Overrings of Prüfer Domains

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Let $D$ be an integral domain with identity having quotient field $K$. By an overring of $D$ we mean any domain between $D$ and $K$. By a quotient ring of $D$ we mean an overring of $D$ of the form $D_N$ for some nonempty multiplicative system $N$ contained in $D - \{0\}$. We say $D_N$ is a prime quotient ring of $D$ if $N = D - P$ for some proper prime ideal $P$ of $D$ and, following the notation of $[10, \text{p. 228}]$, we write $D_P = D_N$ in this case. We say that $D$ has the QR-property if each overring of $D$ is a quotient ring of $D$ [4]. If for each maximal ideal $M$ of $D$, $D_M$ is a rank-one discrete valuation ring, then $D$ is almost Dedekind. If $\Delta$ is the set of maximal ideals of $D$, we say $D$ has property (#) if for $\Delta_1$ and $\Delta_2$ distinct subsets of $\Delta$ we have $\bigcap_{P \in \Delta_1} D_P \neq \bigcap_{P \in \Delta_2} D_P$. In [3], it was conjectured that an almost-Dedekind domain need not have property (#). The validity of this conjecture is established here by Theorem 3, which states that an almost-Dedekind domain satisfying property (#) is a Dedekind domain. In Section 1 we consider property (#) in an arbitrary integral domain $D$ with identity. In Section 2 we consider the case in which $D$ is a Prüfer domain. The examples of Section 3 show that the results obtained are, in most cases, the best possible.

1. Preliminary Results on Property (#)

In this section $D$ denotes an integral domain with identity having quotient field $K$ and $\Delta$ denotes the set of maximal ideals of $D$. We consider consequences of property (#) on $D$.

**Lemma 1.** $D$ has property (#) if and only if for $P \in \Delta$ and $\Delta_P = \Delta - \{P\}$, $\bigcap_{M \in \Delta_P} D_M \neq D_P$.

**Proof.** The condition is obviously necessary in order that $D$ have property (#). And if the condition holds, let $\Delta_1$ and $\Delta_2$ be distinct subsets of $\Delta$, say,

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$P \in A_1 - A_2$. Then $D_P \supseteq \cap_{M \in D_1} D_M$, but $D_P \nsubseteq \cap_{M \in D_2} D_M \subseteq \cap_{M \in D_3} D_M$. Consequently $\cap_{M \in D_2} D_M \neq \cap_{M \in D_3} D_M$. In particular, $\cap_{M \in A_1} D_M \neq \cap_{M \in A_2} D_M$ and $D$ has property ($\#$).

**Lemma 2.** If for $P \in A$, $P \notin \cup_{M \in D} M$, then $D$ has property ($\#$).

**Proof.** If $P \in P - \cup_{M \in D} M$, then $1/p \notin (\cap_{M \in D} D_M) - D_P$. Thus $D$ has property ($\#$) by Lemma 1.

For $D$ one-dimensional, the condition that $P \notin \cap_{M \in D} M$ is equivalent to $P$’s being the radical of a principal ideal, or to the condition that there is a $P$-primary ideal of $D$ which is principal. Thus by taking $D$ to be a Dedekind domain whose class group is not a torsion group [cf. 4, p. 102; 9, p. 146], we see that the conditions of Lemma 2 are not necessary in order that $D$ have property ($\#$). But for $D$ having the QR-property they are necessary as is shown by the following result.

**Lemma 3.** If $D$ has the QR-property, if $\{P_\alpha\}$ is a set of proper prime ideals of $D$ and if $P$ is a proper prime distinct from each $P_\alpha$, the statements “$P \subseteq \bigcup P_\alpha$” and “$D_P \supseteq \bigcap D_{P_\alpha}$” are equivalent.

**Proof.** In the proof of Lemma 2 we have shown that if $D_P \supseteq \bigcap D_{P_\alpha}$ then $P \subseteq \bigcup P_\alpha$. The converse follows immediately from Proposition 1.2 of [4].

