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NODAL VARIABLES FOR COMPLETE CONFORMING FINITE ELEMENTS OF ARBITRARY POLYNOMIAL ORDER

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Abstract—Nodal variables are given for a new family of complete conforming triangular finite elements of arbitrary polynomial order p for use in linear stress analysis. This family has two important properties: (1) hierarchic property, i.e. the elemental stiffness matrix corresponding to an approximation of order p is a submatrix of the elemental stiffness matrix corresponding to an approximation of order $p+1$; (2) the family enforces exactly the degree of smoothness across interelement boundaries which is required by the problem (C^0 continuity for plane elasticity, C^1 continuity for plate bending) even at vertices. It is shown how to use precomputed arrays in an efficient manner in calculating elemental stiffness matrices. Results from a numerical example in plane stress analysis are presented. These results demonstrate the efficiency of a p-convergence procedure which uses the new family of finite elements.

1. INTRODUCTION

In standard approaches to the finite element method the degree of the polynomial approximation to the solution is held fixed while the number of elements is increased in such a way that the maximum diameter, h, of the elements goes to zero. We call this procedure for achieving accuracy h-convergence. It is also possible, however, to obtain convergence by fixing the number of finite elements and allowing the degree p of the approximating polynomial to increase. We call this procedure p-convergence. It has been demonstrated in a variety of cases, including that of a difficult benchmark problem[1] posed by Lockheed, that the second mode of convergence is more rapid[2–5]. The p-convergence procedure also possesses the additional advantage that no change in the number or geometry of the elements is required to achieve accuracy, thereby resulting in manpower savings in data preparation and processing.

In order to implement convergence with respect to polynomial approximations of increasing degree, it is necessary to have available a family of finite element approximations of arbitrary degree p. The constraint formulation for finite element analysis provides just such a family of polynomials. In the constraint method, the total potential energy π is expressed as the sum of the potential energies π^e of the (triangular) elements. Complete pth order polynomial approximations are used[6], and the requirement that the global approximating function is continuous across interelement boundaries (C^0 continuity) or that it and its first normal derivative are continuous across interelement boundaries (C^1 continuity), is expressed by a set of linear equality constraints.

Specifically, the problem is to minimize

$$\pi = \sum_{e=1}^{n} \pi^e = \sum_{e=1}^{n} \left(\tfrac{1}{2}\lfloor a^e \rfloor [S^e]\{a^e\} - \lfloor a^e \rfloor \{Z^e\}\right) \tag{1}$$

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subject to the linear equality constraints:

$$\sum_{e=1}^{n} [P^e]\{a^e\} = \{R\} \tag{2}$$

where π is the total potential energy functional; π^e is the potential energy of the element e; n is the number of finite elements; $[S^e]$ is a positive semi-definite matrix defined over the eth element. It is often called the unconstrained stiffness matrix or the element stiffness matrix; $\{a^e\}$ is the vector of the unknown coefficients of the complete pth degree polynomial approximation over the pth finite element; $\{Z^e\}$ is the vector of the applied loads.

Equation (2) enforces interelement continuity and boundary conditions, $[P^e]$ is a matrix whose rows are polynomial expressions evaluated at specific points along the boundary of the eth element.

Equations (1) and (2) represent a constrained quadratic programming problem. This problem may be solved by separating dependent and independent variables in equation (2) and then substituting back into equation (1) to yield an unconstrained minimization problem. Gaussian elimination is used in [7] to find independent variables among all the coefficients in the set of $\{a^e\}$. This approach, however, is inefficient and unsuitable for large problems. In [8], nodal variables denoted by $\{\delta\}$ are introduced and the constraints given in equation (2) are rewritten in the form

$$[A]\{a\} - [N]\{\delta\} = 0 \tag{3}$$

where $[A]$ is a block diagonal matrix whose ith submatrix defines nodal variables for the ith finite element; $\{a\}$ is the vector of *all* unknown polynomial coefficients for all finite elements; $[N]$ is a nodal matrix. The elements of $[N]$ are either zero or one:

$$n_{ij} = \begin{cases} 1 & \text{if the } i\text{th row in } [A] \text{ corresponds to nodal variable } j \\ 0 & \text{otherwise.} \end{cases}$$

$\{\delta\}$ is the vector of *all* nodal variables.

A special adaptation of the simplex method is then used in [8] for rank analyses and to find basic (dependent) variables and non-basic (independent) variables among the nodal variables and polynomial coefficients. Dependent variables are replaced in equation (1) by their expressions in terms of independent variables in a process which requires multiplication of large matrices and which utilizes sparse matrix multiplication techniques. This process may become very expensive unless the matrices are banded, and bandedness can be obtained only by a careful choice of the dependent variables. This problem does not exist if C^0 continuity only is enforced: All nodal variables are independent; some polynomial coefficients are independent too but can be eliminated by static condensation. Therefore the total potential energy is expressed in terms of nodal variables and an unconstrained minimization problem is formulated directly with respect to these nodal variables. This case corresponds to the standard finite element method and the independent polynomial coefficients can be viewed as internal nodal variables. For C^1 continuity, however, this is no longer possible (and this was one of the motivations of the constraint formulation). This problem is fully investigated in [12, 14, 20] where it is shown how to avoid the need for linear programming. We present here that part of the discussion needed to make this report self-contained. Briefly, the potential energy is again expressed in terms of nodal variables but these nodal variables must now satisfy certain constraint relations. We show how to define the nodal variables so that the constraint relations will have as simple a form as possible. This leads to an *a priori* choice for dependent and independent variables without the use of linear programming. We also show that by using these nodal variables the process of assembling the finite elements is simplified and there is no longer the time-consuming need to multiply large matrices. In either case (C^0 or C^1 continuity) we have a pth order approximating polynomial enforcing the desired level of global continuity for each p.

The most important and unique feature of the nodal variables presented here is that they produce a sequence of hierarchic stiffness matrices and therefore possess the advantages presented in [14]. The hierarchic character of the stiffness matrices is manifested by an embedding property: The elemental stiffness matrix corresponding to a polynomial approximation of order p is a submatrix of the elemental stiffness matrix corresponding to a polynomial approximation of order $p+1$. Therefore when increasing the order of polynomial approximation from p to $p+1$ it is necessary to compute only the *new* rows and columns in the new stiffness matrix.

Other families of approximating polynomials are known. In [9] for example, a hierarchy of polynomials of degree $p = 4m + k(m = 0, 1, \ldots; k = 1, 2, 3, 4)$ is given which generates functions that are m-times continuously differentiable and have piecewise continuous derivatives of order $(m+1)$. Thus, for problems requiring C^1 continuity, the elements are over-conforming when $m \geq 2$. It is important to note, however, that the rapidity of convergence observed with respect to increasing degree of polynomial approximation appears to depend upon enforcement of only the minimal degree of continuity required by the problem. Hence our insistence that our polynomials should not enforce more than C^0 or C^1 continuity, as the case requires. Also, the polynomials in [9] enforce (for $p \geq 5$) at least C^2 continuity at vertices (see also [15]). In some geometries with corners, such as the Lockheed benchmark problem [1], the solution does *not* in fact have C^2 continuity at all the corner vertices. This results in the requirement of a large number of the finite elements given in [9] for an adequate approximation. We have in mind, however, applications where few elements are used (as determined by the geometry) and accuracy is achieved instead by increasing the degree of polynomial approximation. This can best be accomplished by enforcing no more than C^1 continuity at vertices. In [5], this kind of application is shown to yield significant improvements over standard methods for the Lockheed problem. Also, the family given in [9] does not have the hierarchic property referred to earlier. Therefore, when raising the order of polynomial approximation in an element from p to $p+1$, the *entire* elemental stiffness matrix changes and must be recomputed. This is inefficient when seeking convergence by increasing the order of the approximating polynomial, which is the p-convergence procedure that we have in mind.

We note that the definition of nodal variables for the enforcement of C^1 continuity is closely related to a problem posed by Strang [10] concerning the dimension of the space of polynomials of degree p and continuity class C^1. This dimension is, in fact, the number of independent nodal variables remaining in equation (1) after dependent variables have been eliminated using equation (3). Our nodal variables, which are similar to those introduced simultaneously and independently in [11] are actually a nodal basis for this space of polynomials. This is treated in greater detail in [12]. Our main prupose here is to show that these nodal variables are computationally efficient. Additional details on the implementation of these nodal variables in a computer algorithm are given in [18].

In this paper, we first introduce a standard coordinate system for simplicity of computation. Then, we introduce nodal variables and local transformation matrices which enforce C^0 continuity for arbitrary p. After explaining the need for constraint equations in order to enforce C^1 continuity, we introduce nodal variables and local transformation matrices for this case. The simplest form of the constraint equations as well as an explicit choice of independent variables is described. We show how to compute element stiffness matrices and load vectors in standard coordinates by using precalculated arrays, and we discuss the structure of the final unconstrained minimization of the total potential energy. Finally we present a numerical example in plane stress analysis. Other examples are given in [18].

2. COORDINATE SYSTEMS

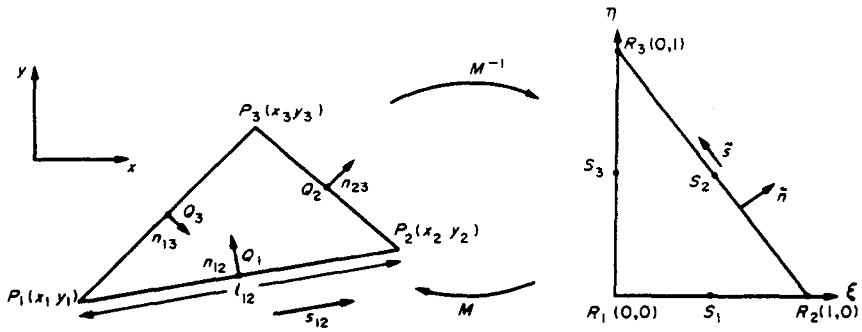
It is convenient and computationally efficient to do many of the calculations in terms of standard triangles. Let (x, y) denote the global coordinates of a point P, let $P_i(x_i, y_i)$ $i = 1, 2, 3$, be the vertices of an arbitrary triangle T in the $x-y$ plane, and let $(\bar{x}, \bar{y}) = (x - x_1, y - y_1)$. We define the transformation M from standard (ξ, η) coordinates to $(\bar{x}, \bar{y})$ coordinates as

$$M: \begin{Bmatrix} \bar{x} \\ \bar{y} \end{Bmatrix} = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \begin{bmatrix} \bar{x}_2 & \bar{x}_3 \\ \bar{y}_2 & \bar{y}_3 \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \begin{bmatrix} l_{12} \cos \theta_{12} & l_{13} \cos \theta_{13} \\ l_{12} \sin \theta_{12} & l_{13} \sin \theta_{13} \end{bmatrix} \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} \tag{4a}$$

where l_{ij} is the length of the side P_iP_j, and where $(\cos\theta_{ij}, \sin\theta_{ij})$ is a unit vector in the direction of P_iP_j. The inverse transformation M^{-1} maps the triangle T in the $x-y$ plane into the standard triangle $\tilde{T}$ with vertices at $R_i(\xi_i, \eta_i)\, i = 1, 2, 3$, in the $\xi-\eta$ plane where $(\xi_1, \eta_1) = (0, 0)$, $(\xi_2, \eta_2) = (1, 0)$, $(\xi_3, \eta_3) = (0, 1)$. This inverse transformation is given by

$$M^{-1}: \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} \bar{y}_3 & -\bar{x}_3 \\ -\bar{y}_2 & \bar{x}_2 \end{bmatrix}\begin{Bmatrix} \bar{x} \\ \bar{y} \end{Bmatrix} \tag{4b}$$

where $\Delta = \frac{1}{2}(\bar{x}_2\bar{y}_3 - \bar{x}_3\bar{y}_2)$; $|\Delta|$ is the area of T.



Let $u(\bar{x}, \bar{y})$ be a complete polynomial of degree p defined on T. Then

$$\begin{aligned} u(\bar{x}, \bar{y}) &= \sum_{0\le i+j\le p} a_{ij}\bar{x}^i\bar{y}^j \\ &= u(\bar{x}_2\xi + \bar{x}_3\eta, \bar{y}_2\xi + \bar{y}_3\eta) \\ &= \tilde{u}(\xi, \eta) = \sum_{0\le i+j\le p} \tilde{a}_{ij}\xi^i\eta^j \end{aligned} \tag{5}$$

that is, the coefficients of u considered as a pth degree polynomial in ξ and η are $\tilde{a}_{ij}$. a_{ij} and $\tilde{a}_{ij}$ can be computed explicitly in terms of each other. The precise expressions are given in Appendix 1. We now give expressions for differentiations along the sides of T in terms of differentiation in standard directions. Let s_{ij} be the direction from P_i to P_j, let n_{ij} be the direction normal to P_iP_j as shown in the figure, let $\tilde{s}$ be the direction $(-1/\sqrt{2}, 1/\sqrt{2})$, and let $\tilde{n}$ be the direction $(1/\sqrt{2}, 1/\sqrt{2})$. Then we have

$$\begin{aligned} \frac{\partial u}{\partial s_{12}} &= \frac{\partial u}{\partial x}\cos\theta_{12} + \frac{\partial u}{\partial y}\sin\theta_{12} = \frac{1}{l_{12}}\left(\bar{x}_2\frac{\partial u}{\partial x} + \bar{y}_2\frac{\partial u}{\partial y}\right) = \frac{1}{l_{12}}\frac{\partial\tilde{u}}{\partial\xi} \\ \frac{\partial u}{\partial s_{13}} &= \frac{\partial u}{\partial x}\cos\theta_{13} + \frac{\partial u}{\partial y}\sin\theta_{13} = \frac{1}{l_{13}}\left(\bar{x}_3\frac{\partial u}{\partial x} + \bar{y}_3\frac{\partial u}{\partial y}\right) = \frac{1}{l_{13}}\frac{\partial\tilde{u}}{\partial\eta} \\ \frac{\partial u}{\partial s_{23}} &= \frac{\partial u}{\partial x}\cos\theta_{23} + \frac{\partial u}{\partial y}\sin\theta_{23} = \frac{1}{l_{23}}\left((\bar{x}_3 - \bar{x}_2)\frac{\partial u}{\partial x} + (\bar{y}_3 - \bar{y}_2)\frac{\partial u}{\partial y}\right) \\ &= \frac{1}{l_{23}}\left(-\frac{\partial\tilde{u}}{\partial\xi} + \frac{\partial\tilde{u}}{\partial\eta}\right) = \frac{\sqrt{2}}{l_{23}}\frac{\partial\tilde{u}}{\partial\tilde{s}} \\ \frac{\partial u}{\partial n_{12}} &= -\sin\theta_{12}\frac{\partial u}{\partial x} + \cos\theta_{12}\frac{\partial u}{\partial y} \\ &= \frac{1}{2\Delta l_{12}}\left(-\bar{y}_2\left(\bar{y}_3\frac{\partial\tilde{u}}{\partial\xi} - \bar{y}_2\frac{\partial\tilde{u}}{\partial\eta}\right) + \bar{x}_2\left(-\bar{x}_3\frac{\partial\tilde{u}}{\partial\xi} + \bar{x}_2\frac{\partial\tilde{u}}{\partial\eta}\right)\right) \\ &= \frac{1}{2\Delta l_{12}}\left(-(\bar{x}_2\bar{x}_3 + \bar{y}_2\bar{y}_3)\frac{\partial\tilde{u}}{\partial\xi} + (\bar{x}_2{}^2 + \bar{y}_2{}^2)\frac{\partial\tilde{u}}{\partial\eta}\right) \\ &= \frac{1}{2\Delta l_{12}}\left(\frac{1}{2}(l_{23}^2 - l_{13}^2 - l_{12}^2)\frac{\partial\tilde{u}}{\partial\xi} + l_{12}^2\frac{\partial\tilde{u}}{\partial\eta}\right) = \alpha_{12}\frac{\partial\tilde{u}}{\partial\xi} + \beta_{12}\frac{\partial\tilde{u}}{\partial\eta} \end{aligned} \tag{6a}$$

