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COMBINATORIAL PROOFS OF SOME LIMIT FORMULAS INVOLVING ORTHOGONAL POLYNOMIALS*

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The object of this paper is to prove combinatorially several (13 of them) limit formulas relating different families of hypergeometric orthogonal polynomials in Askey's chart classifying them. We first find a combinatorial model for Hahn polynomials which, as pointed out by Foata at the ICM (1983), "contains" models for Jacobi, Meixner, Krawtchouk, Laguerre and Charlier polynomials. Seven limit formulas are proved by "looking at surviving structures" when taking the limit. A simple model, T-structures, is then used to prove (using a different technique) four more limit formulas involving Meixner–Pollaczek, Krawtchouk, Laguerre, Charlier and Hermite polynomials. The theory of combinatorial octopuses (of F. Bergeron) is recalled and two more limits are demonstrated using new models of Meixner–Pollaczek, Laguerre, Gegenbauer and Hermite polynomials.

0. Introduction

In [1], Askey suggested a classification of orthogonal hypergeometric polynomials in a chart (see also [13]) in which the arrows are limit formulas relating two families of polynomials. The object of this paper is to show that many of these identities may be proved combinatorially using appropriate combinatorial models for the two polynomials involved.

In Section 1, we find a model for Hahn polynomials and prove the seven arrows of Fig. 1 (see Theorem 1). A simple model for Meixner–Pollaczek, Krawtchouk and Meixner polynomials is introduced in Section 2 and four more limit formulas (Fig. 3 and Theorem 3) demonstrated. Finally we recall the theory of octopuses [2] and prove two last formulas (Fig 7 and Theorem 5).

Notation

For any finite set A, $S[A]$ denotes the set of permutations of A and $|A|$ the cardinality of A. For an arbitrary permutation σ, $d_i(\sigma)$ is the number of its i-cycles. Let $[n]=\{1, 2, \ldots, n\}$ and $(a)_n = a(a+1)\ldots(a+n-1)$. Let ${}_rF_s$

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denote the generalized hypergeometric series

$$ {}_rF_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; x\right] = \sum_{m\geq 0} \frac{(a_1)_m(a_2)_m \ldots (a_r)_m}{(b_1)_m(b_2)_m \ldots (b_s)_m} \frac{x^m}{m!}. $$

1. Hahn configurations

As pointed out by Foata [5], there is a combinatorial model for Hahn polynomials which contains the combinatorics of several families appearing below them in Askey's chart [1, 13] of hypergeometric orthogonal polynomials. All of these models are consistent with the arrows (limit formulas) between them. We would like to make this more explicit. We will describe the models (first the configurations and then their weights) and then prove these limit formulas. Roughly speaking the models are given by the diagram in Fig. 1 (where $A \twoheadrightarrow B$ means an injective map from A to $A+B$ and $\circlearrowright$ a permutation).

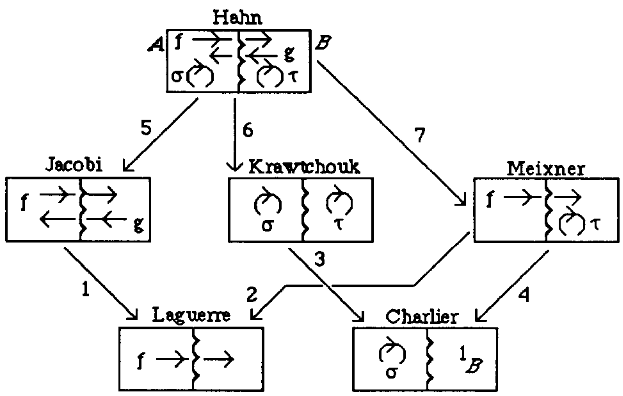


Fig. 1.

(See Theorem 1 for the exact limit formulas.)

Given a pair (A, B) of disjoint finite sets, we define the following combinatorial configurations on (A, B):

(*Charlier configurations*)	$\boldsymbol{C}[A, B] = \boldsymbol{S}[A] \times \{1_B\}$
(*Laguerre configurations*)	$\boldsymbol{L}[A, B] = \{\text{injective map from } A \text{ to } A+B\}$
(*Meixner configurations*)	$\boldsymbol{M}[A, B] = \boldsymbol{L}[A, B] \times \boldsymbol{S}[B]$;
(*Krawtchouk configurations*)	$\boldsymbol{K}[A, B] = \boldsymbol{S}[A] \times \boldsymbol{S}[B]$;
(*Jacobi configurations*)	$\boldsymbol{P}[A, B] = \boldsymbol{L}[A, B] \times \boldsymbol{L}[B, A]$;
(*Hahn configurations*)	$\boldsymbol{Q}[A, B] = \boldsymbol{L}[A, B] \times \boldsymbol{S}[A] \times \boldsymbol{L}[B, A] \times \boldsymbol{S}[B]$
	$= \boldsymbol{M}[A, B] \times \boldsymbol{M}[B, A]$
	$= \boldsymbol{P}[A, B] \times \boldsymbol{K}[A, B]$.

Note that if (f, σ, g, τ) is a Hahn configuration on (A, B) then $(\sigma, 1_B)$ (resp. f, (f, τ), (σ, τ), (f, g)) is a Charlier (resp. Laguerre, Meixner, Krawtchouk, Jacobi) configuration on (A, B).

If $\boldsymbol{T}$ is any of these (in the theory of species (see [9, 10, 12, 24]) they are called 2-species (or bi-species or species of two sorts of points)) and S is any finite set, we write

$$\mathcal{T}[S] = \{(A, B, t) \mid A \cup B = S, A \cap B = \emptyset \text{ and } t \in \boldsymbol{T}[A, B]\}.$$

We say that $\mathcal{T}$ is the species corresponding to $\boldsymbol{T}$.

Let $\mathcal{C}$, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{K}$, $\mathcal{P}$ and $\mathcal{Q}$ be the species corresponding to $\boldsymbol{C}$, $\boldsymbol{L}$, $\boldsymbol{M}$, $\boldsymbol{K}$, $\boldsymbol{P}$ and $\boldsymbol{Q}$ respectively. We make these into weighted species with weights in the ring $\mathbb{Q}[\alpha, \beta, a^{-1}, p, c, x]$ (where α, β, a, p, c are the various parameters of the orthogonal families) by giving every configuration a weight (or valuation) in that ring. The nth-polynomial (or some renormalization of it) will be the total weight of the corresponding configurations on any finite set S with $|S| = n$.

