(i)}^{-1} \pi'(i) + m \right) \right) \},
\]

where \( 1 \leq q_k^{-1}(l), \theta_k^{-1}(l) \leq m \) for all appropriate \( k \) and \( l \). Then if we proceed as in Proposition 4, the result follows.

Recall that \( V(X) = X_{c_0} + R(X_cP\left( g^{-1} \right)) \) for all \( X \in M_n(F) \). Since \( k + g_i(k) = 2k + i - 1 \) for \( i,k = 1,2,\ldots,n \), it follows that \( (X_0)_{k_+(i)} \neq 0 \) and \( (X_c)_{k_-(i)} = 0 \) if and only if \( i \) is odd. Now \( n - g_i(k) + 1 = g_{n-i+1}h(k) \) for \( i,k = 1,2,\ldots,n \), and it follows that for \( i \) even,

\[
(R(X_cP\left( g^{-1} \right)))_{k_+(i)} = \left( X_cP\left( g^{-1} \right) \right)_{k_n-g_i(k)} = \left( X_cP\left( g^{-1} \right) \right)_{k_n-i+1} = \left( X_cP\left( g^{-1} \right) \right)_{k_n-i+1} \]

\[
= x_{k_n-g_i(h(k))},
\]

\[
(R(X_cP\left( g^{-1} \right)))_{k_-(i)} = \left( X_cP\left( g^{-1} \right) \right)_{k_n-g_i(h(k))} = \left( X_cP\left( g^{-1} \right) \right)_{k_n-i+1} = x_{k_n-i}(h(k))
\]

i.e., \( V \) is in \( \mathcal{T}P(D_n,H) \) of type (III) corresponding to

\[
\Omega_0^0 = \Omega_3^0 = \{2i-1: i = 1,2,\ldots,m\}, \quad \Omega_2^0 = \Omega_4^0 = \{2i: i = 1,2,\ldots,m\},
\]

and \( \omega_0 = \omega'_0 = 1, \pi_0(i) = \pi'_0(i) = n - i, i = 1,2,\ldots,n \), or in other words \( \omega_0, \omega'_0, \pi_0, \pi'_0 \) are in \( K_n \). Hence

\[
V(D(g_i)) = D(g_i), \quad V(D(g_ih)) = D(g_ih) \quad \text{if} \quad i \in \Omega_1^0;
\]

\[
V(D(g_i)) = D(g_{\sigma_0(i)}h), \quad V(D(g_ih)) = D(g_{\sigma_0(i)}h) \quad \text{if} \quad i \in \Omega_2^0.
\]

We contend that \( S \in \mathcal{T}P(D_n,H) \) and is of type (III) if and only if \( S = TV \), where \( T \) is of type (I) or of type (II) in \( \mathcal{T}P(D_n,H) \). In fact, if \( T \in \mathcal{T}P(D_n,H) \) and is of type (I) associated with \( \sigma, \mu \in K_n \) or \( K_ng \), i.e.,

\[
T(D(g)) = D(g_{\sigma(i)}), \quad T(D(g_ih)) = D(g_{\mu(i)}h), \quad i = 1,2,\ldots,n \quad (8.2)
\]
then for $i$ odd,

$$TV(D(g_i)) = D(g_{\omega(i)}), \quad TV(D(g_h)) = D(g_{\mu(i)} h).$$

and for $i$ even,

$$TV(D(g_i)) = D(g_{\mu\pi(i)}), \quad TV(D(g_h)) = D(g_{\mu\pi(i)} h).$$

Since $TV$ is in $\mathcal{P}(D_n, H)$ and $\sigma, \mu \in K_n$ if and only if $\mu\pi_0, \sigma\pi_0 \in K_n$, $TV$ is of type (III). If $T$ is of type (II), it can be shown that $TV$ is of type (III) in the same way. On the other hand, suppose $S$ is in $\mathcal{P}(D_n, H)$ and of type (III), i.e.,

$$S(D(g_i)) = D(g_{\omega(i)}), \quad S(D(g_h)) = D(g_{\omega(i)} h), \quad i \in \Omega_1,$$

$$S(D(g_i)) = D(g_{\omega(i)} h), \quad S(D(g_h)) = D(g_{\omega(i)}), \quad i \in \Omega_2,$$