$$\begin{aligned}\frac{\partial u}{\partial n_{13}} &= \sin\theta_{13}\frac{\partial u}{\partial x} - \cos\theta_{13}\frac{\partial u}{\partial y}\\ &= \frac{1}{2\Delta l_{13}}\left(\bar{y}_3\left(\bar{y}_3\frac{\partial\bar{u}}{\partial\xi} - \bar{y}_2\frac{\partial\bar{u}}{\partial\eta}\right) - \bar{x}_3\left(-\bar{x}_3\frac{\partial\bar{u}}{\partial\xi} + \bar{x}_2\frac{\partial\bar{u}}{\partial\eta}\right)\right)\\ &= \frac{1}{2\Delta l_{13}}\left((\bar{x}_3{}^2 + \bar{y}_3{}^2)\frac{\partial\bar{u}}{\partial\xi} - (\bar{x}_2\bar{x}_3 + \bar{y}_2\bar{y}_3)\frac{\partial\bar{u}}{\partial\eta}\right)\\ &= \frac{1}{2\Delta l_{13}}\left(l_{13}^2\frac{\partial\bar{u}}{\partial\xi} + \frac{1}{2}(l_{23}^2 - l_{13}^2 - l_{12}^2)\frac{\partial\bar{u}}{\partial\eta}\right) = \alpha_{13}\frac{\partial\bar{u}}{\partial\xi} + \beta_{13}\frac{\partial\bar{u}}{\partial\eta}\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial n_{23}} &= \sin\theta_{23}\frac{\partial u}{\partial x} - \cos\theta_{13}\frac{\partial u}{\partial y}\\ &= \frac{1}{2\Delta l_{23}}\left((\bar{y}_3 - \bar{y}_2)\left(\bar{y}_3\frac{\partial\bar{u}}{\partial\xi} - \bar{y}_2\frac{\partial\bar{u}}{\partial\eta}\right) - (\bar{x}_3 - \bar{x}_2)\left(-\bar{x}_3\frac{\partial\bar{u}}{\partial\xi} + \bar{x}_2\frac{\partial\bar{u}}{\partial\eta}\right)\right)\\ &= \frac{1}{2\Delta l_{23}}((\bar{x}_3{}^2 + \bar{y}_3{}^2) - (\bar{x}_2\bar{x}_3 + \bar{y}_2\bar{y}_3))\frac{\partial\bar{u}}{\partial\xi} + ((\bar{x}_2{}^2 + \bar{y}_2{}^2) - (\bar{x}_2\bar{x}_3 + \bar{y}_2\bar{y}_3))\frac{\partial\bar{u}}{\partial\eta}\\ &= \frac{1}{2\Delta l_{23}}\left(\frac{1}{2}(l_{23}^2 + l_{13}^2 - l_{12}^2)\frac{1}{\sqrt{2}}\left(-\frac{\partial\bar{u}}{\partial\bar{s}} + \frac{\partial\bar{u}}{\partial\bar{n}}\right)\right.\\ &\quad\left. + \frac{1}{2}(l_{23}^2 - l_{13}^2 + l_{12}^2)\frac{1}{\sqrt{2}}\left(\frac{\partial\bar{u}}{\partial\bar{s}} + \frac{\partial\bar{u}}{\partial\bar{n}}\right)\right)\\ &= \frac{1}{2\sqrt{2}\Delta l_{23}}\left((l_{12}^2 - l_{13}^2)\frac{\partial\bar{u}}{\partial\bar{s}} + l_{23}^2\frac{\partial\bar{u}}{\partial\bar{n}}\right) = \alpha_{23}\frac{\partial\bar{u}}{\partial\bar{s}} + \beta_{23}\frac{\partial\bar{u}}{\partial\bar{n}}\end{aligned}$$

where

$$\begin{aligned}\alpha_{12} &= \frac{1}{4\Delta l_{12}}(l_{23}^2 - l_{13}^2 - l_{12}^2) & \beta_{12} &= \frac{l_{12}}{2\Delta}\\ \alpha_{13} &= \frac{l_{13}}{2\Delta} & \beta_{13} &= \frac{1}{4\Delta l_{13}}(l_{23}^2 - l_{13}^2 - l_{12}^2)\\ \alpha_{23} &= \frac{1}{2\sqrt{2}\Delta l_{23}}(l_{12}^2 - l_{13}^2) & \beta_{23} &= \frac{l_{23}}{2\sqrt{2}\Delta}.\end{aligned} \tag{6b}$$

3. NODAL VARIABLES AND TRANSFORMATION MATRICES FOR C^0 CONTINUITY

We first define the nodal variables used to assure C^0 continuity across interelement boundaries. We refer to these as boundary or external nodal variables. Let Q_1, Q_2, Q_3, be the midpoints of sides P_1P_2, P_2P_3, and P_3P_1 respectively and let the images under the mapping M^{-1} of Q_1, Q_2 and Q_3 be $S_1(1/2, 0)$, $S_2(1/2, 1/2)$ and $S_3(0, 1/2)$ respectively. Then for $p \geq 2$ we define the $3p$ external nodal variables

$$u(P_1), u(P_2), u(P_3); \frac{\partial^2 u}{\partial s_{12}^2}(Q_1), \frac{\partial^2 u}{\partial s_{23}^2}(Q_2), \frac{\partial^2 u}{\partial s_{13}^2}(Q_3); \frac{\partial^3 u}{\partial s_{12}^3}(Q_1),$$
$$\frac{\partial^3 u}{\partial s_{23}^3}(Q_2), \frac{\partial^3 u}{\partial s_{13}^3}(Q_3); \ldots; \frac{\partial^p u}{\partial s_{12}^p}(Q_1), \frac{\partial^9 u}{\partial s_{23}^p}(Q_2), \frac{\partial^p u}{\partial s_{13}^p}(Q_3). \tag{7}$$

In order to show that the external nodal variables in (7) do indeed enforce C^0 continuity along interelement boundaries, we must show that two polynomials with the same values of these nodal variables along a side actually coincide along the side. It suffices, therefore, to show that if the side is of length 1, if $0 \leq s \leq 1$ is distance along the side, $' = \mathrm{d}/\mathrm{d}s$, and if $u(s) = \sum_{i=0}^{p} b_i s^i$, then $u(0) = u(1) = u''(1/2) = u^{(p)}(1/2) = 0$ implies that $b_i = 0, i = 0, 1, \ldots, p$. To show this let

$t = s - 1/2$ and let

$$\hat{u}(t) = u\left(t+\frac{1}{2}\right) = u(s) = \sum_{i=0}^{p} b_i\left(t+\frac{1}{2}\right)^i = \sum_{i=0}^{p} b_i \sum_{j=0}^{i} \binom{i}{j}\left(\frac{1}{2}\right)^{i-j} t^j$$

$$= \sum_{j=0}^{p}\sum_{i=j}^{p} b_i \binom{i}{j}\left(\frac{1}{2}\right)^{i-j} t^j = \sum_{j=0}^{p} \hat{b}_j t^j.$$

Therefore the coefficients $\hat{b}_j$ of u as a function of t are given in terms of the coefficients b_j of $\hat{u}$ as a function of s by

$$\hat{b}_j = \sum_{i=j}^{p} \binom{i}{j}\left(\frac{1}{2}\right)^{i-j} b_i \qquad j = 0, \ldots, p \tag{8}$$

and similarly

$$b_j = \sum_{i=j}^{p} \left(-\frac{1}{2}\right)^{i-j} \binom{i}{j} \hat{b}_i \qquad j = 0, \ldots, p. \tag{9}$$

This means that $b_j = 0$, for $j = 0, \ldots, p$ if and only if $\hat{b}_j = 0$ for $j = 0, \ldots, p$. Now $\hat{u}^{(k)}(0) = k!\, \hat{b}_k = 0$ for $k = 2, \ldots, p$, and

$$0 = u(0) = \hat{u}\left(-\frac{1}{2}\right) = \hat{b}_0 - \frac{1}{2}\hat{b}_1$$

$$0 = u(1) = \hat{u}\left(\frac{1}{2}\right) = \hat{b}_0 + \frac{1}{2}\hat{b}_1$$

so that $\hat{b}_0 = \hat{b}_1 = 0$ also. This proves that our external nodal variables do indeed enforce C^0 continuity.

In order to determine a transformation between the $[(p+1)(p+2)]/2$ polynomial coefficients of u, given by equation (5) and nodal variables, we introduce $\frac{1}{2}(p+1)(p+2) - 3p = \frac{1}{2}(p-1)(p-2)$ additional nodal variables which we call internal nodal variables. These variables are associated only with one element, and when minimizing the potential energy functional π in equation (1) they can be eliminated by static condensation from the elemental potential energy π^e. For this reason we call them internal although they are defined at boundary points (in fact at vertices). For $p = 2$, no internal nodal variables are needed. For $p \geq 3$, the internal nodal variables are chosen as

$$\frac{\partial^3 u}{\partial s_{12}^2 \partial s_{13}}; \frac{\partial^4 u}{\partial s_{12}^3 \partial s_{13}}, \frac{\partial^4 u}{\partial s_{12} \partial s_{13}^3}; \frac{\partial^5 u}{\partial s_{12}^4 \partial s_{13}}, \frac{\partial^5 u}{\partial s_{12}^3 \partial s_{13}^2}, \frac{\partial^5 u}{\partial s_{12} \partial s_{13}^4}; \ldots;$$

$$\frac{\partial^p u}{\partial s_{12}^{p-1} \partial s_{13}}, \frac{\partial^p u}{\partial s_{12}^{p-2} \partial s_{13}^2}, \ldots \frac{\partial^p u}{\partial s_{12} \partial s_{13}^{p-1}} \text{ (for } 3 \leq k \leq p,$$

$$\text{omit } \frac{\partial^k u}{\partial s_{12}^{(k-1)/2} \partial s_{13}^{(k+1)/2}} \text{ if } k \text{ is odd;}$$

$$\text{omit } \frac{\partial^k u}{\partial s_{12}^{k/2} \partial s_{13}^{k/2}} \text{ if } k \text{ is even)} \tag{10}$$

all evaluated at P_1. Thus for each $k, 3 \leq k \leq p$, we introduce $k-2$ derivatives evaluated at the origin, that is, a total of $\frac{1}{2}(p-1)(p-2)$ internal nodal variables.

We now show that these (external and internal) nodal variables form a nodal basis, that is, we show that the transformation mapping polynomial coefficients into these nodal variables is invertible.

Let $\{a\}$ denote the vector of coefficients of the approximating polynomial in a given triangular element T (for convenience we temporarily omit the index e). Then the transformation relating $\{a\}$ to $\{\delta\}$ is

$$[A]\{a\} = \{\delta\} \tag{11}$$

where

$$\lfloor \delta \rfloor = \Big\lfloor u(P_1), u(P_2), u(P_3); \frac{\partial^2 u}{\partial s_{12}^2}(Q_1); \frac{\partial^2 u}{\partial s_{23}^2}(Q_2), \frac{\partial^2 u}{\partial s_{13}^2}(Q_3);$$

$$\frac{\partial^3 u}{\partial s_{12}^3}(Q_1), \frac{\partial^3 u}{\partial s_{23}^2}(Q_2), \frac{\partial^3 u}{\partial s_{13}^3}(Q_3), \frac{\partial^3 u}{\partial s_{12}^2 \partial s_{13}}(P_1)\cdots;$$

$$\frac{\partial^p u}{\partial s_{12}^p}(Q_1), \ldots \frac{\partial^p u}{\partial s_{13}^p}(Q_3); \frac{\partial^p u}{\partial s_{12}^{p-1} \partial s_{13}}(P_1), \ldots \frac{\partial^p u}{\partial s_{12} \partial s_{13}^{p-1}}(P_1)$$

$$\Bigg(\text{for } 3 \le k \le p, \text{ omit } \frac{\partial^k u}{\partial s_{12}^{(k-1)/2} \partial s_{13}^{k/2}} \text{ if } k \text{ is odd}$$

$$\text{omit } \frac{\partial^k u}{\partial s_{12}^{k/2} \partial s_{13}^{k/2}} \text{ if } k \text{ is even}\Bigg)\Big\rfloor. \tag{12}$$

It follows from equation (6a) that under the mapping M

$$\lfloor \delta \rfloor = \Big\lfloor \bar{u}(R_1), \bar{u}(R_2), \bar{u}(R_3); \frac{1}{l_{12}^2} \frac{\partial^2 \bar{u}}{\partial \xi^2}(S_1), \frac{2}{l_{23}^2} \frac{\partial^2 \bar{u}}{\partial \bar{s}^2}(S_2), \frac{1}{l_{13}^2} \frac{\partial^2 \bar{u}}{\partial \eta^2}(S_3);$$

$$\frac{1}{l_{12}^3} \frac{\partial^3 u}{\partial \xi^3}(S_1), \frac{2\sqrt{2}}{l_{23}^3} \frac{\partial^3 \hat{u}}{\partial \bar{s}^3}(S_2), \frac{1}{l_{13}^3} \frac{\partial^3 \bar{u}}{\partial \eta^3}(S_3), \frac{1}{l_{12}^2 l_{13}} \frac{\partial^3 \bar{u}}{\partial \xi^2 \partial \eta}(R_1); \ldots;$$

$$\frac{1}{l_{12}^p} \frac{\partial^p \bar{u}}{\partial \xi^p}(S_1), \ldots, \frac{1}{l_{13}^p} \frac{\partial^p \bar{u}}{\partial \eta^p}(S_3); \frac{1}{l_{12}^{p-1} l_{13}} \frac{\partial^p \bar{u}}{\partial \xi^{p-1} \partial \eta}(R_1) \ldots$$

$$\frac{1}{l_{12} l_{13}^{p-1}} \frac{\partial^p \bar{u}}{\partial \xi \partial \eta^{p-1}}(R_1) \qquad (\text{for } 3 \le k \le p, \text{ omit}$$

$$\frac{1}{l_{12}^{(k-1)/2} l_{13}^{(k+1)/2}} \frac{\partial^k \bar{u}}{\partial \xi^{(k-1)/2} \partial \eta^{(k+1)/2}}(R_1) \text{ if } k \text{ is odd;}$$

$$\text{omit } \frac{1}{l_{12}^{k/2} l_{13}^{k/2}} \frac{\partial^k \bar{u}}{\partial \xi^{k/2} \partial \eta^{k/2}} \text{ if } k \text{ is even})\Big\rfloor$$

$$\lfloor \delta \rfloor = [\bar{\delta}][D^{-1}] \tag{13}$$

where

$$\lfloor \bar{\delta} \rfloor = \Big\lfloor \bar{u}(R_1), \bar{u}(R_2), \bar{u}(R_3); \frac{\partial^2 \bar{u}}{\partial \xi^2}(S_1), \frac{\partial^2 \bar{u}}{\partial \bar{s}^2}(S_2), \frac{\partial^2 \bar{u}}{\partial \eta^2}(S_3);$$

$$\frac{\partial^3 \bar{u}}{\partial \xi^3}(S_1), \frac{\partial^3 \bar{u}}{\partial \bar{s}^3}(S_2), \frac{\partial^3 \bar{u}}{\partial \eta^3}(S_3); \frac{\partial^3 \bar{u}}{\partial \xi^2 \partial \eta}(R_1); \ldots;$$

$$\frac{\partial^p \bar{u}}{\partial \xi^p}(S_1), \ldots \frac{\partial^p \bar{u}}{\partial \eta^p}(S_3), \frac{\partial^p \bar{u}}{\partial \xi^{p-1} \partial \eta}(R_1), \ldots, \frac{\partial^p \bar{u}}{\partial \xi \partial \eta^{p-1}}(R_1)$$

$$\Bigg(\text{for } 3 \le k \le p, \text{ omit } \frac{\partial^k \bar{u}(R_1)}{\partial \xi^{(k-1)/2} \partial \eta^{(k+1)/2}} \text{ if } k \text{ is odd}$$

$$\text{omit } \frac{\partial^k \bar{u}}{\partial \xi^{k/2} \partial \eta^{k/2}}(R_1) \text{ if } k \text{ is even}\Bigg)\Big\rfloor \tag{14}$$

and where $[D]$ is the diagonal matrix

$$[D] = \text{diag}(1, 1, 1; l_{12}^2, l_{23}^2/2, l_{13}^2; l_{12}^3, l_{23}^3/2^{3/2}, l_{13}^3, l_{12}^2 l_{13}; \ldots$$

$$l_{12}^p, \ldots, l_{12}^p, l_{12}^{p-1} l_{13}, \ldots l_{12} l_{13}^{p-1}$$

$$(\text{for } 3 \le k \le p, \text{ omit } l_{12}^{(k-1)/2} l_{13}^{(k+1)/2} \text{ if } k \text{ is odd,}$$

$$\text{omit } l_{12}^{k/2} l_{13}^{k/2} \text{ if } k \text{ is even})). \tag{15}$$

Also we have

$$[\tilde{A}]\{\tilde{a}\} = \{\tilde{\delta}\} \tag{16}$$

where $\{\tilde{\delta}\}$, given in (14), is the vector of nodal variables in the standard ξ–η plane. The transformation matrix $[\tilde{A}]$ is a matrix of constants, that is, it does not depend upon the vertices P_i, and $\{\tilde{a}\}$ is the vector of polynomial coefficients with respect to ξ and η. Now

$$\{\tilde{\delta}\} = [D]\{\delta\}$$

and

$$\{a\} = [B]\{\tilde{a}\}$$

where $[B]$ is the matrix relating polynomial coefficients with respect to ξ, η to those with respect to $\bar{x}, \bar{y}$. $[B]$ and $[B^{-1}]$ are given explicitly in Appendix 1. Substituting into (11), it follows that

$$[A] = [D^{-1}][\tilde{A}][B^{-1}] \tag{17}$$

so that $[A]$ is invertible if and only if the constant matrix $[\tilde{A}]$ is invertible. Moreover$[\tilde{A}^{-1}]$ can be inverted once and for all so that $\{\tilde{a}\}$ can be obtained easily from

$$\{\tilde{a}\} = [\tilde{A}^{-1}]\{\tilde{\delta}\}$$

once $\{\tilde{\delta}\}$ is computed. $\{\delta\}$ can be gotten from $\{\tilde{\delta}\}$ by multiplication by the diagonal matrix $[D]$. (In the case of C^1 continuity $[D]$ becomes block diagonal with the size of the largest block a 2×2 matrix. This will be seen later.)