**Corollary 1.** If $D$ has the QR property and is one-dimensional, then $D$ has property ($\#$) if and only if each maximal ideal of $D$ is the radical of a principal ideal.

Corollary 1 shows that for a wide class of almost-Dedekind domains $J$, $J$ has property ($\#$) if and only if $J$ is Dedekind. This result will be proved in general by Theorem 3. Namely, let $J$ be the integral closure of $Z$ in an infinite algebraic number field $K$. $K$ can be expressed as the union of an ascending sequence of finite algebraic number fields; $K = \bigcup K_i$, and $J = \bigcup Z_i$ where $Z_i$, the integral closure of $Z$ in $K_i$, is well known to be a Dedekind domain with a finite class group, hence a domain with the QR-property [2, p. 200], [4, p. 100]. Thus $J$ is one-dimensional and has the QR-property [4, p. 99]. Consequently, if $J$ is almost Dedekind and has property ($\#$) then Corollary 1 shows that if $M$ is maximal in $J$, $(m)$ is $M$-primary for some $m \in M$. By Theorem 1 of [3], $(m) = M^k$ for some positive integer $k$ so that $M$ is invertible. Then by a well-known result of Nakano [7], $J$ is Dedekind.

**Remark.** By Corollary 1.4 of [4], $J$ obtained as above is almost Dedekind if and only if no maximal ideal of $J$ is idempotent. In [8], Nakano gives necessary and sufficient conditions on the sequence $\{K_i\}$ in order that $J$ contain no indempotent maximal ideals.
2. PRÜFER DOMAINS AND PROPERTY (#)

Our notation in this section is as in the first section except that here we always require that $D$ be a PRÜFER domain. The principal result of the section is Theorem 3 which shows that if $D$ is almost Dedekind and has property (#) then $D$ is Dedekind. Our first result is a generalization to PRÜFER domains of Theorem 4 of [3]. Only part (i) of Theorem 1 is used in the remainder of this paper.

**THEOREM 1.** Suppose $D'$ is an overring of $D$, and let $\Omega$ be the set of prime ideals $P$ of $D$ such that $PD' \subseteq D'$. Then

(i) If $M$ is a maximal ideal of $D'$ and if $P = M \cap D$, then $D_P = D_{M'}$ and $M = P D_P \cap D'$. Therefore $D'$ is PRÜFER.

(ii) For $P$ a proper prime ideal of $D$, $P \in \Omega$ if and only if $D_P \supsetneq D'$. Further, $D' = \bigcap_{P \in \Omega} P D_P$.

(iii) If $A'$ is an ideal of $D'$ and if $A = A' \cap D$, then $A' = A D'$.

(iv) $(PD')_{P \in \Omega}$ is the set of proper prime ideals of $D'$.

*Proof.* In (i) we have $D_{M'} = D_{M \cap D} = D_{M'} = D_{P D_P}$ for some prime ideal $Q$ of $D$. Hence $Q = P \cap D_P$ for some prime ideal $P = M \cap D$. Therefore $MD_{M'} = P D_P$ and $D' = (P D_P)_{P \in \Omega}$. Let $P = M \cap D$. Then $P = M \cap D_P$ and $P = M \cap D_P$. Thus $D_{M'} = D_P$ and $M = MD_{M'} \cap D' = PD_P \cap D'$ as asserted.

The proof of (ii) follows by a slight modification of the proof of part (e) of Theorem 4 of [3]. Also the proof in [3] shows that (iv) is valid as soon as (ii) and (iii) hold. Hence to complete the proof we establish (iii). We may assume $(0) \subseteq A' \subseteq D'$. Obviously $AD' \subseteq A'$. Now $A' = D' = (A' D_{M'} \cap D$), $M_a$ running over all maximal ideals of $D'$ [(11, p. 94)]. If $P_a = M_a \cap D$ we have $D_{M_a} = D_P$ by (i). Hence if $x = A'D_{M_a} = A'D_{P_a}$, then $x = a'/v$ for some $a' \in A'$, $v \in D - P_a$. But $A' \subseteq D$ so $a' = a/u$ for some $a \in D$, $u \in D - P_a$. We then have $a = a'u \in A' \cap D = A$ and $x = a/u \in AD_{P_a} = AD'D_{P_a} = AD'D_{M_a}$. We conclude that $A'D_{M_a} = AD'D_{M_a}$, so for each $\alpha$ so that $A' = \bigcap (A'D_{M_a} \cap D') = \bigcap (AD'D_{M_a} \cap D') = AD'$.

