On double centralizer properties between quantum groups and Ariki–Koike algebras

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Abstract
In this paper, the double centralizer properties for tensor spaces as bimodules over quantum groups and Ariki–Koike algebras as well as their relations with Shoji’s algebra \( \mathcal{H}_h \) (see [J. Algebra 226 (2000) 818–856]) are studied. We proved that, under the separation condition (see [J. Reine Angew. Math. 513 (1999) 53–69]), the natural homomorphism from the Ariki–Koike algebra to the endomorphism algebra of tensor space as module over quantum group is surjective. We also show that the natural homomorphism from Shoji’s algebra \( \mathcal{H}_h \) to that endomorphism algebra is always surjective, and \( \mathcal{H}_h \) is actually isomorphic to a direct sum of some matrix algebras over some Hecke algebras of type \( A \).

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Introduction

Let \( R \) be a commutative ring with 1. Let \( A, B \) be two \( R \)-algebras. Let \( M \) be a free \( R \)-module. Suppose that \( M \) is an \( A-B \)-bimodule in the sense that the left \( A \)-action commutes with the right \( B \)-action on \( M \). Then we have two natural algebra homomorphisms as follows:

\[ \varphi : A \longrightarrow \text{End}_B(M), \quad \psi : B \longrightarrow \text{End}_A(M). \]
If both $\varphi$ and $\psi$ are surjective, we call that the triple $(A, B, M)$ satisfies the double centralizer property.

Such double centralizer property plays a central role in many parts of algebraic Lie theory. For example, let $R = \mathbb{C}$, the complex numbers field; $A = \mathbb{C}\text{GL}_m(\mathbb{C})$, the group algebra (over $\mathbb{C}$) for general linear group; $B = \mathbb{C}\mathfrak{S}_n$, the group algebra (over $\mathbb{C}$) for the symmetric group; and $M = V^\otimes n$, where $V$ is a vector space over $\mathbb{C}$ of dimension $m$. Then the triple $(A, B, M)$ satisfies the above double centralizer property, and this is in fact the classical Schur–Weyl duality which relates the representation theory of general linear groups $\text{GL}_m(\mathbb{C})$ with that of symmetric groups $\mathfrak{S}_n$. There is also a quantized version [17,21,28] of this duality which relates the representation theory of quantized enveloping algebras of type $A$ (or $q$-Schur algebras) with that of Hecke algebras of type $A$. For further examples of such kind of duality, see [11,34].

The Ariki–Koike algebra $H$ is a deformation of the group algebra for complex reflection group of type $G(r, 1, n)$ (see [6,9]). It includes the Iwahori–Hecke algebras of type $A$ and type $B$ as special cases. In recent years these algebras have been extensively studied in a number of literatures, see [1–5,7,15,19,23–27,29,31,32]. These algebras are important not only on their own right, but also because their conjectural connections in the work of Broué and Malle (see [9,10]) with modular representations of finite reductive groups over fields of non-defining characteristic.

Recall that the $q$-Schur algebra, which is defined as Hecke endomorphism algebra of certain permutation modules, plays a central role in determining decomposition matrix [14] for modular representations of finite general linear groups. Those permutation modules can also be realized concretely as subspaces of $n$-tensor space, centralized by an action of quantum groups of type $A$. Such realization is crucial as it relates $q$-Schur algebra directly with the Kazhdan–Lusztig theory. Therefore, it seems very interesting to construct a cyclotomic version of $q$-Schur algebra for Ariki–Koike algebras.\footnote{In [15], Dipper, James, and Mathas introduced the so-called cyclotomic $q$-Schur algebra which is quasi-hereditary in the sense of [12]. However, no connections with quantum groups were found for those algebras.}

One way to do this is to look for some kind of duality between quantum group and Ariki–Koike algebra. The existence of such duality is well-known at least in the classical case. Let $m, n, r$ be three positive integers. Let $m_1, \ldots, m_r \in \mathbb{N}$ be such that $m_1 + \cdots + m_r = m$. For each $1 \leq i \leq r$, let $V_i$ be a vector space over $\mathbb{C}$ with basis $\{v^{(i)}_{j_1}, \ldots, v^{(i)}_{j_{m_i}}\}$. Let $V := V_1 \oplus \cdots \oplus V_r$. Let $\mathfrak{S}_{n,r}$ be the complex reflection group of type $G(r, 1, n)$. It is the subgroup of $\text{GL}_n(\mathbb{C})$ generated by

$$s_0 = \varepsilon E_{1,1} + \sum_{j \neq 1} E_{j,j} \quad \text{and} \quad s_i = E_{i,i+1} + E_{i+1,i} + \sum_{j \neq i, i+1} E_{j,j}, \quad 1 \leq i \leq n-1,$$

where $\varepsilon$ is a primitive $r$th root of unity in $\mathbb{C}$ and $E_{i,j}$ is the $n \times n$ matrix with 1 in the $(i, j)$ position and zero elsewhere. Note that the subgroup generated by $s_1, \ldots, s_{n-1}$ is just the symmetric group $\mathfrak{S}_n$ on $n$ letters. Let $g = gl_{m_1} \oplus \cdots \oplus gl_{m_r}$. One can extend the obvious permutation action of $\mathfrak{S}_n$ to that of $\mathfrak{S}_{n,r}$ by letting $s_0$ acts on $V^\otimes n$ as

$$\left(v^{(j_1)}_{i_1} \otimes \cdots \otimes v^{(j_n)}_{i_n}\right)s_0 = \varepsilon^{j_1} v^{(j_1)}_{i_1} \otimes \cdots \otimes v^{(j_n)}_{i_n}.$$
Then we have

(a) the natural left action of $U(g)$ on $V^\otimes_n$ commutes with the right action of $\mathbb{C}\mathfrak{S}_{n,r}$,

(b) the image of $U(g)$ in $\text{End}_{\mathbb{C}}(V^\otimes_n)$ is equal to $\text{End}_{\mathbb{C}\mathfrak{S}_{n,r}}(V^\otimes_n)$, and the image of $\mathbb{C}\mathfrak{S}_{n,r}$ in $\text{End}_{\mathbb{C}}(V^\otimes_n)$ is equal to $\text{End}_{U(g)}(V^\otimes_n)$,

(c) there is an irreducible $U(g)$-$\mathbb{C}\mathfrak{S}_{n,r}$-bimodules decomposition

$$V^\otimes_n = \bigoplus_{\lambda \in \Lambda^{+}_{m_1, \ldots, m_r}(n)} \Delta_\lambda \otimes S_\lambda,$$

where $\Lambda^{+}_{m_1, \ldots, m_r}(n)$ denote the set of all multipartitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of $n$ such that $\ell(\lambda^{(i)}) \leq m_i$ for each $i$, and $\Delta_\lambda$ (respectively $S_\lambda$) denote the irreducible representation for $U(g)$ (respectively for $\mathbb{C}\mathfrak{S}_{n,r}$) corresponding to $\lambda$.

In [8], Ariki, Terasoma, and Yamada gave a $q$-analogue (i.e., between $U_q(g)$ and the Ariki–Koike algebras) of the above duality in the special case where $m_i = 1$ for all $i$. For general $m_i$ ($1 \leq i \leq r$), such a $q$-analogue was given by Sakamoto and Shoji [36], and independently, by the first author of this paper [25]. The key point in both papers is to give a suitable definition of the action of $T_0$ on tensor space $V^\otimes_n$. The action of $T_0$ given in [36] is a more direct generalization of the one given in [8], while the one given in [25] comes from Dipper–James–Mathas’ cellular basis theory for Ariki–Koike algebra. It turns out that these two actions are “essentially” the same (see Section 1 for precise meaning). This is a result in the second author’s PhD thesis (the detailed proof will be published elsewhere), and also obtained independently by N. Sawada and T. Shoji in a recent preprint [37]. Both actions of $T_0$ could be defined on tensor space over arbitrary base field, though the main results (Schur–Weyl reciprocity) in both papers hold only when the base field is chosen such that the algebras in question are semisimple. Note that in the case of type $A$, the double centralizer properties hold in any situation, semisimple or non-semisimple (see [17]). Based on the work of [36], Shoji also defined [35] a new algebra $\mathcal{H}^b$, which becomes a new presentation of the Ariki–Koike algebra in the semisimple case, and derived a Frobenius formula for the characters of the Ariki–Koike algebras.

It is more interesting to investigate some non-semisimple situation. Using Sakamoto–Shoji’s version of the action of $T_0$ on tensor space, Ariki [3, Theorem 3.2] proved that under the separation condition [3, Section 2], the natural homomorphism from quantum group to tensor space as module over Ariki–Koike algebra is always surjective, which gives one half of the double centralizer property.

The main purpose of this article is to give another half of the double centralizer property. We shall use the version in [25] of the action of $T_0$ on tensor space. We prove that under the same separation condition, the homomorphism from the Ariki–Koike algebra to the endomorphism algebra of tensor space as module over Ariki–Koike algebra is also surjective. Moreover, we show that the natural homomorphism from Shoji’s algebra $\mathcal{H}^b$ to that

3 A different proof for these results for $\mathcal{H}^b$ was also given in [37].
endomorphism algebra is always surjective, and \( H^b \) is actually isomorphic to a direct sum of some matrix algebras over some Hecke algebras of type \( A \).

