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Connected Linear Groups as Differential Galois Groups

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1. INTRODUCTION

In this paper we give a proof of the following

THEOREM 1.1. *Let C be an algebraically closed field of characteristic zero, G a connected linear algebraic group defined over C, and k a differential field containing C as its field of constants and of finite, nonzero transcendence degree over C; then G can be realized as the Galois group of a Picard–Vessiot extension of k.*

Previous work by Kovacic [17, 18] reduced this problem to the case of powers of a simple connected linear algebraic group. Our contribution is to show that one is able to realize any connected semisimple group as a Galois group and, when $k = C(x)$, $x' = 1$, to control the number and types of singularities when one constructs a system $Y' = AY$, $A \in M_n(C(x))$, realizing an arbitrary connected linear algebraic group as its

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Galois group. More precisely, we give a constructive, purely algebraic proof of the following. Let G be a connected linear algebraic group and let R_u be its unipotent radical and P a Levi factor [27]. The group $R_u/(R_u, R_u)$, where (R_u, R_u) is the (closed) commutator subgroup, is a commutative unipotent group and so is isomorphic to a vector group C^n. The group P acts on R_u by conjugation and this action factors to an action on $R_u/(R_u, R_u)$. Therefore we may write $R_u/(R_u, R_u) = U_1^{n_1} \oplus \cdots \oplus U_s^{n_s}$, where each U_i is an irreducible P-module. We shall assume that U_1 is the trivial one-dimensional P-module (and so allow the possibility that $n_1 = 0$). We may write $P = T \cdot H$, where T is a torus and H is a semisimple group. Let $m_i = n_i$ if the action of H on U_i is trivial and let $m_i = n_i + 1$ if the action of H on U_i is nontrivial. Let $N = 0$ if H is trivial and $N = 1$ if H is nontrivial. We define the *defect* $d(G)$ of G to be the number n_1 and the *excess* $e(G)$ of G to be $\max\{N, m_2, \dots, m_s\}$. We note that any two Levi factors are conjugate so that these numbers are independent of the choice of P. Furthermore, one can show that $d(G)$ is the dimension of $R_u/(G, R_u)$ (see the Appendix).

THEOREM 1.2. *Let G be a connected linear algebraic group defined over an algebraically closed field C of characteristic zero. Then G is the Galois group of a Picard–Vessiot extension of $C(x)$ corresponding to a system of the form*

$$Y' = \left(\frac{A_1}{x - \alpha_1} + \dots + \frac{A_{d(G)}}{x - \alpha_{d(G)}} + A_\infty \right) Y$$

where A_i, $i = 1, \dots, d(G)$, are constant matrices and A_∞ is a matrix with polynomial entries of degree at most $e(G)$. In particular, the only possible singularities of this system are $d(G)$ regular singular points in the finite plane and a (possibly irregular) singular point at infinity.

For example, this result implies that for any connected reductive group G, there exists constant matrices A and B such that G is the Galois group of an equation of the form $Y' = (A + xB)Y$.

We now give a brief history of work on the inverse problem in differential Galois theory. An early contribution to this problem is due to Bialynicki-Birula [3] who showed that, for any differential field k of characteristic zero with algebraically closed field of constants C, if the transcendence degree of k over C is finite and nonzero then any connected nilpotent group is a Galois group over k. This result was generalized by Kovacic, who showed that the same is true for any connected solvable group. In [17, 18] Kovacic introduced powerful machinery to solve the inverse problem. In particular, he made extensive use of the logarithmic derivative (see below) and he developed an inductive technique that gave criteria to lift a solution of the inverse problem for G/R_u to a solution for the full group G. Using this, Kovacic showed that to give a complete solution of the inverse problem, one needed only solve the problem for reductive groups (note that G/R_u is reductive). He was able to solve the problem for tori and so could give a solution when G/R_u is such a group (i.e., when G is solvable). He also reduced the problem for reductive groups to the problem for powers of simple groups. We will use logarithmic derivatives and the inductive technique (in a very simple explicit form) below.

When one considers specific fields, more is known. If $K = \mathbf{C}\{x\}[x^{-1}]$, the quotient field of convergent series, Kovacic [17] showed that a necessary and sufficient condition for a connected solvable group G to be a Galois group over K is that the unipotent radical of the center of $G/(R_u, R_u)$ have dimension at most 1, where R_u is the unipotent radical of G. Using analytic techniques, Ramis showed that any connected semisimple group is a Galois group over K (cf. [28]). Recently, Ramis extended this result to show that a necessary and sufficient condition for a linear algebraic group to be a Galois group over K is that it have a *local Galois structure* (cf. [29]), a condition expressed in terms of the Lie algebra of the group. This condition was restated in [26] in more group theoretic terms: a necessary and sufficient condition for a linear algebraic group G to be a Galois group over K is that (i) G/G^0 is cyclic, (ii) $d(G^0) \leq 1$, and (iii) G/G^0 acts trivally on $R_u/(R_u, G^0)$. For connected groups this reduces to the condition that $d(G) \leq 1$. Ramis also shows how solving the inverse problem over K is equivalent to solving the inverse problem on the sphere where the differential equation has a regular singularity at 0 and an arbitrary singularity at ∞.

Tretkoff and Tretkoff [37] have shown that any linear algebraic group is a Galois group over $C(x)$ when $C = \mathbf{C}$, the field of complex numbers. Their result depends on the solution of a weak version of the 21st Hilbert Problem.[1] For arbitrary C, Singer [34] showed that a class of linear algebraic groups (including all connected groups and large classes of nonconnected linear algebraic groups) are Galois groups over $C(x)$. Singer's proof used the result of Tretkoff and Tretkoff and a transfer

[1] Tretkoff and Tretkoff needed the fact that given a finitely generated group H of matrices, there exists a homomorphism of the fundamental group of the sphere minus a finite set of points onto this group and that this representation is the monodromy representation of a Fuchsian equation. Letting one of the generators be the identity matrix, and mapping the homotopy classes of loops around distinct points to distinct generators, one can use the classical solutions of Plemelj or Birkhoff to produce a system with simple poles (they need one of the matrices to be diagonalizable to insure that all poles are simple yet a careful examination of their techniques shows that their results still yield a Fuchsian system, without any hypothesis on the matrices (cf. [1]). This is not in conflict with the recent work of Bolibruch [1].

principle to go from **C** to any algebraically closed field of characteristic zero. Recently, Magid [22] claimed to have proved that all connected groups having no subgroups of codimension one must be Galois groups over $C(x)$. In particular, this would have implied that any connected semisimple group not having $\mathrm{PSL}(2, C)$ as a quotient could be realized as a Galois group over $C(x)$. Regretably, Magid's proof is flawed.[2] Finally, Ramis [29] has shown that any linear algebraic group defined over **C** is the Galois group of a Picard−Vessiot extension of $\mathbf{C}(x)$. Ramis's proof relies on his solution of the inverse problem for the local field $\mathbf{C}\{x\}[x^{-1}]$ and a technique for gluing solutions of the local problem to form a solution of the global problem. Ramis is also able to bound the number and type of singularities.

Another approach to the inverse problem was given by Goldman and Miller. In [10], Goldman developed the notion of a generic differential equation with group G analogous to what E. Noether did for algebraic equations. He showed that many groups have such an equation. In his thesis [23], Miller developed the notion of a differentially Hilbertian differential field and gave a sufficient condition for the generic equation of a group to specialize over such a field to an equation having this group as Galois group. Regrettably, this condition gave a stronger hypothesis than in the analogous theory of algebraic equations. This condition made it difficult to apply the theory and Miller was unable to apply this to any groups that were not already known to be Galois groups.

Finally, many groups have been shown to appear as Galois groups for classical families of linear differential equations. The family of generalized hypergeometric equations has been particularly accessible to computation, either by algebraic methods as in Beukers and Heckmann [2], Katz [14], and Boussel [6], or by mixed analytic and algebraic methods as in Duval and Mitschi [8] or Mitschi [24, 25]. These equations in particular provide classical groups and the exceptional group G_2. Other examples were treated algorithmically, as in Duval and Loday-Richaud [7] or Ulmer and Weil [38] using the Kovacic algorithm for second order equations, or in Singer and Ulmer [35, 36] using a new algorithm for third order equations.

The rest of this paper is organized as follows. In Section 2 we recall some of the results of differential Galois theory and Kovacic's program for solving the inverse problem. In Section 3 we present proofs of the Theorems 1.1 and 1.2.

We thank J. Kovacic for pointing out mistakes in a previous attempt at proving Theorem 1.2 and for showing us that Theorem 1.2 implies Theo-

[2] We had previously given a "proof" of theorem 1.2 based on Magid's ideas and, in particular, on Proposition 7.13 of [22]. Kovacic [20] showed us a counterexample to this proposition invalidating our previous argument.

rem 1.1. We also thank Jean-Pierre Ramis for stimulating conversations concerning the inverse problem and for sharing his ideas with us.