In [17] it is shown that $[\tilde{A}]$ is invertible and a procedure for computing its inverse is given.

It is an important consequence of our choice of higher order derivatives as nodal variables that the matrix of coefficients $[\tilde{A}^{(p)}]$ has the partitioned form

$$[\tilde{A}^{(p)}] = \left[\begin{array}{c:c} \tilde{A}^{(p-1)} & \tilde{A}_{p-1,p} \\ \hdashline 0 & \tilde{A}_{p,p} \end{array}\right]. \tag{18}$$

This means that $[\tilde{A}^{(p)-1}]$ can be computed by partitioning. Let $[\tilde{B}^{(p)}] = [\tilde{A}^{(p)-1}]$ be partitioned similarly

$$[\tilde{B}^{(p)}] = \left[\begin{array}{c:c} \tilde{B}^{(p-1)} & \tilde{B}_{p-1,p} \\ \hdashline 0 & \tilde{B}_{pp} \end{array}\right] \tag{19}$$

Then

$$\begin{aligned} [\tilde{B}^{(p-1)^{-1}}] &= [\tilde{A}^{(p-1)^{-1}}] \\ [\tilde{B}_{p,p}] &= [\tilde{A}_{pp}^{-1}] \\ [\tilde{B}_{p-1,p}] &= -[\tilde{A}^{(p-1)^{-1}}][\tilde{A}_{p-1,p}][\tilde{A}_{p,p}^{-1}]. \end{aligned} \tag{20}$$

Thus, in computing $[\tilde{B}^{(p)}]$ after having computed $[\tilde{B}^{(p-1)}]$ only the new rows and columns in (19) need be calculated. This property is a consequence of the hierarchic character of our nodal variables. We give the transformation matrices $[\tilde{A}^{(p)-1}]$ for C^0 elements explicitly for $p = 2, \ldots, 5$ in Appendix 2. In [12, 14, 20] shape functions for the nodal variables presented here are given in terms of triangular coordinates.

4. NODAL VARIABLES FOR C^1 CONTINUITY

We now consider displacement fields that are required to be continuous together with their normal derivatives across interelement boundaries. Let $w(x, y)$ be such a displacement and assume that $w(x, y)$ is approximated on T by a complete pth degree polynomial,

$$w(\bar{x}, \bar{y}) = \sum_{0\le i+j\le p} a_{ij}\bar{x}^i\bar{y}^j.$$

For the case of C^0 continuity we have defined nodal variables for each element. When these nodal variables are substituted into the potential energy functional π, an unconstrained minimization problem, with these nodal variables as independent variables, results. For C^1 continuity, the situation is considerably more complicated. π again will be expressed in terms of suitably defined nodal variables but, as we show, these nodal variables must now satisfy certain constraint conditions. Thus the choice of nodal variables is more difficult in the C^1 case than in the C^0 case because there are two requirements to be met:

1. Nodal variables should be chosen in such a way that the transformation from polynomial coefficients to nodal variables is as simple as possible.
2. Nodal variables should be chosen in such a way that the constraint equations which they must satisfy are as simple as possible. This is in order to minimize the computational effort required to enforce the constraints.

The enforcement of constraints for C^1 continuity is discussed in detail in [12, 14, 20]. We present here that part of the discussion needed to explain our choice of nodal variables.

A. *Need for constraint equations*

It is important for our discussion to note the following: Suppose $P(x, y)$ and $Q(x, y)$ are polynomials which coincide along a line l and let $\partial/\partial s$ denote differentiation in the direction of l. Then for all points (x, y) on l,

$$\frac{\partial^j P}{\partial s^j}(x, y) = \frac{\partial^j Q}{\partial s^j}(x, y) \qquad j = 0, 1, 2, \ldots$$

Suppose now that the polynomials also satisfy C^1 continuity along l, and let $\partial/\partial n$ denote differentiation in a direction normal to l. Then we have analogously

$$\frac{\partial^j P}{\partial s^{j-1}\partial n}(x, y) = \frac{\partial^j Q}{\partial s^{j-1}\partial n}(x, y) \qquad j = 0, 1, 2, \ldots$$

Similarly if $s = s_1$ and if s_2 is any direction not parallel to s, we have

$$\frac{\partial^j P}{\partial s_1{}^{j-1}\partial s_2}(x, y) = \frac{\partial^j Q}{\partial s_1{}^{j-1}\partial s_2}(x, y) \qquad j = 0, 1, 2, \ldots$$

Let us now restrict ourselves to the case $j = 2$ and denote by $\{\delta\}$ a set of nodal variables which assure C^1 continuity. Let (x_0, y_0) be the common point at which two sides in direction s_1 and s_2 meet. Let $\{\delta\}_{(i)} i = 1, 2$ be a subset of $\{\delta\}$ which enforces C^1 continuity along side (i). The different subsets $\{\delta\}_{(i)}$ may, of course, have elements in common. We have on the one hand

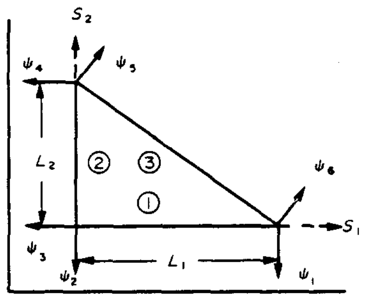
$$\frac{\partial^2 P}{\partial s_1 \partial s_2}(x_0, y_0) = \text{linear combination of } \{\delta\}_{(1)}$$

and on the other hand

$$\frac{\partial^2 P}{\partial s_1 \partial s_2}(x_0, y_0) = \text{linear combination of } \{\delta\}_{(2)}.$$

This implies a linear dependence between components of the subsets $\{\delta\}_{(1)}$ and $\{\delta\}_{(2)}$. Therefore, we can state that the set $\{\delta\}$ of nodal variables enforcing C^1 continuity along the boundary of a polygonal finite element has as many redundancies as the polygon has vertices. In particular when the element is a triangle there are three such redundancies. The rank analysis in the constraint method of equation (3) has revealed, in fact, three dependent equations at each elemental level (see [5] and [8]).

In order to better illustrate what we have stated, consider the following very simple example. Take a second order polynomial and suppose that continuity of the slope is enforced by means of normal derivatives.



$$\{\delta\}_{(1)} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}$$

$$\{\delta\}_{(2)} = \begin{Bmatrix} \psi_3 \\ \psi_4 \end{Bmatrix}$$

$$\text{along side 1:} \quad \frac{\partial^2 P}{\partial s_1 \partial s_2} = \frac{-(\psi_1 - \psi_2)}{L_1} = \frac{\partial^2 P}{\partial x \partial y}$$

$$\text{along side 2:} \quad \frac{\partial^2 P}{\partial s_1 \partial s_2} = \frac{-(\psi_4 - \psi_3)}{L_2} = \frac{\partial^2 P}{\partial x \partial y}.$$

In this simple case where the two sides meet at a right angle the linear relation satisfied by the nodal variables reduces to equality of the mixed second partial derivatives and gives

$$\frac{\psi_1 - \psi_2}{L_1} = \frac{\psi_4 - \psi_3}{L_2}.$$

Another example is given by Zienkiewicz in [19]. In the general case the relation will involve both nodal variables which enforce continuity and nodal variables which enforce continuity of the slope. We give one form of this relation in [12] and [13].

In general we may state that the rank of the matrix which transforms the polynomial coefficients to a minimal set of nodal variables enforcing C^1 continuity along the boundary is equal to the number of its rows minus the number of the vertices of the finite element. There is, however, one important exception which is essential for our choice of nodal variables: If the second order derivatives $\partial^2 P/\partial s_1^2$, $\partial^2 P/\partial s_1 \partial s_2$, and $\partial^2 P/\partial s_2^2$ at vertices are themselves selected as nodal variables, then the subsets $\{\delta\}_{(1)}$ and $\{\delta\}_{(2)}$ coincide since both consist of the three independent second order derivatives. In this case the rank of the transformation can be made equal to the number of its rows. The, by introduction of additional internal nodal variables, the transformation can be made invertible.

The enforcement of an additional continuity of second order derivatives at vertices by choosing them as nodal variables as in [16] is the simplest way to avoid singularities of the transformation matrix. We have already pointed out in the Introduction that we wish to avoid enforcement of C^2 continuity at vertices. Thus, in order to have a nonsingular transformation matrix the set of nodal variables $\{\delta\}$ must be such that C^1 continuity does not hold. Hence, constraints on nodal variables must be imposed separately in order to enforce the desired C^1 continuity.

B. *Nodal variables and transformation matrices for C^1 continuity*

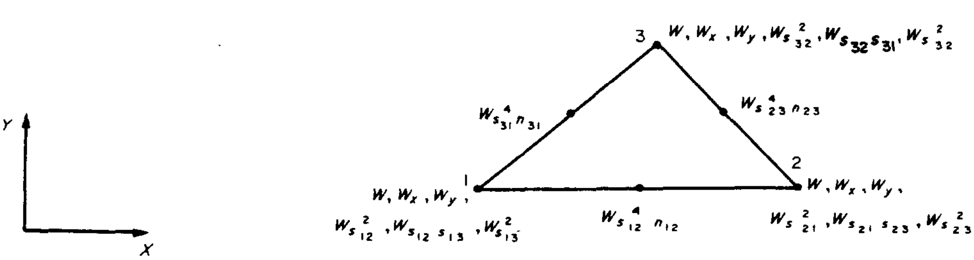
As before, we first define nodal variables which enforce C^1 continuity across interelement boundaries. For $p \geq 5$, define the $6p - 9$ external nodal variables

$$w(P_1),\, w(P_2),\, w(P_3);\, \frac{\partial w}{\partial x}(P_1),\, \frac{\partial w}{\partial y}(P_1),\, \frac{\partial w}{\partial x}(P_2),\, \frac{\partial w}{\partial y}(P_2),$$

$$\frac{\partial w}{\partial x}(P_3),\, \frac{\partial w}{\partial y}(P_3);\, \frac{\partial^2 w}{\partial s_{12}^2}(P_1),\, \frac{\partial^2 w}{\partial s_{12} \partial s_{13}}(P_1),\, \frac{\partial^2 w}{\partial s_{13}^2}(P_1), \ldots,$$

$$\frac{\partial^2 w}{\partial s_{31}^2}(P_3),\, \frac{\partial^2 w}{\partial s_{31} \partial s_{32}}(P_3),\, \frac{\partial^2 w}{\partial s_{32}^2}(P_3);\, \frac{\partial^5 w}{\partial s_{12}^4 \partial n_{12}}(Q_1),\, \frac{\partial^5 w}{\partial s_{23}^4 \partial \eta_{23}}(Q_2),\, \frac{\partial^5 w}{\partial s_{31}^4 \partial n_{13}}(Q_3); \ldots;$$

$$\frac{\partial^p w}{\partial s_{12}^p}(Q_1),\, \frac{\partial^p w}{\partial s_{12}^{p-1} n_{12}}(Q_1),\, \frac{\partial^p w}{\partial s_{23}^p}(Q_2),\, \frac{\partial^p w}{\partial s_{23}^{p-1} n_{23}}(Q_2),\, \frac{\partial^p w}{\partial s_{31}^p}(Q_3),\, \frac{\partial^p w}{\partial s_{31}^{p-1} \partial n_{31}}(Q_3). \tag{21}$$



For example, if $p = 5$ the twenty-one external nodal variables are w, w_x, w_y, w_{ss}, $w_{ss'}$, $w_{s's'}$ at each vertex (s and s' are the directions of the two sides meeting at the vertex), and

$$w_{s_{12}^4 n_{12}}(Q_1),\ w_{s_{23}^4 n_{23}}(Q_2),\ w_{s_{31}^4 n_{31}}(Q_3).$$

We must show that these external nodal variables enforce C^1 continuity across interelement boundaries, that is, we must show that if they are all zero along a side l (with direction s) of length 1, then $w(s) = \sum_{i=0}^{p} b_i s^i$ and $w_n(s) = \sum_{i=0}^{p-1} c_i s^i$ are identically zero for $0 \le s \le 1$. Here $w_n(s)$ denotes the normal slope of w along l.

Let the external variables given by (21) be zero along l. Then

$$\begin{aligned} w(0) = w(1) = w'(0) = w'(1) = w''(0) = w''(1) = w^{(6)}\left(\frac{1}{2}\right) = \dots \\ = w^{(p)}\left(\frac{1}{2}\right) = 0 \\ w_n(0) = w_n(1) = w_n'(0) = w_n'(1) = w_n^{(4)}\left(\frac{1}{2}\right) = \dots\ w_n^{(p-1)}\left(\frac{1}{2}\right) = 0 \end{aligned} \tag{22}$$

We must show that (22) implies that $b_i = 0$, $i = 0, \dots, p$ and $c_i = 0$ $i = 0, \dots, p-1$.

As in (8) let $t = s - 1/2$, $\hat{w}(t) = \sum_{j=0}^{p} \hat{b}_j t^j$, $\hat{w}_n(t) = \sum_{j=0}^{p-1} \hat{c}_j t^j$. Since $\hat{b}_i = 1/i!\ \hat{w}^{(i)}(0) = 0$ for $i = 6, \dots, p$, it follows from (9) that $b_j = 0$ for $j = 6, \dots, p$. Again for $i = 0, 1, 2$ it follows that $b_i(0) = (1/i!) w^{(i)}(0) = 0$. Therefore $w(s) = b_3 s^3 + b_4 s^4 + b_5 s^5$. Finally, note that the system

$$\begin{aligned} 0 &= w(1) = b_3 + b_4 + b_5 \\ 0 &= w'(1) = 3b_3 + 4b_4 + 5b_5 \\ 0 &= w''(1) = 6b_3 + 12b_4 + 20b_5 \end{aligned}$$

is nonsingular. This implies that $b_3 = b_4 = b_5 = 0$, too. Similarly $\hat{c}_i = 1/i!\ \hat{w}_n^{(i)}(0) = 0$ for $i = 4, \dots, p-1$ implies that $c_i = 0$ for $i = 4, \dots, p-1$. Also $c_0 = w_n(0) = 0$, $c_1 = w_n'(0) = 0$, so that $w_n(s) = c_2 s^2 + c_3 s^3$. The nonsingularity of the system

$$\begin{aligned} 0 &= w_n(1) = c_2 + c_3 \\ 0 &= w_n'(1) = 2c_2 + 3c_3 \end{aligned}$$

implies that all c_i are zero $i = 0, \dots, p-1$. Therefore we have shown that these external nodal variables do indeed enforce C^1 continuity across interelement boundaries.

We now introduce additional internal nodal variables in such a way that the transformation from polynomial coefficients to nodal variables is easily inverted. We restrict our attention to $p \ge 5$. For $p = 5$ no internal nodal variables are needed. For $p \ge 6$, $\frac{1}{2}(p+1)(p+2) - (6p-9) = \frac{1}{2}(p-5)(p-4)$ internal nodal variables are required. For each k, $6 \le k \le p$, define the $(k-5)$ internal nodal variables

$$\frac{\partial^k w}{\partial s_{12}^j \partial s_{13}^{k-j}}(P_1) \qquad j = 2, 3, \dots, k-2 \tag{23}$$

omitting any two, say $j'(k)$, $j''(k)$.

