Computers and Mathematics with Applications 62 (2011) 1522–1530

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A generalized fractional KN equation hierarchy and its fractional Hamiltonian structure

Fajun Yu

College of Maths and Systematic Science, Shenyang Normal University, Shenyang 110034, PR China

ARTICLE INFO

Keywords:
Fractional
KN equation hierarchy
Hamiltonian structure

ABSTRACT

A generalized Hamiltonian structure of the fractional soliton equation hierarchy is presented by using differential forms and exterior derivatives of fractional orders. We construct the generalized fractional trace identity through the Riemann–Liouville fractional derivative. An example of the fractional KN soliton equation hierarchy and Hamiltonian structure is presented, which is a new integrable hierarchy and possesses Hamiltonian structure.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

Derivatives and integrals of fractional order [1,2] have found many applications in recent studies in physics. The interest in fractional analysis has been growing continually during the past few years. Fractional analysis has numerous applications: kinetic theories [3–5], statistical mechanics [6,7], dynamics in complex media [8,9], and many others [10–13]. The theory of derivatives of non-integer order goes back to Leibniz, Liouville, Grunwald, Letnikov and Riemann.

In the past few decades, many authors have pointed out that fractional-order models are more appropriate than integer-order models for various real materials. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. The fractional calculus is a generalization of ordinary differentiation and integration to arbitrary order [14–24]. The fractional calculus of variations (FCV) consists an important part of the field of the fractional calculus and it was applied successfully in transport processes, quantum field theory, control theory, plasma physics, wave propagation in complex and porous media, astrophysics, cosmology, potential theory, and so on [25–47]. The corresponding fractional Hamilton's equations were derived and numerous encouraging results were effectively obtained and discussed [42–47]. Recently, the fractional actionlike variational problems have recently gained importance in studying dynamics of nonconservative systems. The multidimensional fractional actionlike problems of the calculus of variations was considered in [48]. In [49], the generalized fractional formalism was introduced through this work includes some interesting features concerning the fractional Euler–Lagrange and Hamilton equations.

In Ref. [50], the fractional Hamiltonian system of the C-KdV soliton equation hierarchy was constructed. In [51], the fractional-order coupled Boussinesq and KdV equations are obtained. In [52], the fractional AKNS soliton equation hierarchy and its Hamiltonian system were considered. In this paper, we construct the generalized fractional trace identity through the Riemann–Liouville fractional derivative. Furthermore, the fractional soliton equation hierarchy is presented by using the fractional trace identity. We consider a fractional generalization of exterior calculus in [53]. It allows us to construct the fractional Hamiltonian systems [54]. Then, we obtain Hamiltonian systems of the fractional soliton equation hierarchy by using differential forms and exterior derivatives of fractional orders.

In Section 2, a brief review of the fractional derivatives and integrals is considered to fix notation. In Section 3, differential forms are suggested. In Section 4, the fractional form spaces are given to fix notations and provide a convenient reference.

E-mail addresses: yfajun@163.com, yufajun888@163.com.

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.04.043

In Section 5, we present a fractional Hamilton systems. In Section 6, an example of the fractional KN equation hierarchy is discussed.

2. Brief review of fractional derivatives and integrals

The derivative of arbitrary real order α can be considered as an interpolation of this sequence of operators. We will use for it the notion suggested and used by Davis [55], namely

$$ {}_a\mathbf{D}_t^{\alpha} f(t). \tag{1}$$

The short name for derivatives of arbitrary order is fractional derivatives.

The subscripts a and t denote the two limits related to the operation of fractional differentiation. Following Ross [56] we will call them the terminals of fractional differentiation. The appearance of the terminals in the symbol of fractional is essential. This helps to avoid ambiguities in applications of fractional derivatives to real problems.

Integrals of arbitrary order $p > 0$

$$ {}_a\mathbf{D}_t^{-p} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{p+k}}{\Gamma(p+k+1)} + \frac{1}{\Gamma(p+k+1)} \int_a^t (t-\tau)^{p+m} f^{(m+1)}(\tau)\mathrm{d}\tau. \tag{2}$$

The formula (2) immediately provides us with the asymptotic of ${}_a\mathbf{D}_t^{-p} f(t)$ at $t = 0$.

Derivatives of arbitrary order

$$\begin{aligned} {}_a\mathbf{D}_t^{p} f(t) &= \lim_{h\to\infty, nh\to t-a} f_n^{(p)}(t) \\ &= \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+k+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau)\mathrm{d}\tau. \end{aligned} \tag{3}$$

The formula (3) has been obtained under the assumption that the derivatives $f^{(k)}(t)$, $(k = 1, 2, \ldots, m+1)$ are continuous in closed interval $[a, t]$ and that m is an integer number satisfying the condition $m > p + 1$. The smallest probable value for m is determined by the inequality $m < p < m + 1$.

From the pure mathematical point of view of such a class of functions is narrow. However this class of functions is very important for applications. Understanding this fact is important for the proper use of the methods of the fractional calculus in applications, especially because of the fact the Riemann–Liouville definition

$$ {}_\alpha\mathbf{D}_t^{p} f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} \int_\alpha^t (t-\tau)^{m-p} f(\tau)\mathrm{d}\tau, \quad (m \le p \le m+1) \tag{4}$$

provides an excellent opportunity to weaken the conditions on the function $f(t)$.