Charlier polynomials: $C_n^{(a)}(x)(a > 0)$ are defined by either:

$$\sum_{n \geq 0} C_n^{(a)}(x) t^n/n! = e^t(1 - t/a)^x. \tag{1.1}$$

$$C_n^{(a)}(x) = {}_2F_0\left[\begin{matrix}-n, -x \\ -\end{matrix}; a^{-1}\right] = \sum_{n \geq i \geq 0} \binom{n}{i} (a^{-1})^i (-x)_i. \tag{1.2}$$

Given $(\sigma, 1_B) \in \boldsymbol{C}[A, B]$, we set

$$w_1(\sigma, 1_B) = (-x)^{\mathrm{cyc}(\sigma)} (a^{-1})^{|A|}, \tag{1.3}$$

where $\mathrm{cyc}(\sigma)$ is the number of cycles of σ.

Notation: For a weighted set X, $|X|$ denotes the total weight of its elements.

Proposition 1. *We have* $|\mathcal{C}[n]| = C_n^{(a)}(x)$.

Proof. The formula follows easily from either (1.1) or (1.2). □

Laguerre polynomials: $L_n^{(\alpha)}(x)$ are defined by either:

$$\sum_{n \geq 0} L_n^{(\alpha)}(x) t^n = (1 - t)^{-1-\alpha} \exp(-xt(1 - t)^{-1}) \tag{1.4}$$

$$n!\, L_n^{(\alpha)}(x) = (\alpha + 1)_n \cdot {}_1F_1\left[\begin{matrix}-n \\ \alpha + 1\end{matrix}; x\right] = \sum_{i+j=n} \binom{n}{i} (\alpha + 1 + j)_i (-x)^j. \tag{1.5}$$

Given $f \in \boldsymbol{L}[A, B]$, we set

$$w_2(f) = (1 + \alpha)^{\mathrm{cyc}(f)} (-x)^{|B|}, \tag{1.6}$$

where $\mathrm{cyc}(f)$ is the number of cycles (in A) of f and $|B|$ is also the number of "chains" of f.

Lemma 1. *We have* $|\boldsymbol{L}[A, B]| = (1+\alpha+|B|)_{|A|}(-x)^{|B|}$.

Proof. *This is a now classical combinatorial lemma (see* [8, *lemma* (2.1)]; [5, *lemma* (3.1)]; [7, *lemma* 3]) *which we will use again and again. See also* [14] *for a short proof using 2-species.* □

Proposition 2. *We have* $|\mathscr{L}[n]| = n!\, L_n^{(\alpha)}(x)$.

Jacobi polynomials: $P_n^{(\alpha,\beta)}(x)$ are defined by either:

$$n!\, P_n^{(\alpha,\beta)}(x) = (\alpha+1)n \cdot {}_2F_1\left[\begin{matrix} -n,\ n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; (1-x)/2\right] \tag{1.7}$$

$$= \sum_{i+j=n} \binom{n}{i} (\alpha+1+j)_i(\beta+1+i)_j((x+1)/2)^i((x-1)/2)^j. \tag{1.8}$$

Given $(f, g) \in \boldsymbol{P}[A, B]$, we set

$$w_3(f, g) = (\alpha+1)^{\mathrm{cyc}(f)}(\beta+1)^{\mathrm{cyc}(g)}((x+1)/2)^{|A|}((x-1)/2)^{|B|}. \tag{1.9}$$

Proposition 3. *We have* $|\mathscr{P}[n]| = n! \cdot P_n^{(\alpha,\beta)}(x)$.

Proof. See [7, 17]. □

Meixner polynomials: $m_n(x; \beta, c)$ are defined by either:

$$\sum_{n\geq 0} m_n(x; \beta, c)t^n = (1-t/c)^x(1-t)^{-x-\beta} \tag{1.10}$$

$$m_n(x; \beta, c) = (\beta)n \cdot {}_2F_1\left[\begin{matrix} -n,\ -x \\ \beta \end{matrix}; 1-c^{-1}\right]$$

$$= \sum_{i+j=n} \binom{n}{i} (\beta+j)_i(-x)_i(c^{-1}-1)^j. \tag{1.11}$$

Given $(f, \tau) \in \boldsymbol{M}[A, B]$, we set

$$w_4(f, \tau) = \beta^{\mathrm{cyc}(f)}(-x)^{\mathrm{cyc}(\tau)}(c^{-1}-1)^{|B|} \tag{1.12}$$

Proposition 4. *We have* $|\mathscr{M}[n]| = m_n(x; \beta, c)$.

Proof. See [6]. □

Krawtchouk polynomials: $K_n(x; p, N)$, for $0 \leq n \leq N$, are defined by:

$$K_n(x; p, N) = {}_2F_1\left[\begin{matrix} -n,\ -x \\ -N \end{matrix}; p^{-1}\right] \quad \text{where} \quad 0<p<1. \tag{1.13}$$

In [23] one finds $(1+qt)^x(1-pt)^{N-x}$, where $p+q=1$, as a generating function for Krawtchouk polynomials. More precisely we have:

$$\sum_{n\geq 0} (-N)_n p^n K_n(x; p, N) t^n/n! = (1+qt)^x (1-pt)^{N-x}. \tag{1.14}$$

This shows that for $(\sigma, \tau) \in \boldsymbol{K}[A, B]$ be setting:

$$w_5(\sigma, \tau) = (-x)^{\mathrm{cyc}(\sigma)}(x-N)^{\mathrm{cyc}(\tau)}(-q/p)^{|A|} \tag{1.15}$$

we obtain a model, $\mathcal{K}$, for Krawtchouk polynomials. More precisely:

Proposition 5. *We have* $|\mathcal{K}[n]| = (-N)_n K_n(x; p, N)$.

Remark. *Several other models for Krawtchouk polynomials are described in* [15].