**COROLLARY 2.** Suppose $D$ is one-dimensional. If $D$ has property (#), so does each overring of $D$.

*Proof.* Let $D'$ be an overring of $D$, let $M$ be maximal in $D'$ and let $\{M_a\}$ be the set of maximal ideals distinct from $M$. If $P = M \cap D$ and $P_a = M_a \cap D$, then $(P, P_a) \subseteq A$ since $D$ is one-dimensional. Theorem 1 shows that the ideals $P, P_a, P_b$ are distinct for $\alpha \neq \beta$, that $D_{M'} = D_P$, and that $D_{M_a} = D_{P_a}$.
for all $\alpha$. Since $D$ has property (#) we then have $D_{M'} = D_P \cap D_{P\alpha} = \cap D_{M\alpha}$.

**Lemma 4.** Let $D^*$ be the integral closure of $D$ in $L$, an algebraic extension of $K$. Then

(a) $D^*$ is Prüfer.

(b) if $P$ is prime in $D$, if $P^*$ is prime in $D^*$, and if $v$ and $v^*$ are the valuations of $K$ and $L$, respectively, associated with the valuation rings $D_P$ and $D_{P^*}$, respectively, the concepts "$v$ extends $v^*$" and "$P^*$ lies over $P$" are equivalent.

(c) if $L$ is of finite degree over $K$, there are only finitely many primes of $D^*$ lying over a given prime $P$ of $D$.

*Proof.* (a) is proved by Krull in [5, p. 555]; (b) follows from Lemma 1 of [II, p. 24]; then (c) follows from (b) and Corollary 4 [II, p. 27].

**Lemma 5.** Let $v$ be a valuation of a field $L_0$ with valuation ring $D_v$ and let $\{L_i\}^\infty_{i=0}$ be an ascending sequence of finite algebraic extensions of $L_0$. Let $F = \cup_{i=0}^\infty L_i$, let $\mathcal{S} = \{v_\lambda\}_{\lambda \in \Lambda}$ be the set of valuations of $F$ which are extensions of $v$, and let $F_\lambda$ be the valuation ring associated with $v_\lambda$. In order that for each $\lambda \in \Lambda$ we have $F_\lambda \supseteq \cap_{v \neq v_\lambda} F_v$, it is necessary and sufficient that $\mathcal{S}$ be finite.

*Proof.* If $\mathcal{S} = \{V_1, V_2, ..., V_n\}$ is finite, then because $F$ is algebraic over $L_0$, $\mathcal{S}$ is an independent set of valuations. That $F_i \supseteq \cap_{v \neq v_i} F_v$ for each $i$ then follows from the approximation theorem for independent valuations [II, p. 41].

Now suppose $\mathcal{S}$ is not a finite set. $v$ has only finitely many extensions $w_{i1}, ..., w_{ij}$ to $L_1$ and for $\lambda \in \Lambda$, $v_\lambda$ is an extension of some $w_{ij}$. Hence there exists an extension $w_1$ of $v$ to $L_1$ such that $w_1$ has infinitely many extensions to $F$. By induction we obtain a sequence $\{w_i\}_{i=1}^\infty$ such that for each $i$, $w_i$ is a valuation on $L_i$ having infinitely many extensions to $F$ and such that for $i \leq j$, $w_j$ is an extension of $w_i$ to $L_j$. The sequence $\{w_i\}_{i=1}^\infty$ then defines a unique valuation $w$ on $F$ such that $w_i$ is the restriction of $w$ to $L_i$ for all $i$. We show that $F_w \supseteq \cap_{v \neq v_i} F_{v_i}$. Thus let $x \in F - F_w$. For some integer $i$ we then have $x \in L_i - F_w$; hence $w_i(x) < 0$. By assumption there exists an extension $v_{\lambda}$ of $w_i$ to $F$ distinct from $w$. Hence $v_{\lambda}(x) = w_\lambda(x) < 0$ so $x \notin \cap_{v \neq v_i} F_{v_i}$, implying our desired conclusion.