Let us look how the Riemann–Liouville definition (4) appears as the result of the unification of the notions of integer-order integration and differentiation. Let us suppose that the function $f(\tau)$ is continuous and integrable in every finite interval (a, t); the function $f(t)$ may have on integrable singularity of order $r < 1$ at the point $\tau = a$

$$\lim_{\tau\to a} (\tau - a)^r f(t) = \mathrm{const}(\neq 0).$$

Then the integral

$$ f^{-1}(t) = \int_\alpha^t f(\tau)\mathrm{d}\tau,$$

exists and has a finite value, namely equal to 0, as $t \to a$.

In the general case we have the Cauchy formula

$$ f^{-n}(t) = \frac{1}{\Gamma(n)} \int_\alpha^t (t-\tau)^{n-1} f(\tau)\mathrm{d}\tau. \tag{5}$$

To extend the notion if n-fold integration to non-integer values of n, we can start with the Cauchy formula (5) and replace the integer n in it by a real $p > 0$,

$$ {}_\alpha\mathbf{D}_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_\alpha^t (t-\tau)^{p-1} f(\tau)\mathrm{d}\tau. \tag{6}$$

In (5) the integer n must satisfy the condition $n \ge 1$; the responding condition for p is weak; for the existence of integral (6) we must have $p > 0$. Moreover, under certain reasonable assumptions

$$\lim_{p\to 0} {}_\alpha\mathbf{D}_t^{-p} f(t) = f(t). \tag{7}$$

So we can put ${}_{\alpha}\mathbf{D}_t^0 f(t) = f(t)$. If $f(t)$ is continuous for $t \geq \alpha$, the integration of arbitrary real order defined by (6) has the follow important property

$$ {}_{\alpha}\mathbf{D}_t^{-p}({}_{\alpha}\mathbf{D}_t^{-q})f(t) = {}_{\alpha}\mathbf{D}_t^{-p-q}f(t). \tag{8}$$

Indeed, we have

$$\begin{aligned} {}_{\alpha}\mathbf{D}_t^{-p}({}_{\alpha}\mathbf{D}_t^{-q})f(t) &= \frac{1}{\Gamma(q)}\int_{\alpha}^{t}(t-\tau)^{q-1}{}_{\alpha}\mathbf{D}_t^{-p}f(\tau)\mathrm{d}\tau \\ &= \frac{1}{\Gamma(p+q)}\int_{\alpha}^{t}(t-\xi)^{p+q-1}f(\xi)\mathrm{d}\xi \\ &= {}_{\alpha}\mathbf{D}_t^{-p-q}f(t). \end{aligned}$$

Obviously, we can interchange p and q, so we have

$$ {}_{\alpha}\mathbf{D}_t^{-p}({}_{\alpha}\mathbf{D}_t^{-q})f(t) = {}_{\alpha}\mathbf{D}_t^{-q}({}_{\alpha}\mathbf{D}_t^{-p})f(t) = {}_{\alpha}\mathbf{D}_t^{-p-q}f(t). \tag{9}$$

The representation (5) for the derivative of an integer order provides an opportunity for extending the notion of differentiation to non-integer n. So that $k - \alpha > 0$, this gives

$$ {}_{\alpha}\mathbf{D}_t^{k-\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}k}{\mathrm{d}t^k}\int_{\alpha}^{t}(t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau, \quad (0 \leq \alpha \leq 1), \tag{10}$$

where the only substantial restriction for α is $\alpha > 0$, which is necessary for the convergence of the integral in (10).

Let us consider some properties of the Riemann–Liouville fractional derivatives. The first and maybe the most important property of the Riemann–Liouville fractional derivative is that for $p > 0$ and $t > \alpha$

$$ {}_{\alpha}\mathbf{D}_t^{-p}({}_{\alpha}\mathbf{D}_t^{p}f(t)) = f(t), \tag{11}$$

which means that the Riemann–Liouville fractional differentiation operator is a left inverse to the Riemann–Liouville fractional integration operator of the same order p.

As with conventional integer-order differentiation and integration, fractional differentiation and integration do not commute.

If the fractional derivative ${}_{\alpha}\mathbf{D}_t^{p}f(t)$, $(k-1 \leq p < k)$, of a function $f(t)$ is integrable, then

$$ {}_{\alpha}\mathbf{D}_t^{-p}({}_{\alpha}\mathbf{D}_t^{p}f(t)) = f(t) - \sum_{j=1}^{k}[{}_{\alpha}\mathbf{D}_t^{p-j}f(t)]_{t=a}\frac{(t-a)^{p-j}}{\Gamma(p-j+1)}. \tag{12}$$

An important particular case must be motioned, if $0 < p < 1$, it gives rise to

$$ {}_{\alpha}\mathbf{D}_t^{-p}({}_{\alpha}\mathbf{D}_t^{p}f(t)) = f(t) - [{}_{\alpha}\mathbf{D}_t^{p-1}f(t)]_{t=\alpha}\frac{(t-a)^{p-1}}{\Gamma(p)}. \tag{13}$$

