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Projective pseudodifferential analysis and harmonic analysis

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Abstract

We consider pseudodifferential operators on functions on $\mathbb{R}^{n+1}$ which commute with the Euler operator, and can thus be restricted to spaces of functions homogeneous of some given degree. The symbols of such restrictions can be regarded as functions on a reduced phase space, isomorphic to the homogeneous space $G_n/H_n = SL(n+1,\mathbb{R})/GL(n,\mathbb{R})$, and the resulting calculus is a pseudodifferential analysis of operators acting on spaces of appropriate sections of line bundles over the projective space $P_n(\mathbb{R})$: these spaces are the representation spaces of the maximal degenerate series $(\pi_{i\lambda,\varepsilon})$ of G_n. This new approach to the quantization of G_n/H_n, already considered by other authors, has several advantages: as an example, it makes it possible to give a very explicit version of the continuous part from the decomposition of $L^2(G_n/H_n)$ under the quasiregular action of G_n. We also consider interesting special symbols, which arise from the consideration of the resolvents of certain infinitesimal operators of the representation $\pi_{i\lambda,\varepsilon}$.

Keywords: Pseudodifferential analysis; Covariant quantization; Para-Hermitian symmetric spaces

Contents

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0. Introduction

This paper is devoted to harmonic analysis on the homogeneous space

$$G_n/H_n = SL(n+1,\mathbb{R})/GL(n,\mathbb{R})$$

and more precisely to a study, on this example, of the interaction between harmonic analysis and pseudodifferential analysis. We here combine two ideas, both of which stem from a long-standing tradition in mathematics or physics: the *superselection method* in pseudodifferential analysis, and the *square-root method* in the analysis of certain second-order differential operators.

Recalling here the definition of a not over-specialized version of pseudodifferential analysis on $\mathbb{R}^{n+1}$ will save space later (the dimension $n+1$ is, of course, meant for coherence with the sequel). A *symbolic calculus* of operators on $\mathbb{R}^{n+1}$ is a linear way to associate linear operators $\mathrm{Op}(H)$ on functions of $n+1$ variables to functions H of $2n+2$ variables: we are not interested in an axiomatization of the concept, but in the following parameter-dependent special case. Given $\kappa \in \mathbb{R}$, consider the defining equation, in which the integration with respect to dy is to be carried first:

$$\big(\mathrm{Op}_\kappa(H)u\big)(x) = \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} H\big((1-\kappa)x + \kappa y, \eta\big) e^{2i\pi\langle x-y,\eta\rangle} u(y)\, d\eta\, dy. \tag{0.1}$$

In the case when $\kappa = \frac{1}{2}$, this is the Weyl calculus of operators, or Weyl pseudodifferential analysis. In the case when $\kappa = 0$, this is the *standard pseudodifferential calculus*, or *convolution-first* calculus, a terminology which the reader will feel justified after he has examined the case when the symbol H decomposes as $H(x,\eta) = h_1(x)h_2(\eta)$: whether this is the case or not, one can reduce the double sign of integration to a simple one with the help of the Fourier transform of u. One cannot introduce the standard calculus without considering, at the same time, the *antistandard calculus*, which is the case $\kappa = 1$ of the formula above, would it be only for the fact that the adjoint of the operator $\mathrm{Op}_0(H)$ is the operator $\mathrm{Op}_1(\overline{H})$.

What we call the superselection method originates from the physicists' superselection rule: we want to devote our interest to a special class of operators, to wit those which commute with some fixed differential operator M with symbol m, with a given self-adjoint realization. It *may* happen that the symbols H of such operators are exactly the ones satisfying the Poisson bracket equation $\{m, H\} = 0$: this will be the case when M is the infinitesimal operator of a one-parameter unitary group lying in the covariance group of the symbolic calculus, a group which contains the metaplectic group when $\kappa = \frac{1}{2}$, and the group $SL(n+1,\mathbb{R})$ of transformations of $L^2(\mathbb{R}^{n+1})$ in all other cases. In significant instances, reducing the phase space $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ so as to account for this equation cannot fail to lead to a geometrically interesting structure: it does, indeed [13], in the case when $n = 3$ and M is the d'Alembert operator $\frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$, since it leads in a canonical way to the basic geometric concepts which occur in the development of special relativity. Moreover, given any positive number μ, the reduced phase space is then the natural choice for a pseudodifferential analysis of operators acting on the space of solutions of the equation $Mu + \mu u = 0$. In the case just alluded to, this spectral equation is the Klein–Gordon equation, and the resulting Klein–Gordon pseudodifferential analysis—a reduction of the Weyl calculus—was developed in [13].

We here consider a reduction of the standard–antistandard pseudodifferential analysis. It goes along the lines of the above general scheme, if one chooses for M the Euler operator $(2i\pi)^{-1}(\sum_{k=1}^{n+1} x_k \frac{\partial}{\partial x_k} + \frac{n+1}{2})$. In other words, we consider operators on functions of $n+1$ variables which preserve the homogeneity—and the parity as well—of functions: these operators then give rise to operators acting on functions defined on the projective space $P_n(\mathbb{R})$. The reduced phase space turns out to be the homogeneous space $\mathcal{X}_n^\bullet = G_n/H_n$, and the corresponding pseudodifferential analysis may also be referred to as a quantization of G_n/H_n.

Such a quantization was studied—in a way independent from the superselection method—in [18] in the case when $n = 1$ and, in the general case, in a series of papers [1,2]. In all cases, the development has to include a description of the decomposition of $L^2(G_n/H_n)$ under the quasiregular representation of G_n in this space, which amounts to a description of the spectral decomposition of the basic G_n-invariant differential operator Δ_n on $\mathcal{X}_n^\bullet = G_n/H_n$. In the one-dimensional case, one could dispense with this task, relying instead on results relative to the harmonic analysis of hyperboloids [11]. In general, in [3,5,8] the problem was dealt with by the use of the H_n-spherical distribution method. We wish to give some idea, in the present introduction, of the *square-root* method adopted here.

It consists in replacing a second-order operator, here Δ_n, by an equivalent, or almost equivalent—it usually provides more information—first-order operator or system of operators. This idea resurfaces in a variety of domains and disguises. Without any attempt at completeness, let us recall the following well known, or not so well known, instances. The first one that springs to mind is Dirac's replacement of the second-order Klein–Gordon equation by his system of four first-order equations; by the way, this can be followed up, again, in the domain of pseudodifferential analysis, leading to the construction of the *Dirac symbolic calculus* of operators [14]. Another circle of ideas, quite close to the one which we will adhere to in the present work, is familiar to harmonic analysts and deals with such objects as the Weyl group and Harish–Chandra's isomorphism. We prefer to come to it in terms most readers will probably not be quite as familiar with, starting from the Lax–Phillips scattering theory for the automorphic wave equation [6]. Automorphic functions are functions in the upper half-plane invariant under the action, by fractional-linear transformations, of some arithmetic group: when at the same time generalized eigenfunctions of the non-Euclidean Laplacian Δ, they are called nonholomorphic modular forms. In the Lax–Phillips scattering theory, pairs of automorphic functions are made to appear as the set of Cauchy data on some hyperboloid for the d'Alembert equation in the three-dimensional forward light-cone. In [15], it was shown that the space of such pairs can be identified with functions on $\mathbb{R}^2$, in such a way that, under the transfer, the operator $\Delta - \frac{1}{4}$ becomes the square of the first-order operator $(2i)^{-1}(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + 1)$. A fully similar idea, replacing Δ by Δ_n, will work here. Let us just mention *en passant* that the concept of automorphic distribution that arose from this transfer made automorphic pseudodifferential analysis [16] possible.

More details follow: the superselection rule present here calls for the consideration of symbols $H = H(x, \xi)$ invariant under the ever-present action $t.(x, \xi) = (tx, t^{-1}\xi)$ of the group $\mathbb{R}^\times$; then, $\mathcal{X}_n^\bullet$ may be realized as the hypersurface, in the corresponding quotient, of equation $\langle x, \xi\rangle = 1$. Next, we consider in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ the operator $\square = \sum_{k=1}^{n+1} \frac{\partial^2}{\partial x_j \partial \xi_j}$. This operator will turn out, in a moment, to be a fundamental one in connection with pseudodifferential analysis. For the time being, its importance stems from what follows: within the domain Ω^+ defined by $\langle x, \xi \rangle > 0$, set $\tau = \log\langle x, \xi \rangle$, which provides an identification of the quotient of Ω^+ by the group $\mathbb{R}^\times$ with the product $\mathcal{X}_n^\bullet \times \mathbb{R}$. Then, under the transformation $H \mapsto H_1 = e^{\frac{n\tau}{2}} H$, the equation $\Box H = 0$ is equivalent to the wave equation $\frac{\partial^2 H_1}{\partial \tau^2} + (\Delta_n - \frac{n^2}{4})H_1 = 0$. This explains a fundamental property of one part at least (the continuous one) of the decomposition of $L^2(\mathcal{X}_n^\bullet) = L^2(G_n/H_n)$, to wit the fact that the generalized eigenvalues always come by pairs $(\rho, -n - \rho)$. Though one can trace this to several possible sources, the following, a continuation of the Lax–Phillips point of view, seems to us especially striking: solutions of the wave equation above can be characterized by their first two traces on $\mathcal{X}_n^\bullet$, not just one. On the other hand, constructing $\mathbb{R}^\times$-invariant solutions, in $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, of the equation $\Box H = 0$ can be achieved by means of a Fourier transformation, starting from functions Φ on $\Sigma_n^\bullet$, the quotient by the now familiar action of $\mathbb{R}^\times$ of the cone Σ_n of equation $\langle y, \eta \rangle = 0$. It is then not surprising that there exists an involution $\mathcal{K}$ of this latter space of functions such that two $\mathcal{K}$-related functions Φ and $\mathcal{K}\Phi$ should always lead to solutions of the above wave equation with the same first trace on $\mathcal{X}_n^\bullet$. Analyzing the involution $\mathcal{K}$ leads without too much difficulty to a full understanding of the continuous part of $L^2(G_n/H_n)$ (Section 2).

Next (Section 1), we decompose functions on $\mathbb{R}^{n+1}$ into their homogeneous components: since the Euler operator commutes with the linear action of G_n, this action decomposes as a continuous sum $(\pi_{i\lambda,\varepsilon})$ with $\lambda \in \mathbb{R}$ and $\varepsilon = 0$ or 1, a "series" of irreducible unitary representations in $L^2(\mathbb{R}^n)$ also arising from the general theory [4] as a maximal degenerate series of representations of the group G_n; then, the Fourier transformation decomposes as the family of intertwining operators relative to this series. The decomposition, along the general lines, of the standard–antistandard pseudodifferential analysis, leads for every pair $(i\lambda, \varepsilon)$ to the definition of two linear maps $\mathrm{Op}_{i\lambda,\varepsilon}$ and $\mathrm{Op}^\vee_{i\lambda,\varepsilon}$ from functions on $\mathcal{X}_n^\bullet$ to linear operators in the space of the representation $\pi_{i\lambda,\varepsilon}$: these two symbolic calculi are, of course, exactly the ones used in the above-given references concerning the quantization of the space G_n/H_n.

The occurrence of the operator $\Box$ above is remarkable since the equation $\Box H = 0$ just means that the operator $\mathrm{Op}_\kappa(H)$ does not depend on κ. In particular, the operator on $\mathbb{R}^{n+1}$ with symbol H is the same, whether one considers H as a standard or antistandard symbol. Now, it is easy to connect the symbol in the $\mathrm{Op}_{i\lambda,\varepsilon}$-calculus of the associated operator to H viewed as a standard symbol, while the symbol in the $\mathrm{Op}^\vee_{i\lambda,\varepsilon}$-calculus of an operator is easily connected to the antistandard symbol of the operator on $\mathbb{R}^{n+1}$ it comes from. In this way, one finds (Section 1) a simple proof of a formula, first given in [1] when $n > 1$, connecting the $\mathrm{Op}_{i\lambda,\varepsilon}$- and $\mathrm{Op}^\vee_{i\lambda,\varepsilon}$-symbols of the same operator. However, the method only works for symbols lying in the continuous part of the decomposition of $L^2(G_n/H_n)$.

To prevent some possible misunderstanding, let us emphasize two points, both related to the fact that the covariance group of the pseudodifferential analysis under discussion is $G_n = SL(n+1, \mathbb{R})$, not $O(n)$. There does not exist on the projective space $P_n(\mathbb{R})$ any measure invariant under G_n: still, a representation such as $\pi_{i\lambda,\varepsilon}$ is unitary because it really acts, rather than on functions on $P_n(\mathbb{R})$, on sections of some line bundle; only, this fact is sometimes blurred by the use on $\mathbb{R}^n \subset P_n(\mathbb{R})$ of affine coordinates. Next, there exists on $P_n(\mathbb{R})$, viewed if so wished as the usual quotient of the sphere S^n, a vast family of pseudodifferential analyses covariant under the action of the orthogonal group: the $\mathrm{Op}_{i\lambda,\varepsilon}$-calculus is covariant under some specific action of G_n, which makes it almost unique, since any two such calculi have to be related under a transformation of functions on G_n/H_n expressing itself, in spectral-theoretic terms, as a function of the operator Δ_n.

The problem of analyzing the *sharp composition*—the terminology is the one in use in pseudodifferential analysis—of symbols, by which is meant the bilinear operation that corresponds to the composition of the associated operators, is a difficult one which, in the case when $n = 1$, was partly solved (for symbols lying in the discrete part of the decomposition of $L^2(G_1/H_1)$) in [18], where the Rankin–Cohen brackets were shown for the first time to have a significant role in pseudodifferential analysis. We do not solve it here for general n, but we seize this opportunity to show that the integral formula for the sharp composition of symbols—which is trivial to obtain—is very far from revealing the more interesting aspects of the operation under consideration. Our main point, in Section 3, will be to do away, on the basis of it, with two related popular misconceptions: one of them consists in pushing too far the concept that the inverse of the parameter λ that specifies an irreducible representation of G_n within its series might be interpreted as a "Planck's constant"; the other one consists in believing that the composition of symbols can be, in some reasonable sense, approximated by a series of bidifferential operators.

Some functions on G_n/H_n, while not in $L^2(G_n/H_n)$ when $n \geqslant 2$ (in the one-dimensional case, these functions lie in the discrete part of the decomposition of $L^2(G_1/H_1)$) are very interesting to consider in view of the role they play in the symbolic calculus: for they provide the symbols of certain operators in the algebra generated by resolvents of elements of the (complexified) space of infinitesimal operators of the representation $\pi_{i\lambda,\varepsilon}$. These symbols are introduced in Section 4, where it is also shown that the above-mentioned formula, linking the two species of symbols of the same operator, continues to hold in this new context. The analysis of individual operators obtained in this way—which played an essential role, when $n = 1$ [18]—can, up to some point, be reduced to the one-dimensional case.

To conclude, let us make it clear that, though the present paper certainly provides more familiarity with the $\mathrm{Op}_{i\lambda,\varepsilon}$-calculus, we are still far from having reached a point where this could be considered as a genuine pseudodifferential analysis in the sense demanded, say, by possible applications to partial differential equations: developments in this direction may prove surprisingly new, in particular in view of the fact that the representations $\pi_{i\lambda,\varepsilon}$ are not square-integrable.

1. Pseudodifferential analysis, from $\mathbb{R}^{n+1}$ to $P_n(\mathbb{R})$

The projective space $P_n(\mathbb{R})$ is the quotient of $\mathbb{R}^{n+1} \setminus \{0\}$ by the equivalence that identifies two vectors when proportional: we denote as $x \mapsto x^\bullet$ the projection map. The vector $x = (x_1, \dots, x_{n+1})$ is called a set of homogeneous coordinates of $x^\bullet$: we shall also represent $x^\bullet$ by the vector $s = x_{n+1}^{-1}(x_1, \dots, x_n)$ in the case when $x_{n+1} \neq 0$. The group $G_n = SL(n+1, \mathbb{R})$ acts on $\mathbb{R}^{n+1}$ in the linear way, which defines an action on $P_n(\mathbb{R})$ too, denoted as $(g, s) \mapsto [g]s$ in inhomogeneous coordinates (note that the use of inhomogeneous coordinates makes this action look like a singular one, which it is not).

We first decompose the Hilbert space $L^2(\mathbb{R}^{n+1})$ under the action $(g, v) \mapsto v \circ g^{-1}$ of G_n. This action preserves the parity of functions, and we denote as $L^2_\varepsilon(\mathbb{R}^{n+1})$, with $\varepsilon = 0$ (respectively 1) the subspace of $L^2(\mathbb{R}^{n+1})$ consisting of even (respectively odd) functions. Given $v = v_0 + v_1 \in L^2_0(\mathbb{R}^{n+1}) \oplus L^2_1(\mathbb{R}^{n+1})$, decompose it as

$$v = \sum_{\varepsilon=0,1} \int_{-\infty}^{\infty} v_{i\lambda,\varepsilon}\, d\lambda, \tag{1.1}$$

where the function

$$v_{i\lambda,\varepsilon}(x)=\frac{1}{2\pi}\int_0^\infty t^{\frac{n-1}{2}+i\lambda}v_\varepsilon(tx)\,dt=\frac{1}{4\pi}\int_{-\infty}^\infty |t|_\varepsilon^{\frac{n-1}{2}+i\lambda}v(tx)\,dt \tag{1.2}$$

is homogeneous of degree and parity $(-\frac{n+1}{2}-i\lambda,\varepsilon)$, a phrasing that we shall adopt for brevity: we here set

$$|t|_\varepsilon^\alpha=|t|^\alpha(\operatorname{sign}t)^\varepsilon\quad \text{for } t\in\mathbb{R}\setminus\{0\},\ \alpha\in\mathbb{C}. \tag{1.3}$$

The function $v_{i\lambda,\varepsilon}$ is, of course, characterized by the function $v^\flat_{i\lambda,\varepsilon}$ on $\mathbb{R}^n$ such that

$$v^\flat_{i\lambda,\varepsilon}(s)=v_{i\lambda,\varepsilon}(s,1) \tag{1.4}$$

since, with $x=(x_*,x_{n+1})$, one has

$$v_{i\lambda,\varepsilon}(x)=|x_{n+1}|_\varepsilon^{-\frac{n+1}{2}-i\lambda}v^\flat_{i\lambda,\varepsilon}\left(\frac{x_*}{x_{n+1}}\right). \tag{1.5}$$

Applying the equation

$$\int_{-\infty}^\infty\left|v_{i\lambda,\varepsilon}(x)\right|^2d\lambda=\frac{1}{2\pi}\int_0^\infty t^n\left|v_\varepsilon(tx)\right|^2dt, \tag{1.6}$$

valid for almost every $x\in\mathbb{R}^{n+1}\setminus\{0\}$, with $x=(s,1)$, and integrating the result with respect to ds, we obtain

$$\|v\|^2_{L^2(\mathbb{R}^{n+1})}=4\pi\sum_{\varepsilon=0,1}\int_{-\infty}^\infty\left\|v^\flat_{i\lambda,\varepsilon}\right\|^2_{L^2(\mathbb{R}^n)}d\lambda. \tag{1.7}$$

Next, given $g\in G_n$ of the form $g=\left(\begin{smallmatrix}M & \boldsymbol{p}\\ \boldsymbol{q}^{\mathrm T} & m\end{smallmatrix}\right)$, where $\boldsymbol{p}\in\mathbb{R}^n$ is a column vector and $\boldsymbol{q}^{\mathrm T}$ is the transpose of the column vector $\boldsymbol{q}\in\mathbb{R}^n$, one has for every $s\in\mathbb{R}^n$, as a consequence of (1.1), the equation

$$(v\circ g)(s,1)=\sum_{\varepsilon=0,1}\int_{-\infty}^\infty v_{i\lambda,\varepsilon}\big(Ms+\boldsymbol{p},\langle\boldsymbol{q},s\rangle+m\big)\,d\lambda \tag{1.8}$$

or, using the homogeneity,

$$(v\circ g)(s,1)=\sum_{\varepsilon=0,1}\int_{-\infty}^\infty\left|\langle\boldsymbol{q},s\rangle+m\right|_\varepsilon^{-\frac{n+1}{2}-i\lambda}v_{i\lambda,\varepsilon}\left(\frac{Ms+\boldsymbol{p}}{\langle\boldsymbol{q},s\rangle+m},1\right)d\lambda. \tag{1.9}$$

It follows that

$$(v \circ g)^{\flat}_{i\lambda,\varepsilon}(s) = \left|\langle \boldsymbol{q}, s\rangle + m\right|_{\varepsilon}^{-\frac{n+1}{2}-i\lambda} v^{\flat}_{i\lambda,\varepsilon}\left(\frac{Ms + \boldsymbol{p}}{\langle \boldsymbol{q}, s\rangle + m}\right). \tag{1.10}$$

Set

$$\pi_{i\lambda,\varepsilon}\left(g^{-1}\right) v^{\flat}_{i\lambda,\varepsilon} = (v \circ g)^{\flat}_{i\lambda,\varepsilon}. \tag{1.11}$$

Since

$$\left|\frac{D([g]s)}{Ds}\right| = \left|\langle \boldsymbol{q}, s\rangle + m\right|^{-n-1}, \tag{1.12}$$

the representation $\pi_{i\lambda,\varepsilon}$ of G_n in $L^2(\mathbb{R}^n)$ so introduced is unitary.

Together with $\pi_{i\lambda,\varepsilon}$, we consider the contragredient representation $\pi^{\sharp}_{i\lambda,\varepsilon}$ defined by the equation (in which $g \mapsto g'$ denotes the matrix transposition)

$$\pi^{\sharp}_{i\lambda,\varepsilon}(g') = \pi_{i\lambda,\varepsilon}\left(g^{-1}\right). \tag{1.13}$$

Although the formal definition of the intertwining operator $\theta_{i\lambda,\varepsilon}$ from the representation $\pi_{i\lambda,\varepsilon}$ to the representation $\pi^{\sharp}_{-i\lambda,\varepsilon}$ is a consequence of the general theory, a better understanding of its properties can be obtained from its definition in terms of the usual Fourier transformation on $L^2(\mathbb{R}^{n+1})$ (cf. [15, p. 28] for the case when $n = 1$).