**Corollary 3.** Suppose $D$ is one-dimensional and has property (#). Let $J$ be the integral closure of $D$ in $L$, an algebraic extension of $K$ which may be expressed as an ascending union of finite algebraic extensions of $K$. Then $J$ has property (#) if and only if for each maximal ideal $M$ of $D$, there are only finitely many maximal ideals of $J$ lying over $M$.
Proof. By Lemma 4, J is one-dimensional and Prüfer. Hence if M is a maximal ideal of J there is associated with J_M a valuation w_M of L. Similarly, for P maximal in D there is a valuation v_P of K associated with D_P. By Lemma 4, M ∩ D = P if and only if w_M extends v_P. Lemma 5 then shows that if J has property (♯) each v_P must have only finitely many extensions to L. Hence for P maximal in D, there can be only finitely many maximal ideals of J lying over P.

Conversely, if each maximal ideal of D lies under only finitely many maximal ideals of J, let M be maximal in J and let M_1, ..., M_n be the other maximal ideals of J lying over P = M ∩ D. If {M_i} is the set of all other maximal ideals of J and if \{P_β\} is the set of maximal ideals of D distinct from P, then D_P \neq \bigcap D_β by hypothesis on D. Hence if \( x \in (\bigcap D_β) - D_P \), then \( x \in J_M - (\bigcap M_i) \). Now \( M \neq \bigcap M_i \) so we choose \( t \in (\bigcap M_i) - M \). Since w_M has rank one for each i, there exists a positive integer k such that w_M(t^k) > w_M(x) for each i. For such a k we then have \( t^k x \in (\bigcap M_i) \cap (\bigcap M_i) \), \( t^k x \notin J_M \). Consequently, J has property (♯).

Lemma 6. Suppose D is one-dimensional. A sufficient condition in order that D have property (♯) is that each maximal ideal of D contain an element which is contained in only finitely many maximal ideals of D.

Proof. The proof is quite similar to that of the converse of Corollary 3. Let M be a maximal ideal of D and let \( \{M_α\} = Δ - \{M\} \). Let \( v \) be the valuation associated with D_M and \( v_α \) the valuation associated with D_M. Let m be a nonzero element of M which is contained in only finitely many maximal ideals of D distinct from M. Let this set be \( \{M_i\}_{i=1}^n \). Then for \( t \in (\bigcap M_i) - M \) and for a suitably chosen positive integer k we have \( v_α(t^k/m) \geq 0 \) for each α while \( v(t^k/m) = -v(m) < 0 \). Hence \( t^k/m \in (\bigcap D_α) - D_M \) and D has property (♯) as asserted.

Note. Example 1 of Section 3 shows that a one-dimensional Prüfer domain in which each nonunit is contained in only finitely many maximal ideals need not be almost Dedekind.

Before proving Theorem 2 we establish the following lemma.

Lemma 7. Suppose D is one-dimensional and that the Jacobson radical of D is nonzero. If \( Δ = \{M_β\} \) and if \( M_α \in Δ \) is the radical of an ideal with two generators, then there exists \( m_α \in M_α \) such that \( 1 - m_α \in M_β \) for each \( β ≠ α \).