3. Brief review of differential forms

The calculus of differential forms is an elegant branch of pure mathematics and a powerful tool in applied mathematics. A clear introduction to the field, with emphasis on applications, is given in [10]. Vector spaces at a point $P \in E^n$ (n dimensional Euclidean space) can be constructed out of expressions of the following type:

$$\text{one forms,} \quad a = \sum_{i=1}^{n} a_i \mathrm{d}x_i, \tag{14}$$

$$\text{two forms,} \quad \beta = \sum_{i,j=1}^{n} b_{ij}\mathrm{d}x_i \wedge \mathrm{d}x_j, \tag{15}$$

$\vdots$

$$n \text{ forms,} \quad \omega = w\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n, \tag{16}$$

where the $\{x_i\}$ are the Cartesian coordinates of E^n. The above sums are taken over all possible values of the indices with the constraint that

$$\mathrm{d}x_i \wedge \mathrm{d}x_j = -\mathrm{d}x_j \wedge \mathrm{d}x_i. \tag{17}$$

The functions a_i, b_{ij}, etc., depend only on P and may be real or complex depending on the application. If a k form, γ, is multiplying an m form, μ, the following would be true:

$$\gamma \wedge \mu = (-1)^{km}\mu \wedge \gamma. \tag{18}$$

The result would be zero if $k + m > n$. The exterior product $\wedge$ is distributive, and antisymmetric. The dimension of the vector space of k forms over $P \in E^n$ is

$$\binom{n}{k} = \frac{n!}{k!(n-k)}$$

which is zero if $k > n$. For the purposes of this paper let $F(k, k, n)$ denote the vector space of k forms over $P \in E^n$. The apparently redundant n in the above notation will be needed later for the fractional form case, as there is some additional freedom.

The exterior derivative is defined as

$$\mathrm{d} = \sum_{i=1}^{n} \mathrm{d}x_i \frac{\partial}{\partial x_i}. \tag{19}$$

The exterior derivative maps k forms into $k+1$ forms and has the following algebraic results. Let γ and λ be k forms, and μ be an m form, then

$$\mathrm{d}(\gamma + \lambda) = \mathrm{d}\gamma + \mathrm{d}\lambda, \tag{20}$$

$$\mathrm{d}(\gamma \wedge \mu) = (\mathrm{d}\gamma) \wedge \mu + (-1)^k \gamma \wedge \mathrm{d}\mu, \tag{21}$$

$$\mathrm{d}(\mathrm{d}\gamma) = 0. \tag{22}$$

The last identity is called the Poincaré lemma. A form γ is called closed if $\mathrm{d}\gamma = 0$. A form γ is called exact if there exists a form μ such that $\mathrm{d}\mu = \gamma$. The order of μ is one less than the order of γ. Exact forms are always closed. Closed forms are not always exact.

Next, the fractional exterior derivative is introduced

$$\mathrm{d}^\alpha = (\mathrm{d}x_i)^\alpha \mathbf{D}^\alpha_{x_i}. \tag{23}$$

A differential 1-form is defined by

$$\omega_\alpha = F^i(x)(\mathrm{d}x_i)^\alpha \tag{24}$$

with the vector field $F^i(x)$ can be represented as $F^i(x) = -\mathbf{D}^\alpha_{x_i} V$ and $V(x)$ is a continuously differentiable function. Using (23) the exact fractional form can be represented as

$$\omega_\alpha = -\mathrm{d}^\alpha V = -(\mathrm{d}x_i)^\alpha \mathbf{D}^\alpha_{x_i} V. \tag{25}$$

Note that Eq. (24) is a fractional generalization of the differential form (19). Obviously that fractional 1-form ω_α can be closed when the differential 1-form $\omega = \omega_1$ is not closed.

4. Fractional form spaces

If the partial derivatives in the definition if the exterior derivative are allowed to assume fractional orders, a fractional exterior derivative can be defined

$$\mathbf{D}^\alpha = \sum_{i=1}^{n} \mathbf{D}_{x_i^\alpha} \frac{\partial^\alpha}{(\partial(x_i - a_i))^\alpha}. \tag{26}$$

Note that the subscript i denote number, the superscript α denotes the order of the fractional coordinate differential, and a_i is the initial point of the derivative. Sometimes the notation ∂_i^α will be used to denote

$$\frac{\partial^\alpha}{(\partial(x_i - a_i))^\alpha}.$$

In two dimension (x, y), the fractional exterior derivative of order α of x^p, with the initial point taken to be the origin, is given by

$$\mathbf{D}^\alpha x^p = \mathbf{D}_{x^\alpha} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha} + \mathbf{D}_{y^\alpha} \frac{x^p}{y^\alpha \Gamma(1-\alpha)}. \tag{27}$$

For specific values of the derivative parameter the following results are obtained:

$$\begin{aligned} &\alpha = 0, \qquad \mathbf{D}^0 x^p = 2x^p, \\ &\alpha = 1, \qquad \mathbf{D}^1 x^p = \mathbf{D}_{x^1} p x^{p-1}, \\ &\alpha = 2, \qquad \mathbf{D}^2 x^p = \mathbf{D}_{x^2} p(p-1) x^{p-2}. \end{aligned} \tag{28}$$