The total number of internal nodal variables is then

$$\sum_{k=6}^{p}(k-5)=\frac{1}{2}(p-5)(p-4).$$

The vector $\{\delta\}$ of all nodal variables now consists of the external nodal variables in (21) and the internal nodal variables in (23). We order these variables in $\{\delta\}$ according to increasing order of derivatives. From (4) and (6), $\{\delta\}$ is given by

$$\begin{aligned}
\lfloor\delta\rfloor=\Big[\ &\bar{w}(R_1),\ \bar{w}(R_2),\ \bar{w}(R_3);\ \frac{1}{2\Delta}(\bar{y}_3\bar{w}_\xi(R_1)-\bar{y}_2\bar{w}_\eta(R_1)),\\
&\frac{1}{2\Delta}(-\bar{x}_3\bar{w}_\xi(R_1)+\bar{x}_2w_\eta(R_1)),\ \frac{1}{2\Delta}([\bar{y}_3-\bar{y}_2]\bar{w}_\xi(R_2)\\
&-\bar{y}_2\sqrt{2}\bar{w}_{\bar{s}}(R_2)),\ \frac{1}{2\Delta}(-[\bar{x}_3-\bar{x}_2]\bar{w}_\xi(R_2)+\bar{x}_2\sqrt{2}\bar{w}_{\bar{s}}(R_2)),\\
&\frac{1}{2\Delta}([\bar{y}_3-\bar{y}_2]\bar{w}_\eta(R_3)-\bar{y}_3\sqrt{2}\bar{w}_{\bar{s}}(R_3)),\ \frac{1}{2\Delta}(-[\bar{x}_3-\bar{x}_2]\bar{w}_\eta(R_3)+\bar{x}_3\sqrt{2}\bar{w}_{\bar{s}}(R_3)),\\
&\frac{1}{l_{12}^2}\bar{w}_{\xi^2}(R_1),\ \frac{1}{l_{12}l_{13}}\bar{w}_{\xi\eta}(R_1),\ \frac{1}{l_{13}^2}\bar{w}_{\eta^2}(R_1),\dots,\frac{1}{l_{13}^2}\bar{w}_{\eta^2}(R_3);\\
&\frac{\sqrt{2}}{l_{13}l_{23}}\bar{w}_{\eta\bar{s}}(R_3),\ \frac{2}{l_{23}^2}\bar{w}_{\bar{s}^2}(R_3);\ \frac{1}{l_{12}^4}(\alpha_{12}\bar{w}_{\xi^5}(S_1)+\beta_{12}\bar{w}_{\xi^4\eta}(S_1)),\\
&\frac{4}{l_{23}^4}(\alpha_{23}\bar{w}_{\bar{s}^5}(S_2)+\beta_{23}\bar{w}_{\bar{s}^4\bar{n}}(S_2)),\ \frac{1}{l_{13}^4}(\alpha_{13}\bar{w}_{\xi\eta^4}(S_3)+\beta_{13}\bar{w}_{\eta^5}(S_3)),\\
&\dots;\ \frac{1}{l_{12}^p}\bar{w}_{\xi^p}(S_1),\ \frac{1}{l_{12}^{p-1}}(\alpha_{12}\bar{w}_{\xi^p}(S_1)+\beta_{12}\bar{w}_{\xi^{p-1}\eta}(S_1)),\\
&\frac{2^{p/2}}{l_{23}^p}\bar{w}_{\bar{s}^p}(S_2),\ \frac{2^{(p-1)/2}}{l_{23}^{p-1}}(\alpha_{23}\bar{w}_{\bar{s}^p}(S_2)+\beta_{23}\bar{w}_{\bar{s}^{p-1}\bar{n}}(S_2)),\\
&\frac{(-1)^p}{l_{13}^p}\bar{w}_{\eta^p}(S_3),\ \frac{(-1)^{p-1}}{l_{13}^{p-1}}(\alpha_{13}\bar{w}_{\xi\eta^{p-1}}(S_3)+\beta_{13}\bar{w}_{\eta^p}(S_3))\Big\rfloor.
\end{aligned}\tag{24}$$

Using notation analogous to that in (14) we have

$$\lfloor\delta\rfloor=\lfloor\bar{\delta}\rfloor[D^{-1}]\tag{25}$$

$$\begin{aligned}
\lfloor\bar{\delta}\rfloor=\lfloor\ &\bar{w}(R_1),\ \bar{w}(R_2),\ \bar{w}(R_3),\ \bar{w}_\xi(R_1),\ \bar{w}_\eta(R_1),\ \bar{w}_\xi(R_2),\ \bar{w}_{\bar{s}}(R_2),\ \bar{w}_\eta(R_3),\ \bar{w}_{\bar{s}}(R_3);\\
&\bar{w}_{\xi^2}(R_1),\ \bar{w}_{\xi\eta}(R_1),\ \bar{w}_{\eta^2}(R_1),\dots,\bar{w}_{\eta^2}(R_3),\ \bar{w}_{\eta\bar{s}}(R_3),\ \bar{w}_{\bar{s}^2}(R_3);\\
&\bar{w}_{\xi^4\eta}(S_1),\ \bar{w}_{\bar{s}^4\bar{n}}(S_2),\ \bar{w}_{\eta^4\xi}(S_3);\dots;\ \bar{w}_{\xi^k}(S_1),\ \bar{w}_{\xi^{k-1}\eta}(S_1),\\
&\bar{w}_{\bar{s}^k}(S_2),\ \bar{w}_{\bar{s}^{k-1}\bar{n}}(S_2),\ \bar{w}_{\eta^k}(S_3),\ \bar{w}_{\xi\eta^{k-1}}(S_3),\ \bar{w}_{\xi^{k-2}\eta^2}(R_1),\\
&\dots,\ \bar{w}_{\xi^3\eta^{k-3}}(R_1),\ \bar{w}_{\xi^2\eta^{k-2}}(R_1)
\end{aligned}$$

(omitting any two for each $k\geq 6$); ...; $\bar{w}_{\xi^p}(S_1),\dots,$

$$\bar{w}_{\xi\eta^{p-1}}(S_3),\dots,\bar{w}_{\xi^2\eta^{p-2}}(R_1)\rfloor\tag{25a}$$

and where $[D]$ is a block diagonal matrix of the form

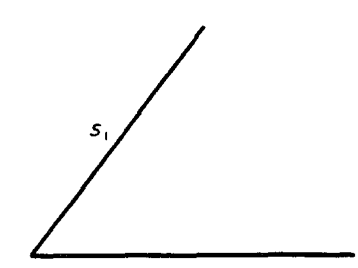
$$
[D]^{-1} =
\begin{bmatrix}
1 \\
& 1 \\
& & 1 \\
& & & \begin{matrix} \dfrac{\bar{y}_3}{2\Delta} & \dfrac{-\bar{y}_2}{2\Delta} \\ \dfrac{-\bar{x}_3}{2\Delta} & \dfrac{\bar{x}_2}{2\Delta} \end{matrix} \\
& & & & \ddots \\
& & & & & \begin{matrix} \dfrac{\bar{y}_3-\bar{y}_2}{2\Delta} & \dfrac{\bar{y}_3\sqrt{2}}{2\Delta} \\ \dfrac{-(\bar{x}_3-\bar{x}_2)}{2\Delta} & \dfrac{\bar{x}_3\sqrt{2}}{2\Delta} \end{matrix} \\
& & & & & & \dfrac{1}{l_{12}^2} \\
& & & & & & & \ddots \\
& & & & & & & & \dfrac{2}{l_{23}^2} \\
& & & & & & & & & \ddots \\
& & & & & & & & & & \begin{matrix} \dfrac{(-1)^p}{l_{13}^p} & 0 \\ \dfrac{(-1)^p\beta_{13}}{l_{13}^{p-1}} & \dfrac{(-1)^p\alpha_{13}}{l_{13}^{p-1}} \end{matrix}
\end{bmatrix}
\tag{25b}
$$

The size of the largest block on the main diagonal of $[D]^{-1}$ is a 2×2 matrix.

In [17] it is shown that the matrix $[\bar{A}]$ in equation (16) is invertible, so from equation (17) it follows that the nodal variables $\{\delta\}$ are a local basis. We give the transformation matrices $[\bar{A}^{(p)-1}]$ for C^1 elements explicitly for $p = 5, 6$ in Appendix 3. In [12, 14, 20] shape functions for the nodal variables presented here are given in terms of triangular coordinates.

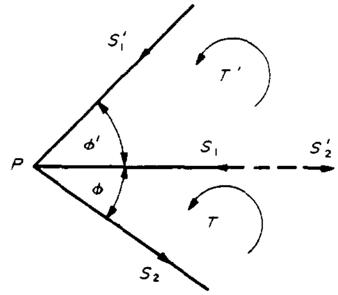
C. *Constraint equations*

An essential point in the determination of nodal variables to enforce C^1 continuity is the selection of the derivatives $\partial^2 w/\partial s_1^2$, $\partial^2 w/\partial s_1 \partial s_2$, and $\partial^2 w/\partial s_2^2$ as nodal variables. In this way, and only in this way, is it possible without enforcing C^2 (or higher) continuity at vertices, to invert the transformation from polynomial coefficients to nodal variables. As shown earlier it is now necessary, however, to enforce continuity of these derivatives at interelement boundaries.



The continuity of $\partial^2 w/\partial s_1^2$ and $\partial^2 w/\partial s_2^2$ is enforced simply by identifying these values when they come from neighboring elements. We now derive an equation which enforces continuity of the mixed derivative $\partial^2 w/\partial s_1 \partial s_2$ at an interelement boundary.

Let T and T' be adjacent triangular elements with a common side. Assume that T and T'



have been given a common orientation (say that their sides are transversed in the counter-clockwise direction). We wish to enforce the continuity of $\partial^2 w/\partial s_1 \partial s_2$ along the common boundary at P, that is, we require

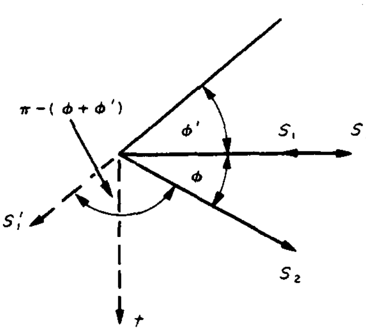
$$\frac{\partial^2 w'}{\partial s_1 \partial s_2}(P) = \frac{\partial^2 w}{\partial s_1 \partial s_2}(P) \tag{26}$$

where w is the approximating polynomial in T and w' in T'. Since $s_2' = -s_1$, we have

$$\frac{\partial^2 w'}{\partial s_1 \partial s_2} = -\frac{\partial^2 w'}{\partial s_2' \partial s_2} \tag{27}$$

and we must now express $\partial/\partial s_2$ in terms of $\partial/\partial s_1'$ and $\partial/\partial s_2'$.

Let the direction t be normal to s_2', then we have



$$\frac{\partial}{\partial s_2} = \sin\phi \frac{\partial}{\partial t} + \cos\phi \frac{\partial}{\partial s_2'}$$

$$\frac{\partial}{\partial s_1'} = \sin\phi' \frac{\partial}{\partial t} - \cos\phi' \frac{\partial}{\partial s_2'}$$

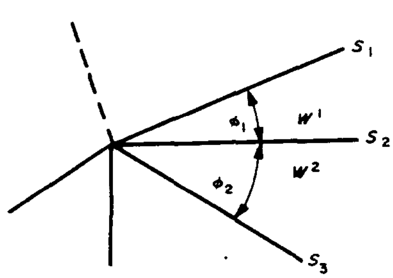
$$\frac{\partial}{\partial t} = \frac{1}{\sin\phi'}\left(\frac{\partial}{\partial s_1'} + \cos\phi' \frac{\partial}{\partial s_2'}\right)$$

$$\begin{aligned}\frac{\partial}{\partial s_2} &= \frac{\sin\phi}{\sin\phi'}\frac{\partial}{\partial s_1'} + \left(\cos\phi + \frac{\sin\phi\cos\phi'}{\sin\phi'}\right) \\ &= \frac{\sin\phi}{\sin\phi'}\frac{\partial}{\partial s_1'} + \frac{\sin(\phi+\phi')}{\sin\phi'}\frac{\partial}{\partial s_2'}\end{aligned}$$

Substituting in equations (26) and (27) we obtain

$$\frac{\partial^2 w}{\partial s_1 \partial s_2} = -\frac{\sin\phi}{\sin\phi'}\frac{\partial^2 w'}{\partial s_1' \partial s_2'} - \frac{\sin(\phi+\phi')}{\sin\phi'}\frac{\partial^2 w'}{\partial s_2'^2}. \tag{28}$$

Now consider a vertex where n sides meet. Then



$$w^2_{s_1 s_2} = -\frac{\sin\phi_2}{\sin\phi_1} w^1_{s_1 s_2} - \frac{\sin(\phi_1+\phi_2)}{\sin\phi_1} w^1_{s_2 s_2}$$

$$\begin{aligned}w^3_{s_1 s_2} &= -\frac{\sin\phi_3}{\sin\phi_2} w^2_{s_1 s_2} - \frac{\sin(\phi_2+\phi_3)}{\sin\phi_2} w^2_{s_2 s_2} \\ &= \frac{\sin\phi_3}{\sin\phi_1} w^1_{s_1 s_2} + \frac{\sin(\phi_1+\phi_2)}{\sin\phi_1 \sin\phi_2}\sin\phi_3\, w^1_{s_2 s_2} \\ &\quad - \frac{\sin(\phi_2+\phi_3)}{\sin\phi_2} w^2_{s_2 s_2}\end{aligned}$$

and in general

$$w^{i+1}_{s_1s_2} = -\frac{\sin\phi_{i+1}}{\sin\phi_i} w^i_{s_1s_2} - \frac{\sin(\phi_i+\phi_{i+1})}{\sin\phi_i} w^i_{s_2s_2}$$

$$= (-1)^i \frac{\sin\phi_{i+1}}{\sin\phi_1} w^1_{s_1s_2} + (-1)^i \sum_{j=1}^{i-1} (-1)^{j+1} \frac{\sin(\phi_j+\phi_{j+1})}{\sin\phi_j \sin\phi_{j+1}} \sin\phi_{i+1}\, w^j_{s_2s_2} - \frac{\sin(\phi_i+\phi_{i+1})}{\sin\phi_i} w^i_{s_2s_2}$$

$$w^{i+1}_{s_1s_2} = (-1)^i \frac{\sin\phi_{i+1}}{\sin\phi_1} w^1_{s_1s_2} + (-1)^i \sin\phi_{i+1} \sum_{j=1}^{i} (-1)^{j+1}(\cot\phi_j + \cot\phi_{j+1}) w^j_{s_2s_2}, \quad i = 1, 2, \ldots \tag{29}$$

Let $i = n$. Then

$$w^{n+1}_{s_1s_2} = w^1_{s_1s_2} = (-1)^n \left[w^1_{s_1s_2} + (\sin\phi_1) \sum_{j=1}^{n} (-1)^{j+1}(\cot\phi_j + \cot\phi_{j+1}) w^j_{s_2s_2} \right]. \tag{30}$$

Therefore, if n is odd

$$w^1_{s_1s_2} = \frac{\sin\phi_1}{2} \sum_{j=1}^{n} (-1)^j (\cot\phi_j + \cot\phi_{j+1}) w^j_{s_2s_2}. \tag{31}$$

Since the numbering of sides is arbitrary, equation (31) can be used to eliminate all mixed partials. Thus the potential energy functional will have n tangential second order partials as independent variables corresponding to each vertex where an odd number n of sides meet.

If n is even, however, then (30) becomes

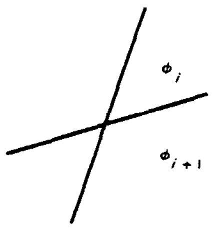
$$\sum_{j=1}^{n} (-1)^{j+1}(\cot\phi_j + \cot\phi_{j+1}) w^j_{s_2s_2} = 0. \tag{32}$$

In this case, then, we can choose any $\bar{j}$ such that $\cot\phi_{\bar{j}} + \cot\phi_{\bar{j}+1} \neq 0$ and solve for $w^{\bar{j}}_{s_2s_2}$ in terms of the other $w^j_{s_2s_2}$, $j \neq \bar{j}$. As another independent variable we may choose one mixed second partial say $w^1_{s_1s_2}$. All other mixed second partials can then be solved in terms of these from (29).

There is one important exception, the case $n = 4$ and $\phi_i = \pi - \phi_{i-1}$, $i = 1, 2, 3, 4$. Equation (32) then reduces to an identity and therefore no pure second derivative can be eliminated. Hence in this case there is one constraint fewer than in the other cases. Such a vertex is called a *singular* vertex and is used in [11] and [12] to determine the total number of independent variables i.e. the dimension of the space S_p^1 consisting of piecewise polynomials of degree p satisfying global C^1 continuity.

We can conclude that

1. The system of constraints given in equation (29) is singular only at vertices where four elements meet and their boundaries are given there by a pair of intersecting straight lines:



Pattern of singular vertex

2. In order to enforce constraints by direct elimination of a set of dependent variables, at each vertex:

(a) if the number n of elements meeting there is odd then all mixed second order derivatives can be selected to be eliminated.

(b) If the number n is even, and provided that the vertex is not singular, one tangential second order derivative w^i_{ss} and $n-1$ mixed second order derivatives can be eliminated. The index i is arbitrary provided that $\cot\phi_i + \cot\phi_{i+1} \neq 0$.

What we have presented here is one choice of dependent and independent variables. In [12, 14, 20] an optimal choice of dependent variables and the reasons for its optimality are discussed in detail.

Note that constraint equations given in equation (28) relate only external nodal variables and, in particular, only the second order derivatives at the vertices. Therefore the internal nodal variables are independent variables in π and the set of internal nodal variables associated with element e is associated with no other element. This means that static condensation can be used to determine internal nodal variables at the elemental level. Also, with our choice of nodal variables once dependent and independent variables have been chosen for a fixed p and the relations between them have been determined, these same constraints relations hold for other p. This means that the determination of dependence in terms of independent variables has to be made only once, as the degree of the approximating polynomial is increased to achieve greater accuracy.