Proof. Let \( x \) be a nonzero element of the Jacobson radical of D. If \( v_β \) is the valuation associated with D_M for each β, then \( v_β(x) > 0 \) for each β. By hypothesis, there exist \( u, t \in M_α \) such that \( M_α = \sqrt{(u, t)} \). Since \( v_α \) has rank one, there is an integer n such that \( v_α(u^n) > v_α(x), v_α(t^n) > v_α(x) \).
Then if \( B = (t^n, u^n, x), \sqrt{B} = M_\alpha \) and the minimum \( v_\alpha \)-value of an element of \( B \) is \( v_\alpha(x) \). Now \( x \in B \) and \( B \) is invertible since \( D \) is Prüfer. Thus \( (x) = AB \) for some ideal \( A \) of \( D \). \( A \) is also invertible so that \( v_\alpha \) attains its minimal value on \( A \), which in this case must be zero because of our observation regarding the \( v_\alpha \)-values of elements of \( B \). That is, \( A \nsubseteq M_\alpha \). Yet for \( \beta \neq \alpha \), \( AB = (x) \subseteq M_\beta \) while \( B \nsubseteq M_\beta \). It follows that \( A \subseteq \bigcap_{\beta \neq \alpha} M_\beta \). We choose \( a \in A - M_\alpha \). Since \( M_\alpha \) is maximal \( m_\alpha + da = 1 \) for some \( m_\alpha \in M_\alpha \), \( d \in D \). Then for \( \beta \neq \alpha \), \( 1 - m_\alpha = du \in M_\beta \).

**Lemma 8.** Let \( R \) be a commutative ring with identity \( e \) and let \( \sum = \{M_\lambda\}_{\lambda \in \Lambda} \) be the set of maximal ideals of \( R \). If for each \( M_\lambda \in \sum \) there exists \( m_\lambda \in M_\lambda \) such that \( e - m_\lambda \in \bigcap_{\mu \neq \lambda} M_\mu \), then \( \Lambda \) is a finite set.

**Proof.** Suppose \( \Lambda \) is not finite. Then there exists a well-ordering \( < \) of \( \Lambda \) under which \( \Lambda \) has no largest element. Then for \( \lambda \in \Lambda \) we define \( A_\lambda = \bigcap_{\beta > \lambda} M_\beta \). By hypothesis, \( e - m_\lambda \in A_\lambda - M_\lambda \) for each \( \lambda \). Hence \( \{A_\lambda\}_{\lambda \in \Lambda} \) is a chain of proper ideals of \( R \). Then \( A - \bigcup A_\lambda \) is again a proper ideal of \( R \) since \( e \notin A \). But by choice of \( A_\lambda \), \( A \) is not contained in any maximal ideal of \( R \), a contradiction. Hence \( \Lambda \) is finite as asserted.

**Theorem 2.** If \( D \) is one-dimensional and if \( \Delta = \{M_\beta\} \), then given \( M_\alpha \in \Delta \), these statements are equivalent.

(a) \( D_{M_\alpha} \nsubseteq \bigcap_{\beta \neq \alpha} D_{M_\beta} \).

(b) \( M_\alpha \) is the radical of an ideal with two generators.

(c) \( M_\alpha \) is the radical of a finitely generated ideal.