By analogy with standard exterior calculus, vector spaces can be constructed using the $\mathrm{d}x_i^\alpha$. Let $F(\alpha, m, n)$ be a vector space at $P \in E^n$. α denotes the sum of the fractional differential orders of the basis elements, m denotes the number of coordinate differentials appearing in the basis element, n the number of coordinates, and $\{x_i\}$ are the Cartesian coordinates for E^n. For example, a basis set for $F(\alpha, 1, n)$ would be $\{dx_1^\alpha, dx_2^\alpha, \ldots, dx_n^\alpha\}$ and arbitrary element of $F(\alpha, 1, n)$ would be expressed as

$$a = \sum_{i=1}^{n} a_i \mathbf{D} x_i^\alpha.$$

For a fixed α this is an n dimensional vector space. Also note that there is a different vector space for each value of α. For $\alpha = 1$ the one forms from exterior calculus are recovered. Now suppose that the basis elements are made up of two coordinate differentials $F(\alpha, 2, n)$. In this case the basis set is more complicated,

$$\{\mathbf{D}x_1^{\mu_{11}} \wedge \mathbf{D}x_1^{\mu_{21}}, \mathbf{D}x_1^{\mu_{11}} \wedge \mathbf{D}x_2^{\mu_{31}}, \ldots, \mathbf{D}x_n^{\mu_{n-1m}} \wedge \mathbf{D}x_n^{\mu_{nm}} | \mu_{ij} + \mu_{kj} = \alpha\}. \tag{29}$$

Note that $dx_1^{\mu_{11}} \wedge dx_1^{\mu_{21}}$ would be zero if and only of $\mu_{11} = \mu_{21}$, etc. An arbitrary element of $F(\alpha, 2, n)$ would be expressed as a sum of the form

$$\beta = \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^\alpha (\beta_{ij}(\alpha_1, \alpha - \alpha_1) \mathbf{D}x_i^{\alpha_1} \wedge \mathbf{D}x_j^{\alpha-\alpha_1}) \mathbf{D}\alpha_1,$$

where $\beta_{ii}(\mu, \mu) = 0$. Unlike the previous vector space, $F(\alpha, 1, n)$, this is clearly infinite dimensional for any value of α. Not only is it infinite but it is uncountably infinite. An arbitrary element of $F(\alpha, 3, n)$ would be expressed as an integral of the form

$$\beta = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \int_0^\alpha \int_0^{\alpha-\alpha_1} (\beta_{ijk}(\alpha_1, \alpha - \alpha_2, \alpha - \alpha_1 - \alpha_2) \mathbf{D}x_i^{\alpha_1} \wedge \mathbf{D}x_j^{\alpha_2} \wedge \mathbf{D}x_k^{\alpha-\alpha_1-\alpha_2}) \mathbf{D}\alpha_2 \mathbf{D}\alpha.$$

With each step up on the middle index of $F(\alpha, m, n)$ another integral and summation is included. A basis set for this vector space would be

$$F(\alpha, m, n) = \left\{ \mathbf{D}x_{\alpha_1}^{\mu_{11}} \wedge \mathbf{D}x_{\alpha_2}^{\mu_{21}} \wedge \cdots \wedge \mathbf{D}x_{\alpha_m}^{\mu_{m1}}, \cdots \left| \sum_{k=1}^{m} \mu_{ij} = \alpha \right. \right\}. \tag{30}$$

The basis elements range over all possible combinations of the fractional coordinate differentials and all possible choices for the μ. Note that m need not be less than or equal to n.

Let $P \in E^n$, and let $A \in F(\alpha, m, n)$ and $B \in F(\mu, k, n)$ at the point P. Then the exterior product of A and B maintains the antisymmetry property of Eq. (18),

$$A \wedge B = (-1)^{km} B \wedge A \in F(\mu + \alpha, k + 1, n). \tag{31}$$

If $k + m > n$, $A \wedge B$ need not be zero. Eq. (20) is also maintained due to the linearity of the fractional derivative. Eq. (21) is not maintained due to the product rule for the fractional derivative. Note also that d^α maps $F(\mu, m, n)$ into $F(\mu + \alpha, k + 1, n)$.

5. A fractional Hamiltonian system

Fractional phase space can be consider as a phase space of the systems that are described by the fractional powers of coordinates and momenta. Let us consider the class of Hamiltonian systems that are constructed in the fractional phase space. Nowadays it is well known that a great number of nonlinear evolution equations, arising form various branches of physics, they admit the zero curvature equation

$$U_t - V_x + [U, V] = 0. \tag{32}$$

Our target is to write the fractional zero curvature equation, which has more excellent instrument than the integer form.

$$\mathbf{D}_t^\alpha U - \mathbf{D}_x^\alpha V^{(n)} + [U, V^{(n)}] = 0. \tag{33}$$

Fractional generalization of the differential form [54]: $\beta = G^i dp_i - F^i dq_i$ (the differential 1-form), which is used in the definition of the Hamiltonian system, can be defined in the following form:

$$\beta_\alpha = G^i (dp_i)^\alpha - F^i (dq_i)^\alpha.$$

Let us consider the canonical coordinates $(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{2n}) = (q^1, \ldots, q^n, p^1, \ldots, p^n)$ in the phase space R^{2n} and a dynamical systems can be defined by the equations

$$\mathbf{D}_t^\alpha \begin{pmatrix} q_i \\ p_i \end{pmatrix} = \begin{pmatrix} \mathbf{D}_{p_i}^\alpha & 0 \\ 0 & -\mathbf{D}_{q_i}^\alpha \end{pmatrix} \begin{pmatrix} H_n \\ H_n \end{pmatrix} = J \begin{pmatrix} H_n \\ H_n \end{pmatrix}, \tag{34}$$

with

$$\mathbf{D}^{\alpha}_{p_i} H_n = G^i(p,q), \qquad \mathbf{D}^{\alpha}_{q_i} H_n = F^i(p,q). \tag{35}$$

The fractional generalization of Hamiltonian system can be defined by the equations generalization of differential forms [53].