Hanh polynomials: $Q_n(x; \alpha, \beta, N)$, for $0 \leq n \leq N$, are defined by:

$$Q_n(x; \alpha, \beta, N) = {}_3F_2\left[\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix}; 1\right]. \tag{1.16}$$

We also have the following expression [5, page 1550]

$$(\alpha+1)_n(-N)_n Q_n(x; \alpha, \beta, N) = \sum_{i+j=n} \binom{n}{i} (\alpha+1+j)_i (\beta+1+i)_j (x-N)_i (-x)_j (-1)^j. \tag{1.17}$$

Given $(f, \sigma, g, \tau) \in \boldsymbol{Q}[A, B]$, we set

$$w_6(f, \sigma, g, \tau) = (\alpha+1)^{\mathrm{cyc}(f)}(\beta+1)^{\mathrm{cyc}(g)}(x-N)^{\mathrm{cyc}(\sigma)}(-x)^{\mathrm{cyc}(\tau)}(-1)^{|B|}. \tag{1.18}$$

Proposition. 6. *We have* $|\mathcal{Q}[n]| = (\alpha+1)_n(-N)_n Q_n(x; \alpha, \beta, N)$.

We are now ready to prove combinatorially the following limit formulas:

Theorem 1. *We have*

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2x\beta^{-1}) = L_n^{(\alpha)}(x) \tag{1}$$

$$\lim_{c\to 1} m_n(cx(1-c)^{-1}; \beta, c) = n!\, L_n^{(\beta-1)}(x) \tag{2}$$

$$\lim_{N\to\infty} K_n(x; aN^{-1}, N) = C_n^{(a)}(x) \tag{3}$$

$$\lim_{\beta\to\infty} m_n(x; \beta, a\beta^{-1})/(\beta)_n = C_n^{(a)}(x) \tag{4}$$

$$(\alpha+1)_n \lim_{N\to\infty} Q_n(Nx; \alpha, \beta, N) = n!\, P_n^{(\alpha,\beta)}(1-2x) \tag{5}$$

$$\lim_{t\to\infty} Q_n(x; pt, qt, N) = K_n(x; p, N) \quad \text{where } p+q=1 \tag{6}$$

$$(\beta_n) \lim_{N\to\infty} Q_n(x; \beta-1, \gamma N, N) = m_n(x; \beta, c) \quad \text{where } \gamma = c^{-1}-1. \tag{7}$$

Remark. These are all proved using the following technique: we consider a model for the left hand side (before taking the limit); when we do take the limit, most configurations are killed (i.e. their weight tends to zero), the only surviving configurations (with their limiting weights) form precisely one of our models for the right hand side. This method, due to Foata, was first used in [6] to prove (2) combinatorially. In other words these limits correspond to the seven arrows in Fig. 1. These arrows are obvious forgetful epimorphisms for which in the inverse image of any given configuration everything is killed except precisely one (degenerate) configuration with the right limiting weight.

Proof of (1). By Proposition 3, Jacobi configurations with weight

$$w_3(A, B, f, g) = (\alpha+1)^{\mathrm{cyc}(f)}(\beta+1)^{\mathrm{cyc}(g)}(1-x\beta^{-1})^{|A|}(-x\beta^{-1})^{|B|}$$

form a model for $P_n^{(\alpha,\beta)}(1-2x\beta^{-1})$. We have $\lim_{\beta\to\infty} w_3(A, B, f, g) = 0$ unless $\mathrm{cyc}\, g = |B|$, i.e. $g = 1_B$, in which case $\lim_{\beta\to\infty} w_3(A, B, f, 1_B) = (\alpha+1)^{\mathrm{cyc}(f)}(-x)^{|B|}$ which is the weight $w_2(f)$ of this Laguerre configuration. □

Proof of (2). Using the same technique, (2) is proved in [6]. □

Proof of (3). We will prove $\lim_{N\to\infty}(-N)^{-n}(-N)_n K_n(x; a/N, N) = C_n^{(a)}(x)$ which is equivalent to (3) since $\lim_{N\to\infty}(-N)_n(-N)^{-n} = 1$. By Proposition 5, Krawtchouk configurations on $[n]$ with weight $w_5(A, B, \sigma, \tau) = (-x)^{\mathrm{cyc}(\sigma)}(x-N)^{\mathrm{cyc}(\tau)} \times (1-N/a)^{|A|}$ form a model for $(-N)_n K_n(x; a/N, N)$. If we put an extra multiplicative weight of $(-N)^{-1}$ on each point and let $N\to\infty$, the weight of (A, B, σ, τ) tends to zero unless $\mathrm{cyc}\,\tau = |B|$ (i.e. $\tau = 1_B$). In this case $(A, B, \sigma, 1_B)$ is really a Charlier configuration (A, σ) of weight $(-x)^{\mathrm{cyc}(\sigma)}(1/a)^{|A|} = \lim_{N\to\infty}(-N)^{-n} w_5(A, B, \sigma, 1_B)$. □

Proof of (4). We will prove $\lim_{\beta\to\infty}\beta^{-n} m_n(x; \beta, a/\beta) = C_n^{(a)}(x)$ which is equivalent to (4) since $\lim_{\beta\to\infty}(\beta)_n\beta^{-n} = 1$. If we put an additional multiplicative weight of $1/\beta$ on each point on the Meixner configurations on $[n]$; the weight of (A, B, f, τ) becomes $\beta^{-n} w_4(A, B, f, \tau) = \beta^{-n}\beta^{\mathrm{cyc}(f)}(-x)^{\mathrm{cyc}(\tau)}(\beta/a-1)^{|B|}$. When $\beta\to\infty$ this tends to zero unless $\mathrm{cyc}(f) + |B| = n$ (i.e. $f = 1_A$ and $\tau \in S[B]$) and $(A, B, 1_A, \tau)$ is really a Charlier configuation (B, τ) of weight $(-x)^{\mathrm{cyc}(\tau)}(1/a)^{|B|} = \lim_{\beta\to\infty} B^{-n} w_4(A, B, 1_A, \tau)$. □

Proof of (5). We will prove $\lim_{N\to\infty}(-N)^{-n}(-N)_n(\alpha+1)_n Q_n(Nx; \alpha, \beta, N) = n!\, P_n^{(\alpha,\beta)}(1-2x)$ which is equivalent to (5) since $\lim_{N\to\infty}(-N)^{-n}(-N)_n = 1$. By Proposition 6, Hahn configuations form a model for $(\alpha+1)_n(-N)_n Q_n(Nx; \alpha, \beta, N)$; we put an additional multiplicative weight of $-1/N$ on each of the n points of the configuration, i.e. the weight of the Hahn configuration $(f, \sigma, g, \tau) \in Q[A, B]$ is now $(\alpha+1)^{\mathrm{cyc}(f)}(\beta+1)^{\mathrm{cyc}(g)}(Nx-N)^{\mathrm{cyc}\,\sigma}(-Nx)^{\mathrm{cyc}\,\tau}(-1)^{|B|} \times$