**Proof.** (a) \( \rightarrow \) (b): Let \( \Delta' = \{M_\lambda\} = \Delta - \{M_\alpha\} \), let \( v_\beta \) be the valuation associated with \( D_{M_\beta} \) for each \( \beta \). Since \( D_{M_\alpha} \nsubseteq \bigcap D_{M_\lambda} \), there exist \( a, b \in D \) such that \( v_\alpha(a) < v_\lambda(b) \) and \( v_\alpha(a) \geq v_\lambda(b) \) for each \( \lambda \). Hence \( v_\alpha(b/a) > 0 \) and \( b/a \in M_{D, M_\lambda} \); say \( b/a = s/t \) where \( s \in M_\alpha \), \( t \in D - M_\alpha \). Then \( v_\alpha(t) = 0 < v_\alpha(s) \) and \( v_\lambda(t) \geq v_\alpha(s) \) for each \( \lambda \). We now let \( \Omega' \) be the set of \( M_\lambda \)'s which contain \( s \) and we let \( \Omega = \Omega' \cup \{M_\lambda\} \). \( \Omega \) is the set of maximal ideals of \( D \) containing \( s \). We note that if \( P \in \Omega \) and if \( P \subseteq \bigcup_{T \in \Omega} T \), then \( P \in \Omega \). For if \( P \notin \Omega \)—that is, if \( s \notin P \), then \( P + ds = 1 \) for some \( p \in P \), \( d \in D \). It then follows that \( p \notin T \) for \( T \in \Omega \). Whence \( p \in P - \bigcup_{T \in \Omega} T \). This observation shows that if \( N = D - \bigcup_{T \in \Omega} T \) and if \( D' = D_N \), then \( \{TD'\}_{T \in \Omega} \) is the set of maximal ideals of \( D' \). \( D' \) is one-dimensional Prüfer by Theorem 1 and \( D_{M_\beta}^{D'} = D_{M_\beta} \nsubseteq \bigcap_{T \in \Omega} D_{T}^{D'} = \bigcap_{T \in \Omega} D_T \) by hypothesis. Also if \( M_\beta \in \Omega, \) \( D_{M_\beta} = D_{M_\beta}^{D'} \) implies \( v_\beta \) is the valuation associated with \( D_{M_\beta}^{D'} \). For each such \( M_\beta \in \Omega' \) we then have \( v_\beta(s) > 0 \) while \( v_\alpha(s) = v_\beta(s) = 0 \). This then implies, as in the proof of Lemma 7 and as previously shown in this proof, that there exists \( u \in M_\alpha D' - \bigcup_{T \in \Omega} (TD') \). There is no loss of generality.
in assuming \( u \in M_\alpha \). We now show that the ideal \( B = (s, u) \) in \( D \) has radical \( M_\alpha \). Clearly \( B \subseteq M_\alpha \). If \( M_\beta \in \Omega - \Omega' \), then \( s \notin M_\beta \) so \( B \notin M_\beta \). Moreover, if \( M_\beta \in \Omega' \), then \( u \notin M_\beta \), implying \( u \notin M_\beta \), again implying \( B \notin M_\beta \). Therefore \( \sqrt{B} = M_\alpha \), and (b) holds.

Obviously (b) \( \rightarrow \) (c).

(c) \( \rightarrow \) (b): We suppose \( M_\alpha = \sqrt{B} \) where \( B = (b_1, b_2, \ldots, b_n) \). Since \( B \) is invertible, \( B \supseteq BM_\alpha \). Thus if \( b \in B - BM_\alpha \) then \( B^2 + (b) = [B^2 + (b)]B^{-1}B = [B + (b)B^{-1}]B \). Hence \( [B^2 + (b)] : B \supseteq B + B^{-1}(b) \) and since \( b \notin M_\alpha B_\alpha \), \( B^{-1}(b) \notin M_\alpha \). Then \( \sqrt{R + B^{-1}(b)} \cap M_\alpha \); therefore \( R + B^{-1}(b) = [B^2 + (b)] : B = D \) and \( B = B^2 + (b) \). Hence for each \( b_i \), there exist \( a_{ij} \in B, \ r_i \in D \) such that

\[
b_i = \sum_{j=1}^{n} a_{ij}b_j + r_id
\]

or such that \( \sum_{j=1}^{n} (\delta_{ij} - a_{ij})b_j = r_id \).

If \( \|\delta_{ij} - a_{ij}\| = u \) is the determinant of this system, then from Cramer’s rule we have \( ub_j \in (d) \) for each \( j \). But \( u \) is of the form \( 1 - t \) for some \( t \in B \) so that we have \( b_j - bt \in (d) \) for each \( j \). Consequently, \( B = (t, d) \) and (b) holds.