A dynamical system (34) on the phase space R^{2n} it called a fractional Hamiltonian system if the fractional differential 1-form $\beta_\alpha = G^i(\mathrm{d}p_i)^\alpha - F^i(\mathrm{d}q_i)^\alpha$ is a closed fractional form $d^\alpha \beta_\alpha = 0$, where d^α is the fractional exterior derivative. A dynamical system is called a fractional non-Hamiltonian system if the fractional differential 1-form β_α is a nonclosed fractional form $d^\alpha \beta_\alpha \neq 0$.

A central and very important subject in the theory of integrable system is to search for a sequence of scalar functions $\{H_n\}$ such that Eq. (33) can be cast in the Hamiltonian form

$$\mathbf{D}^{\alpha}_{t} \begin{pmatrix} q_i \\ p_i \end{pmatrix} = \begin{pmatrix} \mathbf{D}^{\alpha}_{p_i} & 0 \\ 0 & -\mathbf{D}^{\alpha}_{q_i} \end{pmatrix} \begin{pmatrix} H_n \\ H_n \end{pmatrix}.$$

The Hamiltonian $\{H_n\}$ constitute, in fact, an infinite number of conserved densities of the hierarchy (34). Various techniques have been developed to calculate $\{H_n\}$.

In this paper, we construct the generalized fractional trace identity through the Riemann–Liouville fractional derivative.

To fix the notation we recall briefly some basis notions form the theory of Hamiltonian equations. Let S be the Schwartz space over $\mathcal{R}$, $S^M = S \otimes \cdots \otimes S$(M times). The operator $\partial = \frac{\mathrm{d}}{\mathrm{d}x}$ introduces an equivalence relation among elements of S^M,

$$f \sim g \Longleftrightarrow \exists h \quad \text{such that}\, f - g = \partial h (f, g, h \in S^M). \tag{36}$$

The equivalence class that contains f is denoted by $\int f \mathrm{d}x$

$$\int f \mathrm{d}x = \{f + \partial h \mid h \in S^M\}.$$

The scalar product between f and g is defined by

$$(f, g) = \int f \cdot g \mathrm{d}x = \int \sum_i f_i g_i \mathrm{d}x. \tag{37}$$

Let $u = u(x,t) = u(u_1, \ldots, u_M)$ be a smooth function that belong to S^M for any fixed t. A linear and skew-symmetric operator $J = J(u) : S^M \to S^M$ is called symplectic if it holds that

$$(J'(u)[Jf]g, h) + (J'(u)[Jg]h, f) + (J'(u)[Jh]f, g) = 0, \tag{38}$$

for any triplet $f, g, h \in S^M$, where $J'(u)[f]$ represents the Frechet derivative of J,

$$J'(u)[f] = \frac{\mathrm{d}}{\mathrm{d}\epsilon} J(u + \epsilon f)|_{\epsilon=0}. \tag{39}$$

An operator $N(u) : S^M \to S^M$ is called hierarchy if

$$N'(u)[Nf]g - N'(u)[Ng]f = N(N'(u)[f]g - N'(u)[g]f).$$

It is known that if J is symplectic then the bracket

$$\{f, g\} = \left(\frac{\delta f}{\delta u}, J\frac{\delta g}{\delta u}\right) \tag{40}$$

is a well-defined Poisson bracket, and we call the equation $u_t = J\frac{\delta H}{\delta u}$ the Hamiltonian equation.

Our next target is to write the fractional Hamiltonian equation in the fractional form spaces E^n by applying the modified fractional derivatives and integrals.

Let G be a matrix semi-simple Lie algebra. We define that the form $f(A, B)$ is proportional to the trace $\mathrm{tr}(AB)$, $f(A, B) =$ const $\mathrm{tr}(AB)$. For notational convenience, we thus take, in this section,

$$f(A, B) = \mathrm{tr}(AB). \tag{41}$$

The fractional gradient $\bigtriangledown_B f(A, B)$ of the functional $f(A, B)$ is defined by

$$\frac{\partial}{\partial_\epsilon} f(A, B + \epsilon C) = f(\delta_B f(A, B), C), \quad A, B, C \in G, \tag{42}$$

where δ_B is variational derivative with respect to B.

We construct a function with fractional derivative

$$W = f(V, U_\lambda) + f(K, V_x^{(\alpha)} - [U, V]),$$

with the defined fractional variational derivative

$$\frac{\delta W}{\delta K} = V_x^{(\alpha)} - [U, V], \qquad \frac{\delta W}{\delta V} = U_\lambda - K_x^{(\alpha)} + [U, V], \tag{43}$$

according to the above equations, we can obtain

$$[K, V]_x^{(\alpha)} = [U_\lambda, V] + [U, [K, V]]. \tag{44}$$

Suppose we have introduced in some manner the rank for u, D_x^α, λ, and $x \in \tilde{G}$. Under the supposition that the solution V of (43), which is of given homogeneous rank, is unique up to a constant multiplier, it proved that for any solution V of homogeneous rank, there exists a constant γ such that for $\bar{V} = \lambda^\gamma V$, which is again a solution of (43).