$(-1/N)^{|A|}(-1/N)^{|B|}$. When $N \to \infty$, this configuration dies unless $\mathrm{cyc}(\sigma) = |A|$ and $\mathrm{cyc}(\tau) = |B|$ (i.e., $\sigma = 1_A$, $\tau = 1_B$) in which case $(f, \sigma, g, \tau) = (f, 1_A, g, 1_B)$ is really a Jacobi configuration. Moreover the value of the limit is $(\alpha + 1)^{\mathrm{cyc}(f)} \times (\beta + 1)^{\mathrm{cyc}(g)}(1 - x)^{|A|}(-x)^{|B|}$ which is precisely $w_3(f, g)$ (with x replaced by $(1 - 2x)$). □

Proof of (6). By Proposition 6, setting $\alpha = pt$ and $\beta = (1 - p)t = qt$, we have a combinatorial model for $(pt + 1)_n(-N)_n Q_n(x; pt, qt, N)$. Put an additional multiplicative weight of $(pt)^{-1}$ on each of the n points. The weight of the Hahn configuration $(f, \sigma, g, \tau) \in \boldsymbol{Q}[A, B]$ is $(pt + 1)^{\mathrm{cyc}(f)}(qt + 1)^{\mathrm{cyc}(g)}(x - N)^{\mathrm{cyc}(\sigma)} \times (-x)^{\mathrm{cyc}(\tau)}(1/pt)^{|A|}(-1/pt)^{|B|}$. When $t \to \infty$, the only surviving configurations are those with $\mathrm{cyc}(f) = |A|$ and $\mathrm{cyc}(g) = |B|$, i.e. of the form $(1_A, \sigma, 1_B, \tau)$ with limiting weight $(-x)^{\mathrm{cyc}(\tau)}(-N + x)^{\mathrm{cyc}(\sigma)}(-q/p)^{|B|}$. These are Krawtchouk configurations. Since $\lim_{t\to\infty} (pt + 1)_n(pt)^{-n} = 1$, we have $\lim_{t\to\infty} (-N)_n Q_n(x; pt, qt, N) = (-N)_n K_n(x; p, N) = m_n(x; -N, -p/q)$ (the last equality is the classical relationship between Meixner and Krawtchouk polynomials). □

Proof of (7). If we set

$$w_6(f, \sigma, g, \tau) = \beta^{\mathrm{cyc}(f)}(\gamma N + 1)^{\mathrm{cyc}(g)}(x - N)^{\mathrm{cyc}(\sigma)}(-x)^{\mathrm{cyc}(\tau)}(-1)^{|B|},$$

by Proposition 6, Hahn configurations form a model for $(\beta)_n(-N)_n Q_n(x; \beta - 1, \gamma N, N)$. If we add a multiplicative weight of $-1/N$ on each of the n points and let $N \to \infty$, the only surviving Hahn configurations are those with $g = 1_B$ and $\sigma = 1_A$, i.e. Meixner configurations of weight $w_4(f, \tau) = \beta^{\mathrm{cyc}(f)}(-x)^{\mathrm{cyc}(\tau)}\gamma^{|B|}$. These add up to $m_n(x; \beta, c)$ by Proposition 4. □

Remark. Formula (3) is proved similarly in [2] but using a different model for Krawtchouk polynomials.

We can also give a combinatorial proof of the following:

Proposition 7 [23, *page* 381]. *We have*

$$\lim_{\alpha\to\infty} n!\, \alpha^{-n} L_n^{(\alpha)}(\alpha x) = (1 - x)^n \quad \text{and} \quad \lim_{\alpha\to\infty} n!\, \alpha^{-n} P_n^{(\alpha,\beta)}(x) = 2^{-n}(1 + x)^n.$$

2. Pairs of permutations

Given two finite sets A and B, $\boldsymbol{T}[A, B] = \boldsymbol{S}[A] \times \boldsymbol{S}[B]$. For $(\sigma, \tau) \in \boldsymbol{T}[A, B]$, we set $w(\sigma, \tau) = u^{\mathrm{cyc}(\sigma)}v^{\mathrm{cyc}(\tau)}r^{|A|}s^{|B|}$ where u, v, r and s are formal variables. Moreover for any finite set U, let $T[U] = \{(A, B, \sigma, \tau) \mid A \cup B = U, A \cap B = \emptyset, (\sigma, \tau) \in \boldsymbol{T}[A, B]\}$.

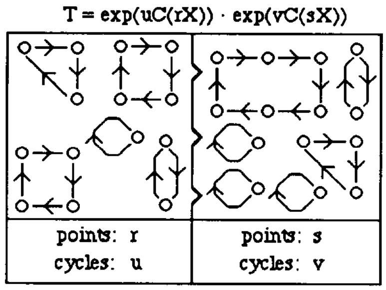


Fig. 2.

Note that $\boldsymbol{T}$ is a 2-species and T the associated species. In fact, these are weighted species with weights in the ring $\mathbb{Z}[u, v, r, s]$. From now on our species will have weights in this ring.

Definition. Let $T_n(r, s; u, v) = T_n = |T[n]|$.

Theorem 2. *We have* $T(t) = (1 - rt)^{-u}(1 - st)^{-v}$ *for the exponential generating series of* T *and* $T_n = \sum_{i+j=n} \binom{n}{i}(u)_i(v)_j r^i s^j$.

Proof. This follows from the fact that T is the species $\exp(uC(rX)) \cdot \exp(vC(sX))$. □

By specifying values for r, s, u and v, we will obtain a combinatorial model for Krawtchouk, Meixner and Meixner–Pollaczek polynomials. A $\boldsymbol{T}$-structure can be thought of as an assembly of blue (those of σ) and red (those of τ) cycles. More generally in Section 3, we will recall the definition of an assembly of blue and red "octupuses" (see [2]) which includes the "classical" models for Hermite, Charlier, Laguerre and Meixner polynomials and gives new one for Krawtchouk and Meixner–Pollaczek polynomials.

Proposition 8. *We have* $T_n(-qp^{-1}, 1; -x, x - N) = (-N)_n K_n(x; p; N)$.

Proof. From Theorem 2 and (1.14). □

Meixner polynomials are also given by (see [3, 6]):

$$m_n(x; \beta, c) = \sum_{i+j=n} \binom{n}{i} (-x)_i (\beta + x)_j c^{-i} \tag{2.1}$$

Proposition 9. *We have* $T_n(c^{-1}, 1; -x, \beta + x) = m_n(x; \beta, c)$.