2. DIFFERENTIAL GALOIS THEORY AND KOVACIC'S PROGRAM

The material of this section comes from [17] and [18]. Let k be a differential field of characteristic zero with algebraically closed field of constants C. Given a connected C-group G we wish to construct Picard–Vessiot extensions of k of the form $k(g)$ where g is a point of G. Such a field is called a *G-primitive extension* of k. We say that a G-primitive extension is *full* if g is a k-generic point of G. Let V be a finite dimensional faithful G-module. We denote by $\mathcal{G} \subset \mathrm{gl}(V)$ the Lie algebra of $G \subset \mathrm{GL}(V)$. For any field extension K of C we denote by $G(K)$ the group of K-points of G and by $\mathcal{G}_K$ the Lie algebra $\mathrm{Lie}_K(G) = \mathrm{Lie}_C(G) \otimes_C K$ of G over K.

The following summarizes some of the material found in Chapters I and II of [17].

PROPOSITION 2.1. *Let G, V, and $\mathcal{G}$ be as above.*

1. *If $k(g)$ is a G-primitive extension of k, then $g'g^{-1} \in \mathcal{G}_k$ and the Galois group H of $k(g)$ over k is an algebraic subgroup of G. The action of H on $k(g)$ is given by $g \mapsto gh$ for any C-rational point h of H. In particular, $k(g)$ is a full G-primitive extension if and only if its Galois group over k is G.*
2. *If $A \in \mathcal{G}_k \subset \mathrm{gl}(V) \otimes_C k$, then there exists a G-primitive extension $k(g)$ of k such that g is a solution of $Y' = AY$.*

Kovacic refers to the element $g'g^{-1}$ as the *logarithmic derivative* of g and denotes it as $l\delta(g)$. He, in fact, shows that if k is a universal field over C_k, the field of constants of k, and $g \in G(k)$, then the map $g \mapsto l\delta(g)$ maps $G(k)$ onto $\mathcal{G}_k$. We shall only need the fact as stated in Proposition 2.1. Note that $l\delta(gh) = l\delta(g) + gl\delta(h)g^{-1}$.

We shall need the following corollary of this result in the next section.

COROLLARY 2.2. *Let $k = C(x)$ and G, V, and $\mathcal{G}$ be as above. Let $A \in \mathcal{G}_k$. If $k(g)$ is a G-primitive extension of k having connected Galois group $H \subset G$ and $g' = Ag$, then there exists an element $\tilde{g} \in G(k)$ such that*

$$\tilde{g}[A] = \tilde{g}'\tilde{g}^{-1} + \tilde{g}A\tilde{g}^{-1} \in \mathcal{H}_k$$

where $\mathcal{H} \subset \mathcal{G}$ is the Lie algebra of H.

Proof. Since k is a cohomologically trivial field, one knows [15, Corollary 1, p. 426] that any Picard–Vessiot extension of k with connected Galois group H is an H-primitive extension of k. Therefore, $k(g) = k(h)$ for some point h of H. The Galois group leaves $\tilde{g} = hg^{-1}$ fixed so $\tilde{g}$ is a $C(x)$-rational point of G. Calculating $(\tilde{g}g)'(\tilde{g}g)^{-1}$ we have

$$\tilde{g}[A] = \tilde{g}'\tilde{g}^{-1} + \tilde{g}A\tilde{g}^{-1} = (\tilde{g}g)'(\tilde{g}g)^{-1} = h'h^{-1}.$$

Proposition 2.1.1 implies that $h'h^{-1} \in \mathscr{H}_k$. ∎

We now recall Kovacic's program. Any connected linear algebraic group can be written as a semidirect product $R_u \rtimes P$ where R_u denotes the unipotent radical of G and P is a maximal reductive subgroup of G. This is called a *Levi decomposition* of G and P is called a *Levi factor*. It is known that any connected reductive subgroup is contained in a Levi factor and that any two Levi factors are conjugate (cf. [27]). Kovacic showed that one can make several reductions of the inverse problem. First, he showed that one can assume that R_u is commutative and so must be of the form C^m. Since P is reductive and acts on C^m, one can further write C^m as a sum of irreducible P-modules. Kovacic further reduced the inverse problem to the case where R_u is the direct sum of copies of a unique irreducible P-module. Second, Kovacic gave a method for selecting a solution of the inverse problem for P that will lift to a solution of the inverse problem for $G = R_u \rtimes P$. In general terms, he showed that a solution of the inverse problem for P will lift provided that an appropriately chosen "inhomogeneous inverse problem" for R_u can be solved. Finally, Kovacic described the obstructions to solving this inhomogeneous inverse problem for R_u. Since we will need to keep a careful accounting of the singularities we introduce in each of these reduction steps in order to prove Theorem 1.2, we shall now describe these reductions in detail. It is convenient to have the following:

DEFINITION 2.3. A connected linear algebraic group G defined over an algebraically closed field C of characteristic zero is *realizable* over a differential field k whose field of constants is C if there exists a full G-primitive extension $k(g)$ of k. Let $A = l\delta(g) \in \mathscr{G}_k$. We say that A realizes G.

Note that Proposition 2.1 implies that given $A \in \mathscr{G}_k$ there is always a G-primitive extension $k(g)$ such that $l\delta(g) = A$. The above definition of realizability requires that the Galois group be the *full* group G.

2.1. *Reduction to Commutative R_u*

Let G be a connected linear algebraic group defined over an algebraically closed field C and let $\overline{G} = G/(R_u, R_u)$, where (R_u, R_u) denotes the (closed) subgroup of commutators of R_u. Let $\mathscr{G}$ be the Lie algebra of G, $\overline{\mathscr{G}}$ be the Lie algebra of $\overline{G}$, and $d\pi : \mathscr{G} \to \overline{\mathscr{G}}$ be the canonical map. Using the fact that the only algebraic subgroup of G that maps surjectively onto $\overline{G}$ is G, Kovacic shows (cf. [17, Lemma 2; 18, Lemma 7, Proposition 18]:

PROPOSITION 2.4. *Let $\overline{A}_1, \dots, \overline{A}_t \in \overline{\mathscr{G}}$; $f_1, \dots, f_t \in k$; and $A_1, \dots, A_t \in \mathscr{G}$ satisfy $d\pi(A_i) = \overline{A}_i$. If $\sum \overline{A}_i \otimes f_i$ realizes $\overline{G}$ over k, then $\sum A_i \otimes f_i$ realizes G over k.*

2.2. *Lifting Solutions of the Inverse Problem*

We shall assume that G is a connected linear algebraic group defined over C and that the unipotent radical of G, which we denote here by U, is abelian. For any Levi decompositions $G = U \rtimes P$ we will describe Kovacic's method for lifting a solution of the inverse problem for the reductive group P to a solution of the inverse problem for G.

Let us first consider what happens when we have a full G-primitive extension $K = k(g)$ of a differential field k (with algebraically closed field of constants). We may write $g = up$ with $u \in U$, $p \in P$. Since U is normal in G and $k(G/U) = k(P)$ is the fixed field of U, we can use Galois correspondence and Proposition 2.1 to show that $k(p)$ is a full P-primitive extension of k. Let $\mathscr{G}$, $\mathscr{U}$, $\mathscr{P}$ be the Lie algebras of G, U, P respectively. Note that $l\delta(g) = A_G \in \mathscr{G}_k$ and that $l\delta(p) = A_P \in \mathscr{P}_k$. Calculating we find

$$\begin{aligned} l\delta(g) &= (up)'(up)^{-1} \\ &= \left(p \cdot p^{-1}up\right)'\left(p \cdot p^{-1}up\right)^{-1} \\ &= p'p^{-1} + p\big(l\delta\left(p^{-1}up\right)\big)p^{-1}. \end{aligned}$$

Kovacic shows that if $\tilde{p}$ is an element of P such that $k(\tilde{p})$ is a P-primitive extension of k with $l\delta(\tilde{p}) = l\delta(p)$, then $\tilde{p}(l\delta(\tilde{p}^{-1}u\tilde{p}))\tilde{p}^{-1} = p(l\delta(p^{-1}up))p^{-1}$. We therefore define

$$l_{A_p}\delta(u) = p\big(l\delta\left(p^{-1}up\right)\big)p^{-1}$$

where p is any element of P such that $l\delta(p) = A_P$ and the constants of $k(p)$ are the same as the constants of k. Note that $\mathscr{U}$ is left fixed by any automorphism of $\mathscr{G}$ so $l_{A_P}\delta(u) \in \mathscr{U}$. The following result of Kovacic shows that one can reverse this process (cf. [17, Proposition 13; 18, Proposition 19]).

PROPOSITION 2.5. *Let G be a connected linear algebraic group defined over an algebraically closed field C and let k be a differential field with field of* *constants C. Assume that the unipotent radical U of G is abelian and let $G = U \rtimes P$ be a Levi factor decomposition. Let $A_P \in \mathscr{P}_k$ realize P and $A_U \in \mathscr{U}_k$ be such that*

1. *there exists an element $p \in P$ with $l\delta(p) = A_P$ such that $k(p)$ is a full P-primitive extension of k,*
2. *there exists an element $u \in U$ with $l_{A_p}\delta(u) = A_U$, such that the field of constants of $k(u, p)$ is C,*
3. *the map $\sigma \mapsto p^{-1}u^{-1}\sigma(u)p$ is a C-isomorphism from the differential Galois group $\mathrm{Gal}(k(up)/k(p))$ onto $U(C)$.*

Then $l\delta(up) = A_U + A_P$ and $k(up)$ is a full G-primitive extension of k.

The condition $l_{A_P}\delta(u) = A_U$ can be described in a simple way. To do this note that U being a commutative unipotent group is the isomorphic image, via the exponential map, of the vector group of its Lie algebra $\mathscr{U}$ (cf. [4, 7.3]) and it is easy to show that $(\exp\gamma)' = \gamma' \exp\gamma$ for any $\gamma \in \mathscr{U}_k$. This implies that $l\delta(\exp\gamma) = \gamma'$ and if we identify U with the vector group of $\mathscr{U}$ via the exponential map, we have that $l\delta(g) = g'$ for all $g \in U(k)$.

Via the identification of U with $\mathscr{U}$ the action of P on U by conjugation induces the (adjoint) representation $\rho : P \to \mathrm{GL}(U)$ and the corresponding representation $d\rho : \mathscr{P} \to \mathrm{gl}(U)$ on the Lie algebras. For A_U, A_P, u, p as in Proposition 2.5, we have

$$\begin{aligned} l_{A_p}\delta(u) &= p\big(l\delta(p^{-1}up)\big)p^{-1} \\ &= \rho(p)\cdot\big(\rho(p^{-1})\cdot u\big)' \\ &= u' - l\delta(\rho(p))\cdot u \\ &= u' - d\rho(A_P)\cdot u. \end{aligned}$$

EXAMPLE 1. Let $k = C(x)$ where $x' = 1$ and C is algebraically closed. Let $G = C \rtimes C^* =$

$$\left\{ \begin{pmatrix} \beta & \alpha \\ 0 & \beta^{-1} \end{pmatrix} \middle| \beta \neq 0, \alpha \in C \right\}.$$

We identify the Lie algebra $\mathscr{U}$ of C with

$$\left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \middle| a \in C \right\}$$

and the Lie algebra $\mathscr{P}$ of C^* with

$$\left\{\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} \middle| b \in C\right\}.$$

We therefore seek elements $A_U = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$ and $A_P = \left(\begin{smallmatrix} b & 0 \\ 0 & -b \end{smallmatrix}\right)$, $a, b \in C(x)$ such that:

1. There is a full C^*-primitive extension $k(p), p = \left(\begin{smallmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{smallmatrix}\right)$ where $l\delta(p) = \left(\begin{smallmatrix} b & 0 \\ 0 & -b \end{smallmatrix}\right)$. This is equivalent to demanding that the Galois group of $k(\beta)$ over k is C^* where $\beta' = b\beta$.

2. There exists an element $u = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ with $I_{A_P}\delta(u) = A_U$, such that the field of constants of $k(u, p)$ is C. Note that

$$\begin{aligned}
l_{A_p}\delta(u) &= \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} l\delta\left(\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}^{-1}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}\right)\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} l\delta\left(\begin{pmatrix} 1 & \beta^{-2}\alpha \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}\begin{pmatrix} 0 & (\beta^{-2}\alpha)' \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} 0 & \beta^2(\beta^{-2}\alpha)' \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \alpha' - 2b\alpha \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Therefore the condition $l_{A_P}\delta(u) = A_U$ is equivalent to $\alpha' - 2b\alpha = a$.

3. The map $\sigma \to \left(\begin{smallmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)^{-1}\sigma\left(\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\right)\left(\begin{smallmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{smallmatrix}\right)$ is a C-isomorphism from $\mathrm{Gal}(k(\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{smallmatrix}\right))/k(\beta))$ onto $U(C)$. This latter condition is equivalent to the condition that the map $\sigma \mapsto \sigma(\alpha) - \alpha$ is a C-isomorphism of $\mathrm{Gal}(k(\alpha\beta)/k(\beta))$ onto C.

Therefore to realize G as a Galois group over $C(x)$, we must find $a, b \in C(x)$ and (α, β) such that

$$\beta' - b\beta = 0$$
$$\alpha' - 2b\alpha = a$$

and such that $k(\beta)$ is a Picard–Vessiot extension of k with Galois group C^* and $k(\alpha\beta)$ is a Picard–Vessiot extension of $k(\beta)$ with Galois group C. If we let $b = x$ and $a = 1$ then $\beta = e^{x^2/2}$ and $\alpha = e^{x^2}\int e^{-x^2}$. We will show in the next section that this selection meets our needs and discuss how this choice was made. Note that in this example the representation ρ is given by $\rho(\beta) = \beta^2$.

Kovacic is able to refine Proposition 2.5 in the following way. Since P is reductive we may write U as a sum of irreducible P-modules. Grouping isomorphic copies, we write $U = U_1^{r_1} \oplus \cdots \oplus U_s^{r_s}$, where the U_i are nonisomorphic P-modules. Let $\rho_i : P \to \mathrm{GL}(U_i)$ be the representation of P on each simple module and $d\rho_i : \mathscr{P} \to \mathrm{gl}(U_i)$ be the representations on their Lie algebras. We denote by $\rho_i^{r_i}$ and $d\rho_i^{r_i}$ the representations on their powers. As before, we shall identify each U_i and its Lie algebra $\mathscr{U}_i$ with some C^{n_i}. Kovacic shows [18, Proposition 19]:

PROPOSITION 2.6. *Let G be a connected linear algebraic group defined over an algebraically closed field C and let k be a differential field with constants C. Assume that $G = (U_1^{r_1} \oplus \cdots \oplus U_s^{r_s}) \rtimes P$ as above. Let $A_P \in \mathscr{P}_k$ and $A_i \in (\mathscr{U}_i^{r_i})_k$ be such that*

1. *there exists an element $p \in P$ with $l\delta(p) = A_P$ such that $k(p)$ is a full P-primitive extension of k,*
2. *there exists an element $u_i \in U_i^{r_i}$ with $u_i' - d\rho_i^{r_i}(A_P)u_i = A_i$, such that the field of constants of $k(u_i, p)$ is C,*
3. *the map $\sigma \mapsto \rho_i^{r_i}(p^{-1}) \cdot (\sigma(u_i) - u_i)$ is a C-isomorphism from* $\mathrm{Gal}(k(u_i p)/k(p))$ *onto $U_i^{r_i}(C)$.*

Then for $u = u_1 + \cdots + u_s$, $l\delta(up) = A_1 + \cdots + A_s + A_P$ and $k(up)$ is a full G-primitive extension.

2.3. *Selecting A_U and A_P*

Let $G = U \rtimes P$ as above. We shall first show that the inverse problem can be reduced to the case where P is a direct product $T \times H$ where T is a torus and H is a semisimple group. We note that any connected reductive linear algebraic group P is of the form $T \cdot H$ where T is a torus and coincides with the center of P, H is semisimple, and $T \cap H$ is finite. We therefore have a homomorphism $\pi : T \times H \to P$ with finite kernel. Let $\mathscr{T}$, $\mathscr{H}$, $\mathscr{P}$ be the Lie algebras of T, H, P respectively and let $d\pi : \mathscr{T} \times \mathscr{H} \to \mathscr{P}$ be the Lie algebra homomorphism associated to π.

PROPOSITION 2.7. *If $A_U + A_{T,H}$ realizes a semidirect product $U \rtimes (T \times T)$ where $A_U \in \mathscr{U}_k$, $A_{T,H} \in (\mathscr{T} \oplus \mathscr{H})_k$, then $A_U + d\pi(A_{T,H})$ realizes $U \rtimes P$ over k.*

The proof of this proposition is contained in the proof of Proposition 9 in [17] (note that the condition concerning the kernel of π in this latter proposition is not needed in the part of the proof that verifies the above proposition). This proposition allows us to assume that our group is of the form $U \rtimes (T \times H)$. The next proposition shows that to realize $T \times H$, we need only realize T and H separately (cf. [17, Proposition 12]).

PROPOSITION 2.8. *Let G_1 and G_2 be connected groups such that the only common homomorphic image of both groups is the trivial group. If A_1 and A_2 realize G_1 and G_2 respectively, then $A_1 + A_2$ realizes $G_1 \times G_2$ (here we identify the Lie algebra of $G_1 \times G_2$ with the direct sum of the Lie algebras of G_1 and G_2).*

COROLLARY 2.9. *With notation as above, if A_1 realizes T and A_2 realizes H, then $A_1 + A_2$ realizes $T \times H$.*

Proof. T and H have no common homomorphic image other than the trivial group. ∎

We do not know a general criterion for realizing a semisimple group over an arbitrary differential field, but in the next section we will show how to realize such a group over $C(x)$. The following is a criterion for realizing tori over any differential field (cf. [17, Proposition 15]). It is a consequence of the Kolchin–Ostrowski theorem (cf. [16]). If T is a torus defined over C, we identify $T(C)$ with the m-fold product $C^* \times \cdots \times C^*$ and the Lie algebra $\mathscr{T}_C$ of $T(C)$ with the m-fold sum $C \oplus \cdots \oplus C$. With this identification, the logarithmic derivative of an element in T becomes $l\delta(\alpha_1, \ldots, \alpha_m) = (\alpha_1'/\alpha_1, \ldots, \alpha_m'/\alpha_m)$.