Consider the communication relationship

$$f([A, B], C) = f(A, [B, C]) \tag{45}$$

we can get a fractional trace identity as follows

$$\begin{aligned}
\frac{\delta f(V, U_\lambda)}{\delta u_i} &= f\left(V, \frac{\partial U_\lambda}{\partial u_i}\right) + f\left([K, V], \frac{\partial U_\lambda}{\partial u_i}\right) = f\left(V, \frac{\partial U_\lambda}{\partial u_i}\right) + f\left(V_\lambda, \frac{\partial U_\lambda}{\partial u_i}\right) + \frac{\gamma}{\lambda} f\left(V, \frac{\partial U}{\partial u_i}\right) \\
&= \frac{\partial}{\partial \lambda} f\left(V, \frac{\partial U}{\partial u_i}\right) + f\left(V_\lambda, \frac{\partial U_\lambda}{\partial u_i}\right) + \left(\lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma\right) f\left(V, \frac{\partial U}{\partial u_i}\right) \\
&= \lambda^{-\gamma} \frac{\partial}{\partial \lambda}\left[\lambda^\gamma f\left(V, \frac{\partial U}{\partial u_i}\right)\right], \quad 0 \le i \le n.
\end{aligned} \tag{46}$$

We have shown that a number of known formulas obtained in the literature by complicated calculations are spectral cases of the above generalized fractional trace identity. To the best of our knowledge, the fractional trace identity and the fractional soliton equation hierarchy are completely new equations in mathematics physics.

6. The generalized fractional KN equation hierarchy

By using the zero curvature equation, many soliton equations hierarchies were presented in [57–65]. To illustrate our method, we want to apply the fractional zero curvature equation and fractional Hamiltonian structure to construct the generalized fractional KN equation hierarchy.

Consider the following matrix spectral problem

$$\mathbf{D}_x^\alpha \phi = U(u, \lambda)\phi, \quad U(u, \lambda) = \begin{pmatrix} -\lambda^2 & \lambda q \\ \lambda r & \lambda^2 \end{pmatrix} = U_0 \lambda^2 + \lambda U_1, \qquad \mathbf{D}_\lambda^\alpha U_0 = \mathbf{D}_\lambda^\alpha U_1 = 0, \tag{47}$$

where λ is a spectral parameter. Because U_0 has multiple eigenvalues, the spectral problem (47) is degenerate. The spectral problem (47) is called KN spectral problem.

To derive an associated soliton hierarchy, we first solve the adjoint equation

$$\mathbf{D}_x^\alpha W = [U, W] \tag{48}$$

of the spectral problem (47) through the generalized Tu scheme [57]. We assume that a solution W is given by

$$W = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}. \tag{49}$$

Then we have

$$[U, W] = \begin{pmatrix} \lambda qc - \lambda rb & -2\lambda^2 b + \lambda qd - \lambda qa \\ 2\lambda^2 c + \lambda ra - \lambda rd & \lambda rb - \lambda qc \end{pmatrix}.$$

Therefore, the adjoint equation (48) is equivalent to

$$\mathbf{D}_x^\alpha a = \lambda qc - \lambda rb, \tag{50a}$$

$$\mathbf{D}_x^\alpha b = -2\lambda^2 b + \lambda qd - \lambda qa, \tag{50b}$$

$$\mathbf{D}_x^\alpha c = 2\lambda^2 c + \lambda ra - \lambda rd. \tag{50c}$$

Let us seek a formal solution of the type

$$W = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{k=0}^{\infty} W_k \lambda^{-k} = \sum_{k=0}^{\infty} \begin{pmatrix} a^{(k)} & b^{(k)} \\ c^{(k)} & -a^{(k)} \end{pmatrix} \lambda^{-k}. \tag{51}$$

Therefore, the condition (48) becomes the following recursion relations:

$$\begin{cases} a^{(0)} = -1, \\ a^{(2n+1)} = b^{(2n)} = c^{(2n)} = 0, \\ \mathbf{D}_x^{\alpha} a^{(2n)} = qc^{(2n+1)} - rb^{(2n+1)}, \\ b^{(2n+1)} = -\dfrac{1}{2}(\mathbf{D}_x^{\alpha} b^{(2n-1)} + qa^{(2n)} - qd^{(2n)}), \\ c^{(2n+1)} = \dfrac{1}{2}(\mathbf{D}_x^{\alpha} c^{(2n-1)} + rd^{(2n)} - ra^{(2n)}). \end{cases} \tag{52}$$

The compatibility condition, i.e., the generalized zero curvature equation $\mathbf{D}_{t_n}^{\alpha} U - \mathbf{D}_x^{\alpha} V^{(n)} + [U, V^{(n)}] = 0$, leads to a system of evolution equations

$$\mathbf{D}_{t_n}^{\alpha} \begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} \mathbf{D}_x^{\alpha} b^{(2j+1)} \\ \mathbf{D}_x^{\alpha} c^{(2j+1)} \end{pmatrix}, \tag{53}$$

which is the generalized fractional version of the KN nonlinear equation hierarchy.