Since the version in [25] of the action of \( T_0 \) on tensor space is "essentially" the same as Sakamoto–Shoji’s version of the action of \( T_0 \) (see Section 1 for precise meaning), the main result of this paper and the result of Ariki [3, Theorem 3.2] indicate that the double centralizer properties always hold in the case of separation condition (for both versions of action of \( T_0 \) on tensor spaces given in [25,36]).

1. Preliminaries

Let \( U_q(\mathfrak{gl}_m) \) be the quantized enveloping algebra of \( \mathfrak{gl}_m \) (with parameter \( q \)) over \( \mathbb{Q}(q) \), where \( q \) is an indeterminate. By definition, \( U_q(\mathfrak{gl}_m) \) is an associative algebra with 1 and has generators

\[
E_i, \quad F_i, \quad K_j^\pm, \quad 1 \leq i \leq m - 1, \quad 1 \leq j \leq m,
\]

and relations

\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = 1 = K_i^{-1} K_i, \\
K_i E_j = q^{\delta_{ij} - \delta_{ij+1}} E_j K_i, \quad K_i F_j = q^{\delta_{ij+1} - \delta_{ij}} F_j K_i, \\
E_i F_j - F_j E_i = \delta_{ij} K_{i,i+1} K_{i,i+1}^{-1} (q - q^{-1}), \\
E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i, \quad \text{if } |i - j| > 1, \\
E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0, \\
F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 = 0, \quad \text{if } |i - j| = 1,
\]

where \( K_{i,i+1} = K_i K_{i+1}^{-1} \). It is a Hopf algebra with comultiplication \( \Delta \) and counit \( \varepsilon \) defined by

\[
\Delta(E_i) = E_i \otimes 1 + K_{i,i+1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_{i,i+1}^{-1} + 1 \otimes F_i, \\
\Delta(K_i) = K_i \otimes K_i, \quad \varepsilon(E_i) = 0 = \varepsilon(F_i), \quad \varepsilon(K_i) = 1.
\]

Let \( A = \mathbb{Z}[q, q^{-1}] \). Let \( U_A \) be the Kostant–Lusztig \( A \)-form [30]. That is, the \( A \)-sub-algebra of \( U_q(\mathfrak{gl}_m) \) generated by

\[
E_i^{(N)} = \frac{E_i^N}{[N]!}, \quad F_i^{(N)} = \frac{F_i^N}{[N]!}, \quad K_j^{\pm 1}, \quad \left[ \frac{K_j}{N} \right] = \prod_{a=1}^{N} \frac{K_j q^{-a+1} - K_j^{-1} q^{a-1}}{q - q^{-1}},
\]

where \( N \in \mathbb{N}, 1 \leq i \leq n - 1, 1 \leq j \leq n \).
Let $V_A$ be a free $A$-module with ordered basis $[v_1, v_2, \ldots, v_m]$. Let $m_1, \ldots, m_r \in \mathbb{N}$ be such that $m_1 + \cdots + m_r = m$. We rewrite the above ordered basis of $V_A$ as $[v^{(1)}_1, \ldots, v^{(1)}_m, \ldots, v^{(r)}_1, \ldots, v^{(r)}_m]$. For each $1 \leq i \leq r$, let $V_{A,i}$ be the free $A$-submodule generated by $\{v_i^{(1)}, \ldots, v_i^{(n_i)}\}$. There is a natural representation of $U_q(\mathfrak{gl}_m)$ on $V := \mathbb{Q}(q) \otimes_A V_A$, i.e., $\rho : U_q(\mathfrak{gl}_m) \rightarrow \text{End} V$, which is defined on generators by

$$\rho(E_i)v_j = \begin{cases} v_{j-1}, & \text{if } j = i + 1, \\ 0, & \text{otherwise}, \end{cases} \quad \rho(F_i)v_j = \begin{cases} v_{j+1}, & \text{if } j = i, \\ 0, & \text{otherwise}, \end{cases} \quad \rho(K_i)v_j = \begin{cases} qv_j, & \text{if } j = i, \\ v_j, & \text{otherwise}. \end{cases}$$

Thus, the $n$-tensor space $V^\otimes n$ is a $U_q(\mathfrak{gl}_m)$-module via the comultiplication $\Delta$.

By [21], $V_A$ naturally becomes an $U_A$-module. For any field $k$ which is an $A$-algebra, let $V_k := k \otimes V_A$, $U_k := k \otimes U_A$. Then $V_k$ naturally becomes an $U_k$-module.

Let $g = \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$. Then $U_q(g)$ can be naturally embedded in $U_q(\mathfrak{gl}_m)$ as a subalgebra generated by $E_i, \ F_i, \ K_j^\pm, \ i \in [1, 2, \ldots, m-1] \setminus \{m_1, m_1 + 1, \ldots, m, \ldots, m \}$.

Similarly, we have the subalgebras $U_{A,1}(g)$ in $U_A$, and $U_k(g)$ in $U_k$. In particular, we get an action of $U_k(g)$ on $V_k$ and hence on $V_k^\otimes n$ for any $A$-algebra $k$.

Let $n, r$ be two positive integers. Let $u_1, \ldots, u_r$ be another $r$ indeterminates. Let $H = H_{n,r} = H(\mathfrak{S}_{n,r})$ be the associative algebra given by generators $T_0, T_1, \ldots, T_{n-1}$ and relations

$$(T_0 - u_1) \cdots (T_0 - u_r) = 0, \quad T_0T_1T_0T_1 = T_4T_0T_1T_0,$$

$$(T_i - q^{-1})T_iT_i + (T_i + q) = 0, \quad \text{for } 1 \leq i \leq n - 1,$$

$$T_i T_{i+1} = T_{i+1} T_i, \quad \text{for } 1 \leq i \leq n - 2,$$

$$T_i T_j = T_j T_i, \quad \text{for } 0 \leq i < j - 1 \leq n - 2.$$  

This algebra is the so-called Ariki–Koike algebra or the Hecke algebra of type $G(r, 1, n)$. It is an algebra defined over $A' := A[u_1, \ldots, u_r]$. Upon setting $q = 1$ and $u_i = \varepsilon^i$ for each $i$ (where $\varepsilon$ is a primitive $r$th root of unity in $\mathbb{C}$), one obtains the group algebra $\mathbb{C}\mathfrak{S}_{n,r}$.

Note that the subalgebra generated by $T_1, \ldots, T_{n-1}$ is isomorphic to the Hecke algebra associated to the symmetric group $\mathfrak{S}_n$. We denote it by $H(\mathfrak{S}_n)$.

There is a right action of $H(\mathfrak{S}_n)$ on $V_A^\otimes n$ which quantizes the classical right $\mathfrak{S}_n$-permutation on $V_A^\otimes n$. We briefly review it here. Let $I(m, n) = \{(i_1, \ldots, i_n) \mid i_t \in [1, 2, \ldots, m], 1 \leq t \leq n\}$. Then the set

$$\{v_I := v_{i_1} \otimes \cdots \otimes v_{i_n} \mid I = (i_1, \ldots, i_n) \in I(m, n)\}$$
forms a basis of $V_A^{\otimes n}$. Now the right $\mathcal{H}(\mathfrak{g})$-action on $V_A^{\otimes n}$ is defined on generators by

$$v_I T_s = \begin{cases} \frac{q}{q-1} v_I, & \text{if } i_a = i_{a+1}, \\ v_{Is}, & \text{if } i_a < i_{a+1}, \\ (q^{-1} - q)v_I + v_{Is}, & \text{if } i_a > i_{a+1}, \end{cases}$$

where $s = (a, a+1)$ and $I = (i_1, \ldots, i_n) s = (i_1, s(i_2), \ldots, i_n s)$. Now the famous quantum Schur–Weyl reciprocity (discovered by Jimbo [28]) states that the left $U_q(\mathfrak{g})$-action on $V_A^{\otimes n}$ commutes with the above right $\mathcal{H}(\mathfrak{g})$-action, and each image of $U_q(\mathfrak{g})$ and of $\mathcal{H}(\mathfrak{g})$ in $\text{End}(V^{\otimes n})$ is the full centralizer algebra of the other.

Let $V_{A'} := A' \otimes_A V_A$. The above Schur–Weyl reciprocity was generalized to the situation between $U_q(\mathfrak{g})$ and $\mathcal{H}_{a,r}$ (in the semisimple case) by Sakamoto and Shoji [36], and independently, by the first author of this paper [25]. The key point in both papers is to give a suitable definition of the action of $T_0$ on tensor space $V^{\otimes n}_A$. In the rest of this section, we shall review these two versions of the action of $T_0$ on $V^{\otimes n}_A$.

To describe the construction given in [36], we need some notations. Following [36], we define a map $b: \{1, 2, \ldots, m\} \to \mathbb{N}$ by $b(j) = i$ whenever $v_j \in V_{A'}$. For each $1 \leq a < n$, we define a linear operator $S_a$ on $V_A^{\otimes n}$ by

$$v_I S_a := \begin{cases} v_I T_a, & \text{if } b(i_a) = b(i_{a+1}), \\ v_{Isa}, & \text{if } b(i_a) \neq b(i_{a+1}), \end{cases}$$

where $I = (i_1, \ldots, i_n)$, $s_a = (a, a+1)$. We now define the action of $T_0$ on $V_{A'}^{\otimes n}$ as follows:

$$v_I \bullet T_0 = u_{b(i_1)} v_I S_1 \cdots S_{n-1} T_{n-1}^{-1} \cdots T_{1}^{-1},$$

where $I = (i_1, \ldots, i_n)$. We call the action $\bullet$ of $T_0$ the modified Sakamoto–Shoji’s action. Our action of $T_i$ ($0 \leq i \leq n-1$) on $V^{\otimes n}_A$ is in fact the converted one from the original Sakamoto–Shoji’s left action in the following sense: if one denote by $\star$ the algebra anti-automorphism of $\mathcal{H}_{a,r}$ defined on generators by $T_i^\star = T_i$ ($0 \leq i \leq n-1$), then any left action of $\mathcal{H}_{a,r}$ on $V^{\otimes n}_A$ can be converted to a right action by defining $v h := h^\star v$ ($v \in V^{\otimes n}, h \in \mathcal{H}_{a,r}$). The original Sakamoto–Shoji’s left action on $V^{\otimes n}_A$ depends on a fixed order of the basis $v_1, \ldots, v_m$. Now we convert the order in a reverse way, such as $v_m, \ldots, v_1$, and using the parameter $q^{-1}$ instead of $q$, we get a modified left action on $V^{\otimes n}_A$. Using the anti-automorphism $\star$, this modified left action is converted to a right action, which is just our modified Sakamoto–Shoji’s (right) action on $V^{\otimes n}_A$. Therefore, all the statements in [36] automatically hold for the present case. In particular, our actions of $T_i$ ($0 \leq i \leq n-1$) naturally extends to a representation of $\mathcal{H}_{a,r}$ on $V^{\otimes n}_A$, such that the Schur–Weyl reciprocity hold between $\mathcal{H}_{a,r}$ and $\mathcal{H}_{a',r'} \otimes_{\mathcal{H}_{a',r'}} U_q(\mathfrak{g})$ and $\mathcal{H}_{a,r} \otimes_{\mathcal{H}_{a',r'}} U_q(\mathfrak{g})$.

Next we are going to describe another way (given in [25]) to define the action of $T_0$ on $V^{\otimes n}_A$. We need some combinatorial notations. A composition of $n$ is a sequence $\lambda := (\lambda_1, \lambda_2, \ldots)$ of non-negative integers such that $|\lambda| = \sum_{i \geq 1} \lambda_i = n$; and $\lambda$ is called
a partition if $\lambda_1 \geq \lambda_2 \geq \cdots$. There are the notions of Young diagrams, dominance order $\triangleright$, (row standard, standard) tableaux, Young subgroups, distinguished right coset representatives, etc. For more details, we refer the readers to [13, 33]. For any composition $\lambda$ of $n$, let $\mathfrak{S}_\lambda$ be the corresponding Young subgroup. We define

$$y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} q^{-\ell(w)} T_w.$$  

Recall that [15] a multicomposition (respectively multipartition) of $n$ (with $r$ components) is an ordered $r$-tuple of compositions (respectively partitions) $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ such that $\sum_{t=1}^r |\lambda^{(t)}| = n$; we write $\lambda \vdash n$ (respectively $\lambda \vdash n$). For each multicomposition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of $n$, the associated $r$-tuple $a = (a_1, \ldots, a_r)$, where $a_k = |\lambda^{(k)}|$ for each $k$, is a composition of $n$. For multicompositions, there are also the notions of Young diagrams, dominance order $\triangleright$, (row standard, standard) tableaux, Young subgroups, distinguished right coset representatives, etc. For their definitions, we refer the readers to [15].

For any multicomposition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of $n$, we denote the associated composition $(\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_1^{(r)}, \lambda_2^{(r)}, \ldots)$ again by $\lambda$. Let $t^\lambda$ be the standard $\lambda$-tableau in which the numbers 1, 2, ..., $n$ appear in order along the rows of the first component, and then along the second component, and so on. For any row standard $\lambda$-tableau $t$, let $d(t) \in \mathfrak{S}_n$ be such that $t^\lambda d(t) = t$, then $d(t)$ is a distinguished right coset representative of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$, and $d(t)^{-1}$ is a distinguished left coset representative of $\mathfrak{S}_\lambda$ in $\mathfrak{S}_n$.

Following [6,15], we define the Murphy operators $L_1, L_2, \ldots, L_n$ by $L_m = T_{m-1} \cdots T_1 T_0 T_1 \cdots T_{m-1}$ for $m = 1, 2, \ldots, n$. For any composition $a = (a_1, \ldots, a_r)$ of $n$, let

\begin{equation}
\begin{align*}
u^a_+ &:= \prod_{j=1}^{r-1} \prod_{m=1}^{a_j+\cdots+a_j} (L_m - u_{j+1}), \\
u^a_- &:= \prod_{j=1}^{r-1} \prod_{m=1}^{a_{j+1}+\cdots+a_r} (L_m - u_j).
\end{align*}
\end{equation}

Note that the element $q$, $T_i$ (where $i \neq 0$) in [15]’s notations are just $q^2$, $\bar{T}_i := -q T_i$ in our notations. Let $\bar{T}_w := (-q)^{\ell(w)} T_w$ for any $w \in \mathfrak{S}_n$, where $\ell(w)$ is the usual length function defined on $\mathfrak{S}_n$.

Let $*$ be the algebra antiautomorphism of $\mathcal{H}_{n,r}$ determined by $T_i^* = T_i$ for all $i$ with $0 \leq i < n$. For any multicomposition $\lambda$ of $n$, let $a$ be its associated $r$-tuple. We define $m_\lambda = \nu^a_+ y_\lambda$, $M^\mu = m_\lambda \mathcal{H}_{n,r}$. For any row standard $\lambda$-tableaux $s$ and $t$, we define $m_{st} = T_{d(t)} m_\lambda T_{d(t)}$. Let $N^\mu$ be the submodule spanned by

$$\left\{ m_{st} \mid s \text{ and } t \text{ are standard } \mu \text{-tableaux for some multipartition } \mu \text{ of } n \text{ with } \mu \triangleright \lambda \right\}.$$
Let $\overline{N^\lambda}$ be the submodule spanned by
\[ \left\{ m_{st} \mid s \text{ and } t \text{ are standard } \mu\text{-tableaux for some multipartition } \mu \text{ of } n \text{ with } \mu \triangleright \lambda \right\}. \]

We have the following proposition.

**Proposition 1.2** [25]. (1) The set
\[ \left\{ m_{st} \mid s \text{ and } t \text{ are standard } \lambda\text{-tableaux for some multipartition } \lambda \text{ of } n \right\} \]
forms a cellular basis (in the sense of [22]) of $\mathcal{H}_{n,r}$.

(2) Suppose that $\lambda$ is a multicomposition of $n$. Then both $N^\lambda$ and $\overline{N^\lambda}$ are two-sided ideals of $\mathcal{H}_{n,r}$.

For any composition $\lambda$ of $n$, we denote by $\ell(\lambda)$ the number of nonzero parts of $\lambda$. Let $\Lambda(m, n)$ (respectively $\Lambda^+(m, n)$) be the set of all compositions (respectively partitions) $\lambda$ of $n$ such that $\ell(\lambda) \leq m$. Let $D_\lambda$ be the set of all distinguished right coset representatives of $S_\lambda$ in $S_n$. For any composition $\lambda \in \Lambda(m, n)$, we define
\[ I_\lambda := (1, \ldots, 1, 2, \ldots, 2, \ldots, m, \ldots, m) \in I(m, n). \]

Let $v_\lambda = v_{I_\lambda}$, and $v_{\lambda,d} = v_{I_\lambda,d}$ for $d \in D_\lambda$. Then the set $\{ v_{\lambda,d} \mid \lambda \in \Lambda(m, n), d \in D_\lambda \}$ forms an $A$-basis of $V_A^\otimes n$.

**Lemma 1.3** [20, (2.1)]. The map $\varphi : \bigoplus_{\lambda \in \Lambda(m, n)} y_\lambda \mathcal{H}(S_n) \to V_A^\otimes n$ which sends $y_\lambda T_d$ to $v_{\lambda,d} = v_{I_\lambda,d}$ for each $d \in D_\lambda$ extends linearly to a right $\mathcal{H}(S_n)$-module isomorphism.

For any $a \in \Lambda(m, n)$, we define
\[ 1_a := \left( (1^{e_1}), (1^{e_2}), \ldots, (1^{e_m}) \right). \]

For any multicomposition $\lambda$ of $n$, let $a$ be its associated $r$-tuple. We define
\[ N^a := N^{1_a}, \quad \tilde{N}^a := \sum_{b \in \Lambda(m, n)} N^b. \]

Then both $N^a, \tilde{N}^a$ are two-sided ideals of $\mathcal{H}_{n,r}$. Set
\[ \tilde{M}^\lambda := M^\lambda \cap \tilde{N}^a, \quad \tilde{M}^\lambda := M^\lambda / \tilde{M}^\lambda. \]

We have the following theorem.
Theorem 1.4 [25, (3.3)]. Let $\mathcal{H}_{A'}(\mathfrak{S}_n) := A' \otimes_A \mathcal{H}(\mathfrak{S}_n)$. Then there is a natural right $\mathcal{H}(\mathfrak{S}_n)_{A'}$-module isomorphism
\[ \tilde{M}^\lambda \downarrow_{\mathcal{H}_{A'}(\mathfrak{S}_n)} \cong y_\lambda \mathcal{H}_{A'}(\mathfrak{S}_n), \]
which maps $u_\mathbf{a} + y_\lambda + \tilde{M}^\lambda$ (where $\mathbf{a}$ is the associated $r$-tuple of $\lambda$) to $y_\lambda$.

Recall the positive integers $m_1, \ldots, m_r$, where $m_1 + \cdots + m_r = m$. We denote by $\Lambda_{m_1, \ldots, m_r}(n)$ (respectively $\Lambda_{m_1, \ldots, m_r}(n)$) the set of all multicompositions (respectively multipartitions) $\lambda = \langle \lambda^{(1)}, \ldots, \lambda^{(r)} \rangle$ of $n$ such that $\ell(\lambda^{(i)}) \leq m_i$ for each $i$. Then there is an obvious bijection between $\Lambda_{m_1, \ldots, m_r}(n)$ and $\Lambda(m, n)$.

Theorem 1.5 [25, (3.4)]. There is a right $\mathcal{H}_{A'}(\mathfrak{S}_n)$-module isomorphism
\[ V_{\mathcal{H}_{A'}(\mathfrak{S}_n)} \cong \bigoplus_{\lambda \in \Lambda_{m_1, \ldots, m_r}(n)} \tilde{M}^\lambda \downarrow_{\mathcal{H}_{A'}(\mathfrak{S}_n)}, \]
In particular, the above identification gives rise to a natural right $\mathcal{H}_{n,r}$-structure on $V_{\mathcal{H}_{A'}(\mathfrak{S}_n)}$, extending its original right $\mathcal{H}_{A'}(\mathfrak{S}_n)$-structure. It follows from [25] that by using this version of action $T_0$ on $V_{\mathcal{H}_{A'}(\mathfrak{S}_n)}$, there is a Schur–Weyl reciprocity between $Q(q, u_1, \ldots, u_r) \otimes_{\mathbb{Q}(q)} U_q(\mathfrak{g})$ and $Q(q, u_1, \ldots, u_r) \otimes_{\mathbb{Q}(q)} \mathcal{H}_{n,r}$. Now one gets (in both [36] and [25]) commuting actions of $U_{A'}(\mathfrak{g}) := U(\mathfrak{g}) \otimes A'$ and $\mathcal{H}(\mathfrak{S}_n, \mathfrak{A})$ on $V_{A'}^{\otimes n}$. For any field $k$ which is an $A'$-algebra, we get commuting actions of $U_k(\mathfrak{g})$ and $\mathcal{H}_k(\mathfrak{S}_n, \mathfrak{A})$ on $V_{A'}^{\otimes n}$. The results on Schur–Weyl reciprocity in [25,36] actually hold whenever both $U_k(\mathfrak{g})$ and $\mathcal{H}_k(\mathfrak{S}_n, \mathfrak{A})$ are semisimple. It is more interesting to ask if the double centralizer property (i.e., (a) and (b)) hold in non semisimple situation. Actually the same things did hold in type $A$ situation.

Definition 1.6 [3]. Let
\[ f_\mathbf{u}(q, u_1, \ldots, u_r) := \prod_{1 \leq i < j \leq r} \prod_{|a| < cn} (q^{2a}u_i - u_j). \]
For any $A'$-algebra $k$, we call the condition $f_\mathbf{u}(q, u_1, \ldots, u_r) \in k^\times$ separation condition.

The following result is due to Ariki [3], where he used Sakamoto–Shoji’s action of $T_0$ on $V_{\mathfrak{A}', n}^{\otimes n}$.

Theorem 1.7 [3, (3.2)]. For any field $k$ which is an $A'$-algebra satisfying separation condition, the natural homomorphism from $U_k(\mathfrak{g})$ to $\text{End}_{\mathcal{H}_k(\mathfrak{S}_n, \mathfrak{A})}(V_k^{\otimes n})$ is surjective.

Remark 1.8. Note that although Ariki used original Sakamoto–Shoji’s action of $T_0$ on $V_{\mathfrak{A}', n}^{\otimes n}$, his proof still goes through without difficulty if we replace it with our modified Sakamoto–Shoji’s action of $T_0$. 

The following interesting result was proved by the second author of this paper (the detailed proof will be published elsewhere), and independently proved by N. Sawada and T. Shoji in a recent preprint [37].

**Theorem 1.9.** The modified Sakamoto–Shoji’s action of $T_0$ is exactly the same as the action of $T_0$ given by [25].

One of the main results in this paper is the following theorem.

**Theorem 1.10.** For any field $k$ which is an $A'$-algebra satisfying separation condition, the natural homomorphism from $H_k(S_{n,r})$ to $\text{End}_{U_k}(g)(V_k^{\otimes n})$ is also surjective.

2. Proof of Theorem 1.10

In this section, we shall give the proof of Theorem 1.10. Following [16], we define the weak Bruhat order $\geq$ on $S_n$ in the following way.

For $u, w \in S_n$, let $u \geq w$ if there is a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_k}$ (where $i_1, \ldots, i_k \in \{1, 2, \ldots, n-1\}$) for $w$ and $u = s_{i_1}s_{i_2}\cdots s_{i_l}$ with $l \leq k$. For any multipartition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ of $n$, let $t_\lambda$ be the standard $\lambda$-tableau in which the numbers $1, 2, \ldots, n$ appear in order along the columns of the last component $t^{(r)}_\lambda$, and then along the second last component $t^{(r-1)}_\lambda$, and so on. Let $w_\lambda \in S_n$ be such that $t_\lambda w_\lambda = t_\lambda$, then $w_\lambda$ is a distinguished right coset representative of $S_{\lambda}$ in $S_n$. Observe that if $t$ is a standard $\lambda$-tableau, then $t \geq t \geq t$. We have the following lemma.

**Lemma 2.1** [16, (3.3)], [32, (5.1)]. Suppose that $\lambda$ is a multipartition of $n$. Then $u \mapsto \lambda u$ gives a bijection between $\{u \in S_n \mid u \geq w_\lambda\}$ and the set of standard $\lambda$-tableaux.

Now for any composition $a = (a_1, a_2, \ldots, a_r)$ of $n$, we regard it as a multipartition (with $r$ components) of $n$. Let $w_a \in S_n$ be such that $t^a w_a = t_a$. For simplicity, we write $h_a$ instead of $T_w^a$. Note that any row standard $a$-tableau is also a standard $a$-tableau. Therefore, we get that

$$D_a = \{u \in S_n \mid u \geq w_a\}. \quad (2.2)$$

For any composition $a = (a_1, \ldots, a_r) \in \Lambda(r, n)$, we define

$$n_a := \frac{n!}{(a_1)! \cdots (a_r)!}. \quad (2.3)$$

It is clear that $n_a = \#D_a$. Let $V_{A,a} := V_{A,1}^{\otimes a_1} \otimes \cdots \otimes V_{A,r}^{\otimes a_r}$. Then it follows from definition that

$$V_{A}^{\otimes a} = \bigoplus_{a \in \Lambda(r, n)} \bigoplus_{d \in D_a} V_{A,a}^{\otimes d} = \bigoplus_{a \in \Lambda(r, n)} \bigoplus_{d \in D_a} V_{A,a}^d = \bigoplus_{a \in \Lambda(r, n)} \bigoplus_{d \in D_a} V_{A,a}^d. \quad (2.4)$$
Hereafter, we shall identify $V_{\Lambda}^{\otimes n}$ with $\bigoplus_{\lambda\in \Lambda_{1},\ldots,n}(n)\tilde{M}^{\lambda}\downarrow_{\Lambda^{r}(\mathbb{S}_{n})}$ by using Theorem 1.5. That means, we shall adopt the definition of the action of $T_{0}$ on $V_{\Lambda}^{\otimes n}$ introduced in [25]. Note that there is an obvious bijection between $\Lambda_{1},\ldots,n(n)$ and $\Lambda(m,n)$. For any composition $\lambda$, this follows immediately from Lemma 1.3.

**Theorem 2.5.** For any composition $a = (a_{1},\ldots,a_{r}) \in \Lambda(r,n)$, with the identification given by Theorem 1.5,

$$\bigoplus_{d \in D_{a}}v_{\lambda}T_{d} = \bigoplus_{\lambda \in \Lambda_{1},\ldots,n(n)}\tilde{M}^{\lambda}.$$ 

**Proof.** This follows immediately from Lemma 1.3. \(\square\)

For any composition $a = (a_{1},a_{2},\ldots,a_{r})$ of $n$, we define $\bar{a} := (a_{r},a_{r-1},\ldots,a_{1})$, which is again a composition of $n$. We need the following lemma cited from [18, (2.7), (2.8), (3.1)] and [15, (3.7)]. Note that our $u_{b}h_{1}u_{b}$ is just $v_{b} = \pi_{b}h_{b}\pi_{b}$ in [18, (3.1)]’s notations (with $u_{i}$ replaced by $u_{r+1-i}$ for each $i$).

**Lemma 2.6** [15,18]. For any $a, b \in \Lambda(r,n)$, we have

1. For any $s_{i} \in \mathbb{S}_{b}$, $T_{i}h_{b} = h_{b}T_{i}w_{b}$.
2. $u_{b}^{+}$ (respectively $u_{b}^{-}$) commute with any element in $\mathcal{H}(\mathbb{S}_{b})$ (respectively any element in $\mathcal{H}(\mathbb{S}_{b})$).
3. $u_{a}^{+}\mathcal{H}(\mathbb{S}_{r},w_{a}) = 0$ unless $b \supseteq a$, and $u_{a}^{-}\mathcal{H}(\mathbb{S}_{r},w_{a}) = 0$ unless $b \triangleleft a$.
4. $u_{b}^{+}h_{b}u_{b}^{+}\mathcal{H}(\mathbb{S}_{r},w_{b}) = u_{b}^{+}h_{b}u_{b}^{+}\mathcal{H}(\mathbb{S}_{r},w_{b})$. Moreover, it is a free module with basis $\{u_{b}^{+}h_{b}u_{b}^{+}T_{w} \mid w \in \mathbb{S}_{n}\}$.
5. $u_{b}^{+}h_{b}u_{b}^{+}\mathcal{H}(\mathbb{S}_{r},w_{b}) = u_{b}^{+}h_{b}u_{b}^{+}\mathcal{H}(\mathbb{S}_{r},w_{b})$. Moreover, it is a free module with basis $\{u_{b}^{+}h_{b}u_{b}^{+}T_{w} \mid w \in \mathbb{S}_{n}\}$.
6. For any $d \in D_{a}$, $u_{a}^{+}T_{a}u_{a}^{-} = 0$ unless $d = w_{a}$.

**Lemma 2.7.** For any composition $b \in \Lambda(r,n)$, there is a unique element $h_{b,+,} \in \mathcal{H}(\mathbb{S}_{n})$ and a unique element $h_{b,-} \in \mathcal{H}(\mathbb{S}_{n})$ such that

$$u_{b}^{+}h_{b}u_{b}^{-} = u_{b}^{+}h_{b,+,} + \sum_{c \supseteq b}g_{c,+,}g_{c,+,}^{+}, \quad u_{b}^{-}h_{b}u_{b}^{+} = u_{b}^{-}h_{b,-} + \sum_{c \supseteq b}g_{c,-}g_{c,-}^{+},$$

for some $g_{c,+,}, g_{c,-}^{+} \in \mathcal{H}(\mathbb{S}_{n})$.

**Proof.** This follows directly from the proof of [15, (3.2)(ii), (3.20)]. \(\square\)
By [18, (3.6)],
\[ u_b h_b u_b^+ h_b u_b^{-} = u_b h_b u_b^+ z_b = z_b u_b h_b u_b^+, \]
where \( z_b \) (respectively \( z_b^* \)) is an element in the center of \( \mathcal{H}(\mathcal{S}_b) \) (respectively of \( \mathcal{H}((\mathcal{S}_b)) \)).

On the other hand, we have (by using Lemmas 2.6(3) and 2.7)
\[ u_b h_b u_b^{-} h_b u_b^+ = u_b h_b u_b^+ h_b + h_b - . \]

It follows that (by Lemma 2.6(4)) \( h_{b,+} h_{b,-} = z_b ).

**Corollary 2.8.** For any field \( k \) which is an \( \mathcal{A}' \)-algebra satisfying separation condition, that is, \( f_n(q, u_1, \ldots, u_r) \neq 0 \) in \( k \), we have both \( h_{b,+} \) and \( h_{b,-} \) are invertible in \( \mathcal{H}_k(\mathcal{S}_b) \). Hence the corollary follows. \( \square \)

For any \( a \in \mathcal{A}(r, n) \), we order the set \( D_a \) as \( 1 = d_1^{(a)}, d_2^{(a)}, \ldots, d_m^{(a)} = a \) such that \( \ell(d_i^{(a)}) < \ell(d_j^{(a)}) \) only if \( i < j \).

For any \( a \in \mathcal{A}(r, n) \), \( 1 \leq s, t \leq n_a, x \in \mathcal{H}(\mathcal{S}_a) \), we define
\[ E_{s,t}^{(a)}(x) = v_f T_{d_{s,t}^{(a)}} = (v_{i_1} \otimes \cdots \otimes v_{i_n}) T_{d_{s,t}^{(a)}} \mapsto \begin{cases} v_f x T_{d_{s,t}^{(a)}}, & \text{if } b = a \text{ and } k = s, \\ 0, & \text{otherwise}, \end{cases} \]
where \( v_f \in V_{\mathcal{A},b} \) and \( 1 \leq s, t \leq n_b \). We extend the above map linearly by using the decomposition given in (2.4) so that we get a linear endomorphism \( E_{s,t}^{(a)}(x) \) on \( V_{\mathcal{A},b}^{\otimes n} \).

Hereafter, we write \( E_{s,t}^{(a)} \) instead of \( E_{s,t}^{(a)}(1) \) and let \( \psi \) be the natural algebra homomorphism \( \psi: \mathcal{H}_n, r \to \text{End}_{\mathcal{A}(r)}(V_{\mathcal{A},b}^{\otimes n}) \).

**Lemma 2.9.** For any \( a, a' \in \mathcal{A}(r, n), x \in \mathcal{H}(\mathcal{S}_a) \) and \( x' \in \mathcal{H}(\mathcal{S}_{a'}) \), \( 1 \leq s, t \leq n_a, 1 \leq s', t' \leq n_{a'} \). We have (as linear operators on \( V_{\mathcal{A},b}^{\otimes n} \))

\begin{enumerate}
\item \( E_{s,t}^{(a)}(x) = E_{s,t}^{(a)}(x) \psi(T_{d_{s,t}^{(a)}}) \).
\item \( E_{s,t}^{(a)}(y) E_{s',t'}^{(a)}(x') = \delta_{a,a'} \delta_{s,s'} E_{s,t}^{(a)}(x x') \).
\item \( E_{s,t}^{(a)}(x) \in \text{End}_{\mathcal{A}(r)}(V_{\mathcal{A},b}^{\otimes n}) \).
\item \( E_{s,t}^{(a)}(x) = \psi(T_{d_{s,t}^{(a)}}) E_{1,1}^{(a)}(x T_{d_{s,t}^{(a)}}) \).
\end{enumerate}

**Proof.** (1) and (2) follows directly from definition. It is enough to prove (3) and (4). Note that \( U_{A} \mathcal{A}(r) \) stabilizes each subspace \( V_{A,b} T_{d_{s,t}^{(a)}} \) and (by definition) \( E_{s,t}^{(a)}(x) \) coincides with right multiplication by \( T_{d_{s,t}^{(a)}}^{-1} x T_{d_{s,t}^{(a)}} \) on \( V_{A,b} T_{d_{s,t}^{(a)}} \). By the classical Schur–Weyl duality, we know that the two actions commute with each other. Hence (3) follows. It remains to prove (4). By (1), it will be enough to prove that \( E_{s,t}^{(a)} = \psi(T_{d_{s,t}^{(a)}}) E_{1,1}^{(a)}(x T_{d_{s,t}^{(a)}}) \).
For any \(a \in \Lambda(r,n)\), \(d, d' \in D_a\), we write
\[
T_d T_{d'}^* = \sum_{d'' \in D_a} x_{d''} T_{d''}.
\]
(2.10)

We claim that \(x_1 \neq 0\) would imply that \(d = d'\) and in that case \(x_1 = 1\). By the definition of \(E_{1,1}^{(a)}\), (4) will follow easily from this claim.

We prove our claim by double induction on \(\ell(d)\) and \(\ell(d')\). If \(\ell(d) = 0\) or \(\ell(d') = 0\), then \(d = 1\) or \(d' = 1\), there is nothing to prove. Now suppose \(\ell(d) > 0\) and \(\ell(d') > 0\). Note that (2.10) is equivalent to
\[
T_d T_{d'}^* = \sum_{d'' \in D_a} T(d'' - 1) y_{d''},
\]
(2.11)
and our claim is equivalent to that \(y_1 \neq 0\) would imply that \(d = d'\) and in that case \(y_1 = 1\).

Therefore, we can assume without loss of generality that \(\ell(d) \leq \ell(d')\). Since \(\ell(d') > 0\), we have \(d' = d''s_i\) for some \(i\) with \(1 \leq i < n\) and \(d'' \in \mathfrak{S}_n\) such that \(\ell(d'') = \ell(d') + 1\). We divide the proof into three cases.

**Case 1.** \(\ell(d s_i) = \ell(d) - 1\). Then we have
\[
T_d T_{d'}^* = T_d s_i T_{s_i(d')}^{-1} = T_d s_i (T_{s_i(d')}^{-1} + (q^{-1} - q) T_{(d')}^{-1})
\]
\[
= T_{d s_i} T_{d'}^* + (q^{-1} - q) T_{d s_i} T_{d'}^*.
\]
By [13, (1.4)], we have \(d'', s_i \in D_a\). By our assumption, \(\ell(d s_i) = \ell(d) - 1 < \ell(d')\). In particular, \(d s_i \neq d'\). Note also that \(d s_i = d''\) if and only if \(d = d'\). Hence our claim follows by induction.

**Case 2.** \(\ell(d s_i) = \ell(d) + 1\) and \(d s_i \in D_a\). Then we have
\[
T_d T_{d'}^* = T_d s_i T_{s_i(d')}^{-1} = T_{d s_i} T_{d'}^*.
\]
Note that \(\ell(d'') < \ell(d')\) and \(d'', s_i \in D_a\), our claim also follows by induction.

**Case 3.** \(\ell(d s_i) = \ell(d) + 1\) and \(d s_i \notin D_a\). In particular, \(d \neq d'\) in this case. By [13, (1.1)], \(d s_i d^{-1} = s_j\) for some \(s_j \in \mathfrak{S}_a\). Then we have
\[
T_d T_{d'}^* = T_d s_i T_{s_i(d')}^{-1} = T_{d s_i d^{-1}} T_{d'}^* = T_{s_j} T_{d'}^*.
\]
Note that \(d \neq d''\) (otherwise \(d s_i = d'' s_i = d' \in D_a\)). Our claim follows again by induction. This completes the proof of our claim in all cases, and (4) follows easily from this claim. \(\square\)
Let \( k \) be a field which is an \( \mathcal{A}' \)-algebra. For any \( k \)-algebra \( A \) and any positive integer \( n \), we denote by \( M_n(A) \) the \( n \times n \) matrix algebra over the \( k \)-algebra \( A \). For any \( x \in A \) and \( 1 \leq s, t \leq n \), we denote by \( E^a_{s,t}(x) \) the \( n \times n \) matrix with \( x \) in the \((s, t)\) position and zero elsewhere. We have the following result which is essentially due to Ariki [3].

**Lemma 2.12** [3]. There is a natural algebra isomorphism

\[
\text{End}_{U_k}(V_k^\otimes n) \cong \bigoplus_{a \in A(r, n)} M_{n_a}(\text{End}_{U_k}(V_k, a))
\]

such that the element \( E^a_{s,t}(x) \) is mapped to \( E^a_{s,t}(V_k^\otimes n)(x) \) for each \( a \in A(r, n) \), \( x \in \mathcal{H}(\mathcal{S}_n) \), and \( 1 \leq s, t \leq n_a \). In particular, \( \text{End}_{U_k}(V_k^\otimes n) \) is generated by the set

\[
\{ E^a_{s,t}, x \mid x \in \mathcal{H}_k(\mathcal{S}_n), \quad a \in A(r, n), \quad 1 \leq s, \ t \leq n_a \}.
\]

**Proof.** This follows immediately from the proof of [3, (5.1)] and Lemma 2.9. Note that the separation condition is not necessary in the proof of [3, (5.1)]. \( \square \)

**Proof of Theorem 1.10.** Let \( \psi \) be the natural algebra homomorphism \( \psi : \mathcal{H}_{n,r} \rightarrow \text{End}_{U_k}(V_k^\otimes n) \). Let \( a_0 := (0, \ldots, 0, n) \). Then \( a_0 \) is the unique minimal element in \( A(r, n) \)

with respect to the dominance order \( \triangleright \). By definition, \( u_{a_0} = 1 \) and \( n_{a_0} = 1 \). We claim that \( E^{(a_0)}_{1,1} = \psi(u_{a_0}^{-1}h_{a_0}^{-1}) \in \text{image}(\psi) \).

In fact, for any \( v_I \in V_k \) and \( 1 \leq k \leq n_{a_0} \), we can rewrite (by using Theorem 2.5) \( v_{I} T_d^{(b)} \)

as \( u_{b_1}^+ \gamma_s T_d^{(b)} u_{b_0}^- + \tilde{M}^k \), where \( d_I \in D_2 \cap \mathcal{S}_n \). Note that (by Lemma 2.6) \( u_{b_1}^+ \gamma_s T_d^{(b)} u_{b_0}^- = y_s T_{d_I} u_{b_1}^+ u_{b_0}^- T_{d_I} \) and by Lemma 2.6(3) and the definition of \( \tilde{M}^k \), \( u_{b_1}^+ \gamma_s T_d^{(b)} u_{b_0}^- \notin \tilde{M}^k \) only if \( a_0 \triangleright b \), and hence (by the fact that \( a_0 \) is minimal and Lemma 2.6(6)) only if \( a_0 = b \) and \( d_k^{(b)} = u_{a_0} = 1 \). In other words, \( v_{I} T_{d_k^{(b)}} u_{a_0}^{-1}h_{a_0}^{-1} = 0 \) unless \( b = a_0 \) and \( d_k^{(b)} = 1 \). Now our claim follows immediately from Lemma 2.7 and Corollary 2.8.

We use induction on \( a \in A(r, n) \). Suppose that for any \( b \in A(r, n) \) with \( b \prec a \) and for any positive integers \( s, t \) with \( 1 \leq s, t \leq n_b \), we have \( E^{(b)}_{s,t} \in \text{image}(\psi) \). Hereafter we fix an element \( X^{(b)}_{s,t} \in \mathcal{H}_{n,r} \) such that \( \psi(X^{(b)}_{s,t}) = E^{(b)}_{s,t} \). We want to prove that for any positive integers \( s, t \) with \( 1 \leq s, t \leq n_a \), \( E^{(a)}_{s,t} \in \text{image}(\psi) \). We divide the proof into two steps:

**Step 1.** We claim that for any positive integer \( s \) with \( 1 \leq s \leq n_a \),

\[ E^{(a)}_{n_a,s} \in \text{image}(\psi). \]
Let $s$ be a fixed positive integer with $1 \leq s \leq n_a$. For any $b \in A(r, n)$ with $b \prec a$ and any positive integer $t$ with $1 \leq t \leq n_b$, we define $Y_t^{(b)}$ to be the unique element in $\mathcal{H}(\mathcal{S}_n)$ such that

$$
 u_b^+ T_{d_t^0} u_a^- h_a^{-1} T_{d_t^0} = u_b^+ Y_t^{(b)} + \sum_{c \prec b} g_{c,+} u_c^+ g_{c,+},
$$

for some $g_{c,+} \in \mathcal{H}(\mathcal{S}_n)$. As a result, for any $\lambda \in \Lambda_{m_1,\ldots,m_r}(n)$ with $|\lambda(i)| = b_i$, $\forall i$ and any $\tilde{d} \in D_\lambda \cap \mathcal{S}_b$, we have

$$
 u_b^+ \gamma_s T_{d_t^0} u_a^- h_a^{-1} T_{d_t^0} + \tilde{M}^\lambda = u_b^+ \gamma_s T_{\tilde{d}_t^0} u_a^- h_a^{-1} T_{\tilde{d}_t^0} + \tilde{M}^\lambda
$$

$$
 = \gamma_s T_{\tilde{d}_t} u_b^+ Y_t^{(b)} + \tilde{M}^\lambda = u_b^+ \gamma_s T_{\tilde{d}_t} Y_t^{(b)} + \tilde{M}^\lambda. \quad (2.13)
$$

Note that every element $d \in D_\lambda$ can be written uniquely as $\tilde{d}_t^{(b)}$ for some $\tilde{d} \in D_\lambda \cap \mathcal{S}_b$ and some $t$ with $1 \leq t \leq n_b$. Therefore, by Theorem 2.5 we get that for any $v_t \in V_b$ and any positive integer $t$ with $1 \leq t \leq n_b$,

$$
 v_t T_{d_t^0} u_a^- h_a^{-1} T_{d_t^0} = v_t Y_t^{(b)}. \quad (2.14)
$$

By our definition of $E_{t,1}^{(b)}$, for any $b' \in A(r, n)$ and any $v_t \in V_{b'}$ and any positive integer $t'$ with $1 \leq t' \leq n_{b'}$,

$$
 v_t T_{d_t^0} T_{t',1}^{(b')} Y_t^{(b')} = \delta_{b,b'} \delta_{t,t'} v_t Y_t^{(b')} \quad (2.15)
$$

Now by Theorem 2.5, Lemma 2.8, Corollary 2.8, (2.14), and (2.15), it is easy to see (as linear operators on $V_{\lambda(n)}^{(b)}$)

$$
 E_{n_a,s}^{(a)} = \psi \left( u_a^- h_a^{-1} T_{d_t^0} - \sum_{b \prec a, 1 \leq t \leq n_b} T_{t,1}^{(b)} Y_t^{(b)} \right).
$$

In particular, $E_{n_a,s}^{(a)} \in \text{image}(\psi)$, as required. This proves our claim.

**Step 2.** By result of Step 1, we know that $E_{n_a,n_a}^{(a)} \in \text{image}(\psi)$. By Lemma 2.9(4),

$$
 E_{1,1}^{(a)} = \psi \left( (T_{d_a^0}^\ast)^{-1} \right) E_{n_a,n_a}^{(a)} \psi \left( T_{d_a^0}^{-1} \right) \in \text{image}(\psi).
$$

Now for any positive integer $s, t$ with $1 \leq s, t \leq n_a$, we have (again by Lemma 2.9(4))

$$
 E_{s,t}^{(a)} = \psi \left( T_{d_s^0}^\ast \right) E_{1,1}^{(a)} \psi \left( T_{d_t^0} \right) \in \text{image}(\psi).
$$
as required. By induction, for any \(a \in A(r, n)\) and any integer \(s, t\) with \(1 \leq s, t \leq n\), \(E^{(a)}_{s,t} \in \text{image}(\psi)\). Hence by Lemma 2.12, \(\psi\) is always surjective in the case of separation condition. This completes the proof of Theorem 1.10.

Suppose that \(m_i \geq n\) for \(1 \leq i \leq r\). Let \(k\) be a field which is an \(A'\)-algebra. Let \(a \in A(r, n)\). By [17, 21], the natural algebra homomorphism \(\mathcal{H}_k(\mathfrak{S}_a) \rightarrow \text{End}_{U_k}(V_{k,a})\) is an isomorphism (as \(m_i \geq n\) for each \(i\)). By Lemma 2.12, we get that

\[
\text{End}_{U_k}(V_{k}^{\otimes n}) \cong \bigoplus_{a \in A(r, n)} M_n(\mathcal{H}_k(\mathfrak{S}_a)). \tag{2.16}
\]

On the other hand, we know (by [36, (5.3)]) that

\[
\dim \text{End}_{U_k}(V_{k}^{\otimes n}) = \dim \mathcal{H}_k(\mathfrak{S}_{n,r}).
\]

Now suppose that the field \(k\) satisfies the separation condition. Then it is easy to see that the natural algebra homomorphism \(\psi\) becomes an isomorphism. Hence we get that (by Theorem 1.10)

**Theorem 2.17.** For any field \(k\) which is an \(A'\)-algebra satisfying separation condition,

\[
\mathcal{H}_k(\mathfrak{S}_{n,r}) \cong \bigoplus_{a \in A(r, n)} M_n(\mathcal{H}_k(\mathfrak{S}_a)).
\]

As a consequence, we obtain the main result of [18] and a special case of the result in [19], which states that the algebra \(\mathcal{H}_k(\mathfrak{S}_{n,r})\) is Morita equivalent to the algebra \(\bigoplus_{a \in A(r, n)} \mathcal{H}_k(\mathfrak{S}_a)\) under the separation condition. Theorem 2.17 is a strong version of that fact, and gives a precise connection between \(\mathcal{H}_k(\mathfrak{S}_{n,r})\) and Hecke algebras of type \(A\).

3. Shoji’s algebra \(\mathcal{H}^b\)

In [35], Shoji defines a new algebra \(\mathcal{H}^b\), which becomes a new presentation of the Ariki–Koike algebra in the semisimple case. In this section, we shall study this algebra and prove that it is actually isomorphic to a direct sum of some matrix algebras over some Hecke algebras of type \(A\).

Throughout this section, we shall mainly follow Shoji’s conventions except that all linear operators still act from right on the tensor space \(V^{\otimes n}\). In particular, we shall adopt the version of \(\mathcal{H}(\mathfrak{S}_n)\)-structure on \(V^{\otimes n}\) used in Shoji’s paper [35, (2.2)].

First, we recall the definition of Shoji’s algebra \(\mathcal{H}^b\) [35]. Let \(u_1, \ldots, u_r\) be \(r\) indeterminates. Let \(A\) be the matrix of degree \(r\) whose \((i, j)\)-entry is equal to \(u^{j-1}_i\) for \(1 \leq i, j \leq r\). Then \(\det(A) = \Delta := \prod_{i \neq j} (u_i - u_j)\) is the Vandermonde determinant. Let \(B := (h_{i,j})_{r \times r}\) be the companion matrix of \(A\). Then \(A^{-1} = \Delta^{-1} B\) and each \(h_{i,j}\) is a
polynomial in \( \mathbb{Z}[u_1, \ldots, u_r] \). Let \( X \) be an indeterminate. For each positive integer \( c \) with \( 1 \leq c \leq r \), we define a polynomial \( F_c(X) \) by

\[
F_c(X) := \sum_{i=0}^{r-1} h_{c,i} X^i.
\]

Let \( A'' := A'[\Delta^{-1}] = \mathbb{Z}[q, q^{-1}, u_1, \ldots, u_r, \Delta^{-1}] \). Following [35], we define an algebra \( H_h \) over \( A'' \) as an associative algebra with generators \( T_1, \ldots, T_{n-1}, \xi_1, \ldots, \xi_n \) and relations

\[
(T_i - q)(T_i + q^{-1}) = 0, \quad \text{for } 1 \leq i \leq n - 1,
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leq i \leq n - 2,
\]

\[
T_i T_j = T_j T_i, \quad \text{for } 0 \leq i < j - 1 \leq n - 2,
\]

\[
(\xi_i - u_1) \cdots (\xi_i - u_r) = 0, \quad \text{for } 1 \leq i \leq n,
\]

\[
T_i \xi_j = \xi_j T_i, \quad \text{for } j \neq i, i + 1,
\]

\[
T_{j-1} \xi_j = \xi_j T_{j-1} + \Delta^{-2} \sum_{c_1 < c_2} (u_{c_2} - u_{c_1}) (q - q^{-1}) F_{c_1} (\xi_{j-1}) F_{c_2} (\xi_j),
\]

\[
T_{j-1} \xi_{j-1} = \xi_j T_{j-1} - \Delta^{-2} \sum_{c_1 < c_2} (u_{c_2} - u_{c_1}) (q - q^{-1}) F_{c_1} (\xi_{j-1}) F_{c_2} (\xi_j).
\]

Note that our \( T_i \) is just \( a_{j+1} \) in [35]’s notation. By [35, (3.7)], \( H^b \) is a free \( A'' \)-module with basis \( \{ \xi_1^{c_1} \cdots \xi_r^{c_r} T_w \mid w \in S_n, 1 \leq c_i < r, \forall i \} \). Furthermore, for any \( A'' \)-algebra \( k \), \( H^b_k := k \otimes H^b \) is also given by the same generators and relations. The subalgebra generated by \( T_1, \ldots, T_{n-1} \) is isomorphic to the Iwahori–Hecke algebra associated to the symmetric group \( S_n \).

Let \( V_A \) be the free \( A \)-module as before (see Section 1). Recall the definition of the function \( \hat{b} \) introduced in Section 1. For each \( 1 \leq a < n \), we define a linear operator \( \omega_a \) on \( V^\otimes_m \) by

\[
(v_1 \otimes \cdots \otimes v_n) \omega_a := u_{b(l_a)} (v_1 \otimes \cdots \otimes v_n).
\]

Recall also the linear operator \( S_a \) introduced in [35, Section 2]. By [35], both \( S_a \) and \( \omega_a \) commute with the action of \( U_A'(g) \) and hence lie in the algebra \( \text{End}_{U_A'(g)}(V^\otimes_m) \).

**Lemma 3.1** [35]. For any \( A'' \)-algebra \( k \), the map which sends \( \xi_a \) (for each \( 1 \leq a \leq n \)) to \( \omega_a \) extends naturally to a representation of \( H^b_k \) on \( V^\otimes_m \), extending its original (the version given in [35, (2.2)]) \( \mathcal{H}_k(S_n) \)-structure on \( V^\otimes_m \).
Proof. Suppose that $m_i \geq n$ for each $i$ with $1 \leq i \leq r$. Then the lemma is a result of T. Shoji (see [35, (3.4), (3.5)]). The case where $m_i < n$ was not considered in [35]. In general, for each $1 \leq i \leq r$, we define

$$
\tilde{V}_{A,i} := \begin{cases} 
V_{A,i} \oplus A \oplus \cdots \oplus A, & \text{if } m_i < n, \\
V_{A,i}, & \text{if } m_i \geq n.
\end{cases}
$$

Let $\tilde{m}_i := \text{rank } \tilde{V}_{A,i}$ for each $i$, and $\tilde{m} = \tilde{m}_1 + \cdots + \tilde{m}_r$. We extend the ordered basis $\{v_1^{(1)}, \ldots, v_{m_i}^{(1)}\}$ of $V_{A,i}$ to an ordered basis $\{v_1^{(1)}, \ldots, v_{\tilde{m}_i}^{(1)}\}$ of $\tilde{V}_{A,i}$. Let $\tilde{V}_{A^r,i} := A' \otimes V_{A,i}$, and $\tilde{V}_{A^r} := \tilde{V}_{A^r,1} \oplus \cdots \oplus \tilde{V}_{A^r,r}$. Let $k$ be an $A'$-algebra. Then the map which sends $\xi_a$ (for each $1 \leq a \leq n$) to $\omega_a$ extends naturally to a representation of $\mathcal{H}_k^n$ on $\tilde{V}_k^{\otimes n}$, extending its original (the version given in [35, (2.2)]) $\mathcal{H}_k(\mathfrak{S}_n)$-structure on $\tilde{V}_k^{\otimes n}$. Note that (by definition) each operators $T_i$, $\omega_a$ stabilize the subspace $V_k^{\otimes n} \subseteq \tilde{V}_k^{\otimes n}$, it follows that the above map also gives rise to a representation of $\mathcal{H}_k^n$ on $V_k^{\otimes n}$ extending its original (the version given in [35, (2.2)]) $\mathcal{H}_k(\mathfrak{S}_n)$-structure on $V_k^{\otimes n}$. \qed

Corollary 3.2 [35, (3.5)]. For $j = 1, \ldots, n - 1$, we have

$$
S_j = T_j - \Delta^{-2}(q - q^{-1}) \sum_{c_1 < c_2} F_{c_1}(\omega_j) F_{c_2}(\omega_{j+1}),
$$

$$
S_j^{-1} = T_j^{-1} - \Delta^{-2}(q - q^{-1}) \sum_{c_1 > c_2} F_{c_1}(\omega_j) F_{c_2}(\omega_{j+1}).
$$

Proof. Note that if $m_i \geq n$ for each $i$ with $1 \leq i \leq r$. Then the first identity is a result of T. Shoji (see [35, (3.5)]). The case where $m_i < n$ was not considered in [35]. In general, we define $\tilde{V}_{A'}$ as in the proof of Lemma 3.1. Then it is easy to see (by definition) both $S_j$ and $S_j^{-1}$ stabilize the submodule $V_{A'}^{\otimes n}$. By [35, (3.5)], as linear operator on $V_{A'}^{\otimes n}$,

$$
S_j = T_j - \Delta^{-2}(q - q^{-1}) \sum_{c_1 < c_2} F_{c_1}(\omega_j) F_{c_2}(\omega_{j+1}).
$$

So it is also true as linear operator on $V_{A'}^{\otimes n}$. This proves the first equality. To prove the second one, we first regard it as linear operator on $V_{A'}^{\otimes n}$. We define $f_j := \sum_{i=1}^n v_{j}^{(i)}$ for each $j$ with $1 \leq j \leq n$, $v_0' := f_1 \otimes \cdots \otimes f_n$. Then

$$
f_1 \otimes \cdots \otimes f_{j-1} \otimes v_{j}^{(c_1)} \otimes v_{j+1}^{(c_2)} \otimes f_{j+2} \otimes \cdots \otimes f_n = v_0' \Delta^{-2} F_{c_1}(\omega_j) F_{c_2}(\omega_{j+1}),
$$

for any $1 \leq j < n$ and $1 \leq c_1, c_2 \leq n$. It follows that
\[ f_1 \otimes \cdots \otimes f_{j-1} \otimes v(c_1) \otimes f_{j+2} \otimes \cdots \otimes f_n \]
\[ = \begin{cases} 
\nu'_0 \Delta^{-2} F_{c_1}(\omega_j) F_{c_2}(\omega_{j+1}) T_j^{-1}, & \text{if } c_1 \leq c_2, \\
\nu'_0 \Delta^{-2} F_{c_1}(\omega_j) F_{c_2}(\omega_{j+1}) \left( T_j^{-1} - (q - q^{-1}) \right), & \text{if } c_1 > c_2.
\end{cases} \]

Note that
\[
\begin{align*}
(f_1 \otimes \cdots \otimes f_{j-1} \otimes f_j \otimes f_{j+2} \otimes \cdots \otimes f_n) S_j^{-1} &= f_1 \otimes \cdots \otimes f_{j-1} \otimes f_j \otimes f_{j+2} \otimes \cdots \otimes f_n,
\end{align*}
\]
and by \([35, (3.5.3)]\),
\[ 1 = \Delta^{-2} \sum_{c_1, c_2} F_{c_1}(\omega_j) F_{c_2}(\omega_{j+1}) \]
It follows that
\[ v'_0 S_j^{-1} = \nu'_0 \left( T_j^{-1} - \Delta^{-2} (q - q^{-1}) \sum_{c_1 > c_2} F_{c_1}(\omega_j) F_{c_2}(\omega_{j+1}) \right). \]

Using the same argument as in \([35, (3.2)]\), one can show that \( \nu'_0 \mathcal{H}^b_{\mathbb{Q}(q,u_1,\ldots,u_r)} \) is isomorphic to the regular (right) \( \mathcal{H}^b_{\mathbb{Q}(q,u_1,\ldots,u_r)} \)-module. Since \( S_j^{-1} \) commutes with the action of \( U_{\mathcal{A}''}(g) \), it follows from \([35, (2.4)]\) that \( S_j^{-1} \) already lies in the image of \( \mathcal{H}^b_{\mathbb{Q}(q,u_1,\ldots,u_r)} \). Hence it follows that
\[ S_j^{-1} = T_j^{-1} - \Delta^{-2} (q - q^{-1}) \sum_{c_1 > c_2} F_{c_1}(\omega_j) F_{c_2}(\omega_{j+1}) \]
as linear operator on \( \tilde{V}^A \otimes \mathcal{A}'' \). By using the same argument as before, this is also true as linear operator on \( V^A \otimes \mathcal{A}'' \). This completes the proof of the second equality. \( \square \)

**Theorem 3.3.** For any field \( k \) which is an \( \mathcal{A}'' \)-algebra, the natural algebra homomorphism from \( \mathcal{H}^b_k \) to \( \text{End}_{U_k(g)}(V^A_k) \) is always surjective.

**Proof.** We denote by \( \psi' \) the natural homomorphism from \( \mathcal{H}^b_k \) to \( \text{End}_{U_k(g)}(V^A_k) \). By Lemmas 2.12 and 2.9(4), it is enough to show that for each \( \mathbf{a} \in \Lambda(r,n) \), \( E^{(a)}_{1,1} \in \text{image}(\psi') \).

Recall that \( V_{\mathcal{A},a} = V^A_{\mathcal{A},1} \otimes \cdots \otimes V^A_{\mathcal{A},r} \). We define
\[ E(\mathbf{a}) := \prod_{i=1}^r \left( \prod_{s=a_i+1}^{a_{i+1}} \prod_{1 \leq c \leq r} \frac{\omega_i - u_c}{u_i - u_c} \right). \]
Clearly, \( E(a) \in \text{image}(\psi') \), and for any \( v_{c_1} \otimes \cdots \otimes v_{c_n} \in V_{k,a} \),

\[
(v_{c_1} \otimes \cdots \otimes v_{c_n}) E(a) = \begin{cases} v_{c_1} \otimes \cdots \otimes v_{c_n}, & \text{if } v_{c_1} \otimes \cdots \otimes v_{c_n} \in V_{k,a}, \\ 0, & \text{otherwise}. \end{cases}
\]

It follows from definition that \( E_{1,1}^{(a)} = E(a) \). In particular, \( E_{1,1}^{(a)} \in \text{image}(\psi') \), as required. \( \square \)

**Remark 3.4.** Let \( j \) be an integer with \( 1 \leq j < n \). For any \( a \in \Lambda(r,n) \), \( 1 \leq i \leq n_a \), we write \( V_{aT_{d_i}} = V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n} \). Then it is easy to see that (as linear operators on \( V^\otimes n \))

\[
E_{i,i}^{(a)} S_j = \begin{cases} E_{i,i}^{(a)} \psi(T_j), & \text{if } b(i,j) \geq b(i,j+1), \\ E_{i,i}^{(a)} \psi(T_j^{-1}), & \text{if } b(i,j) < b(i,j+1), \end{cases}
\]

\[
E_{i,i}^{(a)} S_j^{-1} = \begin{cases} E_{i,i}^{(a)} \psi(T_j), & \text{if } b(i,j) > b(i,j+1), \\ E_{i,i}^{(a)} \psi(T_j^{-1}), & \text{if } b(i,j) \leq b(i,j+1), \end{cases}
\]

and we have

\[
S_j = \sum_{a \in \Lambda(r,n)} E_{i,i}^{(a)} S_j, \quad S_j^{-1} = \sum_{a \in \Lambda(r,n)} E_{i,i}^{(a)} S_j^{-1}.
\]

**Corollary 3.5.** For any field \( k \) which is an \( A'' \)-algebra, the Shoji’s algebra \( \mathcal{H}_k^h \) is always isomorphic to the following algebra:

\[
\bigoplus_{a \in \Lambda(r,n)} M_{n_a}(\mathcal{H}_k(S_a)).
\]

**Proof.** This follows immediately from (2.16) and Theorem 3.3 by taking a suitable \( k \)-linear space \( V \) (such that \( \dim V_i \geq n \) for each \( i \)) and comparing dimensions. \( \square \)

**Remark 3.6.** We remark that a different proof of Theorem 3.3 and Corollary 3.5 was also given in [37].

### 4. Further remark

In this final section, we shall make some further remark that the main results in this paper actually hold over some more general base rings.

Recall the definition of the polynomial \( f_n(q, u_1, \ldots, u_r) \). We define

\[
\mathcal{A}_0' := \mathcal{A}[\left( f_n(q, u_1, \ldots, u_r) \right)^{-1}].
\]
Then an $A'_0$-algebra $k$ is just an $A'$-algebra which satisfies separation condition in Definition 1.6.

Denote by $\varphi$ the natural algebra homomorphism from $U_A(g)$ to $\text{End}(V^\otimes_n)$. By the decomposition in (2.4), $S_A(m_1, \ldots, m_r, n) := \varphi(U_A(g))$ is isomorphic to a direct sum of some tensor products of the well-known Dipper–James $q^2$-Schur algebras (over $A$) introduced in [14]. By [17], for any field $k$ which is an $A$-algebra, $S_k(m_1, \ldots, m_r, n)$ is quasi-hereditary and (by the decomposition in (2.4)) $V^\otimes_k$ is in fact a tilting module over $S_k(m_1, \ldots, m_r, n)$. By [17, (4.4)],

$$\text{End}_{U_k(g)}(V^\otimes_k) = \text{End}_{S_k(m_1, \ldots, m_r, n)}(V^\otimes_k) \cong k \otimes_A \text{End}_{S_A(m_1, \ldots, m_r, n)}(V^\otimes_A) = k \otimes_A \text{End}_{U_A(g)}(V^\otimes_A),$$

for any $A$-algebra $k$ (not necessary to be a field).

By our construction of the homomorphism $\psi$, we know that

$$\psi(\mathcal{H}_k(\mathbb{S}_{n,r})) \subseteq \text{End}_{U_A(g)}(V^\otimes_A),$$

and (by Theorem 1.10) for any field $k$ which is an $A'_0$-algebra, $1 \otimes_{A'_0} \psi$ is surjective. By general facts from commutative algebra, $\psi_{A'_0}$ is already surjective. That is,

$$\psi(\mathcal{H}_{A'_0}(\mathbb{S}_{n,r})) = \text{End}_{U_{A'_0}(g)}(V^\otimes_{A'_0}).$$

Hence for any $A'_0$-algebra $k$ (not necessary to be a field), $1 \otimes_{A'_0} \psi$ is also surjective.

Using the same argument as above, one can show that for any $A''$-algebra $k$ (not necessary to be a field), the natural algebra homomorphism from $\mathcal{H}_k^b$ to $\text{End}_{U_{A''}(g)}(V^\otimes_k)$ is always surjective, and $\mathcal{H}_k^b$ is always isomorphic to the following algebra

$$\bigoplus_{a \in \Lambda(r,n)} M_{n_k}(\mathcal{H}_k(\mathbb{S}_a)).$$

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