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The Dirac Hamiltonian as a Member of a Hierarchy of Matrices*

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"*Of strange combinations out of common things*"—Shelley

We shall give a method of generating a hierarchy of square matrices L_m involving m independent continuous parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that

$$L_m^2 = (\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_m^2)\, I, \tag{1}$$

as m takes values 2, 3,... . We shall show that the L matrices can be expressed as a linear combination of m 'generator' matrices independent of the parameters. The L matrices fall into one of two classes, *saturated* or *unsaturated* according as m is odd or even.

One of the most interesting features of this hierarchy is that the Pauli matrices are recognized to be the generator matrices which saturate L_2, while the Dirac Hamiltonian is an unsaturated L_4.

We start by writing

$$L_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2}$$

and requiring that

$$L_2^2 = \begin{bmatrix} a^2 + bc & (a+d)\, b \\ (a+d)\, c & d^2 + bc \end{bmatrix} = \begin{bmatrix} \lambda_1^2 + \lambda_2^2 & 0 \\ 0 & \lambda_1^2 + \lambda_2^2 \end{bmatrix}. \tag{3}$$

L_2 then falls into *canonical forms* of two distinct types.

Type I

$$L_2 = \begin{bmatrix} 0 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & 0 \end{bmatrix} \tag{4}$$

or

Type II

$$L_2 = \begin{bmatrix} \lambda_2 & \lambda_1 \\ \lambda_1 & -\lambda_2 \end{bmatrix}. \tag{5}$$

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Since the relation $L_2^2 = (\lambda_1^2 + \lambda_2^2) I$ is symmetric in λ_1 and λ_2, we can interchange λ_1 and λ_2 or replace λ_1 or λ_2 by $-\lambda_1$ or $-\lambda_2$, but these operations do not alter the type.

To generate L_3, we can adopt the same procedure as we did to find L_2. We can define

$$L_3 = \begin{bmatrix} 0 & L_2 - i\lambda_3 I \\ L_2 + i\lambda_3 I & 0 \end{bmatrix}, \tag{6}$$

or

$$L_3 = \begin{bmatrix} \lambda_3 I & L_2 \\ L_2 & -\lambda_3 I \end{bmatrix}, \tag{7}$$

or

$$L_3 = \begin{bmatrix} L_2 & \lambda_3 I \\ \lambda_3 I & -L_2 \end{bmatrix}, \tag{8}$$

or

$$L_3 = \begin{bmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{bmatrix}. \tag{9}$$

We notice that while in the first three cases L_3 has double the dimension of L_2, in the last case the dimension of L_3 remains the same as that of L_2. In general, L_{m+1} can be generated from L_m as follows

$$L_{m+1} = \begin{bmatrix} 0 & L_m - i\lambda_{m+1} I \\ L_m + i\lambda_{m+1} I & 0 \end{bmatrix} \tag{10}$$

or

$$L_{m+1} = \begin{bmatrix} \lambda_{m+1} I & L_m \\ L_m & -\lambda_{m+1} I \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} L_m & \lambda_{m+1} I \\ \lambda_{m+1} I & -L_m \end{bmatrix}. \tag{11}$$

In all these three cases, the dimension of L_{m+1} is double that of L_m. However, if L_m is of the form

$$\begin{bmatrix} 0 & L_{m-1} - i\lambda_m \\ L_{m-1} + i\lambda_m & 0 \end{bmatrix}, \tag{12}$$

then L_{m+1} can be generated with the same dimension as L_m by defining

$$L_{m+1} = \begin{bmatrix} \lambda_{m+1} I & L_{m-1} - i\lambda_m \\ L_{m-1} + i\lambda_m & -\lambda_{m+1} I \end{bmatrix}. \tag{13}$$

Thus the most 'economical' way of building L_{m+1} is to 'saturate' the L_m if L_m is 'unsaturated,' i.e., having zeros on the diagonal. Therefore, L_m is of type II while L_{m+1} is saturated. We thus have the table connecting the number of parameters, dimensions of the matrix and type.