According to Eq. (53), we can obtain the Hamiltonian structure of the fractional KN hierarchy

$$H_n = \frac{a_{2n+2}}{n}, \quad n = 1, 2, \ldots, \tag{54}$$

where the Hamiltonian operator J is defined by

$$J = \begin{pmatrix} 0 & \dfrac{1}{2}\mathbf{D}_x^{\alpha} \\ -\dfrac{1}{2}\mathbf{D}_x^{\alpha} & 0 \end{pmatrix}. \tag{55}$$

A technically feasible way to construct the generalized Hamiltonian structure and the application of the theory to the fractional soliton equation hierarchy have been made. The key idea in our target is to establish a new Hamiltonian system in fractional form spaces E^n. There is an open problem. How do we prove that it is Liouville integrable? which is worthwhile studying in the future.

Acknowledgment

This work was supported by the Research Work of Liaoning Provincial Development of Education (L2010513).

References

[1] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives Theory and Applications, Gordon and Breach, New York, 1993.
[2] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[3] G.M. Zaslavsky, Fractional kinetics and anomalous transport, Phys. Rep. 371 (2002) 461–580.
[4] G.M. Zaslavsky, Hamiltonian Chaos and Fractional Dynamics, Oxford University Press, Oxford, 2005.
[5] V.E. Tarasov, G.M. Zaslavsky, Dynamics with low-level fractionality, Physica A 354 (2005) 249–261.
[6] V.E. Tarasov, Fractional Fokker–Planck equation for fractal media, Chaos 14 (2004) 123–127.
[7] V.E. Tarasov, Fractional systems and fractional Bogoliubov hierarchy equations, Phys. Rev. E 71 (2005) 011102.
[8] V.E. Tarasov, Fractional Liouville and BBGKI equations, J. Phys.: Conf. Ser. 7 (2005) 17–33.
[9] R. Nigmatullin, The realization of the generalized transfer in a medium with fractal geometry, Phys. Status Solidi b 133 (1986) 425–430.
[10] H. Flanders, Differential Forms with Applications to Physical Sciences, Dover, New York, 1989.
[11] R. Metzler, J. Klafter, The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics, J. Phys. A 37 (2004) 161–208.
[12] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[13] A. Carpinteri, F. Mainardi, Fractals and Fractional Calculus in Continuum Me-chanics, Springer Verlag, Wien, New York, 1997.
[14] J. Sabatier, O.P. Agrawal, J.A. Tenreiro Machado, Advance in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, Springer, 2007.
[15] D. Baleanu, Z.B. Guvenc, J.A. Machado Tenreiro, New Trends in Nanotechnology and Fractional Calculus Applications, Springer, 2009.
[16] R. Hilfer, Application of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[17] R. Goreno, F. Mainardi, Fractional Calculus: Integral and Differential Equations of Fractional Orders, Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, New York, 1997.
[18] A.A. Kilbas, H.H. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, The Netherlands, 2006.
[19] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic, New York, 1974.
[20] K.S. Miller, B. Ross, An Introduction to the Fractional Integrals and Derivatives-Theory and Application, Wiley, New York, 1993.
[21] I. Podlubny, Fractional Differential Equations, Academic, New York, 1999.
[22] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives Theory and Applications, Gordon and Breach, New York, 1993.
[23] B.J. West, M. Bologna, P. Grigolini, Physics of Fractal Operators, New York, 2003.
[24] G.M. Zaslavsky, Hamiltonian Chaos and Fractional Dynamics, Oxford University Press, 2005.
[25] F. Riewe, Mechanics with fractional derivatives, Phys. Rev. E 55 (1997) 3581–3592.
[26] A.K. Golmankahneh, A.K. Golmankahneh, D. Baleanu, M.C. Baleanu, Hamiltonian structure of fractional first order Lagrangian, Internat. J. Theoret. Phys. 49 (2010) 365–375.
[27] R.L. Bagley, P.J. Torvik, On the fractional calculus model of viscoelastic behavior, J. Rheol. 30 (1) (1986) 133–155.
[28] T. Pritz, Analysis of four-parameter fractional derivative model of real solid materials, J. Sound Vib. 195 (1) (1996) 103–115.