Proof. From Theorem 2 and (2.1). □

Meixner–Pollaczek polynomials: $\mathcal{P}_n^\lambda(x;\varphi)$ are defined by (see [1, 3]):

$$\sum_{n\geq 0} \mathcal{P}_n^\lambda(x;\varphi)t^n = \frac{(1-te^{i\phi})^{-\lambda-ix}}{(1-te^{-i\varphi})^{\lambda+ix}} \tag{2.2}$$

Proposition 10. *We have* $T_n(e^{i\varphi}, e^{-i\varphi}; \lambda - ix, \lambda + ix) = n!\, \mathcal{P}_n^\lambda(x;\varphi)$.

Proof. From Theorem 2 and (2.2). □

Remark. We also have (see [3]):

$$n!\, \mathcal{P}_n^\lambda(x;\varphi) = (2\lambda)_n e^{in\varphi}{}_2F_1\left[\begin{matrix}-n,\ \lambda+ix \\ 2\lambda\end{matrix};1-e^{-2i\varphi}\right]$$

$$= \sum_{k+l=n} \binom{n}{k}(\lambda+ix)_k(2\lambda)_l(2i\sin\varphi)^k e^{il\varphi}.$$

Proposition 11. *We have* $T_n(2i\sin\varphi, e^{i\varphi}; \lambda+ix, 2\lambda) = n!\, \mathcal{P}_n^\lambda(x;\varphi)$.

Proof. From the above remark and Theorem 2. □

Remark. One can prove combinatorially (see [16]) the following recurrence: $T_{n+1} = (n(r+s)+ru+sv)T_n - rsn(u+v+n-1)T_{n-1}$, from which the 3-terms recurrence formulas for Meixner, Krawtchouk, Meixner–Pollaczek and Jacobi polynomilas are obtained.

The next four limit formulas will be proved combinatorially using the following technique. First choose an epimorphism, $\theta: R \twoheadrightarrow S$, between two species. In other words we need a surjective function $\theta_U: R[U] \twoheadrightarrow S[U]$, for each finite set U, such that for all bijection $f: U \to V$. $S[f]\circ\theta_U = \theta_V \circ R[f]$. Give weight functions v and w, for R and S-structures respectively, so that R becomes a combinatorial model for the left hand of the identity (before taking the limit) and S a model for the right hand side. For any S-structure, say s, compute $|\theta^{-1}(s)| = \sum\{v(r) \mid \theta(r) = s\}$ and show that this tends to $w(s)$.

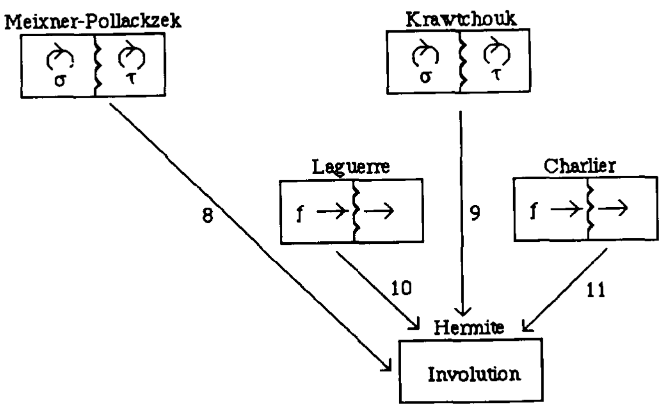


Fig. 3.

Theorem 3. *If $H_n(x)$ is the nth Hermite polynomial, then we have*

$$n! \lim_{\lambda\to\infty} \lambda^{-n/2} \mathscr{P}_n^{\lambda}\left(\frac{\sqrt{\lambda}x - \lambda\cos\varphi}{\sin\varphi}; \varphi\right) = H_n(x) \tag{8}$$

$$(-1)^n\sqrt{2n!\,p^nq^{-n}} \lim_{N\to\infty} \binom{N}{n}^{\frac{1}{2}} K_n(k_x; p, N) = H_n(x) \tag{9}$$

where $k_x = pN + x\sqrt{2pqN}$

$$2^n n! \lim_{\beta\to\infty} \beta^{-n} L_n^{(\beta^2/2)}(\beta^2/2 - \beta x) = H_n(x). \tag{10}$$

Proof of (8). Setting $a = \sqrt{\lambda}$, we write (8) as $\lim_{a\to\infty} a^{-n} n!\, \mathscr{P}_n^{a^2}(y; \varphi) = H_n(x)$ where $y = (ax - a^2\cos\varphi)(\sin\varphi)^{-1}$. Let $\theta : T \twoheadrightarrow S$ be defined by $\theta_U(A, B, \sigma, \tau) = \sigma + \tau \in S[U]$ where $(\sigma+\tau)_{|A} = \sigma$, $(\sigma+\tau)_{|B} = \tau$. By Proposition 10, T is a model for $a^{-n} n!\, \mathscr{P}_n^{a^2}(y; \varphi)$. More precisely: $T_n(e^{i\varphi}a^{-1}, e^{-i\varphi}a^{-1};\ a^2 - iy,\ a^2 + iy) = a^{-n} n!\, \mathscr{P}_n^{a^2}(y; \varphi)$. Recall [14] that "involutions" form a combinatorial model for Hermite polynomials, $H_n(x)$. More precisely: $H_n(x) = |S[n]| = \Sigma\, \{w(\sigma) \mid \sigma \in S[n]\}$ where $w(\sigma)$ is $(2x)^{d_1(\sigma)}(-2)^{d_2(\sigma)}$ if σ is an involution and 0 otherwise. We now computer $A_j = |\theta^{-1}(c_j)|$ where c_j is an arbitrary j-cycle. Since in rebuilding the T-structure above c_j we can either choose a blue (i.e. in A) cycle of weight $a^2 - iy$ where each point has weight $e^{i\varphi}a^{-1}$ or a red (i.e. in B) cycle of weight $a^2 + iy$ where each point has weight $e^{-i\varphi}a^{-1}$, we have:

$$A_1 = (a^2 - iy)e^{i\varphi}a^{-1} + (a^2 + iy)e^{-i\varphi}a^{-1} = 2a^{-1}\,\mathrm{Re}((a^2 - iy)e^{i\varphi}) = 2x$$

$$A_2 = (a^2 - iy)e^{2i\varphi}a^{-2} + (a^2 + iy)e^{-2i\varphi}a^{-2}$$

$$= 2a^{-2}\,\mathrm{Re}((a^2 - iy)e^{2i\varphi}) = -2 + 4a^{-1}x\cos \xrightarrow{a\to\infty} -2$$

and

$$A_k = (a^2 - iy)e^{ki\varphi}a^{-k} + (a^2 + iy)e^{-ki\varphi}a^{-k} \xrightarrow{a\to\infty} 0 \quad \text{for} \quad k \geqslant 3.$$