Applying the Fourier transformation, normalized as

$$(\mathcal{F}v)(x) = \int_{\mathbb{R}^{n+1}} v(y) e^{-2i\pi\langle x,y\rangle}\, dy, \tag{1.14}$$

to both sides of (1.1), and noting that the Fourier transformation sends functions homogeneous of degree $-\frac{n+1}{2} - i\lambda$ to functions homogeneous of degree $-\frac{n+1}{2} + i\lambda$ with the same parity, we obtain

$$\mathcal{F}v = \sum_{\varepsilon=0,1} \int_{-\infty}^{\infty} \mathcal{F}v_{i\lambda,\varepsilon}\, d\lambda = \sum_{\varepsilon=0,1} \int_{-\infty}^{\infty} (\mathcal{F}v)_{-i\lambda,\varepsilon}\, d\lambda. \tag{1.15}$$

We may thus define

$$\theta_{i\lambda,\varepsilon} v^{\flat}_{i\lambda,\varepsilon} = (\mathcal{F}v)^{\flat}_{-i\lambda,\varepsilon}. \tag{1.16}$$

Checking that the operator $\theta_{i\lambda,\varepsilon}$ has the required intertwining property is easy: indeed, given $g \in G_n$, one has, on one hand, applying (1.11) and the definition just given,

$$\theta_{i\lambda,\varepsilon}\pi_{i\lambda,\varepsilon}\left(g^{-1}\right) v^{\flat}_{i\lambda,\varepsilon} = \theta_{i\lambda,\varepsilon}(v \circ g)^{\flat}_{i\lambda,\varepsilon} = \left(\mathcal{F}(v \circ g)\right)^{\flat}_{-i\lambda,\varepsilon} = \left[(\mathcal{F}v) \circ g'^{-1}\right]^{\flat}_{-i\lambda,\varepsilon}, \tag{1.17}$$

on the other hand,

$$\pi_{-i\lambda,\varepsilon}(g')\theta_{i\lambda,\varepsilon}v^{\flat}_{i\lambda,\varepsilon} = \pi_{-i\lambda,\varepsilon}(g')(\mathcal{F}v)^{\flat}_{-i\lambda,\varepsilon} = \left[(\mathcal{F}v)\circ g'^{-1}\right]^{\flat}_{-i\lambda,\varepsilon}. \tag{1.18}$$

Note that, even though (1.16) defines the function $\theta_{i\lambda,\varepsilon}v^{\flat}_{i\lambda,\varepsilon}$ for almost all λ only, it is easy to introduce classes of functions v such that, for every $s\in\mathbb{R}^n$, $v^{\flat}_{i\lambda,\varepsilon}(s)$ has a well-defined meaning for every λ. A simple, useful example is provided by the space $\mathcal{S}_{\text{flat}}(\mathbb{R}^{n+1})$ consisting of all *flat* functions in $\mathcal{S}(\mathbb{R}^{n+1})$, i.e., functions in this latter space every derivative of which is bounded, near 0, by a constant times such power of $|x|$ as one may wish.

We must now compute the operator $\theta_{i\lambda,\varepsilon}$ as an integral operator in terms of the inhomogeneous coordinates on $P_n(\mathbb{R})$. The computations that follow have only formal value, but we plead not guilty on this account: for we only need to compare the perfectly valid definition (1.16) of the intertwining operator to the formal one taken from the general theory. The part homogeneous of degree $-\frac{n+1}{2}+i\lambda$ and of parity ε of the function $x\mapsto e^{-2i\pi\langle x,y\rangle}$ is given by the (divergent) integral, taken from (1.2), to be interpreted as defining a Fourier transform,

$$\begin{aligned}&\frac{1}{4\pi}\int_{-\infty}^{\infty}|t|_{\varepsilon}^{\frac{n-1}{2}-i\lambda}e^{-2i\pi t\langle x,y\rangle}\,dt\\&\quad=\frac{1}{4\pi}(-i)^{\varepsilon}\pi^{-\frac{n}{2}+i\lambda}\frac{\Gamma(\frac{1+n}{4}-\frac{i\lambda}{2}+\frac{\varepsilon}{2})}{\Gamma(\frac{1-n}{4}+\frac{i\lambda}{2}+\frac{\varepsilon}{2})}\left|\langle x,y\rangle\right|_{\varepsilon}^{-\frac{n+1}{2}+i\lambda}.\end{aligned}\tag{1.19}$$

Hence

$$(\mathcal{F}v)^{\flat}_{-i\lambda,\varepsilon}(x)=\frac{1}{4\pi}(-i)^{\varepsilon}\pi^{-\frac{n}{2}+i\lambda}\frac{\Gamma(\frac{1+n}{4}-\frac{i\lambda}{2}+\frac{\varepsilon}{2})}{\Gamma(\frac{1-n}{4}+\frac{i\lambda}{2}+\frac{\varepsilon}{2})}\int_{\mathbb{R}^{n+1}}\left|\langle x,y\rangle\right|_{\varepsilon}^{-\frac{n+1}{2}+i\lambda}v(y)\,dy \tag{1.20}$$

and, for $\sigma\in\mathbb{R}$, setting $y_*=(y_1,\dots,y_n)$,

$$\begin{aligned}\left(\theta_{i\lambda,\varepsilon}v^{\flat}_{i\lambda,\varepsilon}\right)(\sigma)&=(\mathcal{F}v)^{\flat}_{-i\lambda,\varepsilon}(\sigma,1)=(-i)^{\varepsilon}\pi^{-\frac{n}{2}+i\lambda}\frac{\Gamma(\frac{1+n}{4}-\frac{i\lambda}{2}+\frac{\varepsilon}{2})}{\Gamma(\frac{1-n}{4}+\frac{i\lambda}{2}+\frac{\varepsilon}{2})}\\&\quad\times\frac{1}{4\pi}\int_{\mathbb{R}^n}dy_*\int_{-\infty}^{\infty}\left|\langle\sigma,y_*\rangle+y_{n+1}\right|_{\varepsilon}^{-\frac{n+1}{2}+i\lambda}v(y_*,y_{n+1})\,dy_{n+1}.\end{aligned}\tag{1.21}$$

Set $y_*=y_{n+1}s$, transforming the second line into

$$\begin{aligned}&\frac{1}{4\pi}\int_{\mathbb{R}^n}\left|1+\langle s,\sigma\rangle\right|_{\varepsilon}^{-\frac{n+1}{2}+i\lambda}ds\int_{-\infty}^{\infty}|y_{n+1}|_{\varepsilon}^{\frac{n-1}{2}+i\lambda}v\big(y_{n+1}(s,1)\big)\,dy_{n+1}\\&\quad=\int_{\mathbb{R}^n}\left|1+\langle s,\sigma\rangle\right|_{\varepsilon}^{-\frac{n+1}{2}+i\lambda}v^{\flat}_{i\lambda,\varepsilon}(s)\,ds,\end{aligned}\tag{1.22}$$

as seen after another application of (1.2). Thus, the (formal) definition of the intertwining operator $\theta_{i\lambda,\varepsilon}$ is, finally,

$$(\theta_{i\lambda,\varepsilon}u)(\sigma) = C_{i\lambda,\varepsilon} \int_{\mathbb{R}^n} \left|1 + \langle s, \sigma\rangle\right|_{\varepsilon}^{-\frac{n+1}{2}+i\lambda} u(s)\, ds, \quad \text{with} \tag{1.23}$$

$$C_{i\lambda,\varepsilon} = (-i)^{\varepsilon}\pi^{-\frac{n}{2}+i\lambda}\frac{\Gamma(\frac{1+n}{4} - \frac{i\lambda}{2} + \frac{\varepsilon}{2})}{\Gamma(\frac{1-n}{4} + \frac{i\lambda}{2} + \frac{\varepsilon}{2})}, \tag{1.24}$$

just the classical expression of the intertwining operator as obtained from the general theory [4]: the extra phase factor $(-i)^{\varepsilon}\pi^{i\lambda}$, not necessary for unitarity, could be dispensed with in the present context, but is important [15] in modular form theory, where it plays a role in the functional equations.

We are now in a position to introduce the (λ, ε)-dependent pseudodifferential analysis of operators on $L^2(\mathbb{R}^n)$ to be considered in the present paper. Starting with an operator A on $L^2(\mathbb{R}^{n+1})$ commuting with the transformations $x \mapsto tx$, $t \neq 0$, of the argument, so that A preserves the parity of functions and transforms homogeneous functions into functions homogeneous of the same degree, we restrict the operator A to the space of functions homogeneous of a given degree and parity $(-\frac{n+1}{2} - i\lambda, \varepsilon)$, identified with the help of the map $h \mapsto h^{\flat}_{i\lambda,\varepsilon}$ to a space of functions on the projective space.

The following three spaces play a role here:

(i) the space $\Omega = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}\colon \langle x, \xi\rangle \neq 0\}$;

(ii) the quotient $\Omega^\bullet$ of Ω under the equivalence that identifies (x, ξ) to $(tx, t^{-1}\xi)$ for every $t \neq 0$; the image of $(x, \xi) \in \Omega$ under the associated canonical projection is denoted as $(x, \xi)^\bullet$, not to be confused with $(x^\bullet, \xi^\bullet)$;

(iii) finally, the subset $\mathcal{X}_n^\bullet$ of $\Omega^\bullet$ consisting of all points $(x, \xi)^\bullet$ such that $\langle x, \xi\rangle = 1$; $\mathcal{X}_n^\bullet$ can be identified with the quotient of the hypersurface of Ω of equation $\langle x, \xi\rangle = 1$ under the same equivalence as in (ii).

The space $\mathcal{X}_n^\bullet$ can also be identified with the subset of $P_n(\mathbb{R}) \times P_n(\mathbb{R})$ consisting of all points $(x^\bullet, \xi^\bullet)$ such that $\langle x, \xi\rangle \neq 0$, under the embedding $(x^\bullet, \xi^\bullet) \mapsto (x; \frac{\xi}{\langle x,\xi\rangle})^\bullet$ of this latter space into $\Omega^\bullet$. In terms of the (almost always defined only) inhomogeneous coordinates (s, σ), this embedding takes the form

$$(s, \sigma) \mapsto \left(s, 1; \frac{\sigma}{1 + \langle s, \sigma\rangle}, \frac{1}{1 + \langle s, \sigma\rangle}\right). \tag{1.25}$$

Finally, the space $\mathcal{X}_n^\bullet$ can be thought of as the coset space G_n/H_n, where $G_n = SL(n+1, \mathbb{R})$ and H_n is a subgroup of G_n isomorphic to $GL(n, \mathbb{R})$, to wit that made up by the linear transformations that respect the splitting $\mathbb{R}^{n+1} = (\mathbb{R}^n \times \{0\}) \oplus (\{0\} \times \mathbb{R})$. An invariant measure on $\mathcal{X}_n^\bullet$ expresses itself, in terms of the coordinates above, as $|1 + \langle s, \sigma\rangle|^{-n-1}\, ds\, d\sigma$. Let us mention at once that taking quotients under this equivalence will be a fixture of what follows.

Recall from the beginning of the introduction that the standard symbolic calculus Op_0 and the antistandard symbolic calculus Op_1 on $\mathbb{R}^{n+1}$ are defined by the formulas

$$\big(\mathrm{Op}_0(H)v\big)(x) = \int_{\mathbb{R}^{n+1}} H(x;\xi)e^{2i\pi\langle x,\xi\rangle}(\mathcal{F}v)(\xi)\,d\xi \quad \text{and} \tag{1.26}$$

$$\big(\mathcal{F}\,\mathrm{Op}_1(H)v\big)(\xi) = \int_{\mathbb{R}^{n+1}} H(x;\xi)e^{-2i\pi\langle x,\xi\rangle}v(x)\,dx. \tag{1.27}$$

In complete analogy, only replacing the Fourier transformation and its integral kernel by the operator $\theta_{i\lambda,\varepsilon}$ and its integral kernel, one introduces two species of symbols in the (λ,ε)-dependent pseudodifferential calculus on the projective space. The standard symbol f and the antistandard symbol h of some operator A are the functions such that $A = \mathrm{Op}_{i\lambda,\varepsilon}(f)$ or $A = \mathrm{Op}^{\vee}_{i\lambda,\varepsilon}(h)$ according to the definitions that follow.

Definition 1.1. The standard and antistandard symbolic calculi associated with the pair (λ,ε) are defined by the equations

$$\big(\mathrm{Op}_{i\lambda,\varepsilon}(f)u\big)(s) = (-1)^{\varepsilon}C_{-i\lambda,\varepsilon}\int f(s,\sigma)\big|1+\langle s,\sigma\rangle\big|_{\varepsilon}^{-\frac{n+1}{2}-i\lambda}(\theta_{i\lambda,\varepsilon}u)(\sigma)\,d\sigma \quad \text{and} \tag{1.28}$$

$$\big(\theta_{i\lambda,\varepsilon}\,\mathrm{Op}^{\vee}_{i\lambda,\varepsilon}(h)u\big)(\sigma) = C_{i\lambda,\varepsilon}\int h(s,\sigma)\big|1+\langle s,\sigma\rangle\big|_{\varepsilon}^{-\frac{n+1}{2}+i\lambda}u(s)\,ds. \tag{1.29}$$

Since the intertwining operator $\theta_{i\lambda,\varepsilon}$ is unitary, either defining map, after it has been divided by the constant in front of the integral that defines it, sets up an isometry between the Hilbert space $L^2(\mathcal{X}_n^{\bullet}, |1+\langle s,\sigma\rangle|^{-n-1}\,ds\,d\sigma)$ and the space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. It is also immediate that the adjoint of the operator $\mathrm{Op}_{i\lambda,\varepsilon}(f)$ is the operator $\mathrm{Op}^{\vee}_{i\lambda,\varepsilon}(\bar{f})$. The normalization constants in front of the two integrals have been introduced so that the (standard or antistandard) symbol of the identity operator should be the constant 1. An immediate, purely formal, property of the $\mathrm{Op}_{i\lambda,\varepsilon}$-symbolic calculus is its covariance under the representation $\pi_{i\lambda,\varepsilon}$ and the action defined by

$$g.(s,\sigma) = \big([g]s, \big[g'^{-1}\big]\sigma\big) \tag{1.30}$$

of G_n in $P_n(\mathbb{R}) \times P_n(\mathbb{R})$: this means that, for every $g \in G_n$, one has the equation

$$\pi_{i\lambda,\varepsilon}(g)\,\mathrm{Op}_{i\lambda,\varepsilon}(f)\pi_{i\lambda,\varepsilon}\big(g^{-1}\big) = \mathrm{Op}_{i\lambda,\varepsilon}\big(f\circ g^{-1}\big). \tag{1.31}$$

The same holds with the $\mathrm{Op}^{\vee}_{i\lambda,\varepsilon}$-symbolic calculus.

We now connect the $\mathrm{Op}_{i\lambda,\varepsilon}$-calculus (respectively the $\mathrm{Op}^{\vee}_{i\lambda,\varepsilon}$-calculus) to the standard (respectively antistandard) calculus of operators on functions on $\mathbb{R}^{n+1}$. It is necessary to consider symbols $H = H(x,\xi)$ invariant under transformations $(x,\xi)\mapsto(tx,t^{-1}\xi)$, $t\in\mathbb{R}^{\times}$: this condition means that the associated operator (from the standard or antistandard calculus) commutes with the transformations $x\mapsto tx$, $t\neq 0$, of the argument. There is a slight difficulty in relation with the fact that the invariance of H does not permit it to satisfy estimates (relative to its derivatives) of any kind usual in pseudodifferential analysis: a very crude analysis, however, will be sufficient for our purpose. To start with, if H is bounded, $\mathrm{Op}_0(H)$ sends the space $\mathcal{S}(\mathbb{R}^{n+1})$ into the space $\mathcal{B}$ of continuous bounded functions, and $\mathrm{Op}_1(H)$ sends the space $\mathcal{S}(\mathbb{R}^{n+1})$ into the image, under the Fourier transformation, of $\mathcal{B}$. This is not yet satisfactory since, for the analysis to follow, we need to end up in the space $L^2(\mathbb{R}^{n+1})$. To that effect, let us assume that the symbol H is C^∞ and bounded, and that it remains so after it has been applied any operator in the algebra generated by the differential operators $x_j \frac{\partial}{\partial x_k}$ or $\xi_j \frac{\partial}{\partial \xi_k}$: this condition is compatible with the invariance of H. Let $\mathcal{S}_{\text{flat}}(\mathbb{R}^{n+1})$ be the already mentioned space of all rapidly decreasing C^∞ functions on $\mathbb{R}^{n+1}$, flat at the origin. Then, an elementary integration by parts shows that the operator $\text{Op}_0(H)$ sends the image $\mathcal{F}\mathcal{S}_{\text{flat}}(\mathbb{R}^{n+1})$ of the space $\mathcal{S}_{\text{flat}}(\mathbb{R}^{n+1})$ under the Fourier transformation into the space (contained in $L^2(\mathbb{R}^{n+1})$) of continuous functions which remain bounded after they have been multiplied by any polynomial in x. Also, the operator $\text{Op}_1(H)$ sends the space $\mathcal{S}_{\text{flat}}(\mathbb{R}^{n+1})$ into $L^2(\mathbb{R}^{n+1})$: of course, both spaces $\mathcal{S}_{\text{flat}}(\mathbb{R}^{n+1})$ and $\mathcal{F}\mathcal{S}_{\text{flat}}(\mathbb{R}^{n+1})$ are dense in $L^2(\mathbb{R}^{n+1})$.

Proposition 1.2. *Let $H = H(x,\xi)$ be a symbol in the space $C^\infty(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1})$, invariant under the transformations $(x,\xi)\mapsto(tx,t^{-1}\xi)$, $t\neq 0$: assume that it is bounded and remains so after it has been applied any operator in the algebra generated by the differential operators $x_j \frac{\partial}{\partial x_k}$ or $\xi_j \frac{\partial}{\partial \xi_k}$. Let $\text{Op}_0(H)$ be the pseudodifferential operator: $\mathcal{F}\mathcal{S}_{\text{flat}}(\mathbb{R}^{n+1})\to L^2(\mathbb{R}^{n+1})$ with standard symbol H. For every function $v\in\mathcal{S}(\mathbb{R}^{n+1})$ and every (λ,ε), one has*

$$\big(\text{Op}_0(H)v\big)_{i\lambda,\varepsilon} = \text{Op}_0(H)v_{i\lambda,\varepsilon}. \tag{1.32}$$

For any given pair (λ,ε), the operator $A_{i\lambda,\varepsilon}$ on functions of n variables characterized by the property

$$\big(\text{Op}_0(H)v\big)^\flat_{i\lambda,\varepsilon} = A_{i\lambda,\varepsilon}v^\flat_{i\lambda,\varepsilon} \tag{1.33}$$

can be identified with the operator $\text{Op}_{i\lambda,\varepsilon}(f)$ if one defines the function f on $\mathcal{X}_n^\bullet$ in terms of (H,λ,ε) by

$$f\big((x,\xi)^\bullet\big) = (-1)^\varepsilon C^{-1}_{-i\lambda,\varepsilon}\int_{-\infty}^{\infty} H\big((x,t\xi)^\bullet\big)e^{2i\pi t}|t|_\varepsilon^{\frac{n-1}{2}+i\lambda}\,dt. \tag{1.34}$$

In a similar way, the operator $B_{i\lambda,\varepsilon}$ characterized by the equation

$$\big(\text{Op}_1(H)v\big)^\flat_{i\lambda,\varepsilon} = B_{i\lambda,\varepsilon}v^\flat_{i\lambda,\varepsilon} \tag{1.35}$$

can be written as $B_{i\lambda,\varepsilon} = \text{Op}^\vee_{i\lambda,\varepsilon}(h)$ with

$$h\big((x,\xi)^\bullet\big) = C^{-1}_{i\lambda,\varepsilon}\int_{-\infty}^{\infty} H\big((tx,\xi)^\bullet\big)e^{-2i\pi t}|t|_\varepsilon^{\frac{n-1}{2}-i\lambda}\,dt. \tag{1.36}$$

Proof. That the symbol H is even just means that the operator $\text{Op}(H)$ preserves the parity of functions; that it is invariant under the one-parameter group of transformations as defined above with $t>0$ means that the associated operator preserves the space of homogeneous functions of any given degree. Equation (1.32) follows: we now make the operator $A_{i\lambda,\varepsilon}$ explicit, letting $\text{Op}_0(H)$ act on a function v in the space $\mathcal{F}\mathcal{S}_{\text{flat}}(\mathbb{R}^{n+1})$. Setting $\xi=(\xi_*,\xi_{n+1})$, noting that $\mathcal{F}v_{i\lambda,\varepsilon} = (\mathcal{F}v)_{-i\lambda,\varepsilon}$ and using (1.5) and (1.16), we find

$$(\mathcal{F}v_{i\lambda,\varepsilon})(\xi_*,\xi_{n+1}) = (\mathcal{F}v)_{-i\lambda,\varepsilon}(\xi_*,\xi_{n+1}) = |\xi_{n+1}|_\varepsilon^{-\frac{n+1}{2}+i\lambda}(\mathcal{F}v)^\flat_{-i\lambda,\varepsilon}\left(\frac{\xi_*}{\xi_{n+1}}\right)$$
$$= |\xi_{n+1}|_\varepsilon^{-\frac{n+1}{2}+i\lambda}\big(\theta_{i\lambda,\varepsilon}v^\flat_{i\lambda,\varepsilon}\big)\left(\frac{\xi_*}{\xi_{n+1}}\right). \tag{1.37}$$

Since, using definition (1.26) of $\mathrm{Op}_0(H)$,

$$\big(\mathrm{Op}_0(H)v\big)^\flat_{i\lambda,\varepsilon}(s) = \big(\mathrm{Op}_0(H)v\big)_{i\lambda,\varepsilon}(s,1)$$
$$= \int_{\mathbb{R}^{n+1}} H(s,1;\xi)e^{2i\pi(\langle s,\xi_*\rangle+\xi_{n+1})}(\mathcal{F}v_{i\lambda,\varepsilon})(\xi)\,d\xi, \tag{1.38}$$

one sees from Eq. (1.28) that one has $A_{i\lambda,\varepsilon} = \mathrm{Op}_{i\lambda,\varepsilon}(f)$ provided that one defines

$$f(s,\sigma) = (-1)^\varepsilon C^{-1}_{-i\lambda,\varepsilon}\big|1+\langle s,\sigma\rangle\big|_\varepsilon^{\frac{n+1}{2}+i\lambda}\int_{-\infty}^{\infty} H(s,1;t\sigma,t)e^{2i\pi t(1+\langle s,\sigma\rangle)}|t|_\varepsilon^{\frac{n-1}{2}+i\lambda}\,dt$$
$$= (-1)^\varepsilon C^{-1}_{-i\lambda,\varepsilon}\int_{-\infty}^{\infty} H\left(s,1;\frac{t\sigma}{1+\langle s,\sigma\rangle},\frac{t}{1+\langle s,\sigma\rangle}\right)e^{2i\pi t}|t|_\varepsilon^{\frac{n-1}{2}+i\lambda}\,dt, \tag{1.39}$$

an expression which can be identified with (1.34).