Matrix	Number of parameter	Dimension	Character
L_1	1	1	Saturated
L_2	2	2	Unsaturated
·	3	2	Saturated
·	4	$2^2 = 4$	Unsaturated
·	5	$2^2 = 4$	Saturated
L_{2n}	$2n$	2^n	Unsaturated
L_{2n+1}	$2n + 1$	2^n	Saturated

Thus the saturated L matrices involve an odd number of parameters. L_{2n+1} can be obtained from L_{2n-1} bg performing a σ-operation on it defined as

$$L_{2n+1} = \sigma(L_{2n-1}),$$

i.e.,

$$L_{2n+1} = \begin{bmatrix} \lambda_{2n+1}I & L_{2n-1} - i\lambda_{2n}I \\ L_{2n-1} + i\lambda_{2n}I & -\lambda_{2n+1}I \end{bmatrix}. \tag{14}$$

The σ-operation involves the addition of two parameters and the doubling of the dimension.

We shall study these matrices by writing

$$L_{2n+1} = \sum_{i=1}^{2n+1} \lambda_i \mathscr{L}_i^{2n+1}, \tag{15}$$

where $\mathscr{L}_i^{2n+1}$ are $(2n + 1)$ 'generator matrices' independent of λ_i. Then if

$$L_{2n-1} = \sum_{i=1}^{2n+1} \lambda_i \mathscr{L}_i^{2n+1}, \tag{16}$$

we have

$$\mathscr{L}_i^{2n+1} = \begin{bmatrix} 0 & \mathscr{L}_i^{2n-1} \\ \mathscr{L}_i^{2n-1} & 0 \end{bmatrix}, \qquad i = 1, 2,..., 2n - 1, \tag{17}$$

and

$$\mathscr{L}_{2n}^{2n+1} = \begin{bmatrix} 0 & -i\lambda_{2n}I \\ i\lambda_{2n}I & 0 \end{bmatrix}, \qquad \mathscr{L}_{2n+1}^{2n+1} = \begin{bmatrix} \lambda_{2n+1}I & 0 \\ 0 & -\lambda_{2n+1}I \end{bmatrix}. \tag{18}$$

Thus the σ-operation on L_{2n-1} consists of generating $\mathscr{L}_i^{2n+1}$ from $\mathscr{L}_i^{2n-1}$ and adding two matrices $\mathscr{L}_{2n}^{2n+1}$ and $\mathscr{L}_{2n+1}^{2n+1}$.

We now summarize the results we have obtained as follows:

1. A saturated L matrix involves $(2n+1)$, i.e., an odd number of parameters. Its dimension is 2^n. It can be expressed as a linear combination of $(2n+1)$ matrices $\mathscr{L}_i^{2n+1}$, $i = 1, \ldots, (2n+1)$ with λ_i as their coefficients, respectively.

2. An L matrix involving $2n$ (even) parameters is unsaturated. Its dimension is 2^n and it can be expressed as a linear combination of $2n$ matrices, i.e., a set obtained by omitting one of the $2n+1$ matrices.

There are $(2^n)^2$ independent matrices of dimension 2^n and these can be generated either from the $2n+1$ matrices which saturate L or the $2n$ matrices as follows:

The $2n+1$ matrices have the important feature that *their product is 'idempotent.'* More precisely,

$$\mathscr{L}_1^{2n+1}\mathscr{L}_2^{2n+1}\cdots\mathscr{L}_{2n+1}^{2n+1} = i^n I. \tag{19}$$

Hence to generate all the independent matrices we form products of 2, 3,..., n matrices. The product of $(n+r)$ matrices is just equal to the product of $(n-r+1)$ matrices and a numerical factor, and so no independent matrix can be generated by taking products of more than n matrices out of the $2n+1$. The number of independent matrices are

$$\binom{2n+1}{0} + \binom{2n+1}{1} + \cdots + \binom{2n+1}{n} = 2^{2n}. \tag{20}$$

The $2n+1$ matrices anticommute with each other. The idempotent property implies that $2n+1$ is the maximum number of anticommuting matrices in a set of 2^n independent matrices.