(b) \( \rightarrow \) (a): We suppose \( M_\alpha = \sqrt{(r, s)} \) and we fix \( x \in M_\alpha, \ x \neq 0 \). If \( \{M_\alpha\} \) is the set of maximal ideals of \( D \) which contain \( x \), if \( N = D - (\bigcup M_\alpha) \), and if \( D' = D_N \), \( D' \) is one-dimensional Prüfer and \( \{M_\alpha D'\} \) is the set of maximal ideals of \( D' \). Hence the Jacobson radical of \( D' \) contains the nonzero element \( x \) and \( M_\alpha D' = \sqrt{(r, s)}D' \). Lemma 8 then shows that there exists \( t_\alpha \in M_\alpha, \ n \in N \) such that \( t_\alpha/n \in M_\alpha D' \), \( (n - t_\alpha)/n = 1 - (t_\alpha/n) \in M_\alpha D' \) for each \( r \neq x \). Thus \( t_\alpha \in M_\alpha - (\bigcap_{r \neq x} M_r) \) and \( n - t_\alpha \in (\bigcap_{r \neq x} M_r) - M_\alpha \). These observations imply that in \( D'' = D/r_{i=1}^{n} \), \( \{M_r D''\}_{r \neq x} \) is the set of maximal ideals. Further \( n - t_\alpha \in \bigcap_{r \neq x} M_r D'' = \sqrt{x}D'' \) since \( x \) is contained in each \( M_\alpha \). Therefore \( (n - t_\alpha)^k \in xD'' \) for some positive integer \( k \), so that \( (n - t_\alpha)^k/x = \xi \in D'' \). Consequently, \( \nu_\alpha(\xi) \geq 0 \) for each \( M_\alpha \) containing \( x, \ r \neq x \). And if \( x \notin M_\beta \in \Delta \), clearly \( \nu_\beta(\xi) \geq 0 \). Moreover, \( \nu_\alpha(\xi) = k\nu_\alpha(n - t_\alpha) - \nu_\alpha(x) = -\nu_\alpha(x) < 0 \). We then have \( \xi \in (\bigcap_{\beta \neq x} D_{M_\beta}) - D_{M_\alpha} \) and (a) is valid.

Remark. That (c) implies (b) in Theorem 2 is a special case of the following more general result:

If \( A \) is an invertible ideal of \( J \), an integral domain with identity, if \( \{M_\alpha\} \) is the collection of maximal ideals of \( J \) containing \( A \), and if \( A \supseteq \bigcup AM_\alpha \), then \( A \) has a basis of two elements. In particular, if \( A \) is primary for a maximal ideal, \( A \) has a basis of two elements.
THEOREM 3. If $D$ is almost Dedekind and has property ($\#$), $D$ is a Dedekind domain.

Proof. Let $M$ be maximal in $D$. By Theorem 2, there exist $u, v \in D$ such that $\sqrt{(u, v)} = M$. Hence $(u, v)$ is $M$-primary, and therefore a power of $M$: $(u, v) = M^k[3, p. 813]$. Since $D$ is almost Dedekind, $(u, v)$ is invertible and hence $M$ is also invertible. Therefore $D$ is Dedekind as asserted.

Note. Theorem 2, Lemma 7, and Lemma 8 imply that if $D$ is one-dimensional and has property ($\#$), then each nonunit of $D$ is contained in only finitely many maximal ideals of $D$. Hence the conditions of Lemma 6 also are necessary in order that a one-dimensional Prufer domain have property ($\#$).

3. EXAMPLES

The first of the following examples exhibits a one-dimensional Prufer domain with infinitely many prime ideals having property ($\#$) which is not almost Dedekind. The second exhibits an almost Dedekind domain such that all but one of its maximal ideals is the radical of a principal ideal, but such that the domain is not Dedekind.

Example 1. Let $A$ be the domain of all algebraic integers and let $\{p_i\}_{i=1}^\infty$ be the sequence of primes of $\mathbb{Z}$. For each $i$ choose a maximal ideal $M_i$ of $A$ lying over $p_i\mathbb{Z}$, let $N = A - (\bigcup_{i=1}^\infty M_i)$, and let $J_1 = A_N$. $A$ may be expressed as the union of an ascending sequence of Dedekind domains with finite class groups; hence $A$ is one-dimensional and has the QR-property. Consequently, $J_1$ has these same two properties. It is straightforward to check that $\{M_iJ_1\}_{i=1}^\infty$ is the set of maximal ideals of $J_1$ and that each nonunit of $J_1$ is contained in only finitely many maximal ideals. Hence Lemma 6 shows that $J_1$ has property ($\#$). But each maximal ideal of $A$ is known to be idempotent, and this property carries over to $J_1$. Hence $J_1$ is not almost Dedekind [3, p. 814].

