JOURNAL OF COMBINATORIAL THEORY, Series A 43, 146–148 (1986)

Note

On Orthogonal Latin Squares

C. F. WOODCOCK

The Mathematical Institute, University of Kent,
Canterbury, Kent, CT2 7NF, England

Communicated by the Managing Editors

Received October 20, 1985

It is shown that if $n \equiv 15 \bmod 18$ then it is impossible to find a complete set of pairwise-orthogonal $n \times n$ latin squares each obtained from the addition table of a cyclic group of order n by permuting its rows. This result complements the exclusions which can be derived directly from the Bruck–Ryser theorem. © 1986 Academic Press, Inc.

1. Introduction

In [2] Johnson, Dulmage and Mendelsohn considered sets S of pairwise-orthogonal $n \times n$ latin squares obtained from the addition table of a finite group G of order n by permuting its rows. In particular they remarked that Parker had obtained, by computation, the result that it is impossible to find a complete set S (that is, with $|S| = n-1$) if G is the (cyclic) group of order 15.

We show here that, more generally, if $n \equiv 15 \bmod 18$ then it is impossible to find a complete set S when $G = C_n$, the cyclic group of order n. This result complements the exclusions which can be obtained directly from the Bruck–Ryser theorem.

2. Orthogonal Latin Squares (see [1])

Let Z_n denote the ring of integers modulo n $(n > 2)$ so that Z_n, $+$ is a cyclic group of order n. Throughout Section 2 arithmetic will be in Z_n, that is modulo n.

For each permutation $f \colon Z_n \to Z_n$ there is a corresponding $n \times n$ latin square $M(f)$ with (i, j)-entry

$$M_{ij}(f) = f(i) + j \qquad (0 \leqslant i, j \leqslant n-1). \tag{1}$$

0097-3165/86 $3.00

The following lemma, vacuous unless n is odd, is a direct consequence of the definitions:

LEMMA 1. *Let f, g be permutations of Z_n. Then the latin squares $M(f)$ and $M(g)$ are orthogonal if and only if $f-g$ is also a permutation of Z_n (see* [2, *Sect.* 3] *with a different notation).*

Suppose now that $S=\{M(f_1), M(f_2),..., M(f_{n-1})\}$ were a complete set of pairwise-orthogonal latin squares based on the cyclic group Z_n, $+$ (thus n is odd). Then, by Lemma 1, each f_i-f_j ($1 \leqslant i, j \leqslant n-1$, $i \neq j$) is also a permutation of Z_n. Without loss of generality we may suppose that each $f_i(0)=0$ (otherwise just add a suitable constant to each f_i).

We now define an $(n-2)\times(n-1)$ matrix B with entries in Z_n by putting

$$B_{ij}=f_i(j) \qquad \text{for} \quad 1 \leqslant i \leqslant n-2 \text{ and } 1 \leqslant j \leqslant n-1 \tag{2}$$

(that is we disregard f_{n-1} and the zero of Z_n). Thus, by assumption, each row and each difference of rows (modulo n) of B is a "permutation" of $\{1, 2,..., n-1\}$.

It remains therefore to show that no such matrix B can exist if $n \equiv 15 \bmod 18$.

3. The Contradiction

Let $n \equiv 15 \bmod 18$. Throughout Section 3 arithmetic will be modulo 3, that is we now apply the further reduction map $Z_n \to Z_3$ (a field). Now let f, g: $Z_n \to Z_n$ be mappings and put

$$(f, g)=\sum_{m=0}^{n-1} f(m)\, g(m) \in Z_3. \tag{3}$$

Then the following lemma is a simple consequence of the fact that

$$\sum_{m=0}^{n-1} m^2=(n-1)\, n(2n-1)/6=1. \tag{4}$$

LEMMA 2. *If f, g and $f-g$ are permutations of Z_n then $(f, f)=1$ and $(f, g)=-1$.*

From Lemma 2 it follows that

$$BB^t=A \text{ (say)} \tag{5}$$

where B^t denotes the transpose of B and A has off-diagonal entries -1 and diagonal entries $+1$.

Now since $\text{rank}(A+I_{n-2})=1$ and $\text{trace}(A)=1$, it is easily seen that the eigenvalues of A are $+1$ and -1 (with multiplicity $n-3$). Hence $\det(A)=1$ and so, by (5), $\text{rank}(B)=n-2$. Further the columns of B sum to zero since

$$\sum_{m=1}^{n-1} m = n(n-1)/2 = 0. \tag{6}$$

Therefore if C denotes the matrix obtained from B by deleting the last column, then C is non-singular and

$$C^{-1}B = (I_{n-2} \mid v) = D \text{ (say)} \tag{7}$$

where v is a column vector with entries -1.

Thus, from (5),

$$DD' = C^{-1}A(C^{-1})' \tag{8}$$

while, by direct calculation,

$$DD' = -A. \tag{9}$$

Hence, by equating determinants of the right-hand sides of (8) and (9), we obtain:

$$-1 = \det(C^{-1})^2 \in Z_3^{\times 2} = \{0, 1\} \tag{10}$$

the required contradiction.

References

1. J. Dénes and A. D. Keedwell, "Latin Squares and Their Applications," English Universities Press Ltd., London, 1974.
2. D. M. Johnson, A. L. Dulmage, and N. S. Mendelsohn, Orthomorphisms of groups and orthogonal latin squares I, *Canad. J. Math.* **13** (1961), 356–372.

Printed in Belgium