PROPOSITION 2.10. *Let T be as above and F a differential field containing C. Let $(a_1, \ldots, a_m) \in \mathscr{T}_F = F^m$. A necessary and sufficient condition that $(a_1, \ldots, a_m)$ realize T over F is that there exists no relation of the form $n_1 a_1 + \cdots + n_m a_m = f'/f$ with $n_i \in \mathbf{Z}$ not all zero and $f \in F$.*

Finally, Kovacic gives a criterion for finding elements satisfying Proposition 2.6. Let $L_{A_P, \rho} : k^m \to k^m$ be the map defined by $L_{A_p, \rho}(v) = v' - d\rho(A_P) \cdot v$ and let $\pi : k^m \to k^m / L_{A_p, \rho}(k^m)$ be the quotient homomorphism of C-vector spaces. Kovacic shows (cf. [18, Proposition 20]):

PROPOSITION 2.11. *With notations as in Proposition 2.6, assume that $s = 1$. If $A_p \in \mathscr{P}_k$ satisfies condition 1 above, then $A_1 = (a_1, \ldots, a_r) \in \mathscr{U}_1^r$ satisfies conditions 2 and 3 if and only if $\pi a_1, \ldots, \pi a_r$ are linearly independent over C.*

EXAMPLE 1 (bis). Propositions 2.10 and 2.11 allow us to verify the claims at the end of the exposition of Example 1. Using the notation of that example, we apply Proposition 2.10 to $b = x$. Let $nx = f'/f$ for some

$f \in C(x)$. For any finite zero or pole γ of f, f'/f will have a simple pole at γ. Since nx has no poles, we must have that f is a constant and so $n = 0$. Therefore x realizes the torus C^*. To use Proposition 2.11 we must show that $\alpha' - 2x\alpha = 1$ has no rational solutions $\alpha \in C(x)$. Let $\alpha \in C(x)$ be a solution. At any finite pole of α, the order of α' is larger than the order of $2x\alpha$, so there can be no cancellation. Therefore α must be a polynomial. Let n be the degree of this polynomial. The degree of $2x\alpha$ is larger than the degree of α so the degree of the right-hand side of this equation is larger than the degree of the left hand side, a contradiction.

3. PROOFS OF THE THEOREMS

3.1. *Reductive Groups*

We start by showing that any connected semisimple group can be realized over $C(x)$ by a system $Y' = AY$ where the entries of A are polynomials of degree at most one.

LEMMA 3.1. *Let G be a connected semisimple linear group defined over C. There exists a faithful finite-dimensional G-module V such that*

1. *V contains no one-dimensional G-modules.*
2. *Any proper connected closed subgroup $H \subset G$ leaves a one-dimensional subspace $W_H \subset V$ invariant.*

Proof. Recall that Chevalley's theorem (cf. [12, p. 80]) states that for any proper algebraic subgroup $H \subset G$ there is a G-module V such that H is the stabilizer of a line L_H in V. Since G is semisimple, we can write V as the sum of irreducible G-modules. The projection of L_H into one of these irreducible components (of dimension necessarily greater than 1) must be non-trivial, so we can assume that H stabilizes a line in some irreducible G-module V. Note that any subgroup conjugate to H will also stabilize some line in V. Dynkin's theorem [9] implies that there are only a finite number of conjugacy classes of maximal proper connected closed subgroups in G. Selecting a V_i as above for each of these we can now let $V = V_1 \oplus \cdots \oplus V_m$. If need be, we can take the direct sum of this with a faithful irreducible G-module of dimension greater than 1 to assume that V is faithful. ∎

We will refer to a faithful G-module satisfying the conclusions of the above lemma as a *Chevalley module* for G. For example, if $G = \mathrm{SL}(2)$, then any irreducible G-module is a Chevalley module for G since any proper connected subgroup is solvable and will leave some line invariant.

LEMMA 3.2. *Let G be a connected semisimple linear group over C with Lie algebra $\mathscr{G}$ and let V be a Chevalley module for G. Let $A \in \mathscr{G} \otimes_C C[x]$. If $C(x)(g)$ is a G-primitive extension of $C(x)$ with $g' = Ag$ whose Galois group is a proper subgroup of G then there exist $v \in V$, $v \neq 0$, $c \in C(x)$, and $\tilde{g} \in G(C(x))$ such that*

$$\tilde{g}[A] \cdot v = c \cdot v.$$

Proof. Note that since A has polynomial entries the monodromy group is trivial. Gabber's Lemma (cf. [14, Proposition 1.2.5]) implies that the Galois group is connected.[3] Corollary 2.2 implies that there exists a $\tilde{g} \in G$ such that $\tilde{g}[A] \in \mathscr{H}_k$, where $\mathscr{H}$ is the Lie algebra of H. Since V is a Chevalley module for G, there exists a line left invariant by H and therefore by $\mathscr{H}$. The conclusion now follows. ∎

LEMMA 3.3. *Let G be a connected semisimple linear group over C with Lie algebra $\mathscr{G}$ and let V be a Chevalley module for G. Let $A \in \mathscr{G} \otimes_C C[x]$ and let t be the maximum degree of the polynomials appearing in A. If $C(x)(g)$ is a G-primitive extension of $C(x)$ with $g' = Ag$ whose Galois group is a proper subgroup of G then there exists $w \in V \otimes_C C[x]$, $w \neq 0$, and $c \in C[x]$ with $\deg_x c \leq t$ such that*

$$w' - (A - cI)w = 0.$$

Proof. By Lemma 3.2 we know that there exists $v \in V$, $v \neq 0$, $c \in C(x)$, and $\tilde{g} \in G(C(x))$ such that $(\tilde{g}'\tilde{g}^{-1} + \tilde{g}A\tilde{g}^{-1})v = cv$ or, equivalently,

$$\big(\tilde{g}'\tilde{g}^{-1} + \big(\tilde{g}A\tilde{g}^{-1} - cI\big)\big)v = 0.$$

If we differentiate $\tilde{g}\tilde{g}^{-1} = I$ we get that $\tilde{g}'\tilde{g}^{-1} = -\tilde{g}(\tilde{g}^{-1})'$. Therefore we have

$$\big(-\tilde{g}\big(\tilde{g}^{-1}\big)' + \big(\tilde{g}A\tilde{g}^{-1} - cI\big)\big)v = 0.$$

If we let $w = -\tilde{g}^{-1}v$ we get

$$\big(\tilde{g}w' - \big(\tilde{g}A - cI\tilde{g}\big)\big)w = 0.$$

Multiplying through by $\tilde{g}^{-1}$ gives us an equation of the form

$$w' - (A - cI)w = 0.$$

[3]Strictly speaking this is not an algebraic argument but it can be replaced by a tedious algebraic proof.

We now must show that there exists an equation of this form with $w \in V \otimes_C C[x]$, $w \neq 0$, and $c \in C[x]$ with $\deg_x c \leq t$. We first will show that we can assume that c is a polynomial. Let α be a finite pole of c and write

$$c = c_n(x-\alpha)^n + c_{n+1}(x-\alpha)^{n+1} + \dots,$$

$$w = w_m(x-\alpha)^m + w_{m+1}(x-\alpha)^{m+1} + \dots,$$

where $n \leq -1$, $w_i \in V$, and $c_i \in C$. Substituting these expressions into $w' - (A - cI)w = 0$ and comparing leading terms, we see that $n = -1$ and $c_n = -m$. Therefore we have that

$$c = \sum \frac{-m_\alpha}{x-a} + p$$

where p is a polynomial. Letting

$$w_1 = w \cdot \prod (x-\alpha)^{-m_\alpha}$$

we have

$$\begin{aligned} w_1' - (A - pI)w_1 &= w' \cdot \prod (x-\alpha)^{-m_\alpha} + w \cdot \prod (z-\alpha)^{-m_\alpha}\left(\sum \frac{-m_\alpha}{x-\alpha}\right) \\ &\quad - (A - pI)w \cdot \prod (x-\alpha)^{-m_\alpha} \\ &= (w' - (A - cI)w) \prod (x-\alpha)^{-m_\alpha} \\ &= 0. \end{aligned}$$

Therefore we may replace w by w_1 and assume that c is a polynomial. Again by comparing leading terms, we see that the entries of w have no finite poles. Let $c = c_0 + c_1 x + \cdots + c_m x^m$ and $w = w_0 + w_1 x + \cdots + w_n x^n$. If $m > t$ then the term of highest power in x appearing in $w' - (A - cI)w$ is $c_m w_n x^{m+n}$, which is not zero. Therefore, $m \leq t$. ∎

When $t \leq 1$, we can state the conclusion of this result even more concretely. Let $A = A_0 + A_1 x$ and $c = c_0 + c_1 x$ be as above. Any prospective solution w of $w' - (A - cI)w = 0$ is of the form $w = w_m x^m + \cdots + w_0$ where the w_i are in V. Substituting into $w' - (A - cI)w = 0$ we have

$$\begin{aligned} 0 &= w' - (A - cI)w \\ &= m w_m x^{m-1} + (m-1) w_{m-1} x^{m-2} + \dots + w_1 \\ &\quad - (A_0 + xA_1 - (c_0 + c_1 x)I)(w_m x^m + \dots + w_0) \\ &= (c_1 I - A_1) w_m x^{m+1} + [(c_0 I - A_0) w_m + (c_1 I - A_1) w_{m-1}] x^m \\ &\quad + [(c_0 I - A_0) w_{m-1} + (c_1 I - A_1) w_{m-2} + m w_m] x^{m-1} \end{aligned}$$

$$\vdots$$
$$+\left[(c_0 I - A_0)w_1 + (c_1 I - A_1)w_0 + 2w_2\right]x$$
$$+\left[(c_0 I - A_0)w_0 + w_1\right].$$

Therefore Lemma 3.3 states that if the Galois group is a proper connected subgroup of G then there exists a nonnegative integer m and constants c_0, c_1 such that the above system has a solution $(w_m, \dots, w_0)$ in V^{m+1} with $w_m \neq 0$.