To arrive at the computation of the function h such that $B_{i\lambda,\varepsilon} = \mathrm{Op}^\vee_{i\lambda,\varepsilon}(h)$ according to (1.29), we write, using in succession (1.35), (1.16), (1.4) and (1.32), and starting this time from a function $v \in \mathcal{S}_{\mathrm{flat}}(\mathbb{R}^{n+1})$,

$$\big(\theta_{i\lambda,\varepsilon}B_{i\lambda,\varepsilon}v^\flat_{i\lambda,\varepsilon}\big)(\sigma) = \big(\theta_{i\lambda,\varepsilon}\big(\mathrm{Op}_1(H)v\big)^\flat_{i\lambda,\varepsilon}\big)(\sigma) = \big(\big(\mathcal{F}\,\mathrm{Op}_1(H)v\big)^\flat_{-i\lambda,\varepsilon}\big)(\sigma)$$
$$= \big(\big(\mathcal{F}\,\mathrm{Op}_1(H)v\big)_{-i\lambda,\varepsilon}\big)(\sigma,1) = \big(\mathcal{F}\big(\big(\mathrm{Op}_1(H)v\big)_{i\lambda,\varepsilon}\big)\big)(\sigma,1)$$
$$= \big(\mathcal{F}\big(\mathrm{Op}_1(H)v_{i\lambda,\varepsilon}\big)\big)(\sigma,1). \tag{1.40}$$

We then use definition (1.27) of $\mathrm{Op}_1(H)$, expressing what precedes as

$$\int_{\mathbb{R}^{n+1}} H(x_*,x_{n+1};\sigma,1)e^{-2i\pi[\langle x_*,\sigma\rangle+x_{n+1}]}v_{i\lambda,\varepsilon}(x_*,x_{n+1})\,dx_*\,dx_{n+1}$$
$$= \int_{\mathbb{R}^{n+1}} H\left(\frac{ts}{1+\langle s,\sigma\rangle},\frac{t}{1+\langle s,\sigma\rangle};\sigma,1\right)e^{-2i\pi t}|t|_\varepsilon^{\frac{n-1}{2}-i\lambda}v^\flat_{i\lambda,\varepsilon}(s)\big|1+\langle s,\sigma\rangle\big|_\varepsilon^{-\frac{n+1}{2}+i\lambda}\,ds\,dt. \tag{1.41}$$

We must thus take, this time,

$$h(s,\sigma) = C^{-1}_{i\lambda,\varepsilon}\int_{-\infty}^{\infty} H\left(\frac{ts}{1+\langle s,\sigma\rangle},\frac{t}{1+\langle s,\sigma\rangle};\sigma,1\right)e^{-2i\pi t}|t|_\varepsilon^{\frac{n-1}{2}-i\lambda}\,dt, \tag{1.42}$$

which leads finally to (1.36). □

A fully equivalent, more expressive way to write the function f or h on $\mathcal{X}_n^\bullet$ is as the restriction to $\mathcal{X}_n^\bullet$ of the image of H under some operator expressed, in spectral-theoretic terms, as a function of the pair $(\langle \xi, \frac{\partial}{\partial \xi}\rangle, \delta)$, where the first entry denotes the Euler operator $\langle \xi, \frac{\partial}{\partial \xi}\rangle = \sum_{k=1}^{n+1} \xi_k \frac{\partial}{\partial \xi_k}$, and the second one is the usual indicator of parity of functions *with respect to* ξ only. Note that the dissymmetry between the variables x, ξ is only apparent since, as H lives on $\Omega^\bullet$, it satisfies the equation $\sum_{k=1}^{n+1}(x_k \frac{\partial}{\partial x_k} - \xi_k \frac{\partial}{\partial \xi_k})H = 0$; also, it is *globally* even as a function of (x, ξ).

Proposition 1.3. *Under the assumptions of the preceding proposition, observe that the number* $|\varepsilon - \delta|$ *is the number, equal to* 0 *or* 1, *characterized by the congruence* $|\varepsilon - \delta| \equiv \varepsilon + \delta \bmod 2$, *and set*

$$D_1 = i^{-\varepsilon+|\varepsilon-\delta|}\pi^{-\langle \xi, \frac{\partial}{\partial \xi}\rangle} \frac{\Gamma(\frac{1-n}{4} - \frac{i\lambda}{2} + \frac{\varepsilon}{2})}{\Gamma(\frac{1+n}{4} + \frac{i\lambda}{2} + \frac{\varepsilon}{2})} \frac{\Gamma(\frac{n+1}{4} + \frac{1}{2}\langle \xi, \frac{\partial}{\partial \xi}\rangle + \frac{i\lambda}{2} + \frac{|\varepsilon-\delta|}{2})}{\Gamma(\frac{1-n}{4} - \frac{1}{2}\langle \xi, \frac{\partial}{\partial \xi}\rangle - \frac{i\lambda}{2} + \frac{|\varepsilon-\delta|}{2})},$$

$$D_2 = i^{\varepsilon-|\varepsilon-\delta|}\pi^{-\langle \xi, \frac{\partial}{\partial \xi}\rangle} \frac{\Gamma(\frac{1-n}{4} + \frac{i\lambda}{2} + \frac{\varepsilon}{2})}{\Gamma(\frac{1+n}{4} - \frac{i\lambda}{2} + \frac{\varepsilon}{2})} \frac{\Gamma(\frac{n+1}{4} + \frac{1}{2}\langle \xi, \frac{\partial}{\partial \xi}\rangle - \frac{i\lambda}{2} + \frac{|\varepsilon-\delta|}{2})}{\Gamma(\frac{1-n}{4} - \frac{1}{2}\langle \xi, \frac{\partial}{\partial \xi}\rangle + \frac{i\lambda}{2} + \frac{|\varepsilon-\delta|}{2})}. \tag{1.43}$$

Then one has

$$f = D_1 H \big\|_{\mathcal{X}_n^\bullet}, \qquad h = D_2 H \big\|_{\mathcal{X}_n^\bullet}, \tag{1.44}$$

where the double restriction bar indicates that one should first restrict the function under consideration to the hypersurface of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ *of equation* $\langle x, \xi\rangle = 1$, *next use the invariance of the result under the usual action of* $\mathbb{R}^\times$ *to make it a function on the corresponding quotient* $\mathcal{X}_n^\bullet$ *of this hypersurface.*

Proof. On functions H with the parity δ with respect to ξ, one may write the function $(x, \xi) \mapsto H(x, t\xi)$ as $|t|_\delta^{\langle \xi, \frac{\partial}{\partial \xi}\rangle} H$: after having inserted a factor $e^{-\alpha|t|}$ for convergence and letting $\alpha > 0$ go to zero, one may use the Fourier transformation formula

$$\int_{-\infty}^{\infty} |t|_\delta^X e^{2i\pi t} |t|_\varepsilon^{\frac{n-1}{2}+i\lambda}\, dt = i^{|\varepsilon-\delta|}\pi^{-X-\frac{n}{2}-i\lambda} \frac{\Gamma(\frac{n+1}{4} + \frac{X}{2} + \frac{i\lambda}{2} + \frac{|\varepsilon-\delta|}{2})}{\Gamma(\frac{1-n}{4} - \frac{X}{2} - \frac{i\lambda}{2} + \frac{|\varepsilon-\delta|}{2})}, \tag{1.45}$$

which leads to the desired result, starting from (1.39) or (1.42). □

Functions $H = H(x, \xi)$ on Ω invariant under the transformations $(x, \xi) \mapsto (tx, t^{-1}\xi)$, $t \neq 0$, can also be written as functions $h = h(s, \sigma; q)$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, or on $\mathcal{X}_n^\bullet \times \mathbb{R}$ (cf. (1.25)), under the correspondence

$$H(x, \xi) = h\left(\frac{x_*}{x_{n+1}}, \frac{\xi_*}{\xi_{n+1}}; \langle x, \xi\rangle\right) \tag{1.46}$$

with $x = (x_*, x_{n+1})$ and $\xi = (\xi_*, \xi_{n+1})$. Then, one has

$$\langle x,\xi\rangle\sum_{k=1}^{n+1}\frac{\partial^2 H}{\partial x_k\partial\xi_k}=\big(1+\langle s,\sigma\rangle\big)\left[\sum_{j=1}^{n}\frac{\partial^2}{\partial s_j\partial\sigma_j}+\left(\sum_{\ell=1}^{n}s_\ell\frac{\partial}{\partial s_\ell}\right)\left(\sum_{m=1}^{n}\sigma_m\frac{\partial}{\partial\sigma_m}\right)\right]h$$
$$+(n+1)q\frac{\partial h}{\partial q}+q^2\frac{\partial^2 h}{\partial q^2}. \tag{1.47}$$

This brings to light the operator

$$\Delta_n=\big(1+\langle s,\sigma\rangle\big)\left[\sum_{j=1}^{n}\frac{\partial^2}{\partial s_j\partial\sigma_j}+\left(\sum s_\ell\frac{\partial}{\partial s_\ell}\right)\left(\sum\sigma_m\frac{\partial}{\partial\sigma_m}\right)\right], \tag{1.48}$$

the fundamental invariant differential operator on the (non-Riemannian) symmetric space $\mathcal{X}_n^\bullet = G_n/H_n$. One also sees that, if H lies in the null space of the operator $\sum_{k=1}^{n+1}\frac{\partial^2}{\partial x_k\partial\xi_k}$, and if H is, with respect to the variable ξ only, homogeneous of degree ρ, its restriction h to $\mathcal{X}_n^\bullet$ satisfies the eigenvalue equation

$$\Delta_n h=-\rho(n+\rho)h. \tag{1.49}$$

An example is provided by the function $H(x,\xi)=|\langle a,x\rangle\langle b,\xi\rangle|_\delta^\rho$ on $\Omega^\bullet$ with $\langle a,b\rangle=0$, which gives rise to the function $|\phi_{a,b}|_\delta^\rho$ on $\mathcal{X}_n$ with

$$\phi_{a,b}(s,\sigma)=\frac{(a_{n+1}+\langle a_*,s\rangle)(b_{n+1}+\langle b_*,\sigma\rangle)}{1+\langle s,\sigma\rangle}. \tag{1.50}$$

As shown in [1,2], the quasiregular representation of G_n in $L^2(\mathcal{X}_n^\bullet)=L^2(G_n/H_n)$ decomposes into a continuous part and a discrete part, a fact tantamount to the analogous statement regarding Δ_n. We shall take it, temporarily, for granted that—a consequence of the analysis to be developed in the next section—functions on $\mathcal{X}_n^\bullet$ in the continuous part of the decomposition can always be viewed (in many ways) as restrictions to $\mathcal{X}_n^\bullet$ of $\mathbb{R}^\times$-invariant functions satisfying in $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$, in the distribution sense, the equation

$$\Box H:=\sum_{k=1}^{n+1}\frac{\partial^2 H}{\partial x_k\partial\xi_k}=0. \tag{1.51}$$

This condition is remarkable from the point of view of pseudodifferential analysis since it means that the operator $\mathrm{Op}_\kappa(H)$, as defined in (0.1), does not depend on κ. This follows from the equation (cf. [12, p. 15] or do elementary manipulations using the Fourier transformation) $\mathrm{Op}_\kappa(H)=\mathrm{Op}_0(\exp(\frac{\kappa}{2i\pi}\sum_{k=1}^{n+1}\frac{\partial^2}{\partial x_k\partial\xi_k})H)$: in particular, the operators with standard or antistandard symbol H are identical.

The following corollary is one half (that concerning the continuous part $L^2_{\mathrm{cont}}(G_n/H_n)$ of the decomposition of $L^2(G_n/H_n)$) of the last result of the paper [1] by van Dijk and Molchanov.

Corollary 1.4. *The operator $J_{i\lambda,\varepsilon}: L^2_{\mathrm{cont}}(G_n/H_n)\to L^2_{\mathrm{cont}}(G_n/H_n)$ defined by the validity of the equation $\mathrm{Op}^\vee_{i\lambda,\varepsilon}(f)=\mathrm{Op}_{i\lambda,\varepsilon}(J_{i\lambda,\varepsilon}f)$ for every $f\in L^2(G_n/H_n)$ is characterized by the following property: on functions which are generalized eigenfunctions of Δ_n for the eigenvalue*

$-\rho(n+\rho)$ and have, with respect to ξ only, the parity characterized by δ, the operator $J_{i\lambda,\varepsilon}$ coincides with the scalar

$$G_{i\lambda,\varepsilon}(\rho,\delta)=(-1)^{\delta}\frac{\Gamma(\frac{n+1-\mu+\varepsilon}{2})}{\Gamma(\frac{-n+\mu+\varepsilon}{2})}\frac{\Gamma(\frac{-n+\mu-\rho+|\varepsilon-\delta|}{2})}{\Gamma(\frac{n+1-\mu+\rho+|\varepsilon-\delta|}{2})}\frac{\Gamma(\frac{1-\mu+\varepsilon}{2})}{\Gamma(\frac{\mu+\varepsilon}{2})}\frac{\Gamma(\frac{\mu+\rho+|\varepsilon-\delta|}{2})}{\Gamma(\frac{1-\mu-\rho+|\varepsilon-\delta|}{2})}, \qquad (1.52)$$

where $\mu=\frac{n+1}{2}+i\lambda$.

Proof. Apply Proposition 1.2 with a symbol H such that $\Box H=0$, so that $\mathrm{Op}_0(H)=\mathrm{Op}_1(H)$ and $A_{i\lambda,\varepsilon}=B_{i\lambda,\varepsilon}$ for every pair (λ,ε). It then suffices to consider the expression of $D_1D_2^{-1}$ obtained as a consequence of Proposition 1.3, and to use Eq. (1.49): of course, it should be noted that the expression on the right-hand side of (1.52) is invariant under the change $\rho\mapsto -n-\rho$, which makes it a function of $\rho(n+\rho)$. □

Remark. It will be seen in Section 4 that Corollary 1.4 remains valid for certain symbols—important from the point of view of pseudodifferential analysis—which are far from lying in $L^2(G_n/H_n)$.

2. The square root method: the continuous part of the operator $\Delta_n-\frac{n^2}{4}$

It is not our intention to give a complete exposition of the decomposition of the space $L^2(G_n/H_n)$ under the quasiregular action of G_n, already made in the references just recalled. The present section justifies the assertion, made just before Corollary 1.4, concerning the possibility to realize functions in the continuous part of the decomposition of $L^2(G_n/H_n)$ with the help of solutions in $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ of the equation $\Box H=0$. At the same time, it introduces a new construction of this continuous part, which, to our taste at least, makes the whole picture very clear (Theorem 2.2). Rather than continuing with a study of the discrete subspaces of the decomposition of $L^2(G_n/H_n)$, we shall follow, in Section 4, with the study of some interesting special distributions on $\mathcal{X}_n^{\bullet}$ related to symbols of operators in the algebra generated by resolvents of certain infinitesimal operators of the representation $\pi_{i\lambda,\varepsilon}$.

Equation (1.49) shows that generalized eigenvalues of the operator in the title of this section present themselves in the form $-(\rho+\frac{n}{2})^2$: more to the point, the spectral theory of this operator—as developed in [1,2] with the help of H_n-spherical distribution theory—shows that the spectrum of Δ_n has a continuous part, consisting of all numbers $\rho(-n-\rho)$ with $\rho=-\frac{n}{2}+ir\in-\frac{n}{2}+i\mathbb{R}$, and a discrete part consisting of the numbers $(\frac{1-n}{2}+k)(\frac{-1-n}{2}-k)$ with $k\in\mathbb{N}$. Now, the non-negative integer k is certainly uniquely determined by this latter rational number, but making the choice, for every $r^2>0$, of one of the two numbers r and $-r$, would be a very rough way of defining a square root of the continuous part of the operator $\Delta_n-\frac{n^2}{4}$. A better solution consists in making, under some transfer, the latter operator appear as the square of some *differential* operator, defined not on $\mathcal{X}_n^{\bullet}$ but on some other space $\Sigma_n^{\bullet}$, to wit the cone of equation $\langle x,\xi\rangle=0$, divided by the equivalence $(x.\xi)\sim(tx,t^{-1}\xi)$ $(t\in\mathbb{R}^{\times})$, in such a way that the generalized eigenvalue r^2 should split there as the pair $\pm r$.

The way to do this, introduced in [15, Section 18] in the case of the simplest Riemannian symmetric space $SL(2,\mathbb{R})/SO(2)$, is an alternative approach to the spectral theory of the invariant operator Δ under consideration with several advantages: in the situation already experienced, it led to some renewed understanding of the Lax–Phillips scattering theory [6] for the automorphic wave equation, and proved useful in modular form theory.

Let us start from the following analogue of [6, p. 11]: under the change of variables $(x,\xi)\mapsto(s,\sigma;\tau)$ from $\Omega_+^\bullet=\{(x,\xi)^\bullet\in\Omega^\bullet\colon \langle x,\xi\rangle>0\}$ to $\mathcal{X}_n^\bullet\times\mathbb{R}$ defined by the pair of equations (in which $x=(x_*,x_{n+1})$, $\xi=(\xi_*,\xi_{n+1})$)

$$(s,\sigma)=\left(\frac{x_*}{x_{n+1}},\frac{\xi_*}{\xi_{n+1}}\right),\qquad \tau=\log\langle x,\xi\rangle, \tag{2.1}$$

and under the transformation $H\mapsto H_1=e^{\frac{n\tau}{2}}H$, the equation $\square H=0$ inside $\Omega_+^\bullet$ is equivalent to the wave equation

$$\frac{\partial^2 H_1}{\partial\tau^2}+\left(\Delta_n-\frac{n^2}{4}\right)H_1=0. \tag{2.2}$$

Indeed, this follows from Eq. (1.47) if one writes $(n+1)q\frac{\partial h}{\partial q}+q^2\frac{\partial^2 h}{\partial q^2}=(\frac{d}{d\tau}+\frac{n}{2})^2h-\frac{n^2}{4}h$.

Of course, a solution of this equation in $\Omega_+^\bullet$ can be characterized by its first two traces on the hyperplane $\tau=0$: however, only the first one is of interest to us in the present context. It does not change if one replaces H by its transform $\widetilde{H}$ under the inversion map:

$$(\mathrm{Inv}\,H)(x,\xi)=\widetilde{H}(x,\xi)=\big(\langle x,\xi\rangle\big)^{-n}H\left(\frac{x}{\langle x,\xi\rangle},\frac{\xi}{\langle x,\xi\rangle}\right). \tag{2.3}$$

This distribution lies in the null-space of the operator $\square$ within Ω_+ if H does, and is also invariant under the transformations $(x,\xi)\mapsto(tx,t^{-1}\xi)$, $t\in\mathbb{R}^\times$, if H is. In particular, the function H defined just before (1.50) has the same restriction (to be followed by the passage to functions on a quotient set) to $\mathcal{X}_n^\bullet$ as the function $\widetilde{H}(x,\xi)=\frac{H(x,\xi)}{|\langle x,\xi\rangle|^{2\rho+n}}$: note in this case that the degree of homogeneity of $\widetilde{H}$ with respect to ξ is $-n-\rho$ instead of ρ, which was to be expected in view of (1.49). We base our present study of the continuous part of the decomposition of $L^2(\mathcal{X}_n^\bullet)=L^2(G_n/H_n)$ on the construction of a certain map Θ from functions on $\Sigma_n^\bullet$ to $\mathbb{R}^\times$-invariant functions on $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ in the null-space of $\square$, together with an involution $\mathcal{K}$ on the first space of functions, such that the knowledge of the first trace of $H=\Theta\Phi$ on $\mathcal{X}_n^\bullet$ should be equivalent to that of the $\mathcal{K}$-invariant part of Φ on $\Sigma_n^\bullet$.

The role of $\Sigma_n^\bullet$ is also clear from the representation-theoretic point of view. Indeed, the irreducible unitary components of the quasiregular representation of G_n in $L^2(\mathcal{X}_n)$ are subrepresentations of representations induced from a parabolic subgroup of parabolic rank 2. Such representations can be realized on spaces of functions on the homogeneous space G/MN, where N is the Heisenberg group of dimension $2n-1$, a space isomorphic to the cone under consideration.

Our first task is thus to give an efficient construction of all solutions of the wave equation $\square H=0$. This can be done in at least two different ways. The first one—following [15, Section 18]—is based on an extension of the theory of Riesz operators [9] or, more properly said, distributions [10], to the case of the operator $\square$: this was our first choice during the preparation of this paper, and it provides more information than the following one, based on the use of the Fourier transformation; the latter one has the advantage of being more concise.

On $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, we shall use throughout the "symplectic" Fourier transformation $\mathcal{F}_{\text{symp}}$ defined by the equation

$$(\mathcal{F}_{\text{symp}}\mathcal{S})(x,\xi) = \big\langle \mathcal{S}, (y,\eta) \mapsto e^{2i\pi(\langle x,\eta\rangle - \langle y,\xi\rangle)}\big\rangle. \tag{2.4}$$

Note that, when using the symplectic Fourier transformation, one should not consider the "dual" variables as conceptually distinct from the main ones. Then, the symplectic Fourier transform of a distribution $\mathfrak{S}$ satisfying the distribution equation $\langle y,\eta\rangle\mathfrak{S} = 0$ lies in the null-space of $\Box$.