For $\sigma \in S[n]$, an arbitrary permutation of type $1^{d_1(\sigma)}2^{d_2(\sigma)} \cdots n^{d_n(\sigma)}$, we obviously have $\lim_{a\to\infty} |\theta^{-1}(\sigma)| = \lim_{a\to\infty} \prod A_k^{d_k(\sigma)} = w(\sigma)$. □

Proof of (9). First note that $\lim_{N\to\infty} c_n \binom{N}{n}^{\frac{1}{2}} K_n(k_x; p, N) = \lim_{N\to\infty} \alpha^n(-N)_n \times K_n(k_x; p; N)$ where $\alpha = (2pq^{-1}N^{-1})^{\frac{1}{2}}$. By Proposition 8, we get a combinatorial model for $\alpha^n(-N)_n K_n(k_x, p, N)(= T_n(qp^{-1}\alpha, \alpha; -k_x, k_x - N))$. Choose $\theta : T \twoheadrightarrow S$ as before and compute $B_i = \theta^{-1}(c_i)$ for any i-cycle c_i. We have:

$$\lim_{N\to\infty} B_1 = \lim_{N\to\infty} (-qp^{-1}\alpha)(-k_x) + \alpha(-N + k_x) = 2x$$

$$\lim_{N\to\infty} B_2 = \lim_{N\to\infty} (-qp^{-1}\alpha)^2(-k_x) + \alpha^2(-N + k_x)$$

$$= \lim_{\beta\to\infty} -2 + 2xN^{-\frac{1}{2}}(2pq)^{-\frac{1}{2}} = -2$$

and

$$\lim_{N\to\infty} B_k = \lim_{N\to\infty} (-qp^{-1}\alpha)^k(-k_x) + \alpha^k(-N+k_x) = 0 \quad \text{for} \quad k \geq 3.$$

As before, for an arbitrary σ, $\lim_{N\to\infty} |\theta^{-1}(\sigma)| = w(\sigma)$. □

Proof of (10). Since a Laguerre configuration is a set of cycles (of weight $\alpha+1$) and chains (of weight $-x$), and chains and pointed cycles may be identified, we think of a Laguerre configuration as a permutation some of whose cycles are pointed. Giving to unpointed cycles a weight $\beta^2/2+1$ and to pointed cycles a weight $\beta x - \beta^2/2$, Laguerre configurations are a model for $n!\, L_n^{(\beta^2/2)}(\beta^2/2 - \beta x)$. Let $\theta : \mathscr{L}[n] \to \boldsymbol{S}[n]$ be the map that sends a $\mathscr{L}$-structure on its underlying permutation, i.e. θ forgets the pointing. (In fact $\theta : \mathscr{L} \to \boldsymbol{S}$ is a natural epimorphism of species). Set $D_i = |\theta^{-1}(c_i)|$ where c_i is an arbitrary i-cycle. We can easily see that $D_1 = \beta x + 1$, $D_2 = -\beta^2/2 + 2\beta x + 1$ and, more generally, $D_i = (\beta^2/2+1) + i(\beta x - \beta^2/2)$, because in rebuilding the $\mathscr{L}$-structure from c_i we can either take an i-cycle of weight $\beta^2/2+1$ or a pointed i-cycle (i ways to point it) of weight $\beta x - \beta^2/2$. So that we have $\deg_\beta D_1 = 1$, $\deg_\beta D_2 = 2$ and $\deg_\beta D_k = 2$, for $k \geq 3$. Now we put in our model an additional multiplicative weight of $2/\beta$ on each point: $\mathscr{L}$ is then a model for $2^n n!\, \beta^{-n} L_n^{(\beta^2/2)}(\beta^2/2 - \beta x)$. When we take the limit, $\beta \to \infty$, $|\theta^{-1}(\sigma)| = (2/\beta)^n \prod D_i^{d_i(\sigma)} = \prod (2^i D_i/\beta^i)^{d_i(\sigma)}$ goes to zero unless $\forall i \geq 3$, $d_i(\sigma) = 0$. In this case σ is an involution and the value of the limit is $(2x)^{d_1(\sigma)}(-2)^{d_2(\sigma)}$ which is the weight of this Hermite configuration σ. □

Remark. This last limit formula was first proved combinatorially by Strehl [20, Section 4] in a similar but more complicated way. He calls it the "Italian limit formula" because although it appears as an exercise in Szergö's book [23] it was traced back, by Askey, to three Italian authors much earlier. He has also given a combinatorial proof [21] of the so-called Szegö identities.

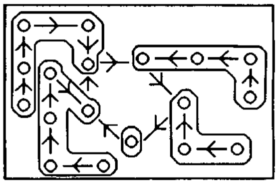
Remark. Applying the well known formula, $n!\, L_n^{(x-n)}(a) = (-a)^n C_n^{(a)}(x)$, to (10) we get:

$$2^n \lim_{\beta\to\infty} (x - \beta/2)^n C_n^{(\beta^2/2 - \beta x)}(\beta^2/2 + n) = H_n(x). \tag{11}$$

3. Octopuses

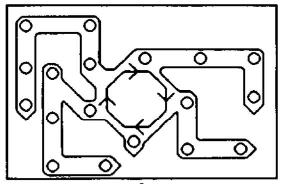
First recall a few results from [2]. A "cyclic permutation of non-empty chains" (Fig. 4) is called an "octopus" (Fig. 5).

Fig. 4.



C ∘ L*

Fig. 5.



O

Now we define the weighted species O of octopuses by the following: if an octopus t has k points on its cycle and has arms (or tentacles) with $j_1, j_2, \ldots, j_k$ points then we set $w(t) = c_k a_{j_1} a_{j_2} \cdots a_{j_k} \in \mathbb{Z}[c_1, c_2, \ldots; a_1, a_2, \ldots]$ where the a's and the c's are just formal variables.