(8.3)

where $\omega, \omega', \pi, \pi' \in K_n$ or $K_n g$. Then if $\Omega_1 = \Omega_1^0, \Omega_2 = \Omega_2^0$, we have

$$SV^{-1}(D(g_i)) = D(g_{\omega(i)}), \quad SV^{-1}(D(g_h)) = D(g_{\omega'(i)})$$

if $i \in \Omega_1^0$, and

$$SV^{-1}(D(g_i)) = D(g_{\omega'\pi_0^{-1}(i)}), \quad SV^{-1}(D(g_h)) = D(g_{\omega'\pi_0^{-1}(i)} h)$$

if $i \in \Omega_2^0$. Clearly $\omega, \omega', \pi, \pi' \in K_n$ if and only if $\omega, \omega', \pi\pi_0^{-1}, \pi'\pi_0^{-1} \in K_n$; and if we define

$$\sigma(i) = \begin{cases} \omega(i) & \text{if } i \in \Omega_1^0, \\ \pi'\pi_0^{-1}(i) & \text{if } i \in \Omega_2^0, \end{cases}$$

$$\mu(i) = \begin{cases} \omega'(i) & \text{if } i \in \Omega_1^0, \\ \pi\pi_0^{-1}(i) & \text{if } i \in \Omega_2^0, \end{cases}$$

then $\sigma, \mu \in K_n$ or $K_n g$. Now $SV^{-1}$ is in $\mathcal{P}(D_n, H)$ and hence is of type (I). If $\Omega_1 = \Omega_2^0, \Omega_2 = \Omega_1^0$, then for $i \in \Omega_1^0$,

$$SV^{-1}(D(g_i)) = D(g_{\sigma(i)} h), \quad SV^{-1}(D(g_h)) = D(g_{\sigma(i)}),$$

$$SV^{-1}(D(g_i)) = D(g_{\mu(i)} h), \quad SV^{-1}(D(g_h)) = D(g_{\mu(i)}).$$
and for $i \in \Omega_2$,

$$SV^{-1}(D(g_i)) = D(g_{\omega \pi_0^{-1}(i)h}), \quad SV^{-1}(D(gh)) = D(g_{\omega \pi_0^{-1}(i)}h).$$

Again $\omega, \omega', \sigma, \pi' \in K_n$ if and only if $\pi, \pi', \omega \pi_0^{-1}, \omega \pi_0^{-1} \in K_n$; and if we define

$$\tau(i) = \begin{cases} \pi(i) & \text{if } i \in \Omega_1^0, \\ \omega \pi_0^{-1}(i) & \text{if } i \in \Omega_2^0, \end{cases}$$

$$\nu(i) = \begin{cases} \pi'(i) & \text{if } i \in \Omega_1^0, \\ \omega \pi_0^{-1}(i) & \text{if } i \in \Omega_2^0, \end{cases}$$

then both $\tau, \nu$ are in $K_n$ or in $K_ng$. Hence $SV^{-1}$ is of type (II). This proves our assertion. Consequently the group $\mathfrak{F}P(D_n,H)$ is equal to $\mathfrak{F}P(D_n,H) \circ \{I, V\}$.

Since $\mathfrak{T}_i P(D_n,H)$ is a subgroup of index 2 in $\mathfrak{T}_{i+1} P(D_n,H)$ for $i = 1, 2, 3$, where $\mathfrak{T}_4 P(D_n,H) = \mathfrak{T}_i P(D_n,H)$, it follows that

$$\mathfrak{T}_1 P(D_n,H) \triangle \mathfrak{T}_2 P(D_n,H) \triangle \mathfrak{T}_3 P(D_n,H) \triangle \mathfrak{T}_i P(D_n,H)$$

is a composition series. We shall show that $\mathfrak{T}_1 P(D_n,H)$ is a normal subgroup of $\mathfrak{T}_i P(D_n,H)$.