Example 2. Denote by $\mathbb{F}$ the field of rational numbers and let $\{p_i\}_{i=1}^\infty$ be the sequence of positive primes in $\mathbb{Z}$. We denote by $\omega_i$ a primitive $p_i$th root of unity for each $i$, and we let $L = \mathbb{F}(\omega_1, \omega_2, \ldots)$. Nakano showed in [8, pp. 426-427] that the integral closure $J$ of $\mathbb{Z}$ in $L$ is an almost-Dedekind domain which is not Dedekind. He further shows that given $p$ a fixed prime of $\mathbb{Z}$, there is an ascending sequence $\{L_i\}_{i=1}^\infty$ of finite algebraic extensions of $\mathbb{F}$ such that $L = \bigcup_{i=1}^\infty L_i$ and such that the following holds: there exists a fixed positive integer $t$ such that in $Z_i$, the integral closure of $Z$ in $L_i$, $pZ_i = (P_{i1}P_{i2} \cdots P_{ikt})^t$ where the $P_{ij}$ are distinct maximal ideals of $Z_i$ and where $P_{ij}$ decomposes into a product of at least two distinct primes in $Z_{i+1}$ for all
These conditions imply that the following construction is possible: Let $v$ be the $p$-adic valuation of $I$. There exist distinct extensions $v_1$ and $w_1$ of $v$ to $L_1$. We let $\mu_1$ be any extension of $w_1$ to $L$. There exist distinct extensions $v_2$ and $w_2$ of $v_1$ to $L_2$. Let $\mu_2$ be any extension of $w_2$ to $L$, etc. Let $M_i$ be the center of $\mu_i$ on $J$ and let $M = \bigcup_{i=1}^{\infty} V_i$ where $V_i$ is the center of $v_i$ on $Z_i$. We let $N = J - (\bigcup_{i=1}^{\infty} M_i)$ and we let $J_2 = J_N$. Since $J$ is almost Dedekind, so is $J_2$. We next show that $\{M_2, M_i J_2\}$ is the collection of maximal ideals of $J_2$. Thus suppose $Q$ is a maximal ideal of $J$ such that $Q \subseteq \bigcup_{i=1}^{\infty} M_i$. Then for any $j$, $Q \cap Z_j \subseteq \bigcup_{i=1}^{\infty} (M_i \cap Z_i)$. For $j > i$, $\mu_j$ extends $v_i$. Hence $\{M_i \cap Z_j\}_{i=1}^{\infty} = \{M_1 \cap Z_j, \ldots, M_{j+1} \cap Z_j\}$, the latter enumeration being into distinct maximal ideals. Thus $Q \cap Z_j = M_r \cap Z_j$ for some $r$. Since $\mu_j$ is the unique extension of $w_j$ to $L$ which is finite on $J_2$, it is apparent that if for some $j$, $Q \cap Z_j = M_i \cap Z_j$ where $r < j + 1$, then $Q \cap Z_i = M_r \cap Z_j$ for all $j$ and $Q = (\bigcup_{j=i}^{\infty} Q \cap Z_i) = (\bigcup_{j=i}^{\infty} M_i \cap Z_i) = M_i$. But if $Q \cap Z_i = M_{j+1} \cap Z_i$ for all $j$, then $Q = \bigcup_{j=i}^{\infty} M_{j+1} \cap Z_i = \bigcup_{j=i}^{\infty} V_j = M$. This proves our assertion concerning the maximal ideals of $J_2$.

Because $M_i \cap Z_{i+1} \subseteq V_{i+1} \cup (\bigcup_{j\neq i} M_j \cap Z_{i+1})$, $M_i \subseteq M \cup (\bigcup_{j\neq i} M_j)$. Therefore $M_i J_2$ is the radical of a principal ideal for each $i$. Finally we note that if $x \in M$, then for some $j$, $x \in V_j$. Hence $x \in M_i$ for all $t > j$. Hence $x$ is contained in infinitely many prime ideals of $J_2$, and $J_2$ is not Dedekind.

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