If on the other hand we had taken $2n$ matrices, we can form products of $2, 3, 4 \cdots 2n$ matrices and we obtain

$$\binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{2n} = 2^{2n} \tag{21}$$

independent matrices.

The following two properties of the L matrices are immediately noticed.

1. If A is a nonsingular matrix then ALA^{-1} is also an L matrix since

$$\begin{aligned}(ALA^{-1})(ALA^{-1}) &= (AL^2A^{-1})\\ &= A\lambda^2 I A^{-1}\\ &= \lambda^2 I.\end{aligned} \tag{22}$$

2. If Λ is a diagonal matrix with half its diagonal elements equal to $\pm\lambda$ then the matrix

$$U = L + \Lambda I \tag{23}$$

satisfies the equation

$$LU = U\Lambda; \tag{24}$$

i.e., the columns of the matrix U are eigenvectors of L with eigenvalues $\pm\lambda$.

Pauli Matrices and the Dirac Hamiltonian

Starting with L_1, which is a number

$$L_1 = \lambda_1 , \tag{25}$$

L_3 is obtained by a σ-operation as

$$L_3 = \begin{bmatrix} \lambda_3 & \lambda_1 - i\lambda_2 \\ \lambda_1 + i\lambda_2 & -\lambda_3 \end{bmatrix} = \sum_i \lambda_i \mathscr{L}_i^3. \tag{26}$$

If σ_x, σ_y, σ_z are the Pauli matrices defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{27}$$

we recognize that

$$\sigma_x = \mathscr{L}_1^3, \qquad \sigma_y = \mathscr{L}_2^3, \qquad \sigma_z = \mathscr{L}_3^3. \tag{28}$$

Performing a σ-operation on L_3 we obtain L_5.

$$L_5 = \sum \lambda_i \mathscr{L}_i^5. \tag{29}$$

If α_x, α_y, α_z, β are the Dirac matrices and γ_5 the product of the four gamma matrices defined as

$$\alpha_x = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}, \qquad \alpha_y = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}, \qquad \alpha_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix}, \tag{30}$$

we recognize that

$$\mathscr{L}_1 = \alpha_x , \qquad \mathscr{L}_2 = \alpha_y , \qquad \mathscr{L}_3 = \alpha_z ,$$
$$\mathscr{L}_4 = -\beta\gamma_5 , \qquad \mathscr{L}_5 = \beta. \tag{31}$$

Thus

$$L_5 u = \left(\sum \lambda_i \mathscr{L}_i^5\right) u = \pm \lambda u \tag{32}$$

is an eigenvector equation for the saturated matrix L_5. From (24) we immediately write the U matrix solution for L_5 as

$$\begin{bmatrix} \lambda + \lambda_5 & 0 & \lambda_3 - i\lambda_4 & \lambda_1 - i\lambda_2 \\ 0 & \lambda + \lambda_5 & \lambda_1 + i\lambda_2 & -\lambda_3 - i\lambda_4 \\ \lambda_3 + i\lambda_4 & \lambda_1 - i\lambda_2 & -\lambda + \lambda_5 & 0 \\ \lambda_1 + i\lambda_2 & -\lambda_3 + i\lambda_4 & 0 & -\lambda + \lambda_5 \end{bmatrix}, \tag{33}$$

where each solumn is the eigenvector of L_5, corresponding to the eigenvalue $+\lambda$ for the first two columns and $-\lambda$ for the last two columns and

$$\lambda = +(\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2}). \tag{34}$$

If we omit L_4 and λ_4 and we obtain the equation for an unsaturated L_4.

$$L_4 u = (\lambda_1 \mathscr{L}_1 + \lambda_2 \mathscr{L}_2 + \lambda_3 \mathscr{L}_3 + \lambda_5 \mathscr{L}_5)\, u = \lambda u. \tag{35}$$

If we write

$$\lambda_1 = p_x, \qquad \lambda_2 = p_y, \qquad \lambda_3 = p_z, \qquad \lambda_5 = m, \qquad \lambda = E, \tag{36}$$

where p_x, p_y, p_z are the components of momenta, m the mass and E the energy, we obtain the eigenvalue equation for the Dirac Hamiltonian.