Let $T \in \mathfrak{T}_1 P(D_n,H)$ and $S$ be of type (III) in $\mathfrak{T}_i P(D_n,H)$, i.e., $T$ satisfies (8.2) with $\sigma, \mu \in K_n$ and $S$ satisfies (8.3). If $\omega, \omega', \sigma, \pi' \in K_n$, then for $i \in \Omega_1$,

$$STS^{-1}(D(g_i)) = D(g_{\omega \sigma \omega^{-1}(i)}), \quad STS^{-1}(D(gh)) = D(g_{\omega \sigma \omega^{-1}(i)}h),$$

and for $i \in \Omega_2$,

$$STS^{-1}(D(g_i)) = D(g_{\omega \pi_0 \mu^{-1}(i)}), \quad STS^{-1}(D(gh)) = D(g_{\omega \pi_0 \mu^{-1}(i)}h).$$

If $\omega, \omega', \sigma, \pi' \in K_ng$, then for $i \in \Omega_1$,

$$STS^{-1}(D(g_i)) = D(g_{\omega \pi_0 \mu^{-1}(i)}), \quad STS^{-1}(D(gh)) = D(g_{\omega \pi_0 \mu^{-1}(i)}h),$$

and for $i \in \Omega_2$,

$$STS^{-1}(D(g_i)) = D(g_{\omega \pi_0 \omega^{-1}(i)}), \quad STS^{-1}(D(gh)) = D(g_{\omega \pi_0 \mu^{-1}(i)}h).$$
It is easily seen that for \( \sigma, \mu \in K_n \) and \( \omega, \omega', \sigma, \sigma' \in K_n \) or \( K_n \), we have \( \omega \sigma \omega^{-1}, \omega' \sigma \omega'^{-1}, \sigma' \sigma^{-1} \in K_n \). Hence \( STS^{-1} \) is in \( \mathcal{T}_1 P(D_n, H) \).

Now let \( T \in \mathcal{T}_1 P(D_n, H) \) and \( S \) be of type (II) in \( \mathcal{T}_1 P(D_n, H) \), i.e., \( S \) satisfies (8.1) and \( T \) satisfies (8.2) with \( \sigma, \mu \in K_n \). Then for \( i = 1, 2, \ldots, n \),

\[
STS^{-1}(D(g_i)) = D(g_{\sigma \mu^{-1}(i)}), \quad STS^{-1}(D(gh)) = D(g_{\sigma \sigma^{-1}(i)}h).
\]

It is clearly that for \( \sigma, \mu \in K_n \) and \( \tau, \nu \in K_n \) or \( K_n \), we have \( \nu \mu \nu^{-1}, \tau \sigma \tau^{-1} \in K_n \). Hence \( STS^{-1} \in \mathcal{T}_1 P(D_n, H) \). Also \( \mathcal{T}_1 P(D_n, H) \triangle \mathcal{T}_2 P(D_n, H) \). Thus we conclude that \( \mathcal{T}_1 P(D_n, H) \triangle \mathcal{T}_2 P(D_n, H) \).

Finally, for \( S \in \mathcal{T}_1 P(D_n, H) \) we write \( \mathcal{T}_1 P(D_n, H) \circ S = \mathcal{T}_1 P(D_n, H) \). Then \( \mathcal{T}_1 P(D_n, H) \circ \mathcal{T}_1 P(D_n, H) = (I, U, R, V, UR, UV, RV, URV) \). By a routine computation \( U, R, V, UR, UV \) are of order 2 and \( UR, RV \) are of order 4, i.e., \( \mathcal{T}_1 P(D_n, H) \circ \mathcal{T}_1 P(D_n, H) \) is isomorphic to \( D_4 \).

This completes the proof of Theorem 4.

