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A combined approximating and interpