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The Verbal Topology of a Group

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1. Introduction

We define for any group G a topology on G called the *verbal topology* of G and study groups which satisfy the minimal condition on the closed sets in the verbal topology, called *min-closed*. The main results of the paper establish with easy proofs that certain classes of groups satisfy min-closed. Since it is shown that groups with min-closed are CZ-groups (see [2] or [5]), these results give some new examples of CZ-groups: for example, finitely generated Abelian-by-nilpotent groups. Also, since centralizers are closed in the verbal topology, another consequence is an easy proof of the fact that finitely generated Abelian-by-nilpotent groups satisfy the minimal condition on centralizers (or, equivalently, the maximal condition on centralizers). This was proved in a stronger form, but with a rather difficult proof, by Lennox and Roseblade [3], answering a question of P. Hall.

The investigation which led to the results mentioned above began with a search for discriminating groups (see Neumann [4]). In this connection we prove that every group with min-closed has a discriminating subgroup of finite index.

2. The Verbal Topology

Let $\langle x\rangle$ be the free cyclic group on a generator x. Let G be any group and let $G * \langle x\rangle$ denote the free product of G and $\langle x\rangle$. Each element of $G * \langle x\rangle$ is a word in x and elements of G. If w is any element of $G * \langle x\rangle$ and g is an element of G then $w(g)$ denotes the element of G obtained by substituting g for x in w: more formally, $w(g)$ is the image of w under the homomorphism ϕ_g from $G * \langle x\rangle$ to G defined by $a\phi_g = a$ for all $a \in G$ and $x\phi_g = g$.

We define the verbal topology on G by specifying the closed subsets of G. First, for each element w of $G * \langle x\rangle$ the set $\{g \in G: w(g) = 1\}$ is called a *primitive solution set* in G. We take the primitive solution sets in G as a sub-basis

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for the closed sets of the verbal topology. Thus an arbitrary closed set is an intersection of sets each of which is a union of finitely many primitive solution sets.

Every one-element subset $\{a\}$ of G is closed: it is the primitive solution set corresponding to the word $x^{-1}a$. For every subset A of G the centralizer of A in G is closed because it is the intersection of the primitive solution sets corresponding to the words $[x, a]$, $a \in A$.

If S is the primitive solution set corresponding to an element $v(x)$ of $G * \langle x \rangle$ and if $w(x)$ is any element of $G * \langle x \rangle$ then the set $\{g \in G: w(g) \in S\}$ is the primitive solution set corresponding to the word $v(w(x))$. Thus the mapping $g \mapsto w(g)$ from G to G is continuous. In particular, for every element a of G the mappings $g \mapsto g^{-1}$, $g \mapsto ag$, $g \mapsto ga$, and $g \mapsto g^{-1}ag$ are continuous. Thus, since every one-element subset of G is closed, G is a C-group in the terminology of [2]. Consequently,

THEOREM 2.1. *If G is a group which satisfies* min-*closed then G is a CZ-group.*

An easy fact which we use later is the following. If H is a subgroup of G then every primitive solution set in H is the intersection with H of a primitive solution set in G. Thus the topology of H relative to the verbal topology of G is a refinement of the verbal topology of H.

As in [2] a topological space is called *irreducible* if it cannot be expressed as the union of two closed proper subsets. (In general this is stronger than connectedness.) Thus a subset is called irreducible if it is irreducible in the relative topology.

In order to prove some results concerned with discriminating groups it is convenient to use the contrapositive form of the usual definition. This may be described as follows. Let F be a free group on generators $x_1, x_2, \ldots$. Any formal expression of the form $w_1 \vee w_2 \vee \cdots \vee w_m$, where the w_i are elements of F, is called a *disjunction*. A group G is said to *satisfy* the disjunction if for every homomorphism $\phi: F \to G$ we have $w_i\phi = 1$ for at least one i. A group G is *discriminating* if whenever G satisfies a disjunction $w_1 \vee w_2 \vee \cdots \vee w_m$ then some w_i is a law of G.

THEOREM 2.2. *Suppose that G is a group which is irreducible in the verbal topology. Then G is discriminating.*

Proof. Suppose that G satisfies the disjunction

$$w_1(x_1, \ldots, x_n) \vee \cdots \vee w_m(x_1, \ldots, x_n).$$

We prove by downward induction on r that if $a_1, \ldots, a_r \in G$ then there exists i such that $w_i(a_1, \ldots, a_r, g_{r+1}, \ldots, g_n) = 1$ for all $g_{r+1}, \ldots, g_n \in G$. The case $r = 0$ then completes the proof by showing that some w_i is a law of G.

The result is true for $r = n$ since G satisfies $w_1 \vee \cdots \vee w_m$. Suppose the result is true for r where $1 \leqslant r \leqslant n$. Let $a_1, \ldots, a_{r-1} \in G$. Then, by the inductive hypothesis, for all $h \in G$ there exists i such that

$$w_i(a_1, \ldots, a_{r-1}, h, g_{r+1}, \ldots, g_n) = 1 \qquad \text{for all} \quad g_{r+1}, \ldots, g_n \in G. \qquad (*)$$

For each i let S_i be the set of all h satisfying $(*)$. Thus $G = S_1 \cup \cdots \cup S_m$. Also, each S_i is an intersection of primitive solution sets, so is closed. Therefore, since G is irreducible, $G = S_i$ for some i. This yields the inductive step.

COROLLARY 2.3. *Suppose that G is a group which satisfies* min-*closed. Then G has a discriminating subgroup of finite index.*

Proof. By Theorem 2.1, G is a CZ-group in the verbal topology. Hence, by [2, Lemmas 4.5 and 8.8] (or by [5, Lemmas 5.2 and 14.3]), G has an irreducible subgroup J of finite index. (Take J to be the connected component of the identity.) Since J is irreducible it is irreducible in its own verbal topology. Thus, by Theorem 2.2, J is discriminating.

3. GROUPS WITH MIN-CLOSED

The object of this section is to establish that certain groups satisfy min-closed. Consequently, by Theorem 2.1 and Corollary 2.3, these groups are CZ-groups and have discriminating subgroups of finite index.

A subset of a group G is called a *solution set* if it is an intersection of primitive solution sets. We use the following criterion.

LEMMA 3.1. *A group satisfies* min-*closed if it satisfies the minimal condition on solution sets.*

This is an immediate consequence of the following result. We give a direct proof although the result can be derived from standard results in topology.

LEMMA 3.2. *Let $\mathscr{T}$ be the set of closed subsets of a topological space X and let $\mathscr{S}$ be a sub-basis of $\mathscr{T}$ which is closed under finite intersections. Suppose that $\mathscr{S}$ satisfies the minimal condition. Then so does $\mathscr{T}$.*

Proof. Clearly there is no harm in assuming that $X \in \mathscr{S}$. Let $\mathscr{T}_0$ denote the set of elements of $\mathscr{T}$ which can be expressed as the union of finitely many elements of $\mathscr{S}$. Then $\mathscr{T}_0$ is closed under finite intersections and every element of $\mathscr{T}$ is an intersection of elements of $\mathscr{T}_0$. Suppose that $\mathscr{T}$ does not satisfy the minimal condition. Then it follows that there is an infinite properly descending chain of elements of $\mathscr{T}_0$. Choose an element S of $\mathscr{S}$ which is minimal subject to containing the first term T_1 of an infinite properly descending chain $T_1 \supset T_2 \supset \cdots$ of elements of $\mathscr{T}_0$. Suppose that $T_2 = S_1 \cup \cdots \cup S_n$ where $S_i \in \mathscr{S}$ for all i. Then, for each i,

$$S \supset S_i \supseteq T_3 \cap S_i \supseteq T_4 \cap S_i \supseteq \cdots,$$

where each term belongs to $\mathscr{T}_0$. By the choice of S, for each i the chain is ultimately stationary. But, for $j \geqslant 2$,

$$T_j = (T_j \cap S_1) \cup \cdots \cup (T_j \cap S_n).$$

Thus the sequence $T_1 \supset T_2 \supset \cdots$ is ultimately stationary, which is a contradiction.

The next two lemmas show that the class of groups satisfying min-closed is closed under the operations of taking subgroups and finite direct products. We then state and prove our main results.

LEMMA 3.3. *Suppose that G is a group which satisfies* min-*closed and let H be a subgroup of G. Then H satisfies* min-*closed.*

Proof. Suppose that $S_1 \cap H \supseteq S_2 \cap H \supseteq \cdots$ where the S_i are closed in G. For each i let T_i denote the closure of $S_i \cap H$ in G. Then $T_i \cap H = S_i \cap H$ and, since G satisfies min-closed, the chain $T_1 \supseteq T_2 \supseteq \cdots$ is ultimately stationary. Thus the chain $S_1 \cap H \supseteq S_2 \cap H \supseteq \cdots$ is ultimately stationary.

LEMMA 3.4. *Suppose that G and H are groups which satisfy* min-*closed. Then $G \times H$ satisfies* min-*closed.*

Proof. By Lemma 3.1 it is enough to establish the minimal condition for solution sets in $G \times H$. But every solution set in $G \times H$ is the direct product of a solution set in G and a solution set in H. The result follows.

THEOREM 3.5. *Every linear group satisfies* min-*closed.*

Proof. Let G be a group of nonsingular matrices over a field K, and consider the Zariski topology on G (see [2] or [5]). Since the minimal condition is satisfied on the closed sets in this topology, it suffices to show that the Zariski topology is a refinement of the verbal topology. So let S be a primitive solution set corresponding to a word w. The condition that $g \in S$, i.e., $w(g) = 1$, is equivalent to a set of polynomial equations, with coefficients in K, in the coordinates of the matrix g. Thus S is closed in the Zariski topology.

THEOREM 3.6. *Let G be a finitely generated group such that every finitely generated group in the variety generated by G satisfies* max-n *(the maximal condition on normal subgroups). Then G satisfies* min-*closed.*

Proof. For each $g \in G$ let ϕ_g denote the homomorphism $w \mapsto w(g)$ from $G * \langle x \rangle$ to G. Let N denote the intersection of the kernels of the ϕ_g. For any solution set S in G we have

$$S = \{g \in G: w(g) = 1 \text{ for all } w \in W\}$$

where W is the intersection of the kernels of those ϕ_g with $g \in S$. Each W/N is a normal subgroup of $(G * \langle x \rangle)/N$ and, by the hypotheses of the theorem, $(G * \langle x \rangle)/N$ satisfies max-n. Thus G satisfies the minimal condition on solution sets. Therefore, by Lemma 3.1, G satisfies min-closed.

COROLLARY 3.7. *Let G be a finitely generated Abelian-by-nilpotent-by-finite group. Then G satisfies* min-*closed.*

Proof. Every finitely generated group in the variety generated by G is Abelian-by-nilpotent-by-finite. (Here we use the fact that every finitely generated group in the variety generated by a finite group is finite: [4, Theorem 15.71].) Such groups satisfy max-n by the results of Hall [1]. Thus the result follows by Theorem 3.6.

The possibility that the next result might be true was suggested to me by C. Higgins, and a proof was arrived at in conversation with him. But the rather simpler proof given here I owe to B.A.F. Wehrfritz.

THEOREM 3.8. *Let G be an Abelian-by-finite group. Then G satisfies* min-*closed.*

Proof. Let A be an Abelian normal subgroup of finite index in G. Then G is the union of finitely many cosets of A. By Lemma 3.1 it suffices to show that the minimal condition is satisfied for solution sets S. Thus it suffices to show that for each t in G the minimal condition is satisfied for the sets $S \cap At$. But $S \cap At = (St^{-1} \cap A)t$ and St^{-1} is a solution set. Thus it suffices to show that the minimal condition is satisfied for sets $S \cap A$.

Let Γ be the group-ring of G/A over the integers. Then A can be regarded as a Γ-module. For each $w \in G * \langle x \rangle$ there is an element δ of Γ and an element g of G such that

$$\{a \in A: w(a) = 1\} = \{a \in A: a^\delta = g\}.$$

If this set is nonempty then it is a coset in A of the subgroup $\{a \in A: a^\delta = 1\}$. It follows that if $S \cap A$ is nonempty it is a coset in A of a subgroup

$$R = \{a \in A: a^\delta = 1 \text{ for all } \delta \in \Delta\}$$

for some $\Delta \subseteq \Gamma$. Clearly we can replace Δ by the set

$$\{\gamma \in \Gamma: a^\gamma = 1 \text{ for all } a \in R\},$$

which is an additive subgroup (in fact a right ideal) of Γ. But Γ is a finitely generated Abelian group and so satisfies the maximal condition on subgroups. Thus the minimal condition is satisfied for the sets $S \cap A$.

It is perhaps worth pointing out some further examples of groups with min-closed. If G is any group of automorphisms of a finitely generated module over a commutative Noetherian ring then, by the main result of Wehrfritz [6], G is quasilinear, i.e., G is isomorphic to a subgroup of the direct product of finitely many linear groups. But, by Lemmas 3.3, 3.4, and Theorem 3.5, every quasilinear group satisfies min-closed.

4. Further Comments

I have not been able to decide whether or not every finite extension of a group with min-closed satisfies min-closed. The corresponding problem for CZ-groups is also unsolved: it is [5, Question 9]. (In contrast with Lemma 3.4 it is not known whether or not the direct product of two CZ-groups is a CZ-group.)

It would be interesting to know whether or not finitely generated Abelian-by-polycyclic groups satisfy min-closed. Unfortunately the proof used for Corollary 3.7 does not extend directly.

In the terminology of [2] any group with its verbal topology is a C-group and, a fortiori, a T_1-group. If it satisfies min-closed it is a Z-group and hence a CZ-group. Thus several results concerning the verbal topology are immediate consequences of results in [2]. For example, by [2, Theorem 8.1], if A is a (normal) subgroup of G then the closure $\bar{A}$ of A in G is also a (normal) subgroup of G.

By [2, Lemma 4.4] the connected component of the identity, $J(G)$ say, in a T_1-group G is a closed normal subgroup. Also, in a T_1-group G it may be verified that if X and Y are irreducible subsets then so is the product XY. From this it follows that if $\{X_\lambda : \lambda \in \Lambda\}$ is a family of irreducible subsets of G such that $1 \in X_\lambda$ for all λ then the subgroup $\langle X_\lambda : \lambda \in \Lambda \rangle$ generated by the sets X_λ is an irreducible subgroup of G. This shows that G has an irreducible subgroup $I(G)$ which contains every irreducible subset X of G with $1 \in X$. Furthermore, it may be shown that $I(G)$ is a closed normal subgroup of G. Clearly $I(G) \subseteq J(G)$.

In the case of the verbal topology, $I(G)$ and $J(G)$ are characteristic subgroups of G because every automorphism of G is a homeomorphism. Also, by Theorem 2.2, $I(G)$ is discriminating. Thus the verbal topology associates with any group G a certain characteristic discriminating subgroup. When G has min-closed then $I(G) = J(G)$ (by [2, Lemma 8.8]) and, as remarked before, $J(G)$ has finite index in G (by [2, Lemma 4.5]). In fact I know of no group G for which $I(G) \neq J(G)$ in the verbal topology.

Some other results applying specifically to the Zariski topology on linear groups may be found in [5]. Some of these extend to the verbal topology. A few examples will illustrate this. For any group G and any set W of elements of a free group F, let $W(G)$ denote the verbal subgroup of G corresponding to W. Then it can be proved that in the verbal topology $W(\bar{A}) \subseteq \overline{W(A)}$ for any subgroup A of G (cf. [5, Exercise 5.8(ii), Lemma 10.7]). Thus if A belongs to a given variety, so does $\bar{A}$. If G is irreducible (or connected) then so is $W(G)$ (cf. [5, Exercise 5.8(i)]). In fact a parallel result concerning discrimination may be proved: *if G is a discriminating group then every verbal subgroup of G is discriminating.* This may be regarded as a generalization of [4, Theorem 17.31] which states that every nontrivial verbal subgroup of a discriminating group is infinite, the connection being the easy fact that a nontrivial finite group is not discriminating ([4, Corollary 17.32]).

Finally, we should give some examples of groups which do not satisfy min-closed. In fact it is easy to construct groups which do not satisfy the minimal condition on centralizers: for example, there are nilpotent groups of class 2 with this property. As far as finitely generated soluble groups are concerned, [3, Theorem G(iii)] shows that there is a 3-generator centre-by-metabelian group which has max-n but fails to satisfy the minimal condition on centralizers.

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