9. EXAMPLE

Let \( n = 6 \). Then \( g_1 = 1, g_2 = (123456), g_3 = (135)(246), g_4 = (14)(25)(36), g_5 = (153)(264), g_6 = (165432); g_1 h = (26)(35), g_2 h = (12)(35)(45), g_3 h = (13)(46), g_4 h = (14)(23)(56), g_5 h = (15)(24), g_6 h = (16)(25)(43), and

\[
\begin{bmatrix}
x_{1g_1}(1) & x_{1g_2}(1) & x_{1g_3}(1) & x_{1g_4}(1) & x_{1g_5}(1) & x_{1g_6}(1) \\
x_{2g_1}(2) & x_{2g_2}(2) & x_{2g_3}(2) & x_{2g_4}(2) & x_{2g_5}(2) & x_{2g_6}(2) \\
x_{3g_1}(3) & x_{3g_2}(3) & x_{3g_3}(3) & x_{3g_4}(3) & x_{3g_5}(3) & x_{3g_6}(3) \\
x_{4g_1}(4) & x_{4g_2}(4) & x_{4g_3}(4) & x_{4g_4}(4) & x_{4g_5}(4) & x_{4g_6}(4) \\
x_{5g_1}(5) & x_{5g_2}(5) & x_{5g_3}(5) & x_{5g_4}(5) & x_{5g_5}(5) & x_{5g_6}(5) \\
x_{6g_1}(6) & x_{6g_2}(6) & x_{6g_3}(6) & x_{6g_4}(6) & x_{6g_5}(6) & x_{6g_6}(6)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
x_{1g_1h}(1) & x_{1g_2h}(1) & x_{1g_3h}(1) & x_{1g_4h}(1) & x_{1g_5h}(1) & x_{1g_6h}(1) \\
x_{2g_1h}(2) & x_{2g_2h}(2) & x_{2g_3h}(2) & x_{2g_4h}(2) & x_{2g_5h}(2) & x_{2g_6h}(2) \\
x_{3g_1h}(3) & x_{3g_2h}(3) & x_{3g_3h}(3) & x_{3g_4h}(3) & x_{3g_5h}(3) & x_{3g_6h}(3) \\
x_{4g_1h}(4) & x_{4g_2h}(4) & x_{4g_3h}(4) & x_{4g_4h}(4) & x_{4g_5h}(4) & x_{4g_6h}(4) \\
x_{5g_1h}(5) & x_{5g_2h}(5) & x_{5g_3h}(5) & x_{5g_4h}(5) & x_{5g_5h}(5) & x_{5g_6h}(5) \\
x_{6g_1h}(6) & x_{6g_2h}(6) & x_{6g_3h}(6) & x_{6g_4h}(6) & x_{6g_5h}(6) & x_{6g_6h}(6)
\end{bmatrix}
\]
Hence
\[ \varphi_1 = \begin{pmatrix} 123456 \\ 135135 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} 123456 \\ 246246 \end{pmatrix}, \quad \varphi_3 = \begin{pmatrix} 123456 \\ 351351 \end{pmatrix}, \]
\[ \varphi_4 = \begin{pmatrix} 123456 \\ 462462 \end{pmatrix}, \quad \varphi_5 = \begin{pmatrix} 123456 \\ 513513 \end{pmatrix}, \quad \varphi_6 = \begin{pmatrix} 123456 \\ 624624 \end{pmatrix}; \]
\[ \theta_1 = \begin{pmatrix} 123456 \\ 153153 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 123456 \\ 264264 \end{pmatrix}, \quad \theta_3 = \begin{pmatrix} 123456 \\ 315315 \end{pmatrix}, \]
\[ \theta_4 = \begin{pmatrix} 123456 \\ 426426 \end{pmatrix}, \quad \theta_5 = \begin{pmatrix} 123456 \\ 531531 \end{pmatrix}, \quad \theta_6 = \begin{pmatrix} 123456 \\ 642642 \end{pmatrix}. \]

Let \( H \in \mathcal{H} \) and \( T \in \mathcal{D}(D_6, H) \) be of type (III) associated with \( \Omega_1 = \Omega_3 = \{1, 3, 5\}, \quad \Omega_2 = \Omega_4 = \{2, 4, 6\}, \quad \omega = (14)(25)(36), \quad \omega' = (14)(23)(56), \quad \pi = \pi' = (16)(23)(45), \quad \lambda_2 = (14), \quad \lambda_4 = (25), \quad \lambda_6 = (14)(36), \quad \kappa_1 = (14)(25), \quad \kappa_3 = (36), \quad \kappa_5 = 1, \quad \Lambda = J. \) Then