EXAMPLE 2. Consider the usual representation of SL(2) in GL(2). We have already noted that this makes C^2 into a Chevalley module for SL(2). Let

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For a given m let us consider the coefficients of x^{m+1}, x^m, and x^{m-1} in the above system. The coefficient of x^{m+1} must vanish so c_1 is an eigenvalue and w_m is an eigenvector of A_1. Let us assume that $c_1 = 1$ (the proof when $c_1 = -1$ is similar). Since we can multiply w by any nonzero element of C, we can furthermore assume that $w_m = (1, 0)^T$. Let $w_{m-1} = (u, v)^T$. Setting the coefficient of x^m equal to zero, we have

$$\begin{pmatrix} c_0 & -1 \\ -1 & c_0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Equating entries, we get $c_0 = 0$ and $v = \frac{1}{2}$. Let $w_{m-2} = (y, z)^T$ and, setting the coefficient of x^{m-1} equal to zero, we have

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} + m\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This implies that $m = \frac{1}{2}$, contradicting the fact that m is an integer.

Therefore we can conclude that the Galois group of the differential system

$$Y' = \begin{pmatrix} x & 1 \\ 1 & -x \end{pmatrix} Y$$

is SL(2).

Note that in the above example A_1 is a regular element of a Cartan subalgebra of SL(2) and A_0 is the sum of generators of the root subspaces of SL(2) with respect to this Cartan algebra. We shall now show how one can generalize this example to deal with any connected semisimple linear group.

Let $\mathscr{G}$ be a semisimple Lie algebra and let

$$\mathscr{G} = \mathscr{H} \oplus \coprod \mathscr{G}_\alpha$$

where $\mathscr{H}$ is a Cartan subalgebra and $\mathscr{G}_\alpha$ are the root spaces. Let V be a $\mathscr{G}$-module and let

$$V = \bigoplus_{i=1}^{s} V_{\beta_i}$$

where the β_i are distinct weights of $\mathscr{H}$ on V and the V_{β_i} are the corresponding weight spaces. Let $\{v_1, \ldots, v_{r_1}\}$ be a basis of V_{β_1}, and $[v_{r_{i-1}+1}, \ldots, v_{r_i}\}$ be a basis of V_{β_i} for $i = 2, \ldots, s$ (so $\dim(V_{\beta_i}) = r_i - r_{i-1}$ where $r_0 = 0$). We then have that $\{v_i\}_{1 \le i \le \dim(V)}$ is a basis of V.

LEMMA 3.4. *Let α be a root of $\mathscr{G}$ and $X_\alpha \in \mathscr{G}_\alpha$, $X_\alpha \neq 0$.*

1. *If β is a weight of $\mathscr{H}$ on V and v_β is an element of V_β then $X_\alpha(v_\beta) \in V_{\alpha+\beta}$.*

2. *The matrix (x_{ij}) of X_α with respect to $\{v_i\}$ has the property that $x_{i,j} = 0$ for each l, $0 \le l \le s$, $r_l \le i \le r_{l+1}$, and $r_l \le j \le r_{l+1}$, that is, (x_{ij}) has s blocks of zeroes along the diagonal.*

Proof. (1) We note that for any $A \in \mathscr{H}$ we have $\alpha(A)X_\alpha = [A, X_\alpha]$. Therefore $\alpha(A)X_\alpha(v_\beta) = AX_\alpha(v_\beta) - X_\alpha A(v_\beta) = AX_\alpha(v_\beta) - \beta(A) X_\alpha(v_\beta)$. This implies that $AX_\alpha(v_\beta) = (\alpha(A) + \beta(A))X_\alpha(v_\beta)$.

(2) We know by (1) that $X_\alpha(v_{i,j})$ lies in $V_{\alpha+\beta_l}$ for every $1 \le l \le s$, $r_l \le i, j \le r_{l+1}$. Since α is a root, $V_{\alpha+\beta_l} \neq V_{\beta_l}$. Therefore, the V_{β_l}-component of $X_\alpha(v_{i,j})$ must be 0. ∎

We now select two special generators of $\mathscr{G}$. Let $A_0 = \sum X_\alpha$ where we sum over all roots of $\mathscr{G}$ and each X_α is a nonzero element of $\mathscr{G}_0$. Let V be a G-module. Lemma 3.4 implies that the matrix of A_0 with respect to the basis $\{v_i\}$ of V, as before, has s blocks of zeroes along the diagonal.

In $\mathrm{gl}(V)$

$$A_0 = \begin{pmatrix} 0 & \dots & 0 & a_{1,r_1+1} & \dots & \dots & \dots & \dots & a_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & a_{r_1,r_1+1} & \dots & \dots & \dots & \dots & a_{r_1,n} \\ a_{r_1+1,1} & \cdots & a_{r_1+1,r_1} & 0 & \cdots & 0 & a_{r_1+1,r_2+1} & \cdots & a_{r_1+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{r_2,1} & \dots & a_{r_2,r_1} & 0 & \dots & 0 & a_{r_2,r_2+1} & \dots & a_{r_2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix}.$$

Let A_1 be a regular element of $\mathscr{H}$ such that distinct roots take distinct, non-zero values on A_1 and furthermore distinct weights (of $\mathscr{H}$ in V) take distinct values on A_1. The set of such elements is Zariski dense in $\mathscr{H}$. Note that there are $d_i \in C$ such that $A_1 = \mathrm{diag}(d_1, \dots, d_n)$ where $d_1 = \cdots = d_{r_1}, d_{r_1+1} = \cdots = d_{r_1+r_2}$, etc. We shall refer to such a pair (A_0, A_1) as a *regular pair of generators of* $\mathscr{G}$. The fact that A_0 and A_1 generate $\mathscr{G}$ is shown in [5, Chap. 8, Sect. 2, Ex. 8, p. 221].

PROPOSITION 3.5. *Let G be a connected semisimple linear algebraic group and $\mathscr{G}$ its Lie algebra. Then there exists a faithful $\mathscr{G}$-module V and a regular pair of generators (A_0, A_1) of the Lie algebra $\mathscr{G} \subset \mathrm{gl}(V)$ such that*

$$Y' = (A_0 + A_1 x)Y$$

has Galois group G.

Proof. Let V be a Chevalley module for G and let (A_0, A_1) be a regular pair of generators of $\mathscr{G}$. We will show that there exist non-zero $t \in C$ such that the Galois group of $Y' = (tA_0 + A_1 x)Y$ is G.

Let $A = tA_0 + A_1 x$, with $t \neq 0$. Lemma 3.3 implies that if the Galois group of $Y' = AY$ is not G, then there exists $x = c_0 + c_1 x$, $c_i \in C$, and $w = w_m x^m + \cdots + w_0$ where the w_i are in V and $w_m \neq 0$, such that $w' - (A - cI)w = 0$. Substituting into this equation we have

$$\begin{aligned} 0 &= w' - (A - cI)w \\ &= m w_m x^{m-1} + (m-1) w_{m-1} x^{m-2} + \cdots + w_1 \\ &\qquad - \big(tA_0 + xA_1 - (c_0 + c_1 x)I\big)\big(w_m x^m + \cdots + w_0\big) \\ &= (c_1 I - A_1) w_m x^{m+1} + \big[(c_0 I - tA_0) w_m + (c_1 I - A_1) w_{m-1}\big] x^m \\ &\qquad + \big[(c_0 I - tA_0) w_{m-1} + (c_1 I - A_1) w_{m-2} + m w_m\big] x^{m-1} \end{aligned}$$

$$\vdots$$

$$+\left[(c_0 I - tA_0)w_1 + (c_1 I - A_1)w_0 + 2w_2\right]x$$

$$+\left[(c_0 I - tA_0)w_0 + w_1\right].$$

Setting the coefficients of x^{m+1}, x^m, and x^{m-1} equal to zero, we have

$$(c_1 I - A_1)w_m = 0 \tag{1}$$

$$(c_0 I - tA_0)w_m + (c_1 I - A_1)w_{m-1} = 0 \tag{2}$$

$$(c_0 I - tA_0)w_{m-1} + (c_1 I - A_1)w_{m-2} + mw_m = 0. \tag{3}$$

Equation (1) implies that w_m is an eigenvector of A_1. We will assume that $c_1 = d_1$. Since any constant multiple of w is again a solution of $w' - (A - cI)w = 0$, we can assume that $w_m = (1, 0, \ldots, 0)^T$. Substituting this into Eq. (2) and letting A_0 be as above, $w_{m-1} = (u_1, \ldots, u_n)^T$, and $r = r_1$, we have

$$\begin{pmatrix} c_0 & 0 & \ldots & 0 & -ta_{1,r+1} & \ldots & -ta_{1n} \\ 0 & c_0 & \ldots & 0 & -ta_{2,r+1} & \ldots & -ta_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & c_0 & -ta_{r,r+1} & \ldots & -ta_{1n} \\ -ta_{r+1,1} & \ldots & \ldots & \ldots & \ldots & \ldots & ta_{r+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -ta_{n1} & \ldots & \ldots & \ldots & \ldots & \ldots & ta_{n,n} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & d_1 - d_{r+1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 & d_1 - d_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_r \\ u_{r+1} \\ \vdots \\ u_n \end{pmatrix} = 0.$$

This yields the system of equations:

$$\begin{aligned} c_0 + 0u_1 &= 0 \\ 0 + 0u_2 &= 0 \\ \vdots \quad &\vdots \quad \vdots \\ 0 + 0u_r &= 0 \\ -ta_{r+1,1} + (d_1 - d_{r+1})u_{r+1} &= 0 \\ \vdots \quad &\vdots \quad \vdots \\ -ta_{n,1} + (d_1 - d_n)u_n &= 0. \end{aligned}$$

This implies that $c_0 = 0$. Since each $d_1 - d_i \neq 0$ for $i > r$, we also have that

$$u_i = \frac{ta_{i1}}{d_1 - d_i}$$

for $i = r + 1, \dots, n$. We now consider Eq. (3)

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -ta_{1,r+1} & \dots & -ta_{1n} \\ 0 & 0 & 0 & \dots & -ta_{2,r+1} & \dots & -ta_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -ta_{r,r+1} & \dots & -ta_{1n} \\ -ta_{r+1,1} & \dots & \dots & \dots & \dots & \dots & ta_{r+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -ta_{n1} & \dots & \dots & \dots & \dots & \dots & ta_{n,n} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_n \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & d_1 - d_{r+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & d_1 - d_n \end{pmatrix} \begin{pmatrix} * \\ \vdots \\ * \\ * \\ \vdots \\ * \end{pmatrix} + \begin{pmatrix} m \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = 0.$$

The first row of this matrix equation is

$$-ta_{1,r+1}u_{r+1} - \cdots - ta_{1n}u_n + m = 0.$$

Substituting the values for u_i we have

$$-t^2\left(a_{1,r+1}\frac{a_{r+1,1}}{d_1 - d_{r+1}} + \cdots + a_{1n}\frac{a_{n1}}{d_1 - d_n}\right) + m = 0.$$

If

$$a_{1,r+1}\frac{a_{r+1,1}}{d_1 - d_{r+1}} + \cdots + a_{1n}\frac{a_{n1}}{d_1 - d_n} = 0,$$

then we must have that $m = 0$. This means that $w \in V$. Since $0 = w' - (A - cI)w = (cI - tA_0 - A_1x)w$, this implies that A_0 and A_1 have a common eigenvector and since A_0 and A_1 generate $\mathscr{G}$, this contradicts the fact that V is a Chevalley module. Therefore, we have that

$$a_{1,r+1}\frac{a_{r+1,1}}{d_1 - d_{r+1}} + \cdots + a_{1,n}\frac{a_{n1}}{d_1 - d_n} \neq 0$$

and again we get a contradiction if we select $t \in C$ such that

$$-t^2\left(a_{1,r+1}\frac{a_{r+1,1}}{d_1 - d_{r+1}} + \cdots + a_{1,n}\frac{a_{n,1}}{d_1 - d_n}\right)$$

is not an integer. The set of such t in $\overline{Q}$, the algebraic closure of the rationals, is a dense open set. For these t the Galois group of $Y' = (tA_0 + A_1 x)$ must be G. ∎

The above result gives a very simple system that realizes a connected semisimple group. In order to realize a Levi factor of a connected linear algebraic group in such a way that we may utilize Propositions 2.5 and 2.11, we may need a system with polynomial entries of higher degree.

PROPOSITION 3.5. *Let G be a connected semisimple linear algebraic group and $\mathcal{G}$ its Lie algebra. Let $n > 1$ be an integer. Then there exists a faithful $\mathcal{G}$-module V and a regular pair of generators (A_0, A_1) of the Lie algebra $\mathcal{G} \subset \mathrm{gl}(V)$ such that*

$$Y' = (A_0 + A_1 x^n)Y$$

has Galois group G.

Proof. We again let V be a Chevalley module for G and (A_0, A_1) be a regular pair of generators of $\mathcal{G}$. If the Galois group of $Y' = (A_0 + A_1 x^n)Y$ is not G, then Lemma 3.3 implies that there exists $c(x) = c_n x^n + \cdots + c_0 \in C[x]$ and $W(x) = w_m x^m + \cdots + w_0 \in V \otimes C[x\}$ such that

$$w' - (A_0 + A_1 x^n - cI)w = 0.$$

We shall deal with three cases:

$n > m$: Setting the coefficients of $x^{m+n}, \ldots, x^n$ equal to zero we have:

$$\begin{aligned}
(c_n I - A_1)w_m &= 0\\
(c_n I - A_1)w_{m-1} + c_{n-1} I w_m &= 0\\
(c_n I - A_1)w_{m-2} + c_{n-2} I w_{m-1} + c_{n-1} I w_m &= 0\\
\vdots \quad &\ \vdots \quad \vdots\\
(c_n I - A)w_1 + c_{n-1} I w_2 + \ldots + c_{n-m+1} I w_m &= 0\\
(c_n I - A_1)w_0 + c_{n-1} I w_1 + \ldots + c_{n-m} I w_m &= 0.
\end{aligned}$$

The first equation implies that c_n is an eigenvalue of A_1 and w_n is an eigenvector of A_1. Let W denote the c_n-eigenspace of A_1. We have that the image of $(c_n I - A_1)$ intersects W trivially. Therefore the second equation implies that $c_{n-1} = 0$ and that $w_{m-1} \in W$. Considering each equation in turn, we have that $c_{n-1} = \cdots = c_{n-m} = 0$ and $w_m, \ldots, w_0 \in$

W. We now set the coefficients of $x^{n-1}, \ldots, x^m$ equal to zero:

$$\begin{array}{rcl} c_{n-1}Iw_0 + \ldots + c_{n-m-1}Iw_m & = & 0 \\ \vdots & \vdots & \vdots \\ c_{m+1}Iw_0 + \ldots + c_1 Iw_m & = & 0 \\ c_m Iw_0 + \ldots + (c_0 I - A_0)w_m & = & 0. \end{array}$$

Considering all but the last of these equations in order, we see that $c_{n-m-1} = \cdots = c_1 = 0$. The final equation then implies that w_m is an eigenvector of A_0. This implies that A_1 and A_0 have a common eigenvector, a contradiction.

$n < m$: Setting the coefficients of $x^{m+n}, \ldots, x^m$ equal to zero, we have:

$$\begin{array}{rcl} (c_n I - A_1)w_m & = & 0 \\ (c_n I - A_1)w_{m-1} + c_{n-1}Iw_m & = & 0 \\ (c_n I - A_1)w_{m-2} + c_{n-2}Iw_{m-1} + c_{n-1}Iw_m & = & 0 \\ \vdots & \vdots & \vdots \\ (c_n I - A_1)w_{m-n} + c_{n-1}Iw_{m-n+1} + \ldots + (c_0 I - A_0)w_m & = & 0. \end{array}$$

The first equation implies that c_n is an eigenvalue of A_1 and w_n is an eigenvector of A_1. Let W denote the c_n-eigenspace of A_1. Arguing as above we see that $c_{n-1} = \cdots = c_1 = 0$ and $w_m, \ldots, w_{n-m} \in W$. Using the form of A_0 established in Lemma 3.4, the last equation implies that $c_0 = 0$ as well. Note that unlike the situation in Proposition 3.5, we can conclude at this point that $w_{m-1} \in W$. We use here the fact that $n > 1$. Setting the coefficient of x^{m-1} equal to zero we have that

$$mw_m + (c_n I - A_1)w_{m-n-1} + (c_0 I - A_0)w_{m-1} = 0.$$

Since $w_{m-1} \in W$ and $c_0 = 0$, we have that the W-component of $(c_0 I - A_0)w_{m-1} = A_0 w_{m-1}$ is 0. Furthermore, the W-component of $(c_n I - A_1)w_{m-n-1}$ must also be 0. Therefore this latter equation implies that $m = 0$. This contradicts the fact that $m > n > 1$.

$m = n$: Setting the coefficients of $x^{m+n}, \ldots, x^m$ equal to zero we have:

$$\begin{array}{rcl} (c_n I - A_1)w_m & = & 0 \\ (c_n I - A_1)w_{m-1} + c_{n-1}Iw_m & = & 0 \\ (c_n I - A_1)w_{m-2} + c_{n-2}Iw_{m-1} + c_{n-1}Iw_m & = & 0 \\ \vdots & \vdots & \vdots \\ (c_n I - A_1)w_0 + c_{n-1}Iw_1 + \ldots + (c_0 I - A_0)w_m & = & 0. \end{array}$$

As before we can conclude that $c_{n-1} = \cdots = c_0 = 0$ and $w_m, \ldots, w_0 \in W$. Setting the coefficient of x^{m-1} equal to zero we have that

$$mw_m + (c_0 I - A_0) w_{m-1} = 0.$$

Since $c_0 = 0$ and the W-component of $A_0 w_{m-1}$ is 0, we must have that $m = 0$. This contradicts the fact that $m = n > 1$. ∎

We now turn to the case of a torus. Let T be a torus of dimension r and $\mathcal{T}$ its Lie algebra. We identify $T(C)$ with $(C^*)^r$ and $\mathcal{T}_C$ with C^r.