We now show how to compute elemental stiffness matrices and elemental load vectors for pth order polynomial elements by using precomputed arrays. More details are given in [18].

5. ELEMENT STIFFNESS MATRICES AND LOAD VECTORS

We assume a displacement vector field $\{\bar{w}(x, y)\}$ of the form

$$\lfloor \bar{w}(x, y) \rfloor = \lfloor u, u_x, u_y; v, v_x, v_y; w, w_x, w_y, w_{xx}, w_{xy}, w_{yy} \rfloor = \lfloor U^T; V^T; W^T \rfloor$$

where

$$\begin{aligned} \lfloor U \rfloor &= \lfloor u, u_x, u_y \rfloor \\ \lfloor V \rfloor &= \lfloor v, v_x, v_y \rfloor \\ \lfloor W \rfloor &= \lfloor w, w_x, w_y, w_{xx}, w_{xy}, w_{yy} \rfloor \end{aligned} \tag{33}$$

$u(x, y)$ and $v(x, y)$ are required to satisfy C^0 continuity and $w(x, y)$ is required to satisfy C^1 continuity.

Let π^e denote the potential energy of the triangular element T. Then

$$\pi^e = \frac{1}{2} \iint_T \lfloor \bar{w}(x, y) \rfloor [S^e] \{\bar{w}(x, y)\} \, dx \, dy - \iint_T \lfloor Z^e \rfloor \{\bar{w}(x, y)\} \, dx \, dy \tag{34}$$

where $[S^e]$ is the symmetric elemental matrix of elastic constants and $[Z^e]$ is the elemental distributed load.

We use the transformation M^{-1} given in equation (4) to map from $x - y$ *to* $\xi - \eta$ coordinates, and we let

$$\tilde{u}(\xi, \eta) = u(x, y), \; \tilde{v}(\xi, \eta) = v(x, y), \; \tilde{w}(\xi, \eta) = w(x, y).$$

Now, since M^{-1} is given by (4b),

$$\{U\} = \begin{Bmatrix} u(x, y) \\ u_x(x, y) \\ u_y(x, y) \end{Bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} 2\Delta & 0 & 0 \\ 0 & \bar{y}_3 & -\bar{y}_2 \\ 0 & -\bar{x}_3 & \bar{x}_2 \end{bmatrix} \begin{Bmatrix} \tilde{u}(\xi, \eta) \\ \tilde{u}_\xi(\xi, \eta) \\ \tilde{u}_\eta(\xi, \eta) \end{Bmatrix} = \frac{1}{2\Delta} [H_1]\{\tilde{U}\} = [G_1]\{\tilde{U}\}$$

$$\{V\} = \frac{1}{2\Delta} [H_1]\{\tilde{V}\} = [G_1]\{\tilde{V}\}$$

$$\{W\} = \begin{Bmatrix} w(x, y) \\ w_x(x, y) \\ w_y(x, y) \\ w_{xx}(x, y) \\ w_{xy}(x, y) \\ w_{yy}(x, y) \end{Bmatrix} = \frac{1}{(2\Delta)^2} \begin{bmatrix} 2\Delta[H_1] & & 0 & \\ & \bar{y}_3^2 & -2\bar{y}_2\bar{y}_3 & \bar{y}_2^2 \\ 0 & -\bar{x}_3\bar{y}_3 & \bar{x}_2\bar{y}_3 + \bar{x}_3\bar{y}_2 & -\bar{x}_2\bar{y}_2 \\ & \bar{x}_3^2 & -2\bar{x}_2\bar{x}_3 & \bar{x}_2^2 \end{bmatrix} \begin{Bmatrix} \tilde{w}(\xi, \eta) \\ \tilde{w}_\xi(\xi, \eta) \\ \tilde{w}_\eta(\xi, \eta) \\ \tilde{w}_{\xi\xi}(\xi, \eta) \\ \tilde{w}_{\xi\eta}(\xi, \eta) \\ \tilde{w}_{\eta\xi}(\xi, \eta) \end{Bmatrix}$$

$$= [G_2]\{\tilde{W}\}.$$

Therefore

$$\{w(x, y)\} = [G]\{\bar{w}(\xi, \eta)\} = [G]\begin{Bmatrix} \bar{U}(\xi, \eta) \\ \bar{V}(\xi, \eta) \\ \bar{W}(\xi, \eta) \end{Bmatrix}$$

where

$$[G] = \begin{bmatrix} G_1 & & \\ & G_1 & \\ & & G_2 \end{bmatrix}. \tag{35}$$

Substituting in equation (34), this gives

$$\pi^e = \frac{1}{2}\left(2|\Delta| \iint_{\bar{T}} \lfloor \bar{w}(\xi, n) \rfloor [\bar{S}^e]\{\bar{w}(\xi, \eta)\, d\xi\, d\eta\right) - \left(2|\Delta| \iint_{\bar{T}} \lfloor \bar{Z}^e \rfloor \{\bar{w}(\xi, \eta)\}\, d\xi\, d\eta\right) \tag{36}$$

where

$$[\bar{S}^e] = [G^T][S^e][G]$$
$$\lfloor \bar{Z}^e \rfloor = \lfloor Z^e \rfloor [G].$$

Now we partition $[\bar{S}^e]$ as

$$[\bar{S}^e] = \left[\begin{array}{c|c|c} \bar{S}^e_{uu} & \bar{S}^e_{uv} & \bar{S}^e_{uw} \\ \hline \bar{S}^e_{vu} & \bar{S}^e_{vv} & \bar{S}^e_{vw} \\ \hline \bar{S}^e_{wv} & \bar{S}^e_{wv} & \bar{S}^e_{ww} \end{array}\right] \tag{37}$$

where $[\bar{S}^e_{uv}]$, $[\bar{S}^e_{uw}]$, $[\bar{S}^e_{vw}]$ gives the (possible) coupling between the different displacement fields. In many applications at least some of these coupling terms will vanish. Then

$$2|\Delta| \iint_{\bar{T}} \lfloor \bar{w} \rfloor [\bar{S}^e]\{\bar{w}\}\, d\xi\, d\eta = 2|\Delta| \iint_{T} (\lfloor \bar{U} \rfloor [\bar{S}^e_{uu}]\{\bar{U}\} + \lfloor \bar{V} \rfloor [\bar{S}^e_{vv}]\{\bar{V}\} + \lfloor \bar{W} \rfloor [\bar{S}^e_{ww}]\{\bar{W}\}$$
$$+ 2(\lfloor \bar{U} \rfloor [\bar{S}^e_{uv}]\{\bar{V}\} + \lfloor \bar{U} \rfloor [\bar{S}^e_{uw}]\{\bar{W}\} + \lfloor \bar{V} \rfloor [\bar{S}^e_{vw}]\{\bar{W}\}))\, d\xi\, d\eta. \tag{38}$$

Now let

$$\{\bar{a}_e\} = \begin{Bmatrix} \bar{a}_u{}^e \\ \bar{a}_v{}^e \\ \bar{a}_w{}^e \end{Bmatrix}, \qquad \{\bar{\delta}_e\} = \begin{Bmatrix} \bar{\delta}_u{}^e \\ \bar{\delta}_v{}^e \\ \bar{\delta}_w{}^e \end{Bmatrix}$$

denote the vector of all polynomial coefficients associated with element e after having been mapped into $\bar{T}$ (partitioned into coefficients associated with $\bar{u}$, $\bar{v}$, and $\bar{w}$ respectively). Let $\{\bar{\delta}_e\}$ denote the vector of all nodal variables partitioned similarly. We have shown that the transformation

$$[\bar{A}]\{\bar{a}_e\} = \{\bar{\delta}_e\}$$

can be inverted to give

$$\{\bar{a}_e\} = [\bar{A}^{-1}]\{\bar{\delta}_e\} = \begin{bmatrix} \bar{A}_u{}^{-1} & & \\ & \bar{A}_v{}^{-1} & \\ & & \bar{A}_w{}^{-1} \end{bmatrix} \begin{Bmatrix} \bar{\delta}_u{}^e \\ \bar{\delta}_v{}^e \\ \bar{\delta}_w{}^e \end{Bmatrix}. \tag{39}$$

The computation for $[\tilde{A}^{-1}]$ in equation (39) is assumed to have been performed and to be available for use. Let

$$\lfloor \tilde{g}^u(\xi,\eta)\rfloor = \lfloor 1,\xi,\eta,\ldots.\xi^{p_u},\ldots,\xi\eta^{p_u-1},\eta^{p_u}\rfloor$$
$$\lfloor \tilde{g}^v(\xi,\eta)\rfloor = \lfloor 1,\xi,\eta,\ldots.\xi^{p_v},\ldots,\xi\eta^{p_v-1},\eta^{p_v}\rfloor$$
$$\lfloor g^w(\xi,\eta)\rfloor = \lfloor 1,\xi,\eta,\ldots.\xi^{p_w},\ldots,\xi\eta^{p_w-1},\eta^{p_w}\rfloor.$$

Then

$$\begin{aligned}\tilde{u}(\xi,\eta) &= \lfloor \tilde{g}^u(\xi,\eta)\rfloor\{\tilde{a}_u\}\\ \tilde{v}(\xi,\eta) &= \lfloor \tilde{g}^v(\xi,\eta)\rfloor\{\tilde{a}_v\}\\ \tilde{w}(\xi,\eta) &= \lfloor \tilde{g}^w(\xi,\eta)\rfloor\{\tilde{a}_w\}.\end{aligned} \tag{40}$$

Let

$$\lfloor \tilde{g}_1{}^w\rfloor = \lfloor \tilde{g}^w\rfloor,\ \lfloor \tilde{g}_2{}^w\rfloor = \left[\frac{\partial \tilde{g}^w}{\partial\xi}\right],\ \lfloor \tilde{g}_3{}^w\rfloor = \left[\frac{\partial \tilde{g}^w}{\partial\eta}\right],$$
$$\lfloor \tilde{g}_4{}^w\rfloor = \left[\frac{\partial^2 g^w}{\partial\xi^2}\right],\ \lfloor \tilde{g}_5{}^w\rfloor = \left[\frac{\partial^2 \tilde{g}^w}{\partial\xi\partial\eta}\right],\ \lfloor \tilde{g}_6{}^w\rfloor = \left[\frac{\partial^2 \tilde{g}^w}{\partial\eta^2}\right]. \tag{42}$$

with similar expressions for $\tilde{g}_i{}^u$ and $\tilde{g}_i{}^v$ for $i = 1, 2, 3$. Substitute in the first term in equation (48) to obtain

$$\begin{aligned}&2|\Delta|\iint_{\bar{T}} \lfloor \tilde{U}\rfloor[\tilde{S}^e_{uu}]\{\tilde{U}\}\,d\xi\,d\eta\\ &= \sum_{1\le i\le j\le 3} d^{uu}_{ij}\lfloor \tilde{a}_u{}^e\rfloor\left(\iint_{\bar{T}} \tfrac{1}{2}(\{\tilde{g}_i{}^u\}\lfloor \tilde{g}_j{}^u\rfloor + \{\tilde{g}_j{}^u\}\lfloor \tilde{g}_i{}^u\rfloor)\,d\xi\,d\eta\right)\{\tilde{a}_u{}^e\}\\ &= \lfloor \tilde{\delta}_u{}^e\rfloor \sum_{1\le i\le j\le 3} d^{uu}_{ij}\left([\tilde{A}_u{}^{-1T}]\iint_{\bar{T}} \tfrac{1}{2}(\{\tilde{g}_i{}^u\}\lfloor \tilde{g}_j{}^u\rfloor + \{\tilde{g}_j{}^u\}\lfloor \tilde{g}_i{}^u\rfloor\,d\xi\,d\eta)[\tilde{A}_u{}^{-1}]\\ &= \lfloor \tilde{\delta}_u{}^e\rfloor \sum_{1\le i\le j\le 3}(d^{uu}_{ij}\,B^{uu}_{ij})\{\tilde{\delta}_u{}^e\}.\end{aligned} \tag{42}$$

The six matrices B^{uu}_{ij}, $1\le i\le j\le 3$, are of size $[(p+1)(p+2)/2]\times[(p+1)(p+2)/2]$. They are computed once and for all and stored for future use. In the calculation of

$$\iint_{\bar{T}} \{\tilde{g}_i{}^u\}\lfloor \tilde{g}_j{}^u\rfloor\,d\xi\,d\eta$$

the following formula is of use:

$$\iint_{\bar{T}} \xi^p\eta^q\,d\xi\,d\eta = \frac{p!q!}{(p+q+2)!}$$

The six constants d^{uu}_{ij}, $1\le i\le j\le 3$ are given by

$$\begin{aligned}d^{uu}_{ii} &= 2(|\Delta|[G_1{}^T][S^e_{uu}][G_1])_{ii} \qquad i = 1,2,3\\ d^{uu}_{ij} &= (4|\Delta|[G_1{}^T][S^e_{uu}][G_1])_{ij} \qquad 1\le i<j\le 3\end{aligned} \tag{43}$$

and are computed for each element.

Similar expressions and computed for d^{vv}_{ij}, $1\le i\le j\le 3$, and d^{ww}_{ij}, $1\le i\le j\le 6$, for each element. Again the matrices B^{vv}_{ij}, B^{ww}_{ij} are computed once and for all and then stored for future use.

The expressions obtained from the coupling terms are

$$4|\Delta| \iint_{\bar{T}} \lfloor \bar{U} \rfloor \lfloor \bar{S}^e_{uv} \rfloor \{\bar{V}\} \, d\xi \, d\eta$$

$$= \lfloor \bar{\delta}_u^e \rfloor \Big(\sum_{1 \le i,j \le 3} d_{ij}^{uv} ([\bar{A}_u^{-1}] \iint_{\bar{T}} \{\bar{g}_i^u\} \lfloor \bar{g}_j^v \rfloor \, d\xi \, d\eta [\bar{A}_v^{-1}] \{\bar{\delta}_v^e\}$$

$$= \lfloor \bar{\delta}_u^e \rfloor \Big(\sum_{1 \le i,j \le 3} d_{ij}^{uv} B_{ij}^{uv} \Big) \{\bar{\delta}_v^e\} \tag{44}$$

$$4|\Delta| \iint_{\bar{T}} \lfloor \bar{U} \rfloor [\bar{S}^e_{uw}] \{\bar{W}\} \, d\xi \, d\eta = \lfloor \bar{\delta}_u^e \rfloor \Big(\sum_{\substack{1 \le i \le 3 \\ 1 \le j \le 6}} d_{ij}^{uw} B_{ij}^{uw} \Big) \{\bar{\delta}_w^e\} \tag{45}$$

$$4|\Delta| \iint_{\bar{T}} \lfloor \bar{V} \rfloor [\bar{S}^e_{vw}] \{\bar{W}\} \, d\xi \, d\eta = \lfloor \bar{\delta}_v^e \rfloor \Big(\sum_{\substack{1 \le i \le 3 \\ 1 \le j \le 6}} d_{ij}^{vw} B_{ij}^{vw} \Big) \{\bar{\delta}_w^e\} \tag{46}$$

and they are obtained analogously

In (42)–(46) there are a total of 78 constants d_{ij} to be calculated for each element and then multiplied by the precomputed matrices B_{ij}. In most applications, however, there will not be complete coupling between u, v, and w, so that not all constants will have to be computed.

Let $\bar{K}^e_{uu} = \Sigma \, d_{ij}^{uu} B_{ij}^{uu}$, $\bar{K}^e_{vv} = \Sigma \, d_{ij}^{vv} B_{ij}^{vv}$, $\bar{K}^e_{ww} = \Sigma \, d_{ij}^{ww} B_{ij}^{ww}$, $\bar{K}^e_{uv} = \Sigma \, d_{ij}^{uv} B_{ij}^{uv}$, $\bar{K}^e_{uw} = \Sigma \, d_{ij}^{uw} B_{ij}^{uw}$, $\bar{K}^e_{vw} = \Sigma \, d_{ij}^{vw} B_{ij}^{vw}$, then from equations (34)–(46) we obtain

$$\frac{1}{2} \iint_T \lfloor \bar{w}(x, y) \rfloor [S^e] \{\bar{w}(x, y)\} \, dx \, dy$$

$$= \frac{1}{2} \lfloor \bar{\delta}_u^{eT} \bar{\delta}_v^{eT} \bar{\delta}_w^{eT} \rfloor \begin{bmatrix} \bar{K}^e_{uu} & \bar{K}^e_{uv} & \bar{K}^e_{uw} \\ \bar{K}^e_{vu} & \bar{K}^e_{vw} & \bar{K}^e_{vw} \\ \bar{K}^e_{wu} & \bar{K}^e_{wv} & \bar{K}^e_{wv} \end{bmatrix} \begin{Bmatrix} \bar{\delta}_u^e \\ \bar{\delta}_v^e \\ \bar{\delta}_w^e \end{Bmatrix}$$

$$= \frac{1}{2} \lfloor \bar{\delta}^e \rfloor [\bar{K}^e] \{\bar{\delta}^e\} = \frac{1}{2} \lfloor \delta^e \rfloor \lfloor D^T \rfloor [\bar{K}^e][D]\{\delta^e\}$$

$$= \frac{1}{2} \lfloor \delta^e \rfloor [K^e] \{\delta^e\} \tag{47}$$

$[\bar{K}^e]$ is the elemental stiffness matrix referred to standard nodal variables, and $[K^e] = [D^{eT}][\bar{K}^e][D^e]$ is the elemental stiffness matrix referred to nodal variables in the xy plane. In equation (47)

$$\{\bar{\delta}^e\} = [D^e]\{\delta^e\} = \begin{bmatrix} D_u^e & o & 0 \\ 0 & D_v^e & 0 \\ 0 & 0 & D_w^e \end{bmatrix} \begin{Bmatrix} \delta_u^e \\ \delta_v^e \\ \delta_w^e \end{Bmatrix}$$

where $[D_u^e]$ and $[D_v^e]$ are the diagonal matrices given in equation (15) and $[D_w^e]$ is the block diagonal matrix given in equation (25b).

Similar calculations give for the distributed load vector:

$$\iint_T \lfloor Z^e(x, y) \rfloor \{\bar{w}(x, y)\} \, dx \, dy$$

$$= 2|\Delta| \Big(\iint_{\bar{T}} \lfloor \bar{Z}_u^e(\xi, \eta) \rfloor \begin{Bmatrix} \lfloor \bar{g}_1^u(\xi, \eta) \rfloor \\ \lfloor \bar{g}_2^u(\xi, \eta) \rfloor \\ \lfloor \bar{g}_3^u(\xi, \eta) \rfloor \end{Bmatrix} d\xi \, d\eta \Big) [\bar{A}_u^{-1}] \bar{\delta}_u^e$$

$$+ 2|\Delta| \Big(\iint_{\bar{T}} \lfloor \bar{Z}_v^e(\xi, \eta) \rfloor \begin{Bmatrix} \lfloor \bar{g}_1^v(\xi, \eta) \rfloor \\ \lfloor \bar{g}_2^v(\xi, \eta) \rfloor \\ \lfloor \bar{g}_3^v(\xi, \eta) \rfloor \end{Bmatrix} d\xi \, d\eta \Big) [\bar{A}_v^{-1}] \{\bar{\delta}_v^e\}$$

$$+2|\Delta|\left(\iint_{\bar{T}} \lfloor \bar{\bar{Z}}_w^e(\xi,\eta)\rfloor \begin{Bmatrix} \lfloor \bar{g}_1^w(\xi,\eta)\rfloor \\ \vdots \\ \lfloor \bar{g}_6^w(\xi,\eta)\rfloor \end{Bmatrix} \mathrm{d}\xi\,\mathrm{d}\eta\right)[\bar{A}_w^{-1}]\{\bar{\delta}_w^e\}$$

$$= 2|\Delta|\left(\iint_{\bar{T}} \lfloor \bar{\bar{Z}}^e(\xi,\eta)\rfloor\{\bar{g}(\xi,\eta)\}\,\mathrm{d}\xi\,\mathrm{d}\eta\right)[\bar{A}^{-1}]\{\bar{\delta}^e\}$$

$$= \lfloor f^e\rfloor\{\bar{\delta}^e\} \tag{48}$$

where

$$\lfloor \bar{\bar{Z}}^e(\xi,\eta)\rfloor = \lfloor \lfloor \bar{Z}_u^e\rfloor, \lfloor \bar{Z}_v^e\rfloor, \lfloor \bar{Z}_w^e\rfloor\rfloor$$

$$\lfloor \bar{g}(\xi,\eta)\rfloor = \lfloor \lfloor \bar{g}_1^u(\xi,\eta)\rfloor,\ldots,\lfloor \bar{g}_3^u(\xi,\eta)\rfloor; \lfloor \bar{g}_1^v(\xi,\eta)\rfloor,\ldots,$$
$$\ldots,\lfloor \bar{g}_3^v(\xi,\eta)\rfloor; \lfloor \bar{g}_1^w(\xi,\eta)\rfloor,\ldots,\lfloor \bar{g}_6^w(\xi,\eta)\rfloor\rfloor$$

$$\lfloor f^e\rfloor = 2|\Delta|\left(\iint_{\bar{T}} \lfloor \bar{\bar{Z}}^e(\xi,\eta)\rfloor\{\bar{g}(\xi,\eta)\}\,\mathrm{d}\xi\,\mathrm{d}\eta\right)[\bar{A}^{-1}]. \tag{49}$$

If $\lfloor \bar{\bar{Z}}^e(\xi,\eta)\rfloor$ is not constant the integration is performed numerically; otherwse, standard formulas are used.

6. THE UNCONSTRAINED MINIMIZATION

The total potential energy π which was given in equation (1) is now expressed in terms of nodal variables as

$$\pi = \sum_e \pi^e = \frac{1}{2}\sum_e \lfloor \delta^e\rfloor[K^e]\{\delta^e\} - \lfloor f^e\rfloor\{\delta^e\}. \tag{50}$$

Independent variables among the set of all $\{\delta^e\}$ consisting of appropriate second order derivatives (either mixed or tangential) have been chosen as described in equation (29) and in the subsequent discussion. The other second order derivatives which are dependent variables are expressed in terms of these explicitly. The set of all independent nodal variables $\{\delta_I\}$ is now determined explicitly. For the case of C^1 continuity it is similar to the set given in [11].

The set of all dependent variables $\{\delta_D\}$ is expressed explicitly in terms of $(\delta_I\}$. These expressions are substituted in equation (50) to give an unconstrained minimization problem. Internal nodal variables are first eliminated at the elemental level via static condensation. It is important to note that the nodal variables associated with a given element e will depend only upon nodal variables of those elements which are adjacent to e. Therefore the final stiffness matrix in the unconstrained optimization will now have a banded structure, the width of the band depending only upon the number of elements adjacent to a given element. This will eliminate the problem of multiplying large matrices referred to in the Introduction. In [12], a scheme is described in detail in which the calculation of the global unconstrained stiffness matrix can be treated as if it were a conventional finite element assembly process. We do not elaborate here.

In any case, form this point on, the problem is one of unconstrained optimization and conventional finite element codes such as Irons' Frontal Solver[21] can be used to determine $\{\delta_I\}$ the vector of all independent nodal variables. $\{\delta_D\}$ is given explicitly in terms of $\{\delta_I\}$ so the entire vector $\{\delta\}$ is known. For each element e, the polynomial coefficient vector $\{\bar{a}^e\}$ is given in terms of $\{\bar{\delta}^e\}$ by means of the explicitly computed inverse $[\bar{A}^{e-1}]$ and $\{\bar{\delta}^e\}$ is easily computed from $\{\delta^e\}$ as described earlier. Hence a typical displacement field $\bar{w}(\xi,\eta)$ is given in terms of its $\xi-\eta$ coordinates. In order to determine $w(x,y)$ one can either compute the coefficients $\{a^e\}$ in terms of $\{\bar{a}^e\}$, or one can map $\bar{x}, \bar{y}$ into ξ, η via the affine transformation M^{-1} and then compute w and its derivatives from equation (35). Similarly for the other displacement fields $u(x,y)$ and $v(x,y)$.

7. A NUMERICAL EXAMPLE—PLANE STRESS SQUARE WITH PARABOLIC END LOADS

We give some numerical results obtained in the solution of a plane stress problem (C^0 continuity). Accuracy was achieved by increasing the orders of polynomial approximation while keeping the triangulation fixed, (p-convergence) as opposed to the conventional approach in which the polynomial orders are fixed and the triangulation is successively refined (h-convergence). Numerical results in other problems, as well as more details concerning the implementation of hierarchic nodal variables on the computer are presented in [18].

To demonstrate the computational efficiency obtainable using hierarchal nodal variables and pre-computed arrays, a computer program was written and a number of test problems solved. The program was executed on an IBM Model 360/65 running under OS/360 MFT II and HASP 3.1 and using Memorex 3330 disks. IBM Level H FORTRAN IV with the OPT = 2 option was used for all routines performing substantial computations; Level G FORTRAN was used elsewhere. No machine language routines were employed. All calculations with real variables were performed with double precision arithmetic. Random access disk files were used for storing the precomputed arrays.

In [22], Cowper described an algorithm (based on precomputed arrays) for the computation of the 18×18 element stiffness matrix for his C^1 plate bending element. He also gave the computation time required to execute the algorithm, and, to indicate the relative speed of his computer, he reported the time required to square an 18×18 matrix. On the computing system used in the present study, an 18×18 matrix can be squared in 0.107 ± 0.007 seconds, a result obtained using the IBM Scientific Subroutine Package [23] routine GMPRD converted to double precision and compiled with optimization level OPT = 2, and averaging over many runs. We will define 0.107 seconds to be an equivalent time unit (e.t.u.), and, to facilitate possible future comparisons with other programs implemented on different computers, all CPU times given in this paper will be given in e.t.u.

The computational effort required to solve the test problems will be indicated in most cases by giving the CPU time (in e.t.u.) consumed in solving the governing system of linear equations. The particular equation solver employed was the frontal solver of Irons [21], modified by the introduction of direct access files (to avoid the BACKSPACE command) and the elimination of all implied DO-loops. Along with the CPU time for a given polynomial order, we will also give the value of N_i, the total number of external and internal nodal variables in the mesh disregarding boundary conditions, since this equals the number of equations processed by the frontal solver. (Boundary conditions are enforced by multiplying appropriate terms on the diagonal of the coefficient matrix by a large number, rather than through omitting the corresponding equations.)

Many investigators in finite element research are reluctant to quote actual machine time, perhaps justifiably, since this may introduce factors irrelevant to evaluating the efficiency of the algorithm, such as particular machine and operating system characteristics or programming skill. To facilitate comparisons which are independent of these factors, in addition to CPU times we will give the number of active degrees of freedom (DOF), equal to N_i minus the number of nodal variables suppressed through boundary conditions. Furthermore, following the suggestion of Abel and Desai [24], a measure of error will be plotted versus NB^2, where N and B are the number of equations and the maximum half bandwidth calculated by disregarding boundary conditions and assuming internal nodal variables have been eliminated through static condensation. Unfortunately, the use of NB^2 to measure computational effort may be misleading, since on the one hand ignoring internal degrees of freedom tends to favor high order elements with many internal nodal variables, while on the other hand ignoring boundary conditions favors low order elements, as was recently shown in [25]. Since these are contrary effects, perhaps they tend to cancel each other; in any event, Abel and Desai's procedure will be adopted here for its convenience.

The strain energy of an isotropic, linearly elastic, plane stress membrane of thickness t is

$$U = \frac{1}{2} \iint \frac{Et}{1+\nu} \left[\frac{1}{1-\nu} (u_x{}^2 + v_y{}^2 + 2u_x v_y) + \frac{1}{2}(u_y + v_x)^2 \right] d\Omega, \tag{51}$$

where E is the elastic modulus, ν is Poisson's ratio, u and v are the x and y displacements (subscripts denote derivatives) and Ω is the area of the membrane. This expression may be written in the form of the first term appearing in the functional of equation (34) if $[S_{uw}] = [S_{vw}] = [S_{wu}] = [S_{wv}] = [S_{ww}] = [0]$ and

$$[S_{uu}] = [S_{vv}] = \frac{Et}{1+\nu}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{1-\nu} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \tag{52}$$

$$[S_{vu}]^T = [S^{12}] = \frac{Et}{1+\nu}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\nu}{1-\nu} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}. \tag{53}$$

As an example, we consider the problem of a square membrane loaded with the parabolically varying edge stresses

$$\sigma_{yy} = \sigma_0(1 - 4x^2/L^2) \tag{54}$$

shown in the inset of Fig. 1. The vertical edges are stress-free. Using the two element mesh shown, to model one quarter of the domain, the values given in Table 1 for various polynomial orders p (the same for u and v) were obtained. Rapid convergence can be observed for all quantities in the table, with some of the computed values agreeing with the analytic values to five or more significant figures. In general, the convergence is not monotonic, for example the most accurate value of v_c is obtained for $p = 4$. The strain energy does converge monotonically, however, and for $p = 8$ agrees to seven digits with the analytic value.

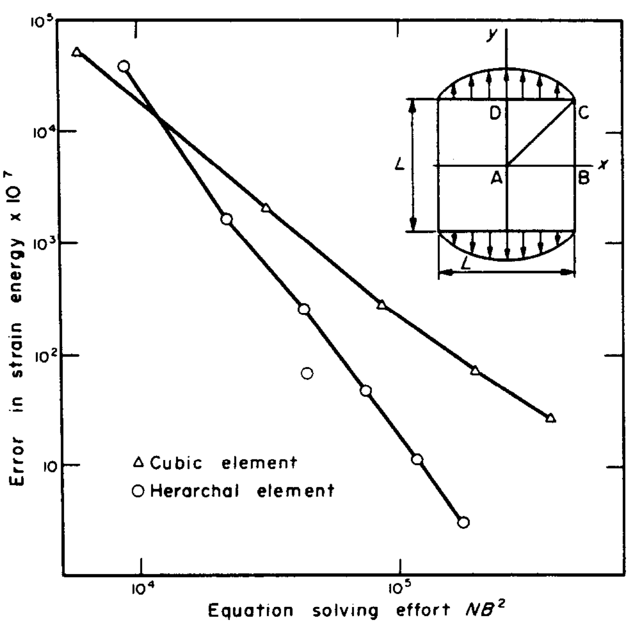


Fig. 1. Comparison of computational efficiency for plane stress problem.

In Fig. 1 the error in strain energy is plotted versus equation solving effort NB^2 for the hierarchal element and for the plane stress element of Cowper *et al.*[26]. The latter element is generally considered quite efficient since it is based on complete third order polynomials and employs both function and first derivative values as vertex nodal variables. Even so the results of Fig. 1 indicate that the conventional approach of mesh refinement with polynomial order fixed (h-convergence) is less efficient than increasing polynomial order with the hierarchal element with the triangulation fixed (p-convergence). Solution trends for the hierarchal element approach are given in Table 2.

Table 1. Results for plane stress square problem

P	DOF	$\frac{10tEu_B}{(1-\nu^2)\sigma_0 L}$	$\frac{10^2Etu_C}{(1-\nu^2)\sigma_0 L}$	$\frac{10Etv_C}{(1-\nu^2)\sigma_0 L}$	$\frac{10Etv_D}{(1-\nu^2)\sigma_0 L}$	$\frac{10\sigma_{xxA}}{\sigma_0}$	$\frac{10\sigma_{yyA}}{\sigma_0}$	$\frac{\sigma_{xxB}}{\sigma_0}$	$\frac{10\sigma_{yyB}}{\sigma_0}$	$\frac{\sigma_{xxC}}{\sigma_0}$	Strain energy $\frac{10Et^2U}{(1-\nu)^2L^2\sigma_0{}^2}$
3	24	−1.524012	1.9310	1.29200	5.069395	−1.20354	8.22596	0.00828	4.51315	0.05357	2.7895590
4	40	−1.519549	1.7839	1.27730	5.073858	−1.39082	8.64721	−0.00167	4.15533	0.00328	2.7934086
5	6–	−1.518966	1.7782	1.27672	5.074441	−1.44156	8.59593	0.00005	4.11341	−0.00664	2.7935428
6	84	−1.519974	1.7808	1.27698	5.073432	−1.39880	8.58848	0.00010	4.10366	−0.00658	2.7935648
7	112	−1.519976	1.7821	1.27712	5.073431	−1.41451	8.59148	−0.00006	4.10611	−0.00451	2.7935684
8	144	−1.519918	1.7829	1.27720	5.073489	−1.40705	8.58991	0.00003	4.10738	−0.00299	2.7935692
Ref. 14 5×5 Mesh	292	−1.519900	1.7852	1.27742	5.073507	−1.40880	8.59120	0.000166	4.10767	−0.00233	2.7935667
Analytic		−1.519928	1.7837	1.27727	5.073478	−1.40954	8.59046	0	4.10670	0	2.7935695

Table 2. CPU time required to solve equations for plane stress square problem

p	N_i	CPU time in e.t.u.
3	32	19
4	50	24
5	72	38
6	98	55
7	128	93
8	162	152

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APPENDIX 1

Transformation from Standard to Non-standard Polynomial Coefficients.

From equations (4) and (5) we have

$$u(\bar{x},\bar{y})=\sum_{k=0}^{p}\sum_{i=0}^{k}a_{k-i,i}\bar{x}^{k-i}\bar{y}^{i} \tag{1-1}$$

$$\bar{u}(\xi,\eta)=u(\bar{x}_2\xi+\bar{x}_3\eta,\bar{y}_2\xi+\bar{y}_3\eta)=\sum_{k=0}^{p}\sum_{i=0}^{k}a_{k-i,i}(\bar{x}_2\xi+\bar{x}_3\eta)^{k-i}(\bar{y}_2\xi+\bar{y}_3\eta)^{i}$$

$$=\sum_{k=0}^{p}\sum_{i=0}^{k}\bar{a}_{k-i,i}\xi^{k-i}\eta^{i}. \tag{1-2}$$

It now follows that if we let $\zeta=\eta/\xi$ then

$$\sum_{i=0}^{k}\bar{a}_{k-i,i}\xi^{k-i}\eta^{i}=\sum_{i=0}^{k}a_{k-i,i}\xi^{k}(\bar{x}_2+\bar{x}_3\zeta)^{k-i}(\bar{y}_2+\bar{y}_3\zeta)^{i}$$

$$=\xi^{k}\sum_{i=0}^{k}a_{k-i,i}\left(\sum_{r=0}^{k-i}\binom{k-i}{r}\bar{x}_2^{k-i-r}(\bar{x}_3\zeta)^{r}\right)\left(\sum_{s=0}^{i}\binom{i}{s}\bar{y}_2^{i-s}(\bar{y}_3\zeta)^{s}\right)$$

$$=\xi^{k}\sum_{i=0}^{k}a_{k-i,i}\sum_{r=0}^{k}\left(\sum_{s=0}^{r}\binom{k-i}{r-s}\bar{x}_2^{k-i-r+s}\bar{x}_3^{r-s}\binom{i}{s}\bar{y}_2^{i-s}\bar{y}_3^{s}\right)\zeta^{r}$$

where the binomial symbol $\binom{n}{m}=0$ if $m>n$. Therefore,

$$\sum_{i=0}^{k}\bar{a}_{k-i,i}\xi^{k-i}\eta^{i}=\sum_{r=0}^{k}\xi^{k-r}\eta^{r}\left(\sum_{i=0}^{k}a_{k-i,i}\sum_{s=0}^{r}\binom{k-i}{r-s}\binom{i}{s}\bar{x}_2^{k-i-r+s}\bar{x}_3^{r-s}\bar{y}_2^{i-s}\bar{y}_3^{s}\right)$$

so that we have the transformation

$$\bar{a}_{k-i,i}=\sum_{r=0}^{k}a_{k-r,r}\left(\sum_{s=0}^{i}\binom{k-r}{i-s}\binom{r}{s}\bar{x}_2^{k-r-i+s}\bar{x}_3^{i-s}\bar{y}_2^{r-s}\bar{y}_3^{s}\right). \tag{1-3}$$

If we let $\{a\}$ and $\{\bar{a}\}$ be the vectors consisting of all the polynomial coefficients in (1-1) and (1-2) respectively then (1-3) gives the matrix $[B^{-1}]$ in

$$\{\bar{a}\}=[B^{-1}]\{a\}. \tag{1-4}$$

Note that $[B^{-1}]$ is block diagonal, each block relating the coefficients of kth degree terms in (1-1) and (1-2). Similarly using equation (4b) we obtain

$$a_{k-i,i}=\frac{1}{J^{k}}\sum_{r=0}^{k}\bar{a}_{k-r,r}\left(\sum_{s=0}^{i}\binom{k-r}{i-s}\binom{r}{s}\bar{y}_3^{k-r-i+s}(-\bar{x}_3)^{i-s}(-\bar{y}_2)^{r-s}\bar{x}_2^{s}\right) \tag{1-5}$$

where $J=\bar{x}_2\bar{y}_3-\bar{x}_3\bar{y}_2=2\Delta$. This gives

$$\{a\}=[B]\{\bar{a}\}. \tag{1-6}$$

Again $[B]$ has block diagonal structure.

APPENDIX 2

Transformation matrices for C^0 elements

We give $[\bar{A}^{(p)-1}]$ by writing $\{\bar{a}^{(p)}\} = [\bar{A}^{(p)-1}]\{\bar{\delta}^{(p)}\}$ explicitly.

Assume that

$$\bar{u}(\xi,\eta) = \sum_{0\le i+j\le p} \bar{a}_{ij}^{(p)}\xi^i\eta^j$$

and let

$$\lfloor \bar{a}^{(p)} \rfloor = \lfloor \bar{a}_{00}^{(p)}, \bar{a}_{10}^{(p)}, \bar{a}_{01}^{(p)}, \bar{a}_{20}^{(p)}, \bar{a}_{11}^{(p)}, \ldots, \bar{a}_{p0}^{(p)}, \bar{a}_{p-1,1}^{(p)}, \ldots, \bar{a}_{1,p-1}^{(p)}, \bar{a}_{0p}^{(p)} \rfloor.$$

For each p we give those expression in $\bar{a}_{ij}^{(p)}$, $0 \le i+j \le p$, which are different from $\bar{a}_{ij}^{(k)}$, for $k < p$.

$p = 2$

$$\lfloor \bar{\delta}^{(2)} \rfloor = \lfloor \bar{u}(R_1), \bar{u}(R_2), \bar{u}(\bar{R}_3); \bar{u}_{\xi\xi}(S_1), \bar{u}_{ss}(S_2), \bar{u}_{\eta\eta}(S_3) \rfloor$$

$$\bar{a}_{00}^{(2)} = \bar{u}(R_1)$$

$$\bar{a}_{10}^{(2)} = -\bar{u}(R_1) + \bar{u}(R_2) - \frac{1}{2}\bar{u}_{\xi\xi}(S_1)$$

$$\bar{a}_{01}^{(2)} = -\bar{u}(R_1) + \bar{u}(R_3) - \frac{1}{2}\bar{u}_{\eta\eta}(S_3)$$

$$\bar{a}_{20}^{(2)} = \frac{1}{2}\bar{u}_{\xi\xi}(S_1)$$

$$\bar{a}_{11}^{(2)} = \frac{1}{2}\bar{u}_{\xi\xi}(S_1) - \bar{u}_{ss}(S_2) + \frac{1}{2}\bar{u}_{\eta\eta}(S_3)$$

$$\bar{a}_{02}^{(2)} = \frac{1}{2}\bar{u}_{\eta\eta}(S_3)$$

$p = 3$

$$\lfloor \bar{\delta}^{(3)} \rfloor = \lfloor \bar{\delta}^{(2)}; \bar{u}_{\xi^3}(S_1). u_{s^3}(S_2), \bar{u}_{\eta^3}(S_3), \bar{u}_{\xi^2\eta}(R_1) \rfloor$$

$$a_{10}^{(3)} = \bar{a}_{10}^{(2)} + \frac{1}{12}\bar{u}_{\xi^3}(S_1)$$

$$a_{01}^{(3)} = \bar{a}_{01}^{(2)} + \frac{1}{12}\bar{u}_{\eta^3}(S_3)$$

$$\bar{a}_{20}^{(3)} = a_{20}^{(2)} - \frac{1}{4}\bar{u}_{\xi^3}(S_1)$$

$$\bar{a}_{11}^{(3)} = \bar{a}_{11}^{(2)} + \frac{1}{12}\bar{u}_{\xi^3}(S_1) + \frac{\sqrt{2}}{6}\bar{u}_{s^3}(S_2) - \frac{1}{12}\bar{u}_{\eta^3}(S_3) - \frac{1}{4}\bar{u}_{\xi^2\eta}(R_1)$$

$$\bar{a}_{02}^{(3)} = \bar{a}_{02}^{(2)} - \frac{1}{4}\bar{u}_{\eta^3}(S_3)$$

$$\bar{a}_{30}^{(3)} = \frac{1}{6}\bar{u}_{\xi^3}(S_1)$$

$$\bar{a}_{21}^{(3)} = \frac{1}{2}\bar{u}_{\xi^2\eta}(R_1)$$

$$\bar{a}_{12}^{(3)} = -\frac{1}{6}\bar{u}_{\xi^3}(S_1) - \frac{\sqrt{2}}{3}\bar{u}_{s^3}(S_2) + \frac{1}{6}\bar{u}_{\eta^3}(S_3) + \frac{1}{2}\bar{u}_{\xi^2\eta}(R_1)$$

$$\bar{a}_{03}^{(3)} = \frac{1}{6}\bar{u}_{\eta^3}(S_3)$$

$p = 4$

$$\lfloor \bar{\delta}^{(4)} \rfloor = \lfloor \bar{\delta}^{(3)}; \bar{u}_{\xi^4}(S_1), \bar{u}_{s^4}(S_2), \bar{u}_{\eta^4}(S_3), \bar{u}_{\xi^3\eta}(R_1), \bar{u}_{\xi\eta^3}(R_1) \rfloor$$

$$\bar{a}_{10}^{(4)} = \bar{a}_{10}^{(3)} - \frac{1}{48} u_{\xi^4}(S_1)$$

$$\bar{a}_{01}^{(4)} = \bar{a}_{01}^{(3)} - \frac{1}{48}\bar{u}_{\eta^4}(S_3)$$

$$\bar{a}_{20}^{(4)} = \bar{a}_{20}^{(3)} + \frac{1}{16}\bar{u}_{\xi^4}(S_1)$$

$$\bar{a}_{11}^{(4)} = \bar{a}_{11}^{(3)} + \frac{1}{48}\bar{u}_{\xi^4}(S_1) - \frac{1}{12}\bar{u}_{s^4}(S_2) + \frac{1}{48}\bar{u}_{\eta^4}(S_3) - \frac{1}{6}\bar{u}_{\xi^3\eta}(R_1)$$

$$\bar{a}_{02}^{(4)} = \bar{a}_{02}^{(3)} + \frac{1}{16}\bar{u}_{\eta^4}(S_3)$$

$$\bar{a}_{30}^{(4)} = \bar{a}_{30}^{(3)} - \frac{1}{12}\bar{u}_{\xi^4}(S_1)$$

$$\bar{a}_{21}^{(4)} = \bar{a}_{21}^{(3)}$$

$$\bar{a}_{12}^{(4)} = \bar{a}_{12}^{(3)} + \frac{1}{6}\bar{u}_{\xi^3\eta}(R_1) - \frac{1}{6}\bar{u}_{\xi\eta^3}(R_1)$$

$$\bar{a}_{03}^{(4)} = \bar{a}_{03}^{(3)} - \frac{1}{12}\bar{u}_{\eta^4}(S_3)$$

$$\bar{a}_{40}^{(4)} = \frac{1}{24}\bar{u}_{\xi^4}(S_1)$$

$$\bar{a}_{31}^{(4)} = \frac{1}{6}\bar{u}_{\xi^3\eta}(R_1)$$

$$\bar{a}_{22}^{(4)} = -\frac{1}{24}\bar{u}_{\xi^4}(S_1) + \frac{1}{6}\bar{u}_{\bar{\xi}^4}(S_2) - \frac{1}{24}\bar{u}_{\eta^4}(S_3) + \frac{1}{6}\bar{u}_{\xi^3\eta}(R_1) + \frac{1}{6}\bar{u}_{\xi\eta^3}(R_1)$$

$$\bar{a}_{13}^{(4)} = \frac{1}{6}\bar{u}_{\xi\eta^3}(R_1)$$

$$\bar{a}_{04}^{(4)} = \frac{1}{24}\bar{u}_{\eta^4}(S_3)$$

$p = 5$

$$\lfloor \bar{\delta}^{(5)} \rfloor = \lfloor \bar{\delta}^{(4)};\ \bar{u}_{\xi^5}(S_1),\ \bar{u}_{\bar{\xi}^5}(S_2),\ \bar{u}_{\eta^5}(S_3),\ \bar{u}_{\xi^4\eta}(R_1)\bar{u}_{\xi^3\eta^2}(R_1),\ \bar{u}_{\xi\eta^4}(R_1) \rfloor$$

$$\bar{a}_{10}^{(5)} = a_{10}^{(4)} + \frac{1}{480}\bar{u}_{\xi^5}(S_1)$$

$$\bar{a}_{01}^{(5)} = \bar{a}_{01}^{(4)} + \frac{1}{480}\bar{u}_{\eta^5}(S_3)$$

$$\bar{a}_{20}^{(5)} = \bar{a}_{20}^{(4)} - \frac{1}{96}\bar{u}_{\xi^5}(S_1)$$

$$\bar{a}_{11}^{(5)} = \bar{a}_{11}^{(4)} + \frac{1}{480}\bar{u}_{\xi^5}(S_1) + \frac{\sqrt{2}}{120}\bar{u}_{\bar{\xi}^5}(S_2) - \frac{1}{480}\bar{u}_{\eta^5}(S_3) - \frac{1}{24}\bar{u}_{\xi^4\eta}(R_1)$$

$$\bar{a}_{02}^{(5)} = \bar{a}_{02}^{(4)} - \frac{1}{96}\bar{u}_{\eta^5}(S_3)$$

$$\bar{a}_{30}^{(5)} = \bar{a}_{30}^{(4)} + \frac{1}{48}\bar{u}_{\xi^5}(S_1)$$

$$\bar{a}_{21}^{(15)} = \bar{a}_{21}^{(4)}$$

$$\bar{a}_{12}^{(5)} = \bar{a}_{12}^{(4)} - \frac{1}{240}\bar{u}_{\xi^5}(S_1) - \frac{\sqrt{2}}{60}\bar{u}_{\bar{\xi}^5}(S_2) + \frac{1}{240}\bar{u}_{\eta^5}(S_3) + \frac{1}{24}\bar{u}_{\xi^4\eta}(R_1) - \frac{1}{24}\bar{u}_{\xi\eta^4}(R_1)$$

$$\bar{a}_{03}^{(5)} = \bar{a}_{03}^{(4)} + \frac{1}{48}\bar{u}_{\eta^5}(S_3)$$

$$\bar{a}_{40}^{(5)} = \bar{a}_{40}^{(4)} - \frac{1}{48}\bar{u}_{\eta^5}(S_1)$$

$$\bar{a}_{31}^{(5)} = \bar{a}_{31}^{(4)}$$

$$\bar{a}_{22}^{(5)} = \bar{a}_{22}^{(4)} - \frac{1}{240}\bar{u}_{\xi^5}(S_1) - \frac{\sqrt{2}}{60}\bar{u}_{\bar{\xi}^5}(S_2) + \frac{1}{240}\bar{u}_{\eta^5}(S_3) + \frac{1}{12}\bar{u}_{\xi^4\eta}(R_1) - \frac{1}{12}\bar{u}_{\xi^3\eta^2}(R_1) + \frac{1}{24}\bar{u}_{\xi\eta^4}(R_1)$$

$$\bar{a}_{13}^{(5)} = \bar{a}_{13}^{(4)}$$

$$\bar{a}_{04}^{(5)} = \bar{a}_{04}^{(4)} - \frac{1}{48}\bar{u}_{\eta^5}(S_3)$$

$$\bar{a}_{50}^{(5)} = \frac{1}{120}\bar{u}_{\eta^5}(S_1)$$

$$\bar{a}_{41}^{(5)} = \frac{1}{24}\bar{u}_{\xi^4\eta}(R_1)$$

$$\bar{a}_{32}^{(5)} = \frac{1}{12}\bar{u}_{\xi^3\eta^2}(R_1)$$

$$\bar{a}_{23}^{(5)} = \frac{1}{120}\bar{u}_{\xi^5}(S_1) + \frac{\sqrt{2}}{30}\bar{u}_{\xi^5}(S_2) - \frac{1}{120}\bar{u}_{\eta^5}(S_3) - \frac{1}{24}\bar{u}_{\xi^4\eta}(R_1) + \frac{1}{12}\bar{u}_{\xi^3\eta^2}(R_1) + \frac{1}{24}\bar{u}_{\xi\eta^4}(R_1)$$

$$\bar{a}_{14}^{(5)} = \frac{1}{24}\bar{u}_{\xi\eta^4}(R_1)$$

$$\bar{a}_{05}^{(5)} = \frac{1}{120}\bar{u}_{\eta^5}(R_3)$$

The matrices $[\bar{A}^{(p)}]$ are not used in the calculations. They are given explicitly, however, in [17].

APPENDIX 3

Transformation matrices for C^1 elements

We give $[\tilde{A}^{(p)-1}]$ by writing $\{\tilde{a}^{(p)}\} = [\tilde{A}^{(p)-1}]\{\tilde{\delta}^{(p)}\}$ explicitly.

Assume that

$$\tilde{w}(\xi, \eta) = \sum_{0 \le i+j \le p} \tilde{a}_{ij}^{(p)} \xi^i \eta^j$$

and let

$$\lfloor \tilde{a}^{(p)} \rfloor = \lfloor \tilde{a}_{00}^{(p)}; \tilde{a}_{10}^{(p)}, \tilde{a}_{01}^{(p)}; \tilde{a}_{20}^{(p)}, \tilde{a}_{11}^{(p)}, \tilde{a}_{02}^{(p)}, \ldots, \tilde{a}_{p0}^{(p)}, \tilde{a}_{p-1,1}^{(p)} \ldots, \tilde{a}_{1,p-1}^{(p)}, \tilde{a}_{0p}^{(p)} \rfloor$$

For each p we give those expressions in $\tilde{a}_{ij}^{(p)}$ $0 \le i+j \le p$ which are different from $\tilde{a}^{(k)}$, $k < p$.

$$p = 5$$

$$\begin{aligned}
\lfloor \tilde{\delta}^{(5)} \rfloor = \lfloor & \tilde{w}(R_1), \tilde{w}(R_2), \tilde{w}(R_3); \tilde{w}_\xi(R_1), \tilde{w}_\eta(R_1), \tilde{w}_\xi(R_2), \tilde{w}_\zeta(R_2), \tilde{w}_\eta(R_3), \tilde{w}_\zeta(R_3); \\
& \tilde{w}_{\xi\xi}(R_1), \tilde{w}_{\xi\eta}(R_1), \tilde{w}_{\eta\eta}(R_1), \tilde{w}_{\xi^2}(R_2), \tilde{w}_{\xi\zeta}(R_2), \tilde{w}_{\bar{\zeta}\zeta}(R_2), \\
& \tilde{w}_{\eta\eta}(R_3), \tilde{w}_{\eta\xi}(R_3), \tilde{w}_{\bar{\zeta}\zeta}(R_3), \tilde{w}_{\xi^4\eta}(S_1), \tilde{w}_{\zeta^4\bar{n}}(S_2), \tilde{w}_{\xi\eta^4}(S_3) \rfloor
\end{aligned}$$

$$\tilde{a}_{00} = \tilde{w}(R_1)$$

$$\tilde{a}_{10} = \tilde{w}_\xi(R_1)$$

$$\tilde{a}_{01} = \tilde{w}_\eta(R_1)$$

$$\tilde{a}_{20} = \frac{1}{2}\tilde{w}_{\xi^2}(R_1)$$

$$\tilde{a}_{11} = \tilde{w}_{\xi\eta}(R_1)$$

$$\tilde{a}_{02} = \frac{1}{2}\tilde{w}_{\eta^2}(R_1)$$

$$\tilde{a}_{30} = -10\tilde{w}(R_1) - 6\tilde{w}_\xi(R_1) - \frac{3}{2}\tilde{w}_{\xi^2}(R_1) + 10\tilde{w}(R_2) - 4\tilde{w}_\xi(R_2) + \frac{1}{2}\tilde{w}_{\xi^2}(R_2)$$

$$\tilde{a}_{21} = -3\tilde{w}_\eta(R_1) - 2\tilde{w}_{\xi\eta}(R_1) + 3\tilde{w}_\xi(R_2) + 3\sqrt{2}\tilde{w}_\zeta(R_2) - \tilde{w}_{\xi^2}(R_2) - \sqrt{2}\tilde{w}_{\xi\zeta}(R_2) + \frac{1}{24}\tilde{w}_{\xi^4\eta}(S_1)$$

$$\tilde{a}_{12} = -3\tilde{w}_\xi(R_1) - 2\tilde{w}_{\xi\eta}(R_1) + 3\tilde{w}_\eta(R_3) - 3\sqrt{2}\tilde{w}_\zeta(R_3) - \tilde{w}_{\eta^2}(R_3) + \sqrt{2}\tilde{w}_{\eta\zeta}(R_3) + \frac{1}{24}\tilde{w}_{\xi\eta^4}(S_3)$$

$$\tilde{a}_{03} = -10\tilde{w}(R_1) - 6\tilde{w}_\eta(R_1) - \frac{3}{2}\tilde{w}_{\eta^2}(R_1) + 10\tilde{w}(R_3) - 4\tilde{w}_\eta(R_3) + \frac{1}{2}\tilde{w}_{\eta^2}(R_3)$$

$$\tilde{a}_{40} = 15\tilde{w}(R_1) + 8\tilde{w}_\xi(R_1) + \frac{3}{2}\tilde{w}_{\xi^2}(R_1) - 15\tilde{w}(R_2) + 7\tilde{w}_\xi(R_2) - \tilde{w}_{\xi^2}(R_2)$$

$$\tilde{a}_{31} = 2\tilde{w}_\eta(R_1) + \tilde{w}_{\xi\eta}(R_1) - 2\tilde{w}_\xi(R_2) - 2\sqrt{2}\tilde{w}_\zeta(R_2) + \tilde{w}_{\xi^2}(R_2) + \sqrt{2}\tilde{w}_{\xi\zeta}(R_2) - \frac{1}{12} w_{\xi^4\eta}(S_1)$$

$$\begin{aligned}
\tilde{a}_{22} = & -30\tilde{w}(R_1) + 15\tilde{w}(R_2) + 15\tilde{w}(R_3) - 6\tilde{w}_\xi(R_1) - 6\tilde{w}_\eta(R_1) - 9\tilde{w}_\xi(R_2) - \frac{3}{\sqrt{2}}\tilde{w}_\zeta(R_2) \\
& - 9\tilde{w}_\eta(R_3) + \frac{3}{\sqrt{2}}\tilde{w}_\zeta(R_3) - \frac{3}{2}\tilde{w}_{\xi^2}(R_1) + 2\tilde{w}_{\xi\eta}(R_1) - \frac{3}{2}\tilde{w}_{\eta^2}(R_1) + 2\tilde{w}_{\xi^2}(R_2) + \sqrt{2}\tilde{w}_{\xi\zeta}(R_2) \\
& + \frac{1}{2}\tilde{w}_{\bar{\zeta}\zeta}(R_2) + 2\tilde{w}_{\eta^2}(R_3) - \sqrt{2}\tilde{w}_{\eta\zeta}(R_3) + \frac{1}{2}\tilde{w}_{\zeta\zeta}(R_3) - \frac{1}{12}\tilde{w}_{\xi^4\eta}(S_1) - \frac{\sqrt{2}}{12}\tilde{w}_{\zeta^4\bar{n}}(S_2) - \frac{1}{12}\tilde{w}_{\xi\eta^4}(S_3)
\end{aligned}$$

$$\tilde{a}_{13} = 2\tilde{w}_\xi(R_1) + \tilde{w}_{\xi\eta}(R_1) - 2\tilde{w}_\eta(R_3) + 2\sqrt{2}\tilde{w}_\zeta(R_3) + \tilde{w}_{\eta^2}(R_3) - \sqrt{2}\tilde{w}_{\eta\zeta}(R_3) - \frac{1}{12}\tilde{w}_{\xi\eta^4}(S_3)$$

$$\tilde{a}_{04} = 15\tilde{w}(R_1) + 8\tilde{w}_\eta(R_1) + \frac{3}{2}\tilde{w}_{\eta^2}(R_1) - 15\tilde{w}(R_3) + 7\tilde{w}_\eta(R_3) - \tilde{w}_{\eta^2}(R_3)$$

$$\tilde{a}_{50} = -6\tilde{w}(R_1) - 3\tilde{w}_\xi(R_1) - \frac{1}{2}\tilde{w}_{\xi^2}(R_1) + 6\tilde{w}(R_2) - 3\tilde{w}_\xi(R_2) + \frac{1}{2}\tilde{w}_{\xi^2}(R_2)$$

$$\tilde{a}_{41} = \frac{1}{24}\tilde{w}_{\xi^4\eta}(S_1)$$

$$\begin{aligned}
\tilde{a}_{32} = & \ 30\tilde{w}(R_1) - 15\tilde{w}(R_2) - 15\tilde{w}(R_3) + 9\tilde{w}_\xi(R_1) + 6\tilde{w}_\eta(R_1) + 9\tilde{w}_\xi(R_2) + \frac{3}{2}\sqrt{2}\tilde{w}_\zeta(R_2) \\
& + 6\tilde{w}_\eta(R_3) + \frac{3}{2}\sqrt{2}\tilde{w}_\zeta(R_3) + \frac{3}{2}\tilde{w}_{\xi^2}(R_1) + \tilde{w}_{\eta^2}(R_1) - \frac{3}{2}\tilde{w}_{\xi^2}(R_2) + \frac{1}{2}\tilde{w}_{\bar{\zeta}\zeta}(R_2) \\
& - \tilde{w}_{\eta^2}(R_3) - \frac{1}{2}\tilde{w}_{\zeta\zeta}(R_3) + \frac{1}{12}\tilde{w}_{\xi^4\eta}(S_1) + \frac{\sqrt{2}}{12}\tilde{w}_{\zeta^4\bar{n}}(S_2) + \frac{1}{24}\tilde{w}_{\xi\eta^4}(S_3)
\end{aligned}$$

$$\begin{aligned}
\tilde{a}_{23} = & \ 30\tilde{w}(R_1) - 15\tilde{w}(R_2) - 15\tilde{w}(R_3) + 6\tilde{w}_\xi(R_1) + 9\tilde{w}_\eta(R_1) + 6\tilde{w}_\xi(R_2) - \frac{3}{2}\sqrt{2}\tilde{w}_\zeta(R_2) \\
& + 9\tilde{w}_\eta(R_3) - \frac{3}{2}\sqrt{2}\tilde{w}_\zeta(R_3) + \tilde{w}_{\xi^2}(R_1) + \frac{3}{2}\tilde{w}_{\eta^2}(R_1) - \tilde{w}_{\xi^2}(R_2) - \frac{1}{2}\tilde{w}_{\bar{\zeta}\zeta}(R_2) - \frac{3}{2}\tilde{w}_{\eta^2}(R_3) \\
& + \frac{1}{2}\tilde{w}_{\bar{\zeta}\zeta}(R_3) + \frac{1}{24}\tilde{w}_{\xi^4\eta}(S_1) + \frac{\sqrt{2}}{12}\tilde{w}_{\zeta^4\bar{n}}(S_2) + \frac{1}{12}\tilde{w}_{\xi\eta^4}(S_3)
\end{aligned}$$

$$\tilde{a}_{14} = \frac{1}{24}\tilde{w}_{\xi\eta^4}(S_1)$$

$$\tilde{a}_{05} = -6\tilde{w}(R_1) - 3\tilde{w}_{\eta}(R_1) - \frac{1}{2}\tilde{w}_{\eta^2}(R_1) + 6\tilde{w}(R_3) - 3\tilde{w}_{\eta}(R_3) + \frac{1}{2}\tilde{w}_{\eta^2}(R_3)$$

$p = 6$

$$\lfloor \tilde{\delta}^{(6)} \rfloor = \lfloor \tilde{\delta}^{(5)};\ \tilde{w}_{\xi^6}(S_1),\ \tilde{w}_{\xi^5\eta}(S_1),\ \tilde{w}_{\xi^6}(S_2),\ \tilde{w}_{\xi^5\eta}(S_2),\ \tilde{w}_{\eta^6}(S_3),\ \tilde{w}_{\eta^5\xi}(S_3),\ \tilde{w}_{\xi^3\eta^3}(R_1) \rfloor$$

$$\tilde{a}_{00}^{(6)} = \tilde{a}_{00}^{(5)}$$
$$\tilde{a}_{10}^{(6)} = \tilde{a}_{10}^{(5)}$$
$$\tilde{a}_{01}^{(6)} = \tilde{a}_{01}^{(5)}$$
$$\tilde{a}_{20}^{(6)} = \tilde{a}_{20}^{(5)}$$
$$\tilde{a}_{11}^{(6)} = \tilde{a}_{11}^{(5)}$$
$$\tilde{a}_{02}^{(6)} = \tilde{a}_{02}^{(5)}$$

$$\tilde{a}_{30}^{(6)} = \tilde{a}_{30}^{(5)} - \frac{1}{720}\tilde{w}_{\xi^6}(S_1)$$

$$\tilde{a}_{21}^{(6)} = \tilde{a}_{21}^{(5)} - \frac{1}{240}\tilde{w}_{\xi^5\eta}(S_1)$$

$$\tilde{a}_{12}^{(6)} = \tilde{a}_{12}^{(5)} - \frac{1}{240}\tilde{w}_{\xi\eta^5}(S_3)$$

$$\tilde{a}_{03}^{(6)} = \tilde{a}_{40}^{(5)} - \frac{1}{720}\tilde{w}_{\eta^6}(S_3)$$

$$\tilde{a}_{40}^{(6)} = \tilde{a}_{40}^{(5)} + \frac{1}{240}\tilde{w}_{\xi^6}(S_1)$$

$$\tilde{a}_{31}^{(6)} = \tilde{a}_{31}^{(5)} + \frac{1}{60}\tilde{w}_{\xi^5\eta}(S_1)$$

$$\tilde{a}_{22}^{(6)} = \tilde{a}_{22}^{(5)} + \frac{1}{720}\tilde{w}_{\xi^6}(S_1) + \frac{1}{240}\tilde{w}_{\xi^5\eta}(S_1) - \frac{1}{90}\tilde{w}_{\xi^6}(S_2) + \frac{1}{720}\tilde{w}_{\eta^6}(S_3) + \frac{1}{240}\tilde{w}_{\xi\eta^5}(S_3) + \frac{1}{72}\tilde{w}_{\xi^3\eta^3}(R_1)$$

$$\tilde{a}_{13}^{(6)} = \tilde{a}_{13}^{(5)} + \frac{1}{60}\tilde{w}_{\xi\eta^5}(S_3)$$

$$\tilde{a}_{04}^{(6)} = \tilde{a}_{04}^{(5)} + \frac{1}{240}\tilde{w}_{\xi^6}(S_3)$$

$$\tilde{a}_{50}^{(6)} = \tilde{a}_{50}^{(5)} - \frac{1}{240}\tilde{w}_{\xi^6}(S_1)$$

$$\tilde{a}_{41}^{(6)} = \tilde{a}_{41}^{(5)} - \frac{1}{48}\tilde{w}_{\xi^5\eta}(S_1)$$

$$\tilde{a}_{32}^{(6)} = \tilde{a}_{32}^{(5)} + \frac{1}{720}\tilde{w}_{\xi^6}(S_1) - \frac{1}{60}\tilde{w}_{\xi^5\eta}(S_1) + \frac{1}{180}\tilde{w}_{\xi^6}(S_2) + \frac{1}{60}\tilde{w}_{\xi^5\eta}(S_2) - \frac{1}{360}\tilde{w}_{\eta^6}(S_3) + \frac{1}{240}\tilde{w}_{\xi\eta^5}(S_3) - \frac{1}{36}\tilde{w}_{\xi^3\eta^3}(R_1)$$

$$\tilde{a}_{23}^{(6)} = \tilde{a}_{23}^{(5)} - \frac{1}{360}\tilde{w}_{\xi^6}(S_1) + \frac{1}{240}\tilde{w}_{\xi^5\eta}(S_1) + \frac{1}{180}\tilde{w}_{\xi^6}(S_2) - \frac{1}{60}\tilde{w}_{\xi^5\eta}(S_2) + \frac{1}{720}\tilde{w}_{\eta^6}(S_3) - \frac{1}{60}\tilde{w}_{\xi\eta^5}(S_3) - \frac{1}{36}\tilde{w}_{\xi^3\eta^3}(R_1)$$

$$\tilde{a}_{14}^{(6)} = \tilde{a}_{14}^{(5)} - \frac{1}{48}\tilde{w}_{\xi\eta^5}(S_3)$$

$$\tilde{a}_{05}^{(6)} = \tilde{a}_{05}^{(5)} - \frac{1}{240}\tilde{w}_{\eta^6}(S_3)$$

$$\tilde{a}_{60}^{(6)} = \frac{1}{720}\tilde{w}_{\xi^6}(S_1)$$

$$\tilde{a}_{51}^{(6)} = \frac{1}{120}\tilde{w}_{\xi^5\eta}(S_1)$$

$$\tilde{a}_{42}^{(6)} = -\frac{1}{360}\tilde{w}_{\xi^6}(S_1) + \frac{1}{80}\tilde{w}_{\xi^5\eta}(S_1) + \frac{1}{180}\tilde{w}_{\xi^6}(S_2) - \frac{1}{60}\tilde{w}_{\xi^5\eta}(S_2) + \frac{1}{720}\tilde{w}_{\eta^6}(S_3) - \frac{1}{240}\tilde{w}_{\xi\eta^5}(S_3) + \frac{1}{72}\tilde{w}_{\xi^3\eta^3}(R_1)$$

$$\tilde{a}_{33}^{(6)} = \frac{1}{36}\tilde{w}_{\xi^3\eta^3}(R_1)$$

$$\tilde{a}_{24}^{(6)} = \frac{1}{720}\tilde{w}_{\xi^6}(S_1) - \frac{1}{240}\tilde{w}_{\xi^5\eta}(S_1) + \frac{1}{180}\tilde{w}_{\xi^6}(S_2) + \frac{1}{60}\tilde{w}_{\xi^5\eta}(S_2) - \frac{1}{360}\tilde{w}_{\eta^6}(S_3) + \frac{1}{80}\tilde{w}_{\xi\eta^5}(S_3) + \frac{1}{72}\tilde{w}_{\xi^3\eta^3}(R_1)$$

$$\tilde{a}_{15}^{(6)} = \frac{1}{120}\tilde{w}_{\xi\eta^5}(S_3)$$

$$\tilde{a}_{06}^{(6)} = \frac{1}{720}\tilde{w}_{\eta^6}(S_3).$$

The matrices $[\tilde{A}^{(p)}]$ are not used in the calculations. They are given explicitly, however, in [17].