If on the other hand we omit L_4, we obtain another unsaturated equation

$$(\boldsymbol{\alpha} \cdot \mathbf{p} - \beta \gamma_5 m)\, u = Eu. \tag{37}$$

If the Dirac equation can be written in the form

$$(\not{p} - m)\, u_D = 0 \tag{38}$$

with

$$\not{p} = \gamma^\mu p_\mu = \gamma \cdot p, \qquad \boldsymbol{\gamma} = \beta \boldsymbol{\alpha}, \qquad \gamma_0 = \beta; \qquad \mu = 0, 1, 2, 3, \tag{39}$$

then the other unsaturated equation can be written as

$$(\not{p} - m)\, u_A = 0 \tag{40}$$

with

$$\not{p} = \gamma^\mu \gamma_5 p_\mu . \tag{41}$$

Solving this equation, we can get the spinor solutions u_A.

We shall now obtain the relation between u_A and u_D. The two spinor solutions u_D for positive energy are given either in

Form I:

$$u_D{}^I = \frac{1}{\sqrt{E+m}} \begin{bmatrix} E+m \\ 0 \\ p_z \\ p_x + ip_y \end{bmatrix}, \quad \frac{1}{\sqrt{E+m}} \begin{bmatrix} 0 \\ E+m \\ p_x - ip_y \\ -p_z \end{bmatrix}; \tag{42}$$

or

Form II:

$$u_D^{II} = \frac{1}{\sqrt{E-m}} \begin{bmatrix} p_z \\ p_x + ip_y \\ E-m \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{E-m}} \begin{bmatrix} p_x - ip_y \\ -p_z \\ 0 \\ E-m \end{bmatrix}. \tag{43}$$

Form II is obtained by operating on Form I with $(-i\gamma_5)$ and replacing m by $-m$. Form B is unsuitable since in the rest system, the normalization factor $1/\sqrt{E-m}$ becomes infinite while the spinor vanishes. On the other hand form B is suitable for negative energy.

In a similar way, we get the two solutions u_A for positive energy in

Form I:

$$u_A{}^I = \frac{1}{\sqrt{E}} \begin{bmatrix} E \\ 0 \\ p_z + im \\ p_x + ip_y \end{bmatrix}, \quad \frac{1}{\sqrt{E}} \begin{bmatrix} 0 \\ E \\ p_x - ip_y \\ -p_z + im \end{bmatrix}; \tag{44}$$

or

Form II:

$$u_A^{II} = \frac{1}{\sqrt{E}} \begin{bmatrix} p_z - im \\ p_x + ip_y \\ E \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{E}} \begin{bmatrix} p_x - ip_y \\ -p_z - im \\ 0 \\ E \end{bmatrix}. \tag{45}$$

We note that interesting feature that both these forms are suitable for the rest system and have the same normalization factor. We also notice that

$$u_A{}^I + iu_A^{II} = \sqrt{\frac{E+m}{E}}\,(1+\gamma_5)\,u_D^I, \tag{46}$$

and $(1 + \gamma_5)\, u_D{}^I$, is a solution of the other unsaturated equation can be seen from the observation that[1]

$$\left(\frac{1 + \gamma_5}{\sqrt{2}}\right)(\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\left(\frac{1 - \gamma_5}{\sqrt{2}}\right) = (\boldsymbol{\alpha} \cdot \mathbf{p} - \beta \gamma_5 m). \tag{47}$$

It should be noticed that $(1 + \gamma_5)/\sqrt{2}$ is nonsingular and has $(1 - \gamma_5)/\sqrt{2}$ as its inverse.

[1] This argument is due to Santhanam.