[29] O.P. Agrawal, Fractional variational calculus in terms of Riesz fractional derivatives, J. Math. Anal. Appl. 272 (2002) 368–379.
[30] L. Yuan, O.P. Agrawal, A numerical scheme for dynamic systems containing fractional derivatives, J. Vib. Acoust. 124 (2002) 321–324.
[31] T.M. Atanackovic, B. Stankovic, An expansion formula for fractional derivatives, Fract. Calc. Appl. Anal. 7 (2004) 365–378.
[32] D. Baleanu, S. Muslih, Lagrangian, formulation of classical fields within Riemann–Liouville fractional derivatives, Phys. Scr. 72 (2005) 119–121.
[33] C.F. Lorenzo, T.T. Hartley, Fractional trigonometry and the spiral functions, Nonlinear Dynam. 38 (2004) 23–60.
[34] G.S.F. Frederico, D.F.M. Torres, A formulation of Noether's theorem for fractional problems of the calculus of variations, J. Math. Anal. Appl. 334 (2007) 834–846.
[35] K. Golmankhaneh Alireza, Fractional Poisson bracket, Turkish J. Phys. 32 (2008) 241–250.
[36] J. Cresson, Fractional embedding of differential operators and Lagrangian systems, J. Math. Phys. 48 (2007) 033504.
[37] K. Klimek, Fractional sequential mechanics-models with symmetric fractional derivatives, Czech. J. Phys. 52 (2002) 1247–1253.
[38] G.S.F. Frederico, D.F.M. Torres, Fractional conservation laws in optimal control theory, Nonlinear Dynam. 53 (3) (2008) 215–222.
[39] M.D. Ortigueira, J.A. Tenreiro Machado, Fractional calculus applications in signals and systems, Signal Process. 86 (10) (2006) 2503–2504. Elsevier (special issue).
[40] M.D. Ortigueira, J.A. Tenreiro Machado, Fractional differentiation and its applications, J. Vib. Control 14 (9–10) (2008) 1253. Sage Pub, (special issue).
[41] M.D. Ortigueira, J.A. Tenreiro Machado, Discontinuous and fractional dynamical systems, ASME J. Comput. Nonlinear Dynam. 3 (2) (2008) doi:10.1115/1.2834905. (special issue).
[42] A.E.H. Mohamed, D. Baleanu, Fractional-order Euler–Lagrange equations and formulation of Hamiltonian equations, Nonlinear Dyn. 58 (2009) 385–391.
[43] R.A. El-Nabulsi, D.F.M. Torres, Fractional action like variational problems, J. Math. Phys. 49 (2008) 053521.
[44] R.A. El-Nabulsi, Modifications at large distances from fractional and fractal arguments, Fractals 18 (2) (2010) 185–190.
[45] R.A. El-Nabulsi, Black hole growth and accretion energy from fractional actionlike variational approach, Fizika B 17 (3) (2008) 369–378.
[46] R.A. El-Nabulsi, On the fractional minimal length Heisenberg–Weyl uncertainty relation from fractional Riccati generalized momentum operator, Chaos Solitons Fractals 42 (2009) 84–88.
[47] R.A. El-Nabulsi, A fractional action-like variational approach of some classical, quantum and geometrical dynamic, Int. J. Appl. Math. 17 (3) (2005) 299–317.
[48] R.A. El-Nabulsi, D.F.M. Torres, Fractional action like variational problems, J. Math. Phys. 49 (2008) 053521.
[49] R.A. El-Nabulsi, Universal fractional Euler–Lagrange equation from a generalized fractional derivate operator, Cent. Eur. J. Phys. 9 (1) (2010) 250–256.
[50] F.J. Yu, H.Q. Zhang, New Hamiltonian structure of the fractional C-KdV soliton equation hierarchy, Chaos Solitons Fractals 37 (2008) 688–697.
[51] F.J. Yu, Integrable coupling system of fractional soliton equation hierarchy, Phys. Lett. A 373 (2009) 3730–3733.
[52] F.J. Yu, H.Q. Zhang, Fractional zero curvature equation and generalized Hamiltonian structure of soliton equation hierarchy, Internat. J. Theoret. Phys. 46 (2007) 3182–3192.
[53] S.K. Cottrill, M. Naber, Fractional differential forms, J. Math. Phys. 42 (2001) 2203–2212.
[54] E. Tarasov, Fractional generalization of gradient and Hamiltonian systems, J. Phys. A 38 (2005) 5929–5943.
[55] H.D. Davis, The Theory of Linear Operators, Principia Press, Bloomington, Indiana, 1936.
[56] B. Ross, Fractional calculus, Math. Mag. 50 (3) (1977) 115–122.
[57] G.T. Tu, The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems, J. Math. Phys. 30 (2) (1989) 330–338.
[58] D.J. Kaup, A.C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys. 19 (1978) 798–801.
[59] T.C. Xia, H. Wang, Y.F. Zhang, The multi-component Tu hierarchy of soliton equations and its multi-component integrable couplings system, Chin. Phys. 14 (2) (2005) 247–250.
[60] Y.F. Zhang, Y.S. Zhang, A type of multi-component integrable hierarchy, Chin. Phys. 13 (8) (2004) 1183–1186.
[61] E.G. Fan, Integrable evolution systems based on Gerdjikov–Ivanov equations, bi-Hamiltonian structure, finite-dimensional integrable systems and N-fold Darboux transformation, J. Math. Phys. 41 (11) (2000) 7769–7782.
[62] W.X. Ma, Enlarging spectral problems to construct integrable couplings of soliton equations, Phys. Lett. A 316 (2003) 72–76.
[63] Y.F. Zhang, A multi-component matrix loop algebra and a unified expression of the multi-component AKNS hierarchy and the multi-component BPT hierarchy, Phys. Lett. A 342 (2005) 82–89.
[64] T.C. Xia, F.J. Yu, Y. Zhang, The multi-component coupled Burgers hierarchy of soliton equations and its multi-component integrable couplings system with two arbitrary functions, Physica A 343 (2004) 238–246.
[65] F.J. Yu, T.C. Xia, H.Q. Zhang, The multi-component TD hierarchy and its multi-component integrable coupling system with five arbitrary functions, Chaos Solitons Fractals 27 (2006) 1036–1041.