Consider the following example. Given $\rho \in \mathbb{C}$ and $\delta = 0$ or 1 with $\rho + \delta \neq -1, -3, \ldots$ and $\rho - \delta \neq 0, 2, \ldots,$ finally, given $a \in \mathbb{R}^{n+1} \setminus \{0\}$, introduce the distribution $M^a_{\rho,\delta}$ on $\mathbb{R}^{n+1}$ defined by the equation

$$\big\langle M^a_{\rho,\delta}, \psi\big\rangle = \int_{-\infty}^{\infty} \psi(ra)|r|_\delta^{-\rho-1}\,dr, \quad \psi \in \mathcal{S}\big(\mathbb{R}^{n+1}\big). \tag{2.5}$$

This is a measure supported by the line $\mathbb{R}a$ in the case when $\operatorname{Re}\rho < 0$, and a well-defined distribution, homogeneous of degree $-n-1-\rho$, whenever the pair (ρ,δ) satisfies the above conditions. If we take for $\mathfrak{S}$ the distribution $M^b_{\rho,\delta} \otimes M^a_{\rho,\delta}$, and if we assume that $\langle a,b\rangle = 0$, so that the equation $\langle y,\eta\rangle\mathfrak{S} = 0$ should hold, we obtain $\mathcal{F}_{\text{symp}}\mathfrak{S} = \mathcal{F}^{-1}M^a_{\rho,\delta} \otimes \mathcal{F}M^b_{\rho,\delta}$, i.e.,

$$(\mathcal{F}_{\text{symp}}\mathfrak{S})(x,\xi) = \pi^{1+2\rho}\left[\frac{\Gamma(\frac{-\rho+\delta}{2})}{\Gamma(\frac{\rho+1+\delta}{2})}\right]^2 \big|\langle a,x\rangle\langle b,\xi\rangle\big|_\delta^\rho; \tag{2.6}$$

we thus get back to the function $|\phi_{a,b}|_\delta^\rho$ considered just before (1.50).

We now need to introduce a few geometric objects. The set $\overline{\Sigma}_n$ is the hypersurface of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ of equation $\langle y,\eta\rangle = 0$: let us warn the reader that, when dealing with functions defined on this space, it is sometimes necessary to switch from the variables (y,η) to the variables (x,ξ), since this space plays a role "on both sides" of the (symplectic) Fourier transformation. Let Σ_n denote the open dense subset of $\overline{\Sigma}_n$ characterized by the conditions $y \neq 0, \eta \neq 0$. The space $\Sigma_n^\bullet$ is the quotient of Σ_n by the group of transformations $(y,\eta) \mapsto (ty, t^{-1}\eta)$ with $t \in \mathbb{R}^\times$: since this action of $\mathbb{R}^\times$ occurs in a consistent way, such notions as $\mathbb{R}^\times$-invariant functions, or quotients by $\mathbb{R}^\times$, will always make reference to this action. On the cone Σ_n, we may use the (singular) coordinates (y,η_*), since $\eta_{n+1} = -\frac{\langle y_*,\eta_*\rangle}{y_{n+1}}$, and the $GL(n+1,\mathbb{R})$-invariant measure $dm(y,\eta_*) = |y_{n+1}|^{-1}\,dy\,d\eta_*$. On its quotient $\Sigma_n^\bullet$, we shall use the coordinates (y_*,η_*) corresponding to the orbit, under the action of $\mathbb{R}^\times$, of the point $(y_*, 1; \eta_*, -\langle y_*,\eta_*\rangle)$. In terms of these coordinates, we set $dm^\bullet(y_*,\eta_*) = dy_*\,d\eta_*$: it is easy to show that this measure is invariant under the action of $GL(n+1,\mathbb{R})$ on $\Sigma_n^\bullet$ coming from the action $g.(y,\eta) = (gy, g'^{-1}\eta)$ of this group on Σ_n. Only, recall that in the case when $g \in SL(n+1,\mathbb{R})$, it can be written as $g = \big(\begin{smallmatrix} M & \boldsymbol{p} \\ \boldsymbol{q}^{\mathrm{T}} & m\end{smallmatrix}\big)$, where $\boldsymbol{p}$ is a column vector and $\boldsymbol{q}^{\mathrm{T}}$ is the transpose of the column vector $\boldsymbol{q}$: the computation of $g.(y_*,\eta_*)$ is trivial in the case when $\boldsymbol{q} = 0$, and it suffices to consider the case when $\boldsymbol{p} = 0$ and $M = I$, in which one has $g.(y_*,\eta_*) = (\frac{y_*}{1+\langle \boldsymbol{q}, y_*\rangle}; (1+\langle \boldsymbol{q}, y_*\rangle)(\eta_* + \langle y_*,\eta_*\rangle \boldsymbol{q}))$. Note that on $\mathcal{X}_n^\bullet$, too, we may still use the coordinates (x_*,ξ_*) corresponding to the orbit, under the action of $\mathbb{R}^\times$, of the point $(x_*, 1; \xi_*, 1 - \langle x_*,\xi_*\rangle)$ and that, in these coordinates, the G_n-invariant measure $|1+\langle s,\sigma\rangle|^{-n-1}\,ds\,d\sigma$ reduces again to $dx_*\,d\xi_*$.

Any C^∞ function $\Phi = \Phi(y_*, \eta_*)$ on $\Sigma_n^\bullet$ can be extended as a function $\widetilde{\Phi}$ on Σ_n in a natural way, setting $\widetilde{\Phi}(y, \eta_*) = \Phi(\frac{y_*}{y_{n+1}}, y_{n+1}\eta_*)$. Then, the distribution $\mathfrak{S} = \widetilde{\Phi}\, dm$ is supported in $\overline{\Sigma}_n$, is $\mathbb{R}^\times$-invariant, and satisfies the distribution equation $\langle y, \eta\rangle \mathfrak{S} = 0$, so that its symplectic Fourier transform lies in the null-space of $\Box$. We are interested in two $\mathbb{R}^\times$-invariant functions, to wit the restrictions of $\Theta\Phi = \mathcal{F}_{\text{symp}}(\widetilde{\Phi}\, dm)$ to the hypersurface of equation $\langle x, \xi\rangle = 1$, or to the cone Σ_n of equation $\langle x, \xi\rangle = 0$: actually, we immediately need to consider the results of these restrictions as living on the quotient $\mathcal{X}_n^\bullet$ or $\Sigma_n^\bullet$ of the corresponding hypersurface by $\mathbb{R}^\times$ and, for clarity—a notation already used in Proposition 1.3—we denote by a double bar the operation of restriction followed by the one of going to the quotient set. With this convention, we set

$$\mathcal{A}\Phi = \mathcal{F}_{\text{symp}}(\widetilde{\Phi}\, dm)\big\|_{\mathcal{X}_n^\bullet}, \qquad \mathcal{H}\Phi = \mathcal{F}_{\text{symp}}(\widetilde{\Phi}\, dm)\big\|_{\Sigma_n^\bullet}. \tag{2.7}$$

We shall first study the operator $\mathcal{H}$, which will turn out to be closely related to the involution $\mathcal{K}$ which we have in mind; then, the operator $\mathcal{A}$, restricted to $\mathcal{K}$-invariant functions, will provide an isomorphism with a dense subspace of the continuous part of $L^2(\mathcal{X}_n^\bullet)$. Under the transfer by this isomorphism, the operator Δ_n will appear as the square of an Euler-type differential operator on $\Sigma_n^\bullet$.

Theorem 2.1. *Let $\mathcal{H}$ be the operator from functions on $\Sigma_n^\bullet$ (say, C^∞ with compact support) to functions on $\Sigma_n^\bullet$ characterized by the identity*

$$\mathcal{H}\Phi = \big(\mathcal{F}_{\text{symp}}(\widetilde{\Phi}\, dm)\big)\big\|_{\Sigma_n^\bullet}. \tag{2.8}$$

It extends as an unbounded self-adjoint operator on $L^2(\Sigma_n^\bullet)$: moreover, denoting as r the self-adjoint operator defined by the equation (in the coordinates (x_*, ξ_*) on $\Sigma_n^\bullet$) $\langle \xi_*, \frac{\partial}{\partial \xi_*}\rangle = -\frac{n}{2} - ir$, one has

$$\mathcal{H}^2 = \mathcal{H}^*\mathcal{H} = \pi\, \frac{\Gamma(ir)\Gamma(-ir)}{\Gamma(\frac{1}{2} + ir)\Gamma(\frac{1}{2} - ir)}. \tag{2.9}$$

Set, assuming that $\Phi = \Phi(y_*, \eta_*)$ has, with respect to η_*, the parity associated with δ,

$$\mathcal{K}\Phi = (-1)^\delta \pi^{-\frac{1}{2} - 2ir}\, \frac{\Gamma(\frac{1}{2} + ir)}{\Gamma(-ir)}\mathcal{H}\Phi. \tag{2.10}$$

Then, the operator $\mathcal{K}$ on $L^2(\Sigma_n^\bullet)$ is a unitary involution and, for every function $\Phi \in C^\infty(\Sigma_n^\bullet)$ with compact support, one has the identity

$$\mathcal{F}_{\text{symp}}(\widetilde{\mathcal{K}\Phi}\, dm) = \operatorname{Inv}\big(\mathcal{F}_{\text{symp}}(\widetilde{\Phi}\, dm)\big). \tag{2.11}$$

Proof. The function $\Theta\Phi = \mathcal{F}_{\text{symp}}(\widetilde{\Phi}\, dm)$ is given by the integral

$$(\Theta\Phi)(x, \xi) = \int e^{2i\pi(\langle x, \eta\rangle - \langle y, \xi\rangle)}\, \Phi\left(\frac{y_*}{y_{n+1}}, y_{n+1}\eta_*\right) \frac{dy\, d\eta_*}{|y_{n+1}|}, \tag{2.12}$$

in which $\eta_{n+1} = -\frac{\langle y_*, \eta_*\rangle}{y_{n+1}}$. In order to make the operator $\mathcal{H}$ introduced in (2.8) explicit, it suffices to set $(x; \xi) = (x_*, 1; \xi_*, -\langle x_*, \xi_*\rangle)$, since this has been our choice of coordinates on $\Sigma_n^\bullet$. After one renames as t the integration variable y_{n+1} and one changes (y_*, η_*) to $(y_{n+1}y_*, \frac{\eta_*}{y_{n+1}})$, one obtains

$$(\mathcal{H}\Phi)(x_*, \xi_*) = \int e^{2i\pi(\frac{\langle x_*-y_*, \eta_*\rangle}{t} + t\langle x_*-y_*, \xi_*\rangle)} \Phi(y_*, \eta_*) \frac{dt}{|t|}\, dy_*\, d\eta_*. \tag{2.13}$$

Consequently, the operator $\mathcal{H}$ is formally self-adjoint as an operator on $L^2(\Sigma_n^\bullet)$. Though it resembles a Fourier transformation, it involves a restriction to some hypersurface and a dual operation, and it is not unitary. Its Fourier-transformed expression is easier to manage.

Denote as $\mathcal{F}_*$ the $(2n)$-dimensional version of the symplectic Fourier transformation, and remark that the parity of Φ with respect to the set of Greek variables is the same as that of $\mathcal{F}_*\Phi$ with respect to the roman ones. Starting from (2.13), one easily obtains

$$\begin{aligned}(\mathcal{F}_*\mathcal{H}\Phi)(y_*, \eta_*) &= \int_{-\infty}^{\infty} |t|^{n-1} e^{2i\pi t\langle y_*, \eta_*\rangle} (\mathcal{F}_*\Phi)\big(-t^2 y_*, \eta_*\big)\, dt \\ &= \int_{-\infty}^{\infty} |t|^{n-1} e^{2i\pi t} \big|\langle y_*, \eta_*\rangle\big|^{-n} (\mathcal{F}_*\Phi)\left(-t^2 \frac{y_*}{(\langle y_*, \eta_*\rangle)^2}, \eta_*\right) dt. \end{aligned} \tag{2.14}$$

Set $\mathcal{F}_* r \mathcal{F}_*^{-1} = \hat{r}$, i.e., $\langle y_*, \frac{\partial}{\partial y_*}\rangle = -\frac{n}{2} + i\hat{r}$. Define

$$(\mathcal{F}_*\mathcal{M}\Phi)(y_*, \eta_*) = \int_{-\infty}^{\infty} |t|^{n-1} e^{2i\pi t} (\mathcal{F}_*\Phi)\big(t^2 y_*, \eta_*\big)\, dt, \tag{2.15}$$

so that

$$\mathcal{F}_*\mathcal{H}\Phi = \mathcal{J}(\mathcal{F}_*\mathcal{M}\Phi) \tag{2.16}$$

if we denote as $\mathcal{J}$ the involution characterized by the equation

$$(\mathcal{J}\Xi)(y_*, \eta_*) = \big|\langle y_*, \eta_*\rangle\big|^{-n}\, \Xi\left(-\frac{y_*}{(\langle y_*, \eta_*\rangle)^2}, \eta_*\right). \tag{2.17}$$

Note that $\mathcal{J}$ anticommutes with $\hat{r}$. One may rewrite (2.15) as

$$\mathcal{F}_*\mathcal{M}\Phi = \pi^{\frac{1}{2}-2i\hat{r}} \frac{\Gamma(i\hat{r})}{\Gamma(\frac{1}{2} - i\hat{r})} \mathcal{F}_*\Phi, \tag{2.18}$$

a result which, when combined with (2.16), leads to

$$\mathcal{F}_*\mathcal{H}^*\mathcal{H}\mathcal{F}_*{}^{-1} = \pi \frac{\Gamma(i\hat{r})\Gamma(-i\hat{r})}{\Gamma(\frac{1}{2} + i\hat{r})\Gamma(\frac{1}{2} - i\hat{r})}, \tag{2.19}$$

then to (2.9).

Definition (2.10) of $\mathcal{K}\Phi$, together with (2.16), leads to

$$\begin{aligned}\mathcal{F}_*\mathcal{K}\Phi &= (-1)^\delta \pi^{-\frac{1}{2}-2i\hat{r}} \frac{\Gamma(\frac{1}{2}+i\hat{r})}{\Gamma(-i\hat{r})} \mathcal{F}_*\mathcal{H}\Phi \\ &= (-1)^\delta \mathcal{J}\left(\pi^{-\frac{1}{2}+2i\hat{r}} \frac{\Gamma(\frac{1}{2}-i\hat{r})}{\Gamma(i\hat{r})} \mathcal{F}_*\mathcal{M}\Phi\right) \\ &= (-1)^\delta \mathcal{J}(\mathcal{F}_*\Phi) \end{aligned} \tag{2.20}$$

in view of (2.18). In terms of the ((2n)-dimensional) symplectic Fourier transform $\mathcal{F}_*\Phi$ of Φ, one may rewrite (2.12) as

$$(\Theta\Phi)(x,\xi) = |x_{n+1}|^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} |t|^{n-1} (\mathcal{F}_*\Phi)(q,p) \times \exp\left(\frac{2i\pi}{x_{n+1}}\big(t^2\langle q,\xi_*\rangle - t\big(\langle q,p\rangle + \langle x,\xi\rangle\big) + \langle x_*,p\rangle\big)\right) dq\,dp\,dt, \tag{2.21}$$

as seen after a perfectly elementary, if somewhat lengthy to write down, computation.

Next, the parity of $\mathcal{F}_*\Phi$ with respect to q is the same as that of Φ with respect to y_*, i.e., δ. Substituting the result of (2.20) into (2.21), one sees that, in order to obtain $(\Theta\mathcal{K}\Phi)(x,\xi)$, it suffices to take the right-hand side of (2.21), replacing $(\mathcal{F}_*\Phi)(q,p)$ by $|\langle q,p\rangle|^{-n}(\mathcal{F}_*\Phi)(\frac{q}{(\langle q,p\rangle)^2},p)$: we shall not display the result, but shall immediately perform the change of variables $(q,p)\mapsto(\frac{q}{(\langle q,p\rangle)^2},p)$, which leads to

$$(\Theta\mathcal{K}\Phi)(x,\xi) = |x_{n+1}|^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} |t|^{n-1} \big|\langle q,p\rangle\big|^{-n} (\mathcal{F}_*\Phi)(q,p) \times \exp\left(\frac{2i\pi}{x_{n+1}}\left(\frac{t^2}{(\langle q,p\rangle)^2}\langle q,\xi_*\rangle - t\left(\frac{1}{\langle q,p\rangle} + \langle x,\xi\rangle\right) + \langle x_*,p\rangle\right)\right) dq\,dp\,dt. \tag{2.22}$$

On the other hand, changing in (2.21) the integration variable t for s one obtains

$$\big(\mathrm{Inv}(\Theta\Phi)\big)(x,\xi) = |x_{n+1}|^{-n} \int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} |s|^{n-1} (\mathcal{F}_*\Phi)(q,p) \times \exp\left(\frac{2i\pi}{x_{n+1}}\big(s^2\langle q,\xi_*\rangle - s\big(\langle q,p\rangle\langle x,\xi\rangle + 1\big) + \langle x_*,p\rangle\big)\right) dq\,dp\,ds. \tag{2.23}$$

Setting $s = \frac{t}{\langle q,p\rangle}$ in the last integral, and comparing the result to (2.22), one obtains the equation $\mathrm{Inv}(\Theta\Phi) = \Theta\mathcal{K}\Phi$, just the same as (2.11). □

We can now state and prove the main result of this section.

Theorem 2.2. *Recall from* (2.7) *that*

$$\mathcal{H}\Phi = \Theta\Phi \|_{\Sigma_n^\bullet}, \qquad \mathcal{A}\Phi = \Theta\Phi \|_{\mathcal{X}_n^\bullet}, \tag{2.24}$$

and that the map $\mathcal{H}$ has been analyzed in Theorem 2.1. *Let $D_{\mathcal{K}}(\mathcal{H}) \subset L^2(\Sigma_n^\bullet)$ be the space of $\mathcal{K}$-invariant functions in the domain of the self-adjoint operator $\mathcal{H}$. The linear map $\mathcal{A}$ is a linear isomorphism from $D_{\mathcal{K}}(\mathcal{H})$ onto the subspace of $L^2(\mathcal{X}_n^\bullet) = L^2(G_n/H_n)$ corresponding to the continuous part of the decomposition of this latter space under the quasiregular action of G_n. One has the identity*

$$\|\mathcal{A}\Phi\|^2 = 2\|\mathcal{H}\Phi\|^2 \tag{2.25}$$

for every $\Phi \in D_{\mathcal{K}}(\mathcal{H})$. Finally, the operator $(\frac{n}{2} + \langle \eta_*, \frac{\partial}{\partial \eta_*}\rangle)^2$ commutes with $\mathcal{K}$ and, for every function $\Phi \in D_{\mathcal{K}}(\mathcal{H})$ such that $(\frac{n}{2} + \langle \eta_*, \frac{\partial}{\partial \eta_*}\rangle)^2\Phi$ lies in $D_{\mathcal{K}}(\mathcal{H})$ as well, the identity

$$\mathcal{A}\left(\frac{n}{2} + \left\langle \eta_*, \frac{\partial}{\partial \eta_*}\right\rangle\right)^2 \Phi = \left(\Delta_n - \frac{n^2}{4}\right)\mathcal{A}\Phi \tag{2.26}$$

holds. It thus reduces the study of the continuous part of the spectral decomposition of Δ_n to the (*trivial*) *spectral theory of an Euler-type operator on the cone $\Sigma_n^\bullet$.*

Proof. On $\mathcal{X}_n^\bullet$ we still use the coordinates (x_*, ξ_*), corresponding this time to the point $(x, \xi) = (x_*, 1; \xi_*, 1 - \langle x_*, \xi_*\rangle)$: these coordinates are related by Eq. (1.25) to the coordinates (s, σ), more useful when dealing with the symbolic calculus; here, there are advantageous, since in particular the G_n-invariant measure $|1 + \langle s, \sigma\rangle|^{-n-1}\, ds\, d\sigma$ on $\mathcal{X}_n^\bullet$ reduces again to $dx_*\, d\xi_*$. With the same computation as the one in the beginning of the proof of Theorem 2.1, we find the following equation, to be compared to (2.13):

$$(\mathcal{A}\Phi)(x_*, \xi_*) = \int e^{2i\pi(\frac{\langle x_* - y_*, \eta_*\rangle}{t} + t\langle x_* - y_*, \xi_*\rangle)} e^{-2i\pi t}\, \Phi(y_*, \eta_*) \frac{dt}{|t|}\, dy_*\, d\eta_*, \tag{2.27}$$

the Fourier-transformed version of which, to be compared to (2.14), is

$$(\mathcal{F}_*\mathcal{A}\Phi)(y_*, \eta_*) = \int_{-\infty}^{\infty} |t|^{n-1} e^{2i\pi t\langle y_*, \eta_*\rangle} e^{-2i\pi t^{-1}} (\mathcal{F}_*\Phi)\big(-t^2 y_*, \eta_*\big)\, dt. \tag{2.28}$$

The n-dimensional analogue of $\Box$, to wit the operator on $\mathbb{R}^n \times \mathbb{R}^n$ defined by the equation $\Box_* = \sum_{j=1}^n \frac{\partial^2}{\partial x_j \partial \xi_j}$, can be interpreted in two different ways, since the coordinates (x_*, ξ_*) can be used on $\mathcal{X}_n^\bullet$ as well as on $\Sigma_n^\bullet$: of course, the two operators obtained are conceptually different, and we shall denote them as $\Box_*^{\mathcal{X}}$ and $\Box_*^{\Sigma}$, respectively. Incidentally, one may prove the equation

$$\Delta_n = \Box_*^{\mathcal{X}} - \left\langle \xi_*, \frac{\partial}{\partial \xi_*}\right\rangle \left(n + \left\langle \xi_*, \frac{\partial}{\partial \xi_*}\right\rangle\right). \tag{2.29}$$

Each of the two operators just defined is a self-adjoint operator with continuous spectrum, and we denote as $P_+^{\mathcal{X}}$ and $P_-^{\mathcal{X}}$ (respectively P_+^{Σ} and P_-^{Σ}) the orthogonal projections onto the positive and negative spaces of $\Box_*^{\mathcal{X}}$ (respectively $\Box_*^{\Sigma}$).

We shall compute the norms of $P_\pm^{\mathcal{X}}\mathcal{A}\Phi$ in terms of Φ separately: since $\mathcal{F}_*\Box_*\mathcal{F}_*{}^{-1} = 4\pi^2\langle y_*, \eta_*\rangle$, one has, for instance,

$$\left\| P_+^{\mathcal{X}}\mathcal{A}\Phi\right\|^2 = \int\limits_{\langle y_*,\eta_*\rangle>0} \left|(\mathcal{F}_*\mathcal{A}\Phi)(y_*, \eta_*)\right|^2 dy_*\, d\eta_*. \tag{2.30}$$

With $y_* = (y_{**}, y_n)$ it is convenient to substitute for (y_*, η_*) the (singular) coordinates $(q, \eta_*; w) = (\frac{y_{**}}{y_n}, \eta_*; \langle y_*, \eta_*\rangle) \in P_{n-1}(\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}$: then,

$$dy_*\, d\eta_* = \frac{|y_n|^n}{|\langle y_*, \eta_*\rangle|}\, dq\, d\eta_*\, dw = \frac{|w|^{n-1}}{|\langle q, \eta_{**}\rangle + \eta_n|^n}\, dq\, d\eta_*\, dw, \tag{2.31}$$

and $i\hat{r} = \frac{n}{2} + w\frac{\partial}{\partial w}$.

In terms of the coordinates just introduced, the operator $\mathcal{B} = \mathcal{F}_*\mathcal{A}\mathcal{F}_*^{-1}$, as given by (2.28), really reduces to the operator on functions of one variable only defined by

$$(\mathcal{B}\phi)(w) = \int_{-\infty}^{\infty} |t|^{n-1} e^{2i\pi tw} e^{-2i\pi t^{-1}} \phi\left(-t^2 w\right) dt, \tag{2.32}$$

since the coordinates (q, η_*) are now simple parameters and have been omitted for clarity; we are interested in $\mathcal{B}$ as an unbounded operator from the space $L^2((-\infty, 0); |w|^{n-1}\, dw)$ to the space $L^2((0, \infty); w^{n-1}\, dw)$. Using (2.20) and (2.17) again, one sees that the symmetry condition $\Phi = \mathcal{K}\Phi$ expresses itself in a way independent of δ: in the new coordinates, and still forgetting the ones that are present as simple parameters, one has

$$\phi(w) = |w|^{-n}\phi\left(\frac{1}{w}\right). \tag{2.33}$$

Set, for real x, y,

$$\chi(y) = e^{ny}\phi\left(-e^{2y}\right), \qquad (\mathcal{D}\chi)(x) = e^{-nx}(\mathcal{B}\phi)\left(e^{-2x}\right), \tag{2.34}$$

so that the function χ is even. Also, one may then rewrite the last equation as

$$\begin{aligned}(\mathcal{D}\chi)(x) &= 2e^{-nx}\int_0^{\infty} t^{n-1}\cos\left(2\pi\left(te^{-2x} - t^{-1}\right)\right)\phi\left(-t^2e^{-2x}\right) dt \\ &= 2\int_{-\infty}^{\infty} \cos\left(4\pi e^{-x}\sinh y\right)\chi(y)\, dy. \end{aligned}\tag{2.35}$$

Now, the inverse Fourier transform of the function $y \mapsto e^{4i\pi e^{-x}\sinh y}$, evaluated at s, can be found from [7, p. 86], after one has inserted a factor $e^{-4\pi\varepsilon\cosh y}$ for convergence and let ε go to zero. The result is the function $s \mapsto 2e^{-\pi^2 s}K_{2i\pi s}(4\pi e^{-x})$: denoting as $\hat{\chi}$ the usual Fourier transform of χ, one finds

$$(\mathcal{D}\chi)(x) = 4\int_{-\infty}^{\infty} \left(\cosh \pi^2 s\right) K_{2i\pi s}\left(4\pi e^{-x}\right)\hat{\chi}(s)\,ds \quad \text{and} \tag{2.36}$$

$$\begin{aligned}\|\mathcal{D}\chi\|^2_{L^2(\mathbb{R};dx)} = 16\int_0^{\infty}\frac{dt}{t}\int_{\mathbb{R}^2} K_{2i\pi s_1}(4\pi t)K_{2i\pi s_2}(4\pi t)\\ \times \left(\cosh\pi^2 s_1\right)\left(\cosh\pi^2 s_2\right)\hat{\chi}(s_1)\overline{\hat{\chi}(s_2)}\,ds_1\,ds_2.\end{aligned} \tag{2.37}$$

This integral can be computed with the help of [15, p. 46], leading to

$$\begin{aligned}\|\mathcal{D}\chi\|^2_{L^2(\mathbb{R};dx)} &= 8\int_{-\infty}^{\infty}\Gamma(2i\pi s)\Gamma(-2i\pi s)\left(\cosh\pi^2 s\right)^2\left|\hat{\chi}(s)\right|^2 ds\\ &= 2\pi\int_{-\infty}^{\infty}\frac{\Gamma(i\pi s)\Gamma(-i\pi s)}{\Gamma(\frac{1}{2}+i\pi s)\Gamma(\frac{1}{2}-i\pi s)}\left|\hat{\chi}(s)\right|^2 ds.\end{aligned} \tag{2.38}$$

Since the multiplication by s corresponds, under the Fourier transformation, to $\frac{1}{2i\pi}\frac{d}{dy}$, and since

$$\frac{1}{2i\pi}\frac{d}{dy}\left(e^{ny}\phi\left(-e^{2y}\right)\right) = \frac{1}{i\pi}e^{ny}\left(\left(\frac{n}{2}+w\frac{d}{dw}\right)\phi\right) \quad \left(w = -e^{2y}\right), \tag{2.39}$$

finally, using the equation $i\hat{r} = \frac{n}{2} + w\frac{\partial}{\partial w}$ mentioned above, one finds

$$\left\|P_+^{\mathcal{X}}\mathcal{A}\Phi\right\|^2 = 2\pi\left(P_-^{\Sigma}\Phi \;\middle|\; \frac{\Gamma(ir)\Gamma(-ir)}{\Gamma(\frac{1}{2}+ir)\Gamma(\frac{1}{2}-ir)}P_-^{\Sigma}\Phi\right). \tag{2.40}$$

The study of $P_-^{\mathcal{X}}\mathcal{A}\Phi$ calls this time for that of $\mathcal{B}$ as an unbounded operator from the space $L^2((0,\infty); w^{n-1}\,dw)$ to the space $L^2((-\infty,0); |w|^{n-1}\,dw)$. We then set

$$\psi(y) = e^{ny}\phi\left(e^{2y}\right), \qquad (\mathcal{C}\psi)(x) = e^{-nx}(\mathcal{B}\phi)\left(-e^{-2x}\right), \tag{2.41}$$

the analogue of (2.35) is the equation

$$(\mathcal{C}\psi)(x) = 2\int_{-\infty}^{\infty}\cos\left(4\pi e^{-x}\cosh y\right)\psi(y)\,dy. \tag{2.42}$$

Using [7, pp. 86, 67, 66], one can see that the inverse Fourier transform of the function $y \mapsto e^{4i\pi e^{-x}\cosh y}$, evaluated at s, is

$$2\lim_{\varepsilon\to 0} K_{2i\pi s}\big(4\pi\big(\varepsilon - ie^{-x}\big)\big) = i\pi e^{-\pi^2 s} H^{(1)}_{2i\pi s}\big(4\pi e^{-x}\big)$$
$$= \pi \frac{e^{-\pi^2 s} J_{-2i\pi s}(4\pi e^{-x}) - e^{\pi^2 s} J_{2i\pi s}(4\pi e^{-x})}{i \sinh 2\pi^2 s}, \tag{2.43}$$

and that of the function $y \mapsto \cos(4i\pi e^{-x}\cosh y)$ is the real function

$$F_s\big(e^{-x}\big) = \pi \frac{J_{-2i\pi s}(4\pi e^{-x}) - J_{2i\pi s}(4\pi e^{-x})}{2i \sinh \pi^2 s}: \tag{2.44}$$

one has

$$(\mathcal{C}\psi)(x) = 2\int_{-\infty}^{\infty} F_s\big(e^{-x}\big)\hat{\psi}(s)\,ds, \tag{2.45}$$

and we need to compute

$$\|\mathcal{C}\psi\|^2_{L^2(\mathbb{R})} = 4\int_0^\infty \frac{dt}{t}\int_{\mathbb{R}^2} F_{s_1}(t)F_{s_2}(t)\hat{\psi}(s_1)\overline{\hat{\psi}(s_2)}\,ds_1\,ds_2$$
$$= \lim_{\rho\to 0}\int_{\mathbb{R}^2} K_\rho(s_1,s_2)\hat{\psi}(s_1)\overline{\hat{\psi}(s_2)}\,ds_1\,ds_2, \tag{2.46}$$

with

$$K_\rho(s_1,s_2) = -\frac{\pi^2}{\sinh \pi^2 s_1 \sinh \pi^2 s_2}\sum_{\epsilon_1^2=\epsilon_2^2=1} \epsilon_1\epsilon_2 \int_0^\infty J_{2i\pi\epsilon_1 s_1}(4\pi t)J_{2i\pi\epsilon_2 s_2}(4\pi t)t^{\rho-1}\,dt. \tag{2.47}$$

We set

$$\langle \epsilon, s\rangle = \epsilon_1 s_1 + \epsilon_2 s_2, \qquad \langle \check{\epsilon}, s\rangle = -\epsilon_1 s_1 + \epsilon_2 s_2 \tag{2.48}$$

and we use [7, p. 99]

$$\int_0^\infty J_{2i\pi\epsilon_1 s_1}(4\pi t)J_{2i\pi\epsilon_2 s_2}(4\pi t)t^{\rho-1}\,dt$$
$$= \frac{1}{2}(2\pi)^{-\rho}\Gamma(1-\rho)\frac{\Gamma(\frac{\rho}{2}+i\pi\langle\epsilon,s\rangle)}{\Gamma(1-\frac{\rho}{2}-i\pi\langle\check{\epsilon},s\rangle)\Gamma(1-\frac{\rho}{2}+i\pi\langle\epsilon,s\rangle)\Gamma(1-\frac{\rho}{2}+i\pi\langle\check{\epsilon},s\rangle)}. \tag{2.49}$$

Note that, for $s_1 \neq \pm s_2$, $K_\rho(s_1, s_2)$ goes to zero as $\rho \to 0$, since

$$K_0(s_1, s_2) = -\frac{1}{4i\pi \sinh \pi^2 s_1 \sinh \pi^2 s_2} \sum_{\epsilon_1^2 = \epsilon_2^2 = 1} \epsilon_1 \epsilon_2 \frac{\sinh \pi^2 \langle \check{\epsilon}, s\rangle}{\langle \epsilon, s\rangle \langle \check{\epsilon}, s\rangle} \tag{2.50}$$

changes to its negative under the change of parameters $(\epsilon_1, \epsilon_2) \mapsto (-\epsilon_1, -\epsilon_2)$. Consequently, as a distribution on $\mathbb{R}^2$, $K_\rho(s_1, s_2)$ has, as $\rho \to 0$, a limit supported in the union of the two lines $s_1 \pm s_2 = 0$. The calculation can be completed in the same way as that in [15, p. 46]. The only contributions to $K_\rho(s_1, s_2)$ which do not vanish in the limit as $\rho \to 0$ come from the pole at zero of the Gamma factor on the top of the right-hand side of (2.49), and we may replace this Gamma factor by its rational equivalent. The terms with $\epsilon_1 = \epsilon_2$ add up to an expression which has the same limit, as $\rho \to 0$, as

$$-\frac{\pi^2}{2 \sinh \pi^2 s_1 \sinh \pi^2 s_2} \left[\Gamma\left(1 - \frac{\rho}{2} - i\pi \langle \check{\epsilon}, s\rangle\right) \Gamma\left(1 - \frac{\rho}{2} + i\pi \langle \check{\epsilon}, s\rangle\right)\right]^{-1} \times \left[\frac{1}{\frac{\rho}{2} + i\pi \langle \epsilon, s\rangle} + \frac{1}{\frac{\rho}{2} - i\pi \langle \epsilon, s\rangle}\right]. \tag{2.51}$$

The function on the second line goes to the distribution $2\delta(s_1 + s_2)$, so that (remembering that $\hat{\psi}$ is an even function) the contribution to (2.46) of the terms with $\epsilon_1 = \epsilon_2$ is the integral of $|\hat{\psi}(s)|^2\, ds$ against the coefficient

$$\frac{\pi^2}{(\sinh \pi^2 s)^2} \left[\Gamma(1 - 2i\pi s)\Gamma(1 + 2i\pi s)\right]^{-1} = \pi \frac{\Gamma(i\pi s)\Gamma(-i\pi s)}{\Gamma(\frac{1}{2} + i\pi s)\Gamma(\frac{1}{2} - i\pi s)}, \tag{2.52}$$

and one obtains the same result from the consideration of the terms with $\epsilon_1 = -\epsilon_2$. Comparing this to (2.38), the net result is that

$$\| P_-^{\mathcal{X}} \mathcal{A}\Phi \|^2 = 2\pi \left(P_+^{\Sigma} \Phi \;\middle|\; \frac{\Gamma(ir)\Gamma(-ir)}{\Gamma(\frac{1}{2} + ir)\Gamma(\frac{1}{2} - ir)} P_+^{\Sigma} \Phi \right). \tag{2.53}$$

The proof of Theorem 2.2 is now complete: for the first part, we only need comparing (2.40) to (2.9); next, Eq. (2.26) is a consequence of (1.49). □

Needless to say, the inversion problem, i.e., the problem of recovering Φ in terms of $\mathcal{A}\Phi$, under the assumption that Φ is $\mathcal{K}$-invariant, is easy. Since the computations can be made by following the same transformations as above, let us satisfy ourselves with the result of the computation:

$$(\mathcal{F}_* \Phi)(y_*, \eta_*) = \frac{|1 - \langle y_*, \eta_* \rangle|}{2} \int_{-\infty}^{\infty} |t|^n e^{2i\pi t(1 + \langle y_*, \eta_* \rangle)} (\mathcal{F}_* \mathcal{A}\Phi)\big(-t^2 y_*, \eta_*\big)\, dt. \tag{2.54}$$

In view of Theorem 2.2, the preceding results, coupled with a Mellin transformation on $\Sigma_n^\bullet$, provide a diagonalization of the continuous part of the spectral decomposition of Δ_n.

3. Composition: the soft approach

The composition problem is a difficult one: we shall have a glimpse of it in the last part of the present paper, in which we shall consider it for some special symbols. In this short section, we explain why the obvious approach to the composition problem does not lead anywhere, in contradiction to what is the case with the symbolic calculus on $\mathbb{R}^{n+1}$. It would be misleading to believe that, in the quantization of symmetric spaces, the composition $f_1 \# f_2$ of two symbols can be, even in a rough way, described by means of a series of differential expressions in the pair of symbols under consideration; also, the integral formula is essentially worthless.

Definition 1.1 gives the integral kernels of the operators $\mathrm{Op}_{i\lambda,\varepsilon}(f)\theta_{i\lambda,\varepsilon}^{-1}$ and $\theta_{i\lambda,\varepsilon}\,\mathrm{Op}_{i\lambda,\varepsilon}^{\vee}(h)$; on the other hand, (1.23) gives the integral kernel of the intertwining operator $\theta_{i\lambda,\varepsilon}$, or of its inverse since it is unitary. It is thus immediate to obtain the following integral formula, analogous to the formula

$$(Jh)(x,\xi) = \int h(y,\eta) e^{2i\pi\langle x-y,\eta-\xi\rangle}\,dy\,d\eta \tag{3.1}$$

which, in the calculus on $\mathbb{R}^{n+1}$, makes it possible to link the antistandard symbol h of some operator to its standard symbol Jh.

Proposition 3.1. *If $h \in L^2(G/H)$, one has*

$$\mathrm{Op}_{i\lambda,\varepsilon}^{\vee}(h) = \mathrm{Op}_{i\lambda,\varepsilon}(f) \tag{3.2}$$

with (setting $d\mu(t,\tau) = |1+\langle t,\tau\rangle|^{-n-1}\,dt\,d\tau$)

$$f(s,\sigma) = |C_{i\lambda,\varepsilon}|^2 \int \left|\frac{(1+\langle s,\sigma\rangle)(1+\langle t,\tau\rangle)}{(1+\langle s,\tau\rangle)(1+\langle t,\sigma\rangle)}\right|_{\varepsilon}^{\frac{n+1}{2}+i\lambda} h(t,\tau)\,d\mu(t,\tau). \tag{3.3}$$

This formula should really be understood as the fact that the function

$$(\sigma,s) \mapsto \left|1+\langle s,\sigma\rangle\right|_{\varepsilon}^{-\frac{n+1}{2}-i\lambda} f(s,\sigma)$$

is $C_{i\lambda,\varepsilon}\bar{C}_{i\lambda,\varepsilon}^{-1}$ times the image of the function

$$(t,\tau) \mapsto \left|1+\langle t,\tau\rangle\right|_{\varepsilon}^{-\frac{n+1}{2}+i\lambda} f(t,\tau)$$

under the operator $\theta_{-i\lambda,\varepsilon} \otimes \theta_{-i\lambda,\varepsilon}$. One may interpret the following integral formula in a similar way.

Proposition 3.2. *Let f_1 and f_2 lie in $L^2(G/H)$. One has*

$$\mathrm{Op}_{i\lambda,\varepsilon}(f_1)\,\mathrm{Op}_{i\lambda,\varepsilon}(f_2) = \mathrm{Op}_{i\lambda,\varepsilon}(f_1 \# f_2) \quad \textit{with} \tag{3.4}$$

$$(f_1 \# f_2)(s,\sigma) = |C_{i\lambda,\varepsilon}|^2 \int \left|\frac{(1+\langle s,\sigma\rangle)(1+\langle t,\tau\rangle)}{(1+\langle s,\tau\rangle)(1+\langle t,\sigma\rangle)}\right|_{\varepsilon}^{\frac{n+1}{2}+i\lambda} f_1(s,\tau) f_2(t,\sigma)\,d\mu(t,\tau). \tag{3.5}$$

For the sake of comparison, we may here recall that the integral formula giving the symbol $f_1 \underset{\mathbb{R}^{n+1}}{\#} f_2$ of the two operators with standard symbols f_1 and f_2 is

$$\big(f_1 \underset{\mathbb{R}^{n+1}}{\#} f_2\big)(x,\xi) = \int f_1(x,\eta) f_2(y,\xi) e^{2i\pi\langle x-y,\eta-\xi\rangle}\, d\eta\, dy. \tag{3.6}$$

We have already come across the operator J which occurs in (3.1): indeed, it has been pointed out, right after (1.51), that one has the equation $J = \exp\frac{\square}{2i\pi}$: writing the exponential as a series, this immediately leads to the expansions

$$Jh \sim \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\square}{2i\pi}\right)^k h \quad \text{and} \tag{3.7}$$

$$\big(f_1 \underset{\mathbb{R}^{n+1}}{\#} f_2\big)(x,\xi) \sim \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\square_{y,\eta}}{2i\pi}\right)^k \big(f_1(x,\xi+\eta) f_2(x+y,\xi)\big) \quad (y=\eta=0). \tag{3.8}$$

These formulas lie at the foundations of classical pseudodifferential analysis: they do not define convergent series except for special symbols—such as those of differential operators—but they are valid, as useful asymptotic expansions, for symbols, or pairs of symbols, lying in large appropriate classes. Let us emphasize that what makes pseudodifferential analysis such a useful tool in partial differential equations is the easy way it makes it possible to define, and handle, *auxiliary* operators: you certainly do not need it to compose differential operators. Harmonic analysts may find an extra, immediate, satisfaction in the fact that if one introduces from the start, in the usual way, a Planck constant in the definition of the (say, standard) symbolic calculus, this constant will appear, in (3.7) or (3.8), as a coefficient in front of $\square$: thus, these two asymptotic expansions may also be viewed as series expansions with respect to Planck's constant.

As will be shown on the example which is the subject of this paper, this feature of pseudodifferential analysis in Euclidean space does not survive in the quantization of symmetric spaces, whether you wish to interpret the would-be analogues of (3.7) and (3.8) as asymptotic expansions or as series in the parameter λ^{-1}, sometimes viewed as some kind of analogue of Planck's constant.

We view the developments in the present section as necessary for a good understanding of the nature of quantization: however, they are, to a certain extent, of a negative nature, and we shall be as brief as possible. Using the coordinates (s,σ) on the phase space $\mathcal{X}_n^\bullet$ (cf. (1.25)), we note that the point (s,σ) is the image of the point $(0,0)$ under the matrix

$$g_{s,\sigma} = \begin{pmatrix} I & \frac{s}{1+\langle s,\sigma\rangle} \\ -\sigma^{\mathrm{T}} & \frac{1}{1+\langle s,\sigma\rangle} \end{pmatrix}.$$

Concentrating on the composition formula—since the operator $h \mapsto f$ from Proposition 4.1 has already been studied—we may use covariance to reduce the pointwise study of $(f_1 \# f_2)(s,\sigma)$ to that of $(f_1 \# f_2)(0,0)$. In perfect analogy with the way (3.8) was derived from (3.6), it suffices to compute the Fourier transform of a certain distribution to obtain the following.