PROPOSITION 3.7. *Let* $\{c_1, \ldots, c_r\} \subset C$ *be linearly independent over* $\mathbf{Q}$. *For any* $n \geq 0$, *the element* $(c_1 x^n, \ldots, c_r x^n) \in \mathcal{T}_{C(x)}$ *realizes* T *over* $C(x)$.

Proof. By Proposition 2.10, it is enough to show that if $n_1 c_1 x^n + \ldots + n_r c_r x^n = f'/f$ with $n_i \in \mathbf{Z}$ and $f \in C(x)$, then all the $n_i = 0$. If such a relation exists, let $f = \prod (x - d_j)^{m_j}$, $d_j \in C$, $m_j \in \mathbf{Z}$. We then have that $f'/f = \sum (m_j/(x - d_j))$. Therefore, each $m_j = 0$. Since the c_i are $\mathbf{Q}$-linearly independent, we have that each $n_i = 0$. ∎

We now combine Propositions 3.6 and 3.7 with Proposition 2.4, Proposition 2.7, and Corollary 2.9 to yield:

PROPOSITION 3.8. *Let* G *be a connected reductive linear algebraic group. Let* $m > 1$ *be an integer. Then there exists a differential equation of the form*

$$Y' = (A + Bx^m) Y,$$

with A, B *constant matrices, with Galois group* G.

For later use, we recall that if $G = T \times H$, where T is a torus and H is a semisimple group, then we can select $B = D + E$ where (A, D), for some A, is a regular pair of generators of the Lie algebra of H and E is a diagonal matrix whose non-zero entries are linearly independent over $\mathbf{Q}$. Note that we can select a basis so that both D and E (and therefore B) are diagonal.

3.2. *Proof of Theorem* 1.2

In order to use the lifting results of Kovacic we will need the following technical lemma.

LEMMA 3.9. *Let* A, B *be* $s \times s$ *square matrices with entries in an algebraically closed field* C. *Assume that*

1. B *is diagonal.*
2. *If* $(C^s)_0$ *is the eigenspace of* B *corresponding to eigenvalue* 0, *then for any* $w \in (C^s)_0$ *the* $(C^s)_0$*-component of* Aw *is zero.*
3. A *and* B *share no nonzero eigenvectors.*

Then for any integer $m > 0$ there exists a nonzero subspace V of C^s such that if $c_{m-2}, \ldots, c_0 \in V$ and

$$Y' - (A + Bx^m)Y = c_{m-2}x^{m-2} + \cdots + c_0 \tag{4}$$

has a solution $T \in (C(x))^s$, then $c_{m-2} = \cdots = c_0 = 0$.

Proof. Note that by comparing the orders of poles, one can see that any solution Y of (4) must lie in $(C[x])^s$. Now assume that $(C^s)_0 = \{0\}$ and let $V = C^s$. Let $Y = w_n x^n + \cdots + w_0$, $w_i \in C^s$, be a solution of (4). Comparing coefficients of x^{m+n} we see that $Bw_n = 0$ so $w_n = 0$. Therefore $Y = 0$ and so $c_{m-2} = \cdots = c_0 = 0$.

Now assume that $(C^s)_0 \neq \{0\}$ and let $V = (C^s)_0$. Let $Y = w_n x^n + \cdots + w_0$, $w_i \in C^s$, $w_n \neq 0$ be a solution of (4). We will deal with three cases:

$m > n$: Comparing coefficients of x^{n+m} in (4) we see that $Bw_n = 0$ so $w_n \in (C^s)_0$. Comparing coefficients of x^n in (4), we see that $Aw_n = -c_n$. Since $c_n \in (C^s)_0$ and the $(C^s)_0$-component of Aw_n is 0, we have that $c_n = 0$ so $Aw_n = 0$. Therefore w_n would be a common eigenvector of A and B, a contradiction.

$m = n$: Comparing coefficients of $x^{m+n} = x^{2m}$ in (4) we see that $Bw_m = 0$. Similarly, comparing coefficients of $x^{m+n-1} = x^{2m-1}$ in (4) we see that $Bw_{m-1} = 0$. Comparing coefficients of x^{m-1} in (4) we see that $Aw_{m-1} + mw_m = 0$. Since $w_m \in (C^s)_0$ and the $(C^s)_0$-component of Aw_{m-1} is 0, we must have $m = 0$, a contradiction since $m > 0$.

$m < n$: Considering the coefficients of x^{m+n} and x^{m+n-1} as above, we see that $w_n, w_{n-1} \in (C^s)_0$. Comparing coefficients of x^{n-1}, we see that $nw_n + Bw_{n-m-1} + Aw_{n-1} = 0$. Since the $(C^s)_0$-component of Bw_{n-m-1} and Aw_{n-1} is 0, we have that $n = 0$, contradicting that $n > m > 0$. ∎

We now give the proof of Theorem 1.2. Let G be a connected linear algebraic group defined over C with defect d and excess e. Proposition 2.4 implies that we may assume that R_u is commutative (note that the defect and excess of the groups G and $G/(R_u, R_u)$ are the same). Furthermore, Proposition 2.7 implies that we may assume that $G = U \rtimes (T \times H)$ where U is a commutative unipotent group, T is a torus, and H is a semisimple group (note that e and d are unchanged under $\mathrm{id} \times \pi$). We identify the Lie algebra $\mathscr{T}_C$ of $T(C)$ with C^l and let $(c_1, \ldots, c_l) \in \mathscr{T}_C$, where the c_i are $\mathbf{Q}$-linearly independent. Proposition 3.7 implies that $(c_1 x^e, \ldots, c_l x^e)$ realizes T over $C(x)$. If (A_0, A_1) is a regular pair of generators of the Lie algebra $\mathscr{H}_C$ of $H(C)$, then Propositions 3.5 and 3.6 imply that $A_0 + x^e A_1$ realize H. Therefore Corollary 2.9 implies that $A_P = (c_1 x^e, \ldots, c_l x^e) + A_0 + x^e A_1$ realizes $T \times H$. We shall now use Propositions 2.6 and 2.11 to realize G. Let $U = U_1^{r_1} \oplus \cdots \oplus U_s^{r_s}$ be the unipotent radical where each U_i

is an irreducible $P = T \times H$-module and assume that U_1 is the one-dimensional trivial module. Fix some i and let $U = U_i$, $r = r_i$, and $\rho = \rho_i$. Proposition 2.11 implies that it is sufficient to show that for the above choice of A_P, there exists $(a_1, \ldots, a_r) \in \mathscr{U}^r_{C(x)}$ such that if $v' - d\rho(A_P) \cdot v = \alpha_1 a_1 + \cdots + \alpha_r a_r$ has a solution $v \in C(x)$, for some $\alpha_j \in C$, then all α_j must be zero.

Let us first assume that $i = 1$, i.e., that U is the trivial one-dimensional module and $r = d$. In this case $d\rho \equiv 0$. Let $\gamma_1, \ldots, \gamma_d$ be distinct elements of C. If any $v \in C(x)$ satisfies

$$v' = \frac{\alpha_1}{x - \gamma_1} + \cdots + \frac{\alpha_d}{x - \gamma_d},$$

then clearly each $\alpha_j = 0$. Therefore $A_1 = (1/(x - \gamma_1), \ldots, 1/(x - \gamma_d))$ satisfies Proposition 2.11.

We now consider the case where $i > 1$. In this case U is a nontrivial irreducible $T \times H$-module. Since T is diagonalizable and H commutes with T, we have that H preserves each eigenspace of T. Therefore, each element of T acts as a constant matrix on U. This furthermore implies that U is an irreducible H-module. We now consider two subcases.

The first is the case when U is the one-dimensional trivial H-module. In this case, $d\rho(A_0) = d\rho(A_1) = 0$ and $d\rho(c_1 x^e, \ldots, c_l x^e) = (q_1 c_1 + \cdots + q_l c_l) x^e$ for some integers q_i, not all zero. Note that our assumptions on the c_i imply that this latter expression is nonzero. Let $a_1 = 1, \ldots, a_r = x^{r-1}$ and note that $r - 1 < e$. We must show that if

$$v' - (q_1 c_1 + \cdots + q_l c_l) x^e \cdot v = \alpha_1 + \cdots + \alpha_r x^{r-1}$$

has a solution $v \in C(x)$, then all the $\alpha_j = 0$. To see this note that a solution v cannot have finite poles and so must be a polynomial. Comparing degrees we get $r = 0$ so $A_i = (1, \ldots, x^{r-1})$ satisfies Proposition 2.11.