Lemma 2. *We have* $O(t) = \sum_{n\geq 1} n^{-1} c_n (\sum_{k\geq 1} a_k t^k)^n$.

Proof. We have $O(t) = C(t) \circ L^*(t)$, $C(t) = \sum_{n\geq 1} n^{-1} c_n t^n$ and $L^*(t) = \sum_{k\geq 1} a_k t^k$. □

Definition. Let e^O be species of "assemblies of octopuses" where $w\{t_1, t_2, \ldots, t_r\} = \prod w(t_i)$.

Lemma 3. *We have* $(e^O)(t) = \exp\{\sum_{n\geq 1} n^{-1} c_n (\sum_{k\geq 1} a_k t^k)^n\}$.

Definition. Let $B = e^{O_b + O_r}$ be the species of "assemblies of blue (O_b) or red (O_r) octopuses; the weight of a blue octopus being defined using variables, $c_1, c_2, \ldots; a_1, a_2 \ldots,$ and the weight of a red octopus, variables $c'_1, c'_2, \ldots; a'_1, a'_2, \ldots.$

Lemma 4. *We have* $B(t) = \exp\{\sum_{n\geq 1} n^{-1} c_n (\sum_{k\geq 1} a_k t^k)^n + \sum_{n\geq 1} n^{-1} c'_n \times (\sum_{k\geq 1} a'_k t^k)^n\}$.

By choosing specific values for these variables, known models for Hermite, Charlier, Laguerre, Meixner polynomials, and new models for Krawtchouk, Meixner–Pollaczek, Gegenbauer and Tchebicheff (1st and 2nd kind) polynomials are constructed.

Theorem 4. ([2]). *The value of* $|B[n]|$ *is*:

(a) $H_n(x)$ (*Hermite polynomial*)
if $c_1 = 2x$; $a_1 = 1$; $c_j = a_j = 0$ *for* $j \geq 2$; $c'_1 = 0$; $c'_2 = -2$; $a'_1 = 1$; $a'_j = c'_{j+1} = 0$ *for* $j \geq 2$.

(b) $C_n^{(a)}(x)$ (*Charlier polynomial*)
if $c_j = -x$, $\forall j \geq 1$; $a_1 = a^{-1}$; $a_j = 0$ *for* $j \geq 2$; $c'_1 = 1$; $c'_j = 0$ *for* $j \geq 2$; $a'_1 = 1$; $a'_j = 0$ *for* $j \geq 2$.

(c) $n!\, L_n^{(\alpha)}(x)$ (*Laguerre polynomial*)
if $c_j = \alpha + 1$, $\forall j \geq 1$; $a_1 = 1$; $a_j = 0$ *for* $j \geq 2$; $c'_1 = -x$; $c'_j = 0$ *for* $j \geq 2$; $a'_j = 1$, $\forall j \geq 1$.

(d) $m_n(x;\beta, c)$ (*Meixner polynomial of the first kind*)
if $c_j=\beta$, $\forall j \geqslant 1$; $a_1 = 1$; $a_j = 0$ *for* $j \geqslant 2$; $c'_j = -x$, $\forall j \geqslant 1$; $a'_j = c^{-1} - 1$, $\forall j \geqslant 1$.

(e) $M_n(x;\delta, 2a)$ (*Meixner polynomial of the second kind*)
if $c_j = a$, $\forall j \geqslant 1$; $a_1 = -2\delta$; $a_2 = -(1+\delta^2)$; $a_j = 0$ *for* $j \geqslant 3$; $\forall j \geqslant 1$, $c'_{2j} = 0$, $c'_{2j-1} = -ix$, $a'_j = i(-\delta)^{j-1}$ (*where* $i = \sqrt{-1}$).

Proof. For (a) and (b) see [2, 14]; for (c) see [2, 8, 14, 16]; for (d) see [2, 5, 6, 15]. In order to prove (e) recall [3, page 179] that Meixner polynomials of the second kind are defined by:

$$\sum_{n \geqslant 0} M_n(x;\delta, 2a)\frac{t^n}{n!} = (1 + 2\,\delta t + (1+\delta^2)t^2)^{-a} \exp\left\{x \tan^{-1}\left(\frac{t}{1+\delta t}\right)\right\} \tag{2.3}$$

and are related to Meixner–Pollaczed polynomials by [3, page 180]:

$$n!\,\mathscr{P}_n^a(x,\varphi) = (\sin\varphi)^n M_n(2x;\delta, 2a) \qquad \text{where } \delta = \operatorname{cotg}\varphi. \tag{2.4}$$

Now (e) follows from (2.3) by a careful application of Lemma 4. □

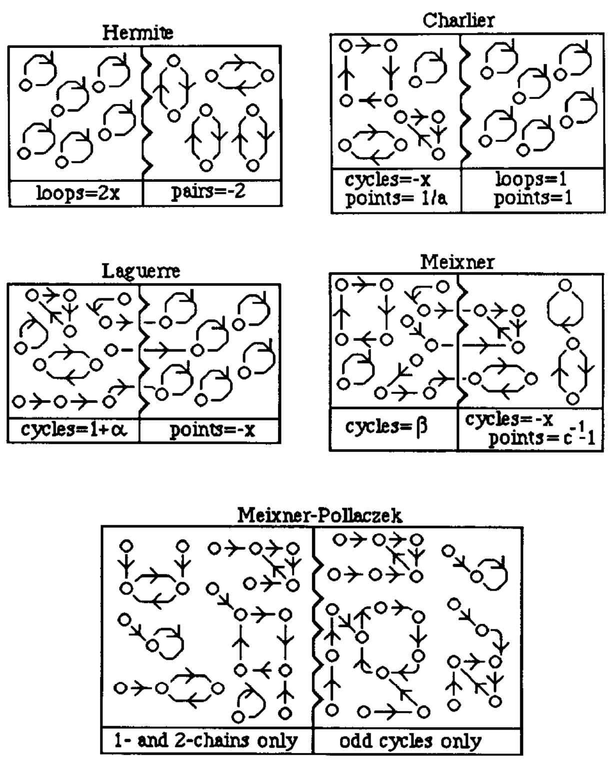


Fig. 6.