Proposition 3.3. *Denote as $F_{i\lambda,\varepsilon}$ the function of one real variable defined as follow*:

$$F_{i\lambda,\varepsilon}(\beta) = \frac{2(2\pi)^{\frac{1-n}{2}-i\lambda}}{\Gamma(\frac{1-n}{2}-i\lambda)}\beta^{\frac{1-n}{4}-\frac{i\lambda}{2}}K_{\frac{n-1}{2}+i\lambda}\big(4\pi\beta^{\frac{1}{2}}\big) \tag{3.9}$$

if $\beta > 0$, and

$$\begin{aligned} &F_{i\lambda,\varepsilon}(\beta) \\ &\quad= (2\pi)^{\frac{1-n}{2}-i\lambda}\Gamma\left(\frac{n+1}{2}+i\lambda\right)|\beta|^{\frac{1-n}{4}-\frac{i\lambda}{2}}\big[J_{\frac{n-1}{2}+i\lambda}\big(4\pi|\beta|^{\frac{1}{2}}\big) - (-1)^{\varepsilon}J_{\frac{1-n}{2}-i\lambda}\big(4\pi|\beta|^{\frac{1}{2}}\big)\big] \end{aligned} \tag{3.10}$$

if $\beta < 0$. Then one has

$$(f_1 \# f_2)(0,0) = \left[F_\lambda\left(-\frac{1}{4\pi^2}\sum\frac{\partial^2}{\partial t_j\partial\tau_j}\right)\big(f_1(0,\tau)f_2(t,0)\big)\right] \quad \big((t,\tau)=(0,0)\big). \tag{3.11}$$

Proof. The right-hand side of the equation

$$(f_1 \# f_2)(0,0) = |C_{i\lambda,\varepsilon}|^2\int\big|1+\langle t,\tau\rangle\big|_\varepsilon^{-\frac{n+1}{2}+i\lambda}f_1(0,\tau)f_2(t,0)\,dt\,d\tau \tag{3.12}$$

can be interpreted as the value at $(0,0)$ of a convolution product on $\mathbb{R}^n\times\mathbb{R}^n$, to wit that of the function $G_{i\lambda,\varepsilon}(t,\tau) = |C_{i\lambda,\varepsilon}|^2|1+\langle t,\tau\rangle|_\varepsilon^{-\frac{n+1}{2}+i\lambda}$ by the function $(t,\tau)\mapsto f_1(0,\tau)f_2(t,0)$. In a more expressive way, one may write

$$(f_1 \# f_2)(0,0) = \left[(\mathcal{F}G_{i\lambda,\varepsilon})\left(\frac{1}{2i\pi}\frac{\partial}{\partial t},\frac{1}{2i\pi}\frac{\partial}{\partial\tau}\right)\big(f_1(0,\tau)f_2(t,0)\big)\right](0,0). \tag{3.13}$$

We thus compute the (usual, not symplectic, this time) Fourier transform of the distribution $(t,\tau)\mapsto|1+\langle t,\tau\rangle|_\varepsilon^{-\frac{n+1}{2}+i\lambda}$, denoting as (ρ,r) a pair of variables dual of (t,τ). With $\nu = \frac{n+1}{2} - i\lambda$, one has for real R the equation

$$|R|_\varepsilon^{-\nu} = \frac{(-i)^\varepsilon}{2\pi}\Gamma\left(\frac{1-\varepsilon+\nu}{2}\right)\Gamma\left(\frac{1+\varepsilon-\nu}{2}\right)\big[(0-iR)^{-\nu}+(-1)^\varepsilon(0+iR)^{-\nu}\big]. \tag{3.14}$$

For $a > 0$, one has

$$\big[a+i\big(1+\langle t,\tau\rangle\big)\big]^{-\nu} = \frac{(2\pi)^\nu}{\Gamma(\nu)}\int_0^\infty e^{-\frac{2\pi}{h}[a+i(1+\langle t,\tau\rangle)]}h^{-\nu-1}\,dh \tag{3.15}$$

and, since

$$\mathcal{F}\big(e^{-\frac{2i\pi}{h}\langle t,\tau\rangle}\big)(\rho,r) = h^n e^{2i\pi h\langle r,\rho\rangle}, \tag{3.16}$$

one obtains

$$\mathcal{F}\big[0+i\big(1+\langle t,\tau\rangle\big)\big]^{-\nu}=\lim_{a\to 0}\frac{(2\pi)^{\nu}}{\Gamma(\nu)}\int_0^\infty h^{n-\nu-1}e^{-\frac{2\pi}{h}(a+i)}e^{2i\pi h\langle r,\rho\rangle}\,dh. \tag{3.17}$$

This is an integral giving, classically [7, p. 85], the function $K_{n-\nu}$, but since one of the exponents, to wit $2i\pi h\langle r,\rho\rangle$, is pure imaginary, it must be interpreted as $-(0-2i\pi h\langle r,\rho\rangle)$. Being careful with phases, one then finds

$$\begin{aligned}\mathcal{F}\big[0+i\big(1+\langle t,\tau\rangle\big)\big]^{-\nu}&=2^{\nu+1}\frac{\pi^{\nu}}{\Gamma(\nu)}\big|\langle r,\rho\rangle\big|^{\frac{\nu-n}{2}}\exp\left(\frac{i\pi(n-\nu)}{4}\big(1+\operatorname{sign}\langle r,\rho\rangle\big)\right)\\&\quad\times K_{n-\nu}\left(4\pi\big|\langle r,\rho\rangle\big|^{\frac12}\exp\left(\frac{i\pi}{4}\big(1-\operatorname{sign}\langle r,\rho\rangle\big)\right)\right).\end{aligned} \tag{3.18}$$

To finish the computation, one notes [7, p. 67] that, for every $\mu\in\mathbb{C}$, if $x>0$ one has the equalities

$$\begin{aligned}K_\mu\big(xe^{-\frac{i\pi}{2}}\big)+K_\mu\big(xe^{\frac{i\pi}{2}}\big)&=\frac12\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{2-\mu}{2}\right)\big(-J_\mu(x)+J_{-\mu}(x)\big),\\K_\mu\big(xe^{-\frac{i\pi}{2}}\big)-K_\mu\big(xe^{\frac{i\pi}{2}}\big)&=\frac{i}{2}\Gamma\left(\frac{1+\mu}{2}\right)\Gamma\left(\frac{1-\mu}{2}\right)\big(J_\mu(x)+J_{-\mu}(x)\big).\end{aligned} \tag{3.19}$$

Hence, the Fourier transform of the distribution $(t,\tau)\mapsto|1+\langle t,\tau\rangle|_\varepsilon^{-\frac{n+1}{2}+i\lambda}$ is a distribution in the variables (r,ρ) which coincides, when $\langle r,\rho\rangle\neq 0$, with the function $H_{i\lambda,\varepsilon}(\langle r,\rho\rangle)$ defined by

$$H_{i\lambda,\varepsilon}(\beta)=\frac{2(2\pi)^{\frac{n+1}{2}-i\lambda}}{\Gamma(\frac{n+1}{2}-i\lambda)}\frac{\Gamma(\frac{n+3}{4}-\frac{i\lambda}{2}-\frac{\varepsilon}{2})\Gamma(\frac{1-n}{4}+\frac{i\lambda}{2}+\frac{\varepsilon}{2})}{\Gamma(\frac{n+1}{4}+\frac{i\lambda}{2}+\frac{\varepsilon}{2})\Gamma(\frac{3-n}{4}-\frac{i\lambda}{2}-\frac{\varepsilon}{2})}\beta^{\frac{1-n}{4}-\frac{i\lambda}{2}}K_{\frac{n-1}{2}+i\lambda}\big(4\pi\beta^{\frac12}\big) \tag{3.20}$$

if $\beta>0$, and

$$\begin{aligned}H_{i\lambda,\varepsilon}(\beta)&=\Gamma\left(\frac{n+3}{4}-\frac{i\lambda}{2}-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{1-n}{4}+\frac{i\lambda}{2}+\frac{\varepsilon}{2}\right)\Gamma\left(\frac{n-1}{4}+\frac{i\lambda}{2}+\frac{\varepsilon}{2}\right)\\&\quad\times\Gamma\left(\frac{5-n}{4}-\frac{i\lambda}{2}-\frac{\varepsilon}{2}\right)\frac{(2\pi)^{\frac{n-1}{2}-i\lambda}}{\Gamma(\frac{n+1}{2}-i\lambda)}|\beta|^{\frac{1-n}{4}-\frac{i\lambda}{2}}\\&\quad\times\big[J_{\frac{1-n}{2}-i\lambda}\big(4\pi|\beta|^{\frac12}\big)-(-1)^\varepsilon J_{\frac{n-1}{2}+i\lambda}\big(4\pi|\beta|^{\frac12}\big)\big]\end{aligned} \tag{3.21}$$

if $\beta<0$.

On the other hand, according to (1.24), one has

$$|C_{i\lambda,\varepsilon}|^2=\pi^{-n}\frac{\Gamma(\frac{1+n}{4}-\frac{i\lambda}{2}+\frac{\varepsilon}{2})\Gamma(\frac{1+n}{4}+\frac{i\lambda}{2}+\frac{\varepsilon}{2})}{\Gamma(\frac{1-n}{4}+\frac{i\lambda}{2}+\frac{\varepsilon}{2})\Gamma(\frac{1-n}{4}-\frac{i\lambda}{2}+\frac{\varepsilon}{2})}. \tag{3.22}$$

Using the duplication formula

$$\Gamma\left(\frac{1\pm n}{4}-\frac{i\lambda}{2}+\frac{\varepsilon}{2}\right)\Gamma\left(\frac{3\pm n}{4}-\frac{i\lambda}{2}-\frac{\varepsilon}{2}\right)=(2\pi)^{\frac{1}{2}}2^{\frac{1\mp n}{2}+i\lambda}\Gamma\left(\frac{1\pm n}{2}-i\lambda\right), \quad (3.23)$$

one, finally, obtains the function $F_{i\lambda,\varepsilon}(\beta)$ as characterized in the proposition. □

Remark. Using the series expansion of the Bessel functions, one obtains in all cases the following expression (involving Pochhammer's symbols)

$$F_{i\lambda,\varepsilon}(\beta)=\sum_{m\geqslant 0}\frac{(4\pi^2\beta)^m}{m!(\frac{n+1}{2}+i\lambda)_m} + \left(4\pi^2\right)^{\frac{1-n}{2}-i\lambda}\frac{\Gamma(\frac{n-1}{2}+i\lambda)}{\Gamma(\frac{1-n}{2}-i\lambda)}|\beta|_\varepsilon^{\frac{1-n}{2}-i\lambda}\sum_{m\geqslant 0}\frac{(4\pi^2\beta)^m}{m!(\frac{3-n}{2}-i\lambda)_m}. \quad (3.24)$$

Plugging this expansion into Eq. (3.11), or in its generalized version

$$(f_1 \# f_2)(s,\sigma) = \left[F_{i\lambda,\varepsilon}\left(-\frac{1}{4\pi^2}\sum\frac{\partial^2}{\partial t_j\partial\tau_j}\right)\left(f_1\left(s,\left[g'^{-1}_{s,\sigma}\right]\tau\right)f_2\left([g_{s,\sigma}]t,\sigma\right)\right)\right] \quad \left((t,\tau)=(0,0)\right), \quad (3.25)$$

one sees that the series which is first term of (3.24) contributes to $(f_1 \# f_2)(s,\sigma)$ a series of differential operators applied to the tensor product $f_1 \otimes f_2$, evaluated at (s,σ). However, the second term of (3.24) cannot be neglected. It is a ramified function of β at the origin, which shows that, in reality, no convergent, or simply asymptotic, series of differential operators applied to the tensor product $f_1 \otimes f_2$ evaluated at (s,σ) can produce a satisfactory approximation of the sharp product $f_1 \# f_2$. This is, of course, in contrast with what happens with the usual (standard, antistandard or Weyl) symbolic calculi on $\mathbb{R}^{n+1}$.

A fully similar phenomenon appeared in [17], in relation with the quantization of the upper half-plane $SL(2,\mathbb{R})/SO(2)$: indeed, the function $E(z)$ that occurs there, in Theorem 5.1, has a comparable type of singularity at the origin, to wit a ramified part. The main difference is that the exponent of $|\beta|$, to wit $-i\lambda$ in the present context, is to be replaced, in the former reference, by λ so that, as λ increases, the ramified term is, in some sense, pushed away. This explains why, indeed—as was shown in [17]—the symbolic calculus developed there has better properties for increasing values of λ, and the reason why a limit of the calculus as $\lambda \to \infty$ could be found (the Fuchs calculus). Nothing of the sort can work in the present situation, which is worse in this respect. So far as series expansions with respect to λ^{-1} are concerned, they never occur in the quantization of symmetric spaces, since the true functions of λ involved always have an essential singularity at infinity.

To put an end to this section, let us observe that, assuming that the symbols H_1 and H_2 of two pseudodifferential operators A_1 and A_1 on $\mathbb{R}^{n+1}$ both satisfy Eq. (1.51) (which means that the standard and antistandard symbols of each of the two operators under consideration agree) does not imply that the same holds for the composition A_1A_2: it suffices to consider the two symbols $x_1\xi_2$ and $x_2\xi_3$, the sharp composition of which, in the standard or antistandard calculus, is, respectively, $x_1x_2\xi_2\xi_3 + \frac{1}{2i\pi}x_1\xi_3$ or $x_1x_2\xi_2\xi_3$. This makes it impossible to reduce the study of the composition of symbols in the $\pi_{i\lambda,\varepsilon}$-calculus to the usual one on $\mathbb{R}^{n+1}$.

Another reason, even more decisive, why pseudodifferential analysis on projective space cannot be fully reduced (despite Proposition 1.2) to the (standard, or Weyl) pseudodifferential analysis on $\mathbb{R}^{n+1}$, as currently developed, has to do with more technical aspects. Even though, for their applications to partial differential equations, miscellaneous classes of symbols and associated operators have been considered, translations of the phase space always play a role there, albeit a local one: in contrast, it is only through its action by (local) conjugations that the group $GL(n+1,\mathbb{R})$ or, more generally (in the case of the Weyl calculus), the symplectic group $\mathrm{Sp}(n+1,\mathbb{R})$, plays a role in the definition of such classes of symbols. To give but one example, using the space $\mathcal{S}(\mathbb{R}^{n+1})$ of C^∞ vectors of the Heisenberg representation and its dual space $\mathcal{S}'(\mathbb{R}^{n+1})$, one finds immediately a very large class of symbols, to wit $\mathcal{S}'(\mathbb{R}^{2n+2})$, all of which give rise to meaningful operators; but giving a characterization of, say, the standard symbols of linear operators from the space of C^∞ vectors of the quasiregular representation of $GL(n+1,\mathbb{R})$ in $L^2(\mathbb{R}^{n+1})$ to the dual space is a problem in harmonic analysis—possibly a not too difficult one—which PDE people would probably find no reason to consider.

4. Some special symbols

In this last section, we try to familiarize ourselves with the calculus by an analysis of the operators the symbols of which are integral powers (the exponents can be of any sign, but negative ones are more interesting) of functions of the species $\langle a,x\rangle\langle b,\xi\rangle$, with $\langle a,b\rangle = 0$: these functions already appeared in (1.50) in the case when $a,b\in\mathbb{R}^{n+1}$ but, here, they will be complex vectors. In the case when $n=1$ (the study of which was made in [18]), the symbols under consideration generate the discrete spaces of the decomposition of $L^2(G_1/H_1)$: moreover, the Hilbert sum of (one half of) these spaces is closed under the sharp product of symbols, in the $\mathrm{Op}_{i\lambda,\varepsilon}$-calculus, and the composition formulas were made explicit with the help of the so-called Rankin–Cohen brackets. In the case when $n\geqslant 2$, these functions do not lie in $L^2(G_n/H_n)$ any more. Our interest in them lies in the fact that they are the symbols of integral powers, with positive exponents, of resolvents of certain infinitesimal operators of the representation $\pi_{i\lambda,\varepsilon}$.

We need to introduce the infinitesimal operators of the representation $\pi_{i\lambda,\varepsilon}$: these are defined by the equation

$$\big(d\pi_{i\lambda,\varepsilon}(X)u\big)(s) = \frac{d}{dt}\bigg|_{t=0}\big(\pi_{i\lambda,\varepsilon}(\exp tX)u\big)(s), \qquad X\in\mathfrak{g}_n, \tag{4.1}$$

where $\mathfrak{g}_n$ is the Lie algebra of G_n. As a linear basis of $\mathfrak{g}_n$, we choose the set $(E_{jk})_{(j,k)\neq(n+1,n+1)}$ defined as follows: if $j\neq k$, $E_{jk}=e_j\otimes e_k^*$ is the matrix such that $(E_{jk})_{\ell,m}=\delta_\ell^j\delta_m^k$; next, E_{jj} is the diagonal matrix with diagonal $\{0,\dots,0,1,0,\dots,0,-1\}$, where the 1 occupies the jth place. A fixture of the developments to come will be the operator $\sum_m s_m\frac{\partial}{\partial s_m}+\frac{n+1}{2}+i\lambda$. It is convenient to set

$$\mu = \frac{n+1}{2}+i\lambda, \qquad D_\mu = \left\langle s,\frac{\partial}{\partial s}\right\rangle + \mu. \tag{4.2}$$

Applying (1.10), (1.11), one finds the equations

$$d\pi_{i\lambda,\varepsilon}(E_{j,n+1}) = -\frac{\partial}{\partial s_j}, \qquad d\pi_{i\lambda,\varepsilon}(E_{n+1,k}) = s_k D_\mu,$$

$$d\pi_{i\lambda,\varepsilon}(E_{jk}) = -s_k\frac{\partial}{\partial s_j} \quad (j,k,n+1 \text{ distinct}), \qquad d\pi_{i\lambda,\varepsilon}(E_{jj}) = -D_\mu - s_j\frac{\partial}{\partial s_j}. \tag{4.3}$$

Note that, so far as the *formal* infinitesimal operators only are considered, there is no difference between the representations associated with the same value of λ but distinct values of ε: there is, of course, a considerable difference when the self-adjoint extensions of the operators under consideration are concerned. We also denote as $d\pi_{i\lambda,\varepsilon}$ the extension of this map to the enveloping algebra $\mathcal{U}(\mathfrak{g}_n)$ of $\mathfrak{g}_n$.

Under the assumption that not only some operator $\mathrm{Op}_{i\lambda,\varepsilon}(f)$ but also the result of its composition on the left by the image, under $d\pi_{i\lambda,\varepsilon}$, of any element of $\mathcal{U}(\mathfrak{g}_n)$, is a Hilbert–Schmidt endomorphism of the space $L^2(\mathbb{R}^n)$, one can, with the help of (1.26) and of the preceding equations, compute the symbol of the operator $d\pi_{i\lambda,\varepsilon}(X)\,\mathrm{Op}_{i\lambda,\varepsilon}(f)$ for any vector $X \in \mathfrak{g}$, getting after a trivial computation the set of relations (in which $j,k \neq n+1$)

$$d\pi_{i\lambda,\varepsilon}(E_{j,n+1})\,\mathrm{Op}_{i\lambda,\varepsilon}(f) = \mathrm{Op}_{i\lambda,\varepsilon}\left(-\frac{\partial f}{\partial s_j} + \mu\frac{\sigma_j}{1+\langle s,\sigma\rangle}f\right),$$

$$d\pi_{i\lambda,\varepsilon}(E_{n+1,k})\,\mathrm{Op}_{i\lambda,\varepsilon}(f) = \mathrm{Op}_{i\lambda,\varepsilon}\left(s_k\left\langle s,\frac{\partial f}{\partial s}\right\rangle + \mu\frac{s_k}{1+\langle s,\sigma\rangle}f\right),$$

$$d\pi_{i\lambda,\varepsilon}(E_{jk})\,\mathrm{Op}_{i\lambda,\varepsilon}(f) = \mathrm{Op}_{i\lambda,\varepsilon}\left(-s_k\frac{\partial f}{\partial s_j} + \mu\frac{s_k\sigma_j}{1+\langle s,\sigma\rangle}f\right),$$

$$d\pi_{i\lambda,\varepsilon}(E_{jj})\,\mathrm{Op}_{i\lambda,\varepsilon}(f) = \mathrm{Op}_{i\lambda,\varepsilon}\left(-\left\langle s,\frac{\partial f}{\partial s}\right\rangle - s_j\frac{\partial f}{\partial s_j} + \mu\frac{s_j\sigma_j - 1}{1+\langle s,\sigma\rangle}f\right). \tag{4.4}$$

Our main concern, in this section, has to do with the operators the symbols of which are integral powers of the function

$$\phi_{a,b}\big((x,\xi)^\bullet\big) = \langle a,x\rangle\langle b,\xi\rangle, \quad (x,\xi)^\bullet \in \mathcal{X}_n^\bullet, \tag{4.5}$$

or, in inhomogeneous coordinates, with $a = (a_1,\ldots,a_{n+1}) = (a_*,a_{n+1})$ and $b = (b_*,b_{n+1})$,

$$\phi_{a,b}(s,\sigma) = \frac{(a_{n+1}+\langle a_*,s\rangle)(b_{n+1}+\langle b_*,\sigma\rangle)}{1+\langle s,\sigma\rangle}. \tag{4.6}$$

It is assumed that a and b lie in $\mathbb{C}^{n+1}$: when these two vectors are real, this function has already been considered in (1.50). The case when $\langle a,b\rangle = 0$ will be of special interest. Since Eqs. (4.4) give in particular (setting $f = 1$) the symbols of the infinitesimal operators of the representation $\pi_{i\lambda,\varepsilon}$, one can verify that, *in this case*, the function $\phi_{a,b}$ is the symbol of the operator $\mu^{-1}d\pi_{i\lambda,\varepsilon}(X_{a,b})$, with

$$X_{a,b} = \sum_{(j,k)\neq(n+1,n+1)} a_k b_j E_{jk}. \tag{4.7}$$

We first make a quick study of the operator with symbol $\phi_{a,b}^p$ with $p = 0,1,\ldots$: we are more interested in the same symbols with $p = -1,-2,\ldots$, but this will require some preparation.

Even in the case when $p \in \mathbb{N}$, this symbol is associated to the function $(x, \xi) \mapsto (\langle a, x\rangle\langle b, \xi\rangle)^p$ on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, certainly not a bounded function so that Proposition 1.2 does not apply, even though the application of differential operators such as $x_j \frac{\partial}{\partial x_k}$ or $\xi_j \frac{\partial}{\partial \xi_k}$ does not make the symbol any worse. In this section, we shall sometimes extend the meaning of $\mathrm{Op}_{i\lambda,\varepsilon}$ beyond the domain in which full justifications have been carried, keeping in mind that the following basic property of the calculus should continue to hold: in the case when a symbol f_1 depends only on s, or when f_2 depends only on σ, the product $f_1 f_2$ must be the symbol of the composition $\mathrm{Op}_{i\lambda,\varepsilon}(f_1)\,\mathrm{Op}_{i\lambda,\varepsilon}(f_2)$; of course, the situation has to be reversed when dealing with the $\mathrm{Op}^{\vee}_{i\lambda,\varepsilon}$-calculus. Also, we shall take advantage of Eqs. (4.4).

First, let us deal with powers of the symbol $1 + \langle s, \sigma\rangle$.

Lemma 4.1. *With $\mu = \frac{n+1}{2} + i\lambda$, consider the operator $D_\mu = \langle s, \frac{\partial}{\partial s}\rangle + \mu$, an endomorphism of the space $H^\infty_{i\lambda,\varepsilon}$ of C^∞ vectors of the representation $\pi_{i\lambda,\varepsilon}$, and set*

$$\Lambda_p = \mathrm{Op}_{i\lambda,\varepsilon}\big(\big(1 + \langle s, \sigma\rangle\big)^{-p}\big). \tag{4.8}$$

For $p = 0, 1, \ldots,$ one has

$$\Lambda_p = \frac{D_\mu(D_\mu + 1)\cdots(D_\mu + p - 1)}{\mu(\mu + 1)\cdots(\mu + p - 1)}. \tag{4.9}$$

Proof. Set $D_0 = \langle s, \frac{\partial}{\partial s}\rangle$. From (4.4), applied with $f = 1$, one has

$$d\pi_{i\lambda,\varepsilon}\Big(\sum E_{jj}\Big) = -(n+1)D_0 - n\mu, \tag{4.10}$$

so that D_μ is indeed an endomorphism of the space $H^\infty_{i\lambda,\varepsilon}$. On the other hand, by (4.4) again, given any symbol f, the symbol of the operator $d\pi_{i\lambda,\varepsilon}(\sum E_{jj})\,\mathrm{Op}_{i\lambda,\varepsilon}(f)$ is the function

$$-(n+1)\left\langle s, \frac{\partial f}{\partial s}\right\rangle + \mu\frac{\langle s, \sigma\rangle - n}{1 + \langle s, \sigma\rangle} f; \tag{4.11}$$

this leads to the equation

$$d\pi_{i\lambda,\varepsilon}\Big(\sum E_{jj}\Big)\Lambda_p = \big(\mu + (n+1)p\big)\Lambda_p - (n+1)(p+\mu)\Lambda_{p+1}, \tag{4.12}$$

which can also be written, using (4.10), as

$$\left(\left\langle s, \frac{\partial f}{\partial s}\right\rangle + \mu + p\right)\Lambda_p = (\mu + p)\Lambda_{p+1}, \tag{4.13}$$

from which (4.9) follows. □

We need to introduce some right inverse D_μ^{-1} of D_μ and more generally, for later purposes, a resolvent $(D_\mu - \rho)^{-1}$ of this operator: we define it by the equation

$$\big((D_\mu - \rho)^{-1}u\big)(s) = \int_0^1 u(ts)t^{\mu-\rho-1}\,dt, \tag{4.14}$$

and observe first (this is one of the so-called Hardy's inequalities) that, when $\operatorname{Re}\rho < \frac{1}{2}$, it extends as a bounded operator on $L^2(\mathbb{R}^n)$: indeed, it suffices to write

$$\begin{aligned}\big(v \mid \big((D_\mu - \rho)^{-1}u\big)\big) &= -\int_{\mathbb{R}^n} \bar{v}(s)\,ds \int_1^\infty u(ts)t^{\mu-\rho-1}\,dt \\ &= -\int_1^\infty t^{\mu-\rho-1}\,dt \int_{\mathbb{R}^n} \bar{v}(s)u(ts)\,ds,\end{aligned} \tag{4.15}$$

where the last integral, by the Cauchy–Schwarz inequality, is bounded by $t^{-\frac{n}{2}}\|v\|_{L^2(\mathbb{R}^n)}\|u\|_{L^2(\mathbb{R}^n)}$. Note that, even for large values of $\operatorname{Re}\rho$, the integral (4.14) makes sense when u is flat enough at $s=0$.

Lemma 4.2. *Given a and $b \in \mathbb{C}^{n+1}$, the operators with symbols $(s,\sigma) \mapsto a_{n+1} + \langle a_*, s\rangle$ and $(s,\sigma) \mapsto \mu\frac{b_{n+1}+\langle b_*,\sigma\rangle}{1+\langle s,\sigma\rangle}$ are, respectively, the operator S_a of multiplication by the function $s \mapsto a_{n+1} + \langle a_*, s\rangle$ and the operator*

$$T_b = b_{n+1}D_\mu - \left\langle b_*, \frac{\partial}{\partial s}\right\rangle, \tag{4.16}$$

where $D_\mu = \langle s, \frac{\partial}{\partial s}\rangle + \mu$. In the case when $\langle a, b\rangle = 0$, the two operators under consideration generate a Lie algebra isomorphic to that of the one-dimensional affine group.

Proof. From (4.4), then (4.3),

$$\operatorname{Op}_{i\lambda,\varepsilon}\left(\mu\frac{\langle b_*, \sigma\rangle}{1+\langle s,\sigma\rangle}\right) = \sum_{j=1}^n b_j\,d\pi_{i\lambda,\varepsilon}(E_{j,n+1}) = -\left\langle b_*, \frac{\partial}{\partial s}\right\rangle; \tag{4.17}$$

on the other hand, it has been found in Lemma 4.1 that $\operatorname{Op}_{i\lambda,\varepsilon}(\frac{\mu}{1+\langle s,\sigma\rangle}) = D_\mu$. Next, it is immediate to verify that

$$[T_b, S_a] = b_{n+1}\langle a_*, s\rangle - \langle a_*, b_*\rangle = b_{n+1}S_a - \langle a, b\rangle. \qquad \square \tag{4.18}$$

We have recalled that the symbol of an operator such as $\operatorname{Op}_{i\lambda,\varepsilon}(f_1)\operatorname{Op}_{i\lambda,\varepsilon}(f_2)$ reduces to $f_1 f_2$ whenever the symbol f_1 on the left-hand side depends only on the variable s, or when the symbol f_2 on the right-hand side depends only on the variable σ. This has the consequence that, with the notation introduced in Proposition 4.3, one has

$$\mathrm{Op}_{i\lambda,\varepsilon}(\phi_{a,b}) = \mu^{-1} S_a T_b. \tag{4.19}$$

The operator T_b can be decomposed further as the product

$$T_b = \mathrm{Op}_{i\lambda,\varepsilon}\left(\frac{\mu}{1+\langle s,\sigma\rangle}\right)\mathrm{Op}_{i\lambda,\varepsilon}\big(b_{n+1} + \langle b_*,\sigma\rangle\big); \tag{4.20}$$

with the help of Lemma 4.1, this leads to the equation

$$T_b = D_\mu\,\mathrm{Op}_{i\lambda,\varepsilon}\big(b_{n+1} + \langle b_*,\sigma\rangle\big), \tag{4.21}$$

which can be inverted as

$$\mathrm{Op}_{i\lambda,\varepsilon}\big(b_{n+1} + \langle b_*,\sigma\rangle\big) = D_\mu^{-1} T_b. \tag{4.22}$$

In all that precedes, the condition $\langle a,b\rangle = 0$ was not needed. It is, however, needed in the following proposition, which stresses the "reproducing" property of the symbols $\phi_{a,b}$ under consideration.

Proposition 4.3. *Under the assumption that $\langle a,b\rangle = 0$, one has, for $p = 1,2,\ldots,$*

$$\big(\mathrm{Op}_{i\lambda,\varepsilon}(\phi_{a,b})\big)^p = \frac{\mu(\mu+1)\cdots(\mu+p-1)}{\mu^p}\,\mathrm{Op}_{i\lambda,\varepsilon}\big(\phi_{a,b}^p\big). \tag{4.23}$$

Proof. We abbreviate in this proof S_a and T_b as S and T. It is no loss of generality to assume that $b_{n+1} = 1$ (in the case when $b_{n+1} = 0$, one may use to that effect the covariance of the calculus), so as to simplify the commutation relation (4.18). This immediately leads to

$$\big(\mathrm{Op}_{i\lambda,\varepsilon}(\phi_{a,b})\big)^p = \big(\mu^{-1}ST\big)^p = \mu^{-p}S^pT(T+1)\cdots(T+p-1). \tag{4.24}$$

On the other hand, using the basic property of the calculus, Lemma 4.1 and (4.22), one obtains

$$\mathrm{Op}_{i\lambda,\varepsilon}\big(\phi_{a,b}^p\big) = S^p\,\frac{D_\mu(D_\mu+1)\cdots(D_\mu+p-1)}{\mu(\mu+1)\cdots(\mu+p-1)}\big[D_\mu^{-1}T\big]^p. \tag{4.25}$$

The equation to be shown thus reduces to

$$T(T+1)\cdots(T+p-1) = D_\mu(D_\mu+1)\cdots(D_\mu+p-1)\big[D_\mu^{-1}T\big]^p. \tag{4.26}$$

One first verifies the commutation relation

$$[D_\mu, T] = D_\mu - T, \tag{4.27}$$

from which one gets

$$(T+1)D_\mu = (D_\mu+1)T \tag{4.28}$$

and, by induction,

$$(T+k+1)(D_\mu+k)=(D_\mu+k+1)(T+k) \tag{4.29}$$

for $k=0,1,\ldots$. First simplifying the right-hand side of (4.26) as

$$(D_\mu+p-1)\cdots(D_\mu+1)T\big[D_\mu^{-1}T\big]^{p-1}, \tag{4.30}$$

we show by induction on k $(0\leqslant k\leqslant p-1)$ that it can also be written as

$$(D_\mu+p-1)\cdots(D_\mu+k+1)(T+k)\cdots T\big[D_\mu^{-1}T\big]^{p-k-1} \tag{4.31}$$

(an expression that reduces to the left-hand side of (4.26) when $k=p-1$): the step from k to $k+1$ is managed with the help of (4.29), writing

$$\begin{aligned}(D_\mu+k+1)(T+k)\cdots T &= (T+k+1)(D_\mu+k)(T+k-1)\cdots T\\ &= (T+k+1)(T+k)(D_\mu+k-1)(T+k-2)\cdots T\\ &\ \ \vdots\\ &= (T+k+1)\cdots(T+1)D_\mu. \qquad \square\end{aligned} \tag{4.32}$$

Corollary 4.4. *Let $a,b\in\mathbb{C}^{n+1}$ satisfy $\langle a,b\rangle=0$. When $f=\phi_{a,b}^p$, $p\in\mathbb{N}$, the equation $\mathrm{Op}^\vee_{i\lambda,\varepsilon}(f)=\mathrm{Op}_{i\lambda,\varepsilon}(J_{i\lambda,\varepsilon}f)$ from Corollary* 1.4 *extends, only replacing, in the expression* (1.52) *of the function $G_{i\lambda,\varepsilon}(\rho,\delta)$, ρ by p and δ by p* mod 2.

Proof. Assuming without loss of generality that $b_{n+1}=1$, we first note that $S^*_{\bar a}=S_a$ and $T^*_{\bar b}=1-T_b$, and that $(1-T_b)S_a=-S_aT_b$: as a consequence, starting from the equation $\mathrm{Op}_{i\lambda,\varepsilon}(\phi_{a,b})=\mu^{-1}S_aT_b$,

$$\mathrm{Op}^\vee_{i\lambda,\varepsilon}(\phi_{a,b})=\mathrm{Op}_{i\lambda,\varepsilon}(\phi_{\bar a,\bar b})^*=\bar\mu^{-1}(1-T_b),\qquad S_a=-\bar\mu^{-1}S_aT_b, \tag{4.33}$$

so that

$$\mathrm{Op}^\vee_{i\lambda,\varepsilon}(\phi_{a,b})=-\frac{\mu}{\bar\mu}\,\mathrm{Op}_{i\lambda,\varepsilon}(\phi_{a,b}). \tag{4.34}$$

We then obtain from Corollary 4.4 that

$$\begin{aligned}\mathrm{Op}^\vee_{i\lambda,\varepsilon}\big(\phi^p_{a,b}\big)=\mathrm{Op}_{i\lambda,\varepsilon}\big(\phi^p_{\bar a,\bar b}\big)^* &= \frac{\bar\mu^p}{\bar\mu\cdots(\bar\mu+p-1)}\big[\mathrm{Op}_{i\lambda,\varepsilon}(\phi_{\bar a,\bar b})^*\big]^p\\ &= (-1)^p\frac{\mu(\mu+1)\cdots(\mu+p-1)}{\bar\mu(\bar\mu+1)\cdots(\bar\mu+p-1)}\,\mathrm{Op}_{i\lambda,\varepsilon}\big(\phi^p_{a,b}\big).\end{aligned} \tag{4.35}$$

On the other hand, going back to (1.52), one finds

$$G_{i\lambda,\varepsilon}(p,p \bmod 2)=(-1)^p\frac{F(\mu)}{F(\bar\mu)} \tag{4.36}$$

with

$$F(\mu)=\frac{\Gamma(\frac{1-\mu+\varepsilon}{2})}{\Gamma(\frac{1-\mu-p+|\varepsilon-\delta|}{2})}\cdot\frac{\Gamma(\frac{\mu+p+|\varepsilon-\delta|}{2})}{\Gamma(\frac{\mu+\varepsilon}{2})}. \tag{4.37}$$

Since both numbers $p\pm|\varepsilon-\delta|-\varepsilon$ are non-negative even numbers, one may interpret each of the two factors above as a Pochhammer's symbol, which leads after a case-by-case computation to the equation

$$F(\mu)=(-1)^p\frac{\mu(\mu+1)\cdots(\mu+p-1)}{2^p}, \tag{4.38}$$

and to the identification of the coefficient in front of the right-hand side of (4.35) with $G_{i\lambda,\varepsilon}(p, p \bmod 2)$. □

As a preparation towards some calculations related to the composition of operators with certain special symbols, we compute the symbol of the resolvent operator $(D_\mu-\rho)^{-1}$. Since, according to definition (4.14), one has

$$(D_\mu-\rho)^{-1}=\int_0^1 t^{\langle s,\frac{\partial}{\partial s}\rangle+\mu-\rho-1}\,dt, \tag{4.39}$$

and since, from (1.28), the symbol in the $\mathrm{Op}_{i\lambda,\varepsilon}$-calculus of the operator $t^{\langle s,\frac{\partial}{\partial s}\rangle}$ is immediately seen to be the function

$$f_t(s,\sigma)=\frac{|1+\langle s,\sigma\rangle|_\varepsilon^\mu}{|1+t\langle s,\sigma\rangle|_\varepsilon^\mu}, \tag{4.40}$$

the symbol of the operator $(D_\mu-\rho)^{-1}$ is the function

$$h_\rho(s,\sigma)=\big|1+\langle s,\sigma\rangle\big|_\varepsilon^\mu\int_0^1\big|1+t\langle s,\sigma\rangle\big|_\varepsilon^{-\mu}t^{\mu-\rho-1}\,dt. \tag{4.41}$$

There is no need to display the elementary calculations, based on the splitting of the integral into two parts in the case when $\langle s,\sigma\rangle<-1$, which lead to the explicit formula

$$h_\rho(s,\sigma)=(\mu-\rho)^{-1}\big(1+\langle s,\sigma\rangle\big)^\mu{}_2F_1\big(\mu,\mu-\rho;\mu+1-\rho;-\langle s,\sigma\rangle\big) \tag{4.42}$$

when $\langle s,\sigma\rangle>-1$, and to

$$\begin{aligned}h_\rho(s,\sigma)={}&(-1)^\varepsilon\frac{\Gamma(\mu-\rho)\Gamma(1-\mu)}{\Gamma(1-\rho)}\big|\langle s,\sigma\rangle\big|^{\rho-\mu}\big|1+\langle s,\sigma\rangle\big|^\mu\\&+(1-\mu)^{-1}\big|\langle s,\sigma\rangle\big|^{\rho-\mu}\big|1+\langle s,\sigma\rangle\big|\\&\times{}_2F_1\big(\rho+1-\mu,1-\mu;2-\mu;1+\langle s,\sigma\rangle\big)\end{aligned} \tag{4.43}$$

when $\langle s,\sigma\rangle < -1$. Note that, when $\rho = 1, 2, \dots$, $h_\rho(s,\sigma)$ reduces to a polynomial of degree $\rho - 1$ in $\langle s,\sigma\rangle$, independent of ε. One may mention the following formula (more easily verified with the help of (4.4)): for $p = 0, 1, \dots,$

$$(D_\mu - 1)^{-1}(D_\mu - 2)^{-1}\cdots(D_\mu - p)^{-1} = \mathrm{Op}_{i\lambda,\varepsilon}\left(\frac{(1+\langle s,\sigma\rangle)^p}{(\mu-1)(\mu-2)\cdots(\mu-p)}\right); \tag{4.44}$$

the factors $(D_\mu - j)^{-1}$ on the left-hand side do not act within the space $L^2(\mathbb{R}^n)$, but their composition still makes sense if interpreted (using a decomposition into simple elements) as the sum

$$\sum_{j=1}^{p} \frac{(-1)^{p-j}}{(j-1)!(p-j)!}(D-j)^{-1}.$$

The following lemma, in which we allow $i\lambda$ to be replaced by a complex number no longer pure imaginary, will be needed soon. Note that, if $v \in \mathcal{S}(\mathbb{R}^n)$, the function $\rho \mapsto v^\flat_{\rho,\varepsilon}$ extends as a meromorphic function, valued in the space $C^\infty(\mathbb{R}^n)$, in the whole complex plane, with simple poles only at points $-(\frac{n+1}{2} + k)$, where $k = 0, 1, \dots$ and $k \equiv \varepsilon \bmod 2$: this also makes it possible to define the intertwining operator $\theta_{\rho,\varepsilon}$ in general.

Lemma 4.5. *On functions in $\mathbb{R}^n$ with a parity associated to δ, one has*

$$\theta_{i\lambda,\varepsilon} = \theta_{i\lambda+1,\varepsilon}\cdot\frac{1}{\pi}\cdot\frac{\Gamma(\frac{1-D_\mu+|\varepsilon-\delta|}{2})\Gamma(\frac{1+D_\mu+|\varepsilon-\delta|}{2})}{\Gamma(\frac{D_\mu+|\varepsilon-\delta|}{2})\Gamma(\frac{-D_\mu+|\varepsilon-\delta|}{2})}. \tag{4.45}$$

Proof. We may pretend that we are testing both sides of the identity to be proven on a given function $s \mapsto v^\flat(s)$ homogeneous of degree and parity $(-\frac{n}{2} - i\nu, \delta)$, keeping in mind, however, that we really deal with *nice* integral superpositions of such functions. Under the map (1.5), such a function $v^\flat$ lifts to $\mathbb{R}^{n+1}$ as the function

$$v(x) = |x_{n+1}|_{|\varepsilon-\delta|}^{-\frac{1}{2}+i(\nu-\lambda)} v^\flat(x_*), \tag{4.46}$$

the Fourier transform of which is

$$\begin{aligned}\big(\mathcal{F}^{(n)}v\big)(x) = (-1)^{|\varepsilon-\delta|}\pi^{i(\lambda-\nu)}\frac{\Gamma(\frac{1}{4}+\frac{i(\nu-\lambda)}{2}+\frac{|\varepsilon-\delta|}{2})}{\Gamma(\frac{1}{4}-\frac{i(\nu-\lambda)}{2}+\frac{|\varepsilon-\delta|}{2})}|x_{n+1}|_{|\varepsilon-\delta|}^{-\frac{1}{2}-i(\nu-\lambda)} \\ \times\big(\mathcal{F}^{(n-1)}v^\flat\big)(x_*).\end{aligned} \tag{4.47}$$

It has been deemed prudent, here, to emphasize, as a superscript, the dimension of the Fourier transform under consideration: the same precaution will be used, presently, in connection with the intertwining operators or quantizing maps Op to be considered, as well as when using the constants $C_{i\lambda,\varepsilon}$ as defined in (1.24). Then, (1.16) yields

$$\big(\theta_{i\lambda,\varepsilon}v^\flat\big)(\sigma) = (-1)^{|\varepsilon-\delta|}\pi^{i(\lambda-\nu)}\frac{\Gamma(\frac{1}{4}+\frac{i(\nu-\lambda)}{2}+\frac{|\varepsilon-\delta|}{2})}{\Gamma(\frac{1}{4}-\frac{i(\nu-\lambda)}{2}+\frac{|\varepsilon-\delta|}{2})}\big(\mathcal{F}^{(n-1)}v^\flat\big)(\sigma), \tag{4.48}$$

from which Eq. (4.45) follows, once it has been observed that, on a function such as $v^\flat$, D_μ reduces to the multiplication by $\frac{1}{2} + i(\lambda - \nu)$. □

We shall also need the following lemma, in which the variable $s \in \mathbb{R}^n$ is split as $s = (s_1, s_*) \in \mathbb{R} \times \mathbb{R}^{n-1}$: note that the subscript $*$ here concerns the *last* $n-1$ variables.

Lemma 4.6. *If $u = u(s_1, s_*)$ is homogeneous of degree and parity $(\frac{1-n}{2} - i\nu, \delta)$ with respect to the variables s_*, one has*

$$\big(\theta^{(n)}_{i\lambda,\varepsilon} u\big)(\sigma_1, \sigma_*) = (-1)^{|\varepsilon-\delta|} C^{(1)}_{i(\nu-\lambda),|\varepsilon-\delta|} \big(\big(\theta^{(1)}_{i(\lambda-\nu),|\varepsilon-\delta|} \otimes \theta^{(n-1)}_{i\lambda+\frac{1}{2},\varepsilon}\big) u\big)(\sigma_1, \sigma_*). \tag{4.49}$$

Proof. Even though the genuine proof depends again on the lifting, depending on $(i\lambda, \varepsilon)$, from functions on $\mathbb{R}^n$ to homogeneous functions on $\mathbb{R}^{n+1}$, we shall satisfy ourselves with a shorter formal proof based on (1.23). Starting from this equation, performing the change of variables $s_* \mapsto (1 + s_1\sigma_1)s_*$ and using the homogeneity, one obtains

$$\begin{aligned}\big(\theta^{(n)}_{i\lambda,\varepsilon} u\big)(\sigma_1, \sigma_*) &= C^{(n)}_{i\lambda,\varepsilon} \int_{-\infty}^{\infty} |1 + s_1\sigma_1|^{i(\lambda-\nu)-1}_{|\varepsilon-\delta|} ds_1 \int_{\mathbb{R}^{n-1}} \big|1 + \langle s_*, \sigma_*\rangle\big|_{\varepsilon}^{-\frac{n+1}{2}+i\lambda} u(s_1, s_*)\, ds_* \\ &= \frac{C^{(n)}_{i\lambda,\varepsilon}}{C^{(1)}_{i(\nu-\lambda),|\varepsilon-\delta|} C^{(n-1)}_{i\lambda-\frac{1}{2},\varepsilon}} \big(\big(\theta^{(1)}_{i(\lambda-\nu),|\varepsilon-\delta|} \otimes \theta^{(n-1)}_{i\lambda-\frac{1}{2},\varepsilon}\big) u\big)(\sigma_1, \sigma_*).\end{aligned} \tag{4.50}$$

From Lemma 4.5, we may substitute for $\theta^{(n-1)}_{i\lambda-\frac{1}{2},\varepsilon}$ the product of $\theta^{(n-1)}_{i\lambda+\frac{1}{2},\varepsilon}$ by the number

$$\frac{1}{\pi} \frac{\Gamma(\frac{1}{2} + \frac{i(\nu-\lambda)}{2} + \frac{|\varepsilon-\delta|}{2})\Gamma(\frac{1}{2} - \frac{i(\nu-\lambda)}{2} + \frac{|\varepsilon-\delta|}{2})}{\Gamma(-\frac{i(\nu-\lambda)}{2} + \frac{|\varepsilon-\delta|}{2})\Gamma(\frac{i(\nu-\lambda)}{2} + \frac{|\varepsilon-\delta|}{2})} = (-1)^{|\varepsilon-\delta|} C^{(1)}_{i(\lambda-\nu),|\varepsilon-\delta|} C^{(1)}_{i(\nu-\lambda),|\varepsilon-\delta|}, \tag{4.51}$$

which leads to the result indicated, if one notes also that

$$C^{(n)}_{i\lambda,\varepsilon} = C^{(n-1)}_{i\lambda-\frac{1}{2},\varepsilon}. \qquad \square \tag{4.52}$$

Recalling our present interest in symbols such as $\phi^p_{a,b}$, with $p \in \mathbb{Z}$ and $\langle a, b\rangle = 0$, we first show how, using covariance, the analysis of operators with such a kind of symbols can be reduced to a seemingly specialized class.

Lemma 4.7. *Let $a, b \in \mathbb{C}^{n+1}$ be such that $\langle a, b\rangle = 0$, $\langle a, \bar{b}\rangle \neq 0$. There exists a matrix $g \in G_n$ such that the vectors $g'a$ and $g^{-1}b$ are both linear combinations, with complex coefficients, of the vectors e_1 and e_{n+1} from the canonical basis of $\mathbb{R}^{n+1}$.*

Proof. There is no change in the statement if one substitutes for b any multiple νb with $\nu \in \mathbb{C}^\times$, so that we may assume, without loss of generality, that $\langle a, \bar{b}\rangle = 2i$. Let us decompose the complex vectors involved as $a = p + iq$, $b = r + is$, so that

$$\langle p, r\rangle = \langle q, s\rangle = 0, \qquad \langle q, r\rangle = 1, \qquad \langle p, s\rangle = -1. \tag{4.53}$$

Since the matrix $\begin{pmatrix} \langle q,r\rangle & -\langle p,r\rangle \\ \langle q,s\rangle & -\langle p,s\rangle \end{pmatrix}$ is the identity matrix, it is possible to find a positive-definite symmetric $(n+1)\times(n+1)$ matrix h such that $hr=q$, $hs=-p$. Let $h^{\frac{1}{2}}$ be the positive-definite square-root of h. As

$$\begin{aligned} \big\|h^{-\frac{1}{2}}q\big\|^2 &= \big\langle h^{-1}q,q\big\rangle = \langle r,q\rangle = 1, \\ \big\|h^{-\frac{1}{2}}p\big\|^2 &= \big\langle h^{-1}p,p\big\rangle = -\langle s,p\rangle = 1, \\ \big\langle h^{-\frac{1}{2}}q,h^{-\frac{1}{2}}p\big\rangle &= \big\langle h^{-1}q,p\big\rangle = \langle r,p\rangle = 0, \end{aligned} \tag{4.54}$$

one can find $\omega\in \mathrm{O}(n+1)$ such that $\omega e_1 = \pm h^{-\frac{1}{2}}q$, $\omega_2 = h^{-\frac{1}{2}}p$: assuming $n+1\geqslant 3$ (if $n=1$, the lemma is trivial), one may take for ω a rotation matrix. Then, setting $g=h^{-\frac{1}{2}}\omega$, one has $p=g'^{-1}e_1$, $q=g'^{-1}e_{n+1}$ and $r=h^{-1}q=ge_{n+1}$, finally $s=-h^{-1}p=-ge_1$, so that

$$g'a = e_1 + ie_{n+1}, \qquad g^{-1}b = -ie_1 + e_{n+1}. \qquad \square \tag{4.55}$$

As made possible by the lemma that precedes, we now specialize to the case when the symbols $\phi^p_{a,b}$, $p\in\mathbb{Z}$, to be considered together with their integral superpositions, all correspond to the case when a and b are linear combinations of e_1 and e_{n+1}: in this way, the situation is, up to some point, reduced to that obtained when $n=1$. Not quite, though, in view of the ever-present occurrence of the operator $D_\mu = \langle s,\frac{\partial}{\partial s}\rangle + \frac{n+1}{2} + i\lambda$. However, in this case, setting $s=(s_1,s_*)$ with $s_*=(s_2,\dots,s_n)$, the only operators we shall have to deal with commute with the partial Euler operator $\langle s_*,\frac{\partial}{\partial s_*}\rangle + \frac{n-1}{2}$ (the extra constant makes i times this operator a self-adjoint operator on $L^2(\mathbb{R}^{n-1})$): it is thus possible to decompose functions $u=u(s)$ as integrals of functions $u_{*,i\nu,\delta}$ homogeneous of degree and parity $(\frac{1-n}{2}-i\nu,\delta)$ with respect to the variables s_* only. On such a function, the operator D_μ reduces to $s_1\frac{d}{ds_1}+1+i(\lambda-\nu)$: this is just the analogue of the operator D_μ in a one-dimensional pseudodifferential calculus $\mathrm{Op}^{(1)}_{i(\lambda-\nu),\varepsilon}$. The recipe for reducing our present analysis to the one-dimensional case thus essentially calls for replacing λ by $\lambda-\nu$.

To be more specific, let us recall some facts relative to the discrete terms of the decomposition of $L^2(G_1/H_1)$. With the help of the (singular) coordinates (s_1,σ_1) on $\mathcal{X}_1^\bullet$ introduced in (1.25), we associate to each complex number $z\in\Pi$, the upper half-plane, the function ϕ_z such that

$$\phi_z(s_1,\sigma_1) = \frac{(s_1-\bar z)(1+\bar z\sigma_1)}{1+s_1\sigma_1}. \tag{4.56}$$

An alternative expression, in terms of the homogeneous coordinates (x,ξ) (with $\langle x,\xi\rangle=1$) of the point of $\mathcal{X}_1^\bullet$ considered, is

$$\phi_z(s_1,\sigma_1) = \langle a,x\rangle\langle b,\xi\rangle \quad \text{with } a=\begin{pmatrix}1\\ -\bar z\end{pmatrix},\ b=\begin{pmatrix}\bar z\\ 1\end{pmatrix}, \tag{4.57}$$

an expression which may be compared to (2.6).

Given $k=0,1,\dots$, denote as E_{k+1} the closed subspace of $L^2(\mathcal{X}_1^\bullet)=L^2(\mathbb{R}^2;\frac{ds\,d\sigma}{(1+s\sigma)^2})$ generated by the functions ϕ_z^{-k-1} with $z\in\Pi$: this is an irreducible space of the quasiregular representation of $G_1 = SL(2,\mathbb{R})$ in $L^2(\mathcal{X}_1^\bullet)$. It makes up half the eigenspace of Δ_1 for the eigenvalue $-k(k+1)$: the other half is obtained with the help of the similar functions related to the lower half-plane. On the other hand, let us recall that the representation π_{2k+2} taken from the holomorphic discrete series of G_1 can be realized in the space $\mathcal{D}_{2k+2}$ consisting of all holomorphic functions f on Π such that

$$\|f\|_{2k+2}^2 = \int_{\Pi} \left|f(z)\right|^2 (\operatorname{Im} z)^{2k+2}\, d\mu(z) < \infty \tag{4.58}$$

with $d\mu(z) = (\operatorname{Im} z)^{-2} d\operatorname{Re} z \wedge d\operatorname{Im} z$. One has

$$\left(\pi_{2k+2}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) f\right)(z) = (-cz+a)^{-2k-2} f\left(\frac{dz-b}{-cz+a}\right). \tag{4.59}$$

One may then recall [18, Proposition 2.2] the following. Set $\alpha_{k+1} = 2^{-2k}\binom{2k}{k}\pi^2$, and define the operator T_{k+1} by

$$(T_{k+1}h)(z) = \alpha_{k+1}^{-1} \int_{\mathcal{X}_1^\bullet} h(s_1,\sigma_1)\bar{\phi}_z^{-k-1}(s_1,\sigma_1) \frac{ds_1\, d\sigma_1}{(1+s_1\sigma_1)^2} \tag{4.60}$$

for every $h \in L^2(\mathcal{X}_1^\bullet)$ and $z \in \Pi$. Then, the operator $(\frac{(2k+1)\alpha_{k+1}}{4\pi})^{\frac{1}{2}} T_{k+1}$ is an isometry from E_{k+1} onto $\mathcal{D}_{2k+2}$. It acts as an intertwining operator between the quasiregular action of G_1 in E_{k+1} and the representation π_{2k+2} of G_1 in $\mathcal{D}_{2k+2}$. Its inverse is given by the formula

$$h(s_1,\sigma_1) = \frac{2k+1}{4\pi} \int_{\Pi} (T_{k+1}h)(z)\phi_z^{-k-1}(s_1,\sigma_1)(\operatorname{Im} z)^{2k+2}\, d\mu(z). \tag{4.61}$$

It has been shown in [18, Proposition 2.2] that the Hilbert sum of the spaces E_{k+1} is an algebra for the sharp composition of symbols, the sharp products expressing themselves in terms of Rankin–Cohen brackets of the T_{k+1}-transforms of the terms from the decompositions of the two symbols under consideration.

If a symbol f lies in E_{k+1}, so that it is an integral superposition of symbols $(\langle a, x\rangle\langle b,\xi\rangle)^{-k-1}$ with $a = \binom{1}{-\bar{z}}$, $b = \binom{\bar{z}}{1}$, and where $x = \binom{x_1}{x_2}$, $\xi = \binom{\xi_1}{\xi_2}$, we can turn it to a symbol $\tilde{f}$ in the $\mathrm{Op}_{i\lambda,\varepsilon}$-calculus in n variables, substituting for the two-dimensional vectors above the $(n+1)$-dimensional ones

$$a = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -\bar{z} \end{pmatrix}, \qquad b = \begin{pmatrix} \bar{z} \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix}, \qquad \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{n+1} \end{pmatrix}. \tag{4.62}$$

Taking integral superpositions, with respect to $z \in \Pi$, of such symbols, we obtain symbols which can be written, in the (s,σ)-coordinates on $\mathcal{X}_n$, as

$$\tilde{f}(s,\sigma) = \left(\frac{1+\langle s,\sigma\rangle}{1+s_1\sigma_1}\right)^{k+1} f(s_1,\sigma_1). \tag{4.63}$$

As a final topic in this paper, we analyse the operator with symbol $\tilde{f}$. In view of the different kind of dependence of the latter with respect to the two groups of variables involved, one may start from a decomposition of the function $u = u(s_1, s_*)$ to which the operator is applied into homogeneous components.

Proposition 4.8. *Let $n \geqslant 2$, and assume that $\tilde{f}$ is given by* (4.63). *On functions of $s = (s_1, s_*)$ homogeneous of degree and parity $(\frac{1-n}{2} - i\nu, \delta)$ with respect to the variables s_*, one has*

$$\big(\mathrm{Op}^{(n)}_{i\lambda,\varepsilon}(\tilde{f})u\big)(s) = \frac{\Gamma(\frac{n+1}{2}+i\lambda)}{\Gamma(\frac{n-1}{2}+i\lambda-k)} \frac{\Gamma(i(\lambda-\nu)-k)}{\Gamma(i(\lambda-\nu)+1)} \mathrm{Op}^{(1)}_{i(\lambda-\nu),|\varepsilon-\delta|}(f) \quad \big(s_1 \mapsto u(s_1,s_*)\big). \tag{4.64}$$

Proof. Changing $\sigma_* = (\sigma_2,\dots,\sigma_n)$ to $(1+s_1\sigma_1)\sigma_*$ in the integral (1.28), and using the fact that the function $(\theta^{(n)}_{i\lambda,\varepsilon}u)(\sigma_1,\sigma_*)$ is homogeneous of degree and parity $(\frac{1-n}{2}+i\nu,\delta)$ with respect to σ_*, we obtain

$$\begin{aligned}\big(\mathrm{Op}^{(n)}_{i\lambda,\varepsilon}(\tilde{f})u\big)(s) = (-1)^\varepsilon C^{(n)}_{-i\lambda,\varepsilon} \int f(s_1,\sigma_1)|1+s_1\sigma_1|^{-1+i(\nu-\lambda)}_{|\varepsilon-\delta|}\big[1+\langle s_*,\sigma_*\rangle\big]^{k+1} \\ \times \big|1+\langle s_*,\sigma_*\rangle\big|_\varepsilon^{-\frac{n+1}{2}-i\lambda}\big(\theta^{(n)}_{i\lambda,\varepsilon}u\big)(\sigma_1,\sigma_*)\,d\sigma_1\,d\sigma_*.\end{aligned} \tag{4.65}$$

Expressing $(\theta^{(n)}_{i\lambda,\varepsilon}u)(\sigma_1,\sigma_*)$ with the help of Lemma 4.6, one may interpret this as

$$\frac{(-1)^\varepsilon C^{(n)}_{-i\lambda,\varepsilon}}{(-1)^{|\varepsilon-\delta|}C^{(1)}_{-i(\lambda-\nu),|\varepsilon-\delta|}\cdot(-1)^\varepsilon C^{(n-1)}_{-i\lambda-\frac12,\varepsilon}} \cdot (-1)^{|\varepsilon-\delta|} C^{(1)}_{i(\nu-\lambda),|\varepsilon-\delta|} \tag{4.66}$$

$$\times \big(\mathrm{Op}^{(1)}_{i(\lambda-\nu),|\varepsilon-\delta|}(f) \otimes \big(\mathrm{Op}^{(n-1)}_{i\lambda+\frac12,\varepsilon}\big(\big[1+\langle s_*,\sigma_*\rangle\big]^{k+1}\big)\big)u\big)(s_1,s_*). \tag{4.67}$$

Now, the constant above reduces to 1 in view of (4.52). On the other hand, the operator with symbol $[1+\langle s_*,\sigma_*\rangle]^{k+1}$ has been made explicit in (4.44): note that $\mu = \frac{n+1}{2}+i\lambda$ does not change if the pair $(n, i\lambda)$ is replaced by $(n-1, i\lambda+\frac12)$ and that, in our case, $\langle s_*, \frac{\partial}{\partial s_*}\rangle + \mu$ reduces to $1+i(\lambda-\nu)$, which leads to the result indicated. □

Remark. The operator under consideration is not bounded in $L^2(\mathbb{R}^n)$ in view of the pole at $\nu = \lambda$ of the second Gamma factor on top of the first line of the right-hand side of (4.64): but it becomes bounded when composed with the spectral projection, relative to the self-adjoint operator $i(\langle s_*, \frac{\partial}{\partial s_*}\rangle + \frac{n-1}{2})$, corresponding to the complementary, in the real line, of any neighborhood of the point λ.

The following proposition extends Proposition 4.3 to negative integral exponents: let us warn the reader that, though a formal proof, shorter than the one developed below, can be obtained as a consequence of equations (4.4), it is only as an application of Proposition 4.8 that a meaning is given to the operator with symbol $\phi_{a,b}^{-p}$, and that it would be just as much work to extend to this case the validity of the quoted equations.

Proposition 4.9. *Assume $n \geqslant 2$. Let $a, b \in \mathbb{C}^{n+1}$ be such that $\langle a, b\rangle = 0$, $\langle a, \bar{b}\rangle \neq 0$. Recalling that $\phi_{a,b}$ has been defined in* (4.6)*, one has for $p = 0, 1, \ldots$ the equation*

$$\big(\mathrm{Op}_{i\lambda,\varepsilon}(\phi_{a,b})\big)^{-p} = \frac{\mu^p}{(\mu-1)\cdots(\mu-p)} \mathrm{Op}_{i\lambda,\varepsilon}\big(\phi_{a,b}^{-p}\big). \tag{4.68}$$

Proof. According to Lemma 4.7, it is no loss of generality to assume that $a = e_1 + ie_{n+1}$, $b = -ie_1 + e_{n+1}$, in which case, with the notation in (4.63), one has $\phi_{a,b}^{-p} = \tilde{f}_p$ if one sets

$$f_p(s_1, \sigma_1) = \left(\frac{(s_1 + i)(1 - i\sigma_1)}{1 + s_1\sigma_1}\right)^{-p}. \tag{4.69}$$

Our aim is to prove the equation

$$\mathrm{Op}_{i\lambda,\varepsilon}\big(\phi_{a,b}^{-p-1}\big) = \frac{\mu - p - 1}{\mu} \mathrm{Op}_{i\lambda,\varepsilon}\big(\phi_{a,b}^{-1}\big)\, \mathrm{Op}_{i\lambda,\varepsilon}\big(\phi_{a,b}^{-p}\big), \tag{4.70}$$

using the equations (from Proposition 4.8)

$$\begin{aligned}\big(\mathrm{Op}_{i\lambda,\varepsilon}^{(n)}\big(\phi_{a,b}^{-p}\big)u\big)(s) &= \frac{\Gamma(\frac{n+1}{2} + i\lambda)}{\Gamma(\frac{n+1}{2} + i\lambda - p)} \frac{\Gamma(i(\lambda - \nu) - p + 1)}{\Gamma(i(\lambda - \nu) + 1)} \\ &\quad \times \mathrm{Op}_{i(\lambda-\nu),|\varepsilon-\delta|}^{(1)}(f_p)\big(s_1 \mapsto u(s_1, s_*)\big),\end{aligned} \tag{4.71}$$

valid when applied to functions $u = u(s_1, s_*)$ which are homogeneous of degree and parity $(\frac{1-n}{2} - i\nu, \delta)$ with respect to the variables s_*: the formula reduces to a formula in the one-dimensional $\mathrm{Op}_{i(\lambda-\nu),|\varepsilon-\delta|}^{(1)}$-calculus, to wit

$$f_p \,\#\, f_1 = \frac{i(\lambda - \nu)}{i(\lambda - \nu) - p} f_{p+1}. \tag{4.72}$$

Of course, when $n = 1$, symbols such as f_p with $p = 1, 2, \ldots$ are square-integrable, so that the composition is easier to analyze. A detailed proof of (4.72) is to be found in [18, Proposition 4.1], but here is some help towards sorting-out the notation: there, (s, σ) was denoted as $(s, -t^{-1})$ so that f_p would have been denoted as $(-1)^p g_i^p$; finally, only the case when $|\varepsilon - \delta| = 0$ was explicitly considered in this reference, but no change whatsoever occurs when dealing only with symbols such as f_p, taken from the discrete spaces of the decomposition of $L^2(G_1/H_1)$. □

Remark. More generally, with the help of the results of [18, Proposition 4.1], together with Proposition 4.8, one can make a composition such as $\tilde{f}_1 \,\#\, \tilde{f}_2$, with $f_1 \in E_{k_1+1}$ and $f_2 \in E_{k_2+1}$, fully explicit. We may come back to the more general composition problem at some later time.

Let us just mention, without (the lengthy) proof, the following result, an analogue of the last proposition, concerned this time with symbols that occur in the continuous part of the decomposition of $L^2(G_n/H_n)$.

Proposition 4.10. *Let $a, b \in \mathbb{R}^{n+1}$ satisfy $\langle a, b\rangle = 0$. Set $R_{a,b} = i\mu\,\mathrm{Op}_{i\lambda,\varepsilon}(\phi_{a,b})$: this is an (unbounded) self-adjoint operator in $L^2(\mathbb{R}^n)$ with a purely continuous spectrum, to wit the real line. Denote as $(\Pi_{a,b})_\pm$ the projection operators corresponding to the positive and negative parts of the spectrum of $R_{a,b}$, and set, with $\rho \in \mathbb{C}, \operatorname{Re}\rho = -\frac{n}{2}, \delta = 0$ or 1,*

$$(R_{a,b})_\pm = \pm R_{a,b}(\Pi_{a,b})_\pm, \qquad |R_{a,b}|_\delta^\rho = (R_{a,b})_+^\rho + (-1)^\delta (R_{a,b})_-^\rho. \tag{4.73}$$

Then, one has

$$\mathrm{Op}_{i\lambda,\varepsilon}\big(|\phi_{a,b}|_\delta^\rho\big) = (-1)^\varepsilon i^\delta 2^{-\rho} \frac{\Gamma(\frac{\mu+\varepsilon}{2})}{\Gamma(\frac{1-\mu+\varepsilon}{2})} \frac{\Gamma(\frac{1-\mu-\rho+|\varepsilon-\delta|}{2})}{\Gamma(\frac{\mu+\rho+|\varepsilon-\delta|}{2})} |R_{a,b}|_\delta^\rho. \tag{4.74}$$

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