The second case is the case when the action of H on U is non-trivial. In this case, both $d\rho(A_0)$ and $d\rho(A_1)$ are non-zero and the matrices $A = A_0$ and $B = d\rho((c_1, \ldots, c_l)) + d\rho(A)$ satisfy the hypotheses of Lemma 3.9. Let $a_1 = 1, \ldots, a_r = x^{r-1}$. Since $r - 1 \leq e - 2$, Lemma 3.9 implies that $A_{U_i} = (1, \ldots, x^{r-1})$ satisfies Proposition 2.11.

In all cases we have introduced no more than d finite simple poles and polynomials of degree at most e.

EXAMPLE 3. Consider the group $G = (C \times C) \rtimes C^*$, where the action of G on the first factor of $C \times C$ is trivial and the action of $c \in C^*$ on the second factor is given by multiplication by c^2. In concrete terms this is the group:

$$\left\{ \begin{pmatrix} 1 & b_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b_2 \\ 0 & 0 & 0 & a^{-1} \end{pmatrix} \middle| a, b_1, b_2 \in C, a \neq 0 \right\}.$$

The unipotent radical is $C \times C$ and $d = 1$ and $e = 1$. We can let $A_1 = 1$, $A_2 = 1$, and $A_P = x$. In terms of matrices, we see that

$$Y' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x & 1 \\ 0 & 0 & 0 & -x \end{pmatrix} Y$$

realizes this group. The solution of this equation is

$$Y = \begin{pmatrix} \log x \\ 1 \\ e^{x^2/2} \int e^{-x^2} \\ e^{-x^2/2} \end{pmatrix}. \qquad \blacksquare$$

We end this section by showing that the bound of $d(G) + 1$ singular points is sharp. We do not know if the bound on the degrees of the entries of A_∞ is sharp.

PROPOSITION 3.10. *If $G = C^r$, then $d(G) = r$ and G cannot be realized by a differential equation over $C(x)$ with fewer than $r + 1$ singular points.*

Proof. Assume not and let $k = C(x)$ and K be a Picard–Vessiot extension corresponding to the system. We know that $K = k(G)$ so we can write $k(G) = k(t_1, \dots, t_r)$ where the action of the Galois group is given by $\sigma(t_i) = t_i + c_{ig}$ for $\sigma \in G$ and $c_{ig} \in C$. This implies that each t_i' is left invariant by the Galois group G and so must be in k. Therefore each t_i is of the form $t_i = \sum_j c_{ij} \log(f_j) + g_i$, where $f_j, g_i \in k$. The assumption on the singularities implies that any element in $k(G)$ has singularities (except for poles) in some fixed set $\{p_1, \dots, p_{r-1}, p_r\}$. Making a change of variables we can assume that $p_r = \infty$. Therefore we may write $\log(f_j) = \sum_{i=1}^{r-1} n_{ij} \log(x - p_i)$. This gives us r linear equations in the $r - 1$ logarithms and so the t_i will be algebraically dependent over $C(x)$, a contradiction. $\blacksquare$

3.3. *Proof of Theorem* 1.1

Theorem 1.1 follows from the following proposition, whose statement and proof are due to Kovacic [19]. In [18] he attacked the inverse problem for arbitrary connected (not necessarily linear) algebraic groups. He showed that one could reduce this problem to the inverse problem for connected linear groups and for abelian varieties. Kovacic also showed that if G is an abelian variety and F is a differential field of finite but nonzero transcendence degree over its algebraically closed field of constants, then there exists a strongly normal extension of F whose differential Galois group is G (*strongly normal*) generalizes the notion of Picard–Vessiot, cf. [15, 17, 18]; a strongly normal extension whose Galois group is linear is a Picard–Vessiot extension). The following proposition is stated and proved in such a way that it holds in this generality but so that readers unfamiliar with strongly normal extensions can restrict themselves to Picard–Vessiot extensions and follow the proof as well. In general, a connected algebraic group is realizable over F if there is a strongly normal extension K of F whose Galois group is isomorphic to the given group.

PROPOSITION 3.11. *Let H be a connected C-group such that for any natural number n, H^n is realizable over $C(x)$. Then for any differential field F of finite but non-zero transcendence degree over C, the group H is realizable over F.*

Proof. Choose any x in F not in C and note that x is transcendental over C. Let $\bar{d}$ be the derivation $\bar{d} = (dx)^{-1}d$, where d is the derivation on F. Then any $\bar{d}$-extension $\bar{K}$ (a differential extension of F, with derivation $\bar{d}$) is also a d-extension K. We also have that $\bar{K}$ is strongly normal if and only if K is, and $\mathrm{Gal}(\bar{K}/F) = \mathrm{Gal}(K/F)$. Thus we may replace d by $\bar{d}$ and thereby assume that $C(x) \subset F$.

Let n be the transcendence degree of F over C. Choose an extension K of $C(x)$ that realizes H^n (K need not contain F). Since $C(x)$ is cohomologically trivial, there exists $a = (a_1, \dots, a_n) \in H^n$ such that $K = C(x)(a)$ and $l\delta a \in \mathrm{Lie}_{C(x)}(H^n)$. Note that $\mathrm{trdeg}(C(x)(a)/C(x)) = n \dim H$.

Choose $b = (b_1, \dots, b_n) \in H^n$ such that $l\delta b = l\delta a$ and such that the field of constants of $F(b)$ is C. Then there exists c in $H(L)^n$ (where L is the field of constants of some universal differential field) with $b = ac$. Since $\mathrm{Gal}(C(x)(a)/C(x)) = H^n$, there is a differential isomorphism, σ, of $C(x)(a)$ over $C(x)$ such that $c = a^{-1}\sigma(a)$. So $b = \sigma(a)$. Thus $\mathrm{tr\,deg}(C(x)(b)/C(x)) = n \dim H$.

Evidently, for each i, $F(b_i)$ is a strongly normal extension of F. We claim that $\mathrm{Gal}(F(b_i)/F) = H$ for at least one i. Suppose not. Then $\mathrm{tr\,deg}\, F(b_i)/F \leq \dim H - 1$, for all $i = 1, \dots, n$. So $\mathrm{tr\,deg}\, F(b)/F \leq n \dim H - n$ and $\mathrm{tr\,deg}\, F(b)/C(x) \leq n \dim H - 1$ (since $\mathrm{tr\,deg}\, F/C(x) = n - 1$). But $C(x)(b) \subset F(b)$ and $\mathrm{tr\,deg}\, C(x)(b)/C(x) = n \dim H$. This contradiction proves the theorem. ∎

COROLLARY 3.12. *Let F be of finite, but non-zero, transcendence degree over C. Let G be any connected C-group. Then G is realizable over F.*

APPENDIX

Let G be a connected linear algebraic group and let R_u be its unipotent radical and P a Levi factor. We shall verify that $d(G)$ is the dimension of $R_u/(G, R_u)$. We shall use the notation of the Introduction and write $R_u/(R_u, R_u) = U_1^{n_1} \oplus \cdots \oplus U_s^{n_s}$ where each U_i is an irreducible P-module and U_1 is the trivial one-dimensional P-module. Since $(R_u, R_u) \subset (G, R_u)$ we have a canonical surjective homomorphism $\pi : R_u/(R_u, R_u) \to R_u/(G, R_u)$. We shall show that the kernel of this homomorphism is $U_2^{n_2} \oplus \cdots \oplus U_s^{n_s}$ and so $d(G) = n_1 = \dim R_u/(G, R_u)$.

First note that the kernel of π is $(G, R_u)/(R_u, R_u)$ and that this latter group is $(P, R_u)/(R_u, R_u)$. To see this second statement note that for $g \in G$ we may write $g = pu$, $p \in P$, $u \in R_u$. For any $w \in R_u$, $gwg^{-1}w^{-1} = puwu^{-1}p^{-1}w^{-1} = p(uwu^{-1})p^{-1}(uw^{-1}u^{-1})(uwu^{-1}w^{-1})$. Therefore $(G, R_u)/(R_u, R_u) \subset (P, R_u)/(R_u, R_u)$. The reverse inclusion is clear.

Next we will show that $(P, R_u)/(R_u, R_u) = U_2^{n_2} \oplus \cdots \oplus U_s^{n_s}$. The group $(P, R_u)/(R_u, R_u)$ is the subgroup of $U_1^{n_1} \oplus \ldots \oplus U_s^{n_s}$ generated by the elements $pvp^{-1} - v$ where $p \in P$, $v \in U_1^{n_1} \oplus \cdots \oplus U_s^{n_s}$. Since the action of P on U_1 via conjugation is trivial, we see that any element $pvp^{-1} - v$ as above must lie in $U_2^{n_2} \oplus \cdots \oplus U_s^{n_s}$. Furthermore, note that for each i, the image of the map $P \times U_i \to U_i$ given by $(p, u) \mapsto pup^{-1} - u$ generates a P-invariant subspace of U_i. For $i \geq 2$, this image is nontrivial so, since U_i is an irreducible P-module, we have that the image generates all of U_i. This implies that the elements $pvp^{-1} - v$, where $p \in P$, $v \in U_1^{n_1} \oplus \cdots \oplus U_s^{n_s}$, generate all of $U_2^{n_2} \oplus \cdots \oplus U_s^{n_s}$, and completes the proof.

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