Gegenbauer polynomials: $P_n^{(\lambda)}(x)$ are defined by (see [3, 23]):

$$\sum_{n\geqslant 0} P_n^{(\lambda)}(x)t^n = (1-2xt+t^2)^{-\lambda} \tag{2.5}$$

and are related to Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, by:

$$P_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{(\lambda+\frac{1}{2})_n} P_n^{(\lambda-\frac{1}{2},\,\lambda-\frac{1}{2})}(x).$$

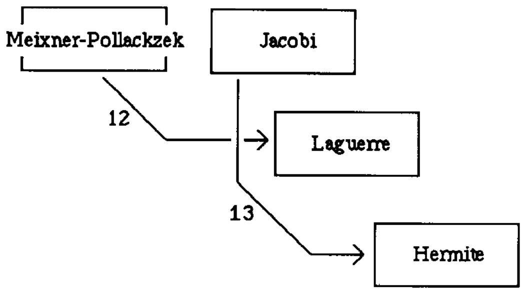


Fig. 7.

Theorem 5. *We have*

$$\lim_{\varphi\to 0} (-1)^n \mathcal{P}_n^{(1+\alpha)/2}\left(\frac{(1+\alpha)(1-\cos\varphi)-x}{2\sin\varphi}\,;\varphi\right) = L_n^{(\alpha)}(x) \tag{12}$$

$$n!\lim_{\lambda\to\infty} \lambda^{-n/2} P_n^{(\lambda)}(x\lambda^{-\frac{1}{2}}) = H_n(x). \tag{13}$$

Before proving (12), we introduce a new model for Laguerre polynomials.

Proposition 12. *We have* $|B[n]| = n!\,L_n^{(\alpha)}(x)$ *if* $c_j = (1+\alpha)/2$, $\forall j\geqslant 1$; $a_1 = 2$, $a_2 = -1$, $a_j = 0$ *for* $j\geqslant 3$; $c_1' = -x$, $c_j' = 0$ *for* $j \geqslant 2$; $a_j' = 1$ *for all* $j\geqslant 1$.

Proof. This follows from Lemma 4 by writing:

$$\sum_{n\geqslant 0} L_n^{(\alpha)}(x)t^n = (1-t)^{-1-\alpha}\exp(-xt(1-t)^{-1})$$
$$= (1-(2t-t^2))^{-(1+\alpha)/2}\exp(-xt(1-t)^{-1}) \quad \square \tag{2.7}$$

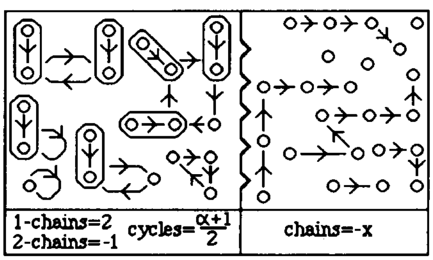


Fig. 8.

Proof of (12). In this case, B is a model for both $(-1)^n n!\,\mathscr{P}_n^a(y,\varphi)$, with $a=(\alpha+1)/2$, $y=(2\sin\varphi)^{-1}((1+\alpha)(1-\cos\varphi)-x)$, and $n!\,L_n^{(\alpha)}(x)$. More explicitly, we know $|B[n]|$ is:

(i) $(-1)^n n!\,\mathscr{P}_n^a(y;\varphi)$ if $\forall j\geqslant 1$, $c_j=a$; $a_1=2\cos\varphi$, $a_2=-1$; $a_j=0$ for $j\geqslant 3$; $\forall j\geqslant 1$, $c'_{2j}=0$, $c'_{2j-1}=-2iy$ and $a'_j=i\sin\varphi(\cos\varphi)^{j-1}$ (by Theorem 4(e) and (2.4)).

(ii) $n!\,L_n^{(\alpha)}(x)$ if $\forall j\geqslant 1$, $c_j=(\alpha+1)/2$; $a_1=2$, $a_2=1$; $a_j=0$ for $j\geqslant 3$; $c'_1=-x$; $c'_j=0$ for $j\geqslant 2$; $a'_j=1$ for $j\geqslant 1$ (by Lemma 4 and (2.7)).

When $\varphi\to 0$, in the model for $(-1)^n n!\,\mathscr{P}_n^a(y;\varphi)$ the only surviving structures (with their limiting weights) are those forming the model for $n!\,L_n^{(\alpha)}(x)$.

Proof of (13). We have $|e^O[n]|=n!\,P_n^{(\lambda)}(x)$ where in the species e^O of "assemblies of octopuses" the weight is defined by setting: $\forall j$, $c_j=\lambda$; $a_1=2x$, $a_2=-1$, $a_j=0$ for $j\geqslant 3$. Which follows from Lemma 3 by writing: $-\lambda\log(1-(2x-t^2))=\sum_{n\geqslant 0} n^{-1}\lambda((2x)t+(-1)t^2)^n$. The rest is similar to the proof of (12), taking e^O with $c_1=1$, $b_1=2x$, $b_2=-1$, and $c_j=b_{j+1}=0$ for $j\geqslant 2$ as a combinatorial model for Hermite polynomials. □

There is nice isomorphism Φ (due to Foata [6]) between the species e^O of "assemblies of octopuses" (i.e. permutations of non-empty chains) and "bicolored permutations".

Let $e^O \overset{\theta}{\twoheadrightarrow} S$ be the composition of Φ with the forgetful map (which forgets the coloring).

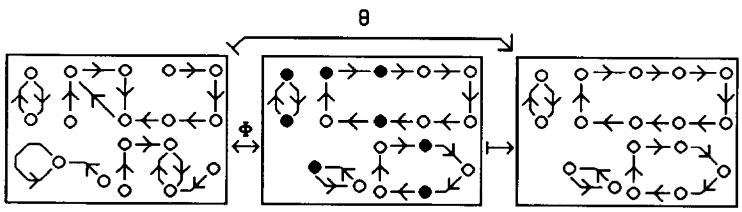


Fig. 9.

Note that limit formulas (8), (10), (11) and (13) can also be proved by computing $|\theta^{-1}(\sigma)|$ for any permutation σ, using $e^{O_b+O_r}$ as a model for $\mathscr{P}_n^a(x;\varphi)$ (see Theorem 4(e)), $L_n^{(\alpha)}(x)$ (see Theorem 4(c)), $K_n(x;p,N)$ (see [2]) and e^O as a model for $P_n^{(\lambda)}(x)$ as above.

4. Conclusion

We now have several combinatorial models for classical orthogonal polynomials appearing in R. Askey's chart (see [14, 15, 16]). Most formulas, including limit formulas and three terms recurrences, can be proved combinatorially using these models which all have a natural combinatorial q-analogue (see [4]).

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