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \lambda_{\omega(i)} \varphi_{\omega(i)}^{-1} \omega' \varphi_i )</th>
<th>( g_{\omega(i)} \lambda_{\omega(i)} \varphi_{\omega(i)}^{-1} \omega' \varphi_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(2356)</td>
<td>(14)(2653)</td>
</tr>
<tr>
<td>3</td>
<td>(1245)(36)</td>
<td>(2356)</td>
</tr>
<tr>
<td>5</td>
<td>(1346)</td>
<td>(14)(2356)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \kappa_{\pi(i)} \theta_{\pi(i)}^{-1} \pi' \varphi_i )</th>
<th>( g_{\pi(i)} \kappa_{\pi(i)} \theta_{\pi(i)}^{-1} \pi' \varphi_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(23)(56)</td>
<td>(132)(465)</td>
</tr>
<tr>
<td>4</td>
<td>(23)(56)</td>
<td>(156)(234)</td>
</tr>
<tr>
<td>6</td>
<td>(14)(2356)</td>
<td>(14)(25)</td>
</tr>
</tbody>
</table>

and

\[
T = \begin{pmatrix}
    x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\
    x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\
    x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\
    x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} \\
    x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\
    x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66}
\end{pmatrix} = \begin{pmatrix}
    x_{43} & x_{44} & x_{45} & x_{46} & x_{41} & x_{42} \\
    x_{53} & x_{54} & x_{55} & x_{56} & x_{51} & x_{52} \\
    x_{63} & x_{64} & x_{65} & x_{66} & x_{61} & x_{62} \\
    x_{13} & x_{14} & x_{15} & x_{16} & x_{11} & x_{12} \\
    x_{23} & x_{24} & x_{25} & x_{26} & x_{21} & x_{22} \\
    x_{33} & x_{34} & x_{35} & x_{36} & x_{31} & x_{32} \\
    x_{43} & x_{44} & x_{45} & x_{46} & x_{41} & x_{42} \\
    x_{53} & x_{54} & x_{55} & x_{56} & x_{51} & x_{52} \\
    x_{63} & x_{64} & x_{65} & x_{66} & x_{61} & x_{62}
\end{pmatrix},
\]

\( x_{ij} \in F. \)
Hence

\[ T \begin{bmatrix} x_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{66} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & x_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{66} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & 0 & 0 \\ 0 & 0 & x_{33} & 0 & 0 & 0 \\ 0 & 0 & x_{55} & 0 & 0 & 0 \end{bmatrix}, \]

\[ T \begin{bmatrix} 0 & x_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & x_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{56} \\ x_{61} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & x_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{34} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{45} \\ 0 & 0 & 0 & 0 & 0 & x_{61} \\ 0 & 0 & 0 & 0 & x_{56} & 0 \end{bmatrix}, \]

\[ T \begin{bmatrix} x_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{26} \\ 0 & 0 & 0 & 0 & x_{35} & 0 \\ 0 & 0 & 0 & x_{44} & 0 & 0 \\ 0 & 0 & x_{53} & 0 & 0 & 0 \\ 0 & x_{62} & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & x_{11} & 0 & 0 \\ 0 & 0 & x_{26} & 0 & 0 & 0 \\ 0 & x_{62} & 0 & 0 & 0 & 0 \\ x_{44} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{53} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{35} \end{bmatrix}, \]

\[ T \begin{bmatrix} 0 & x_{12} & 0 & 0 & 0 & 0 \\ x_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{36} \\ 0 & 0 & 0 & 0 & x_{45} & 0 \\ 0 & 0 & x_{54} & 0 & 0 & 0 \\ 0 & x_{63} & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & x_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{36} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{21} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{45} \\ x_{63} & 0 & 0 & 0 & 0 & 0 \\ 0 & x_{54} & 0 & 0 & 0 & 0 \end{bmatrix}, \]

etc.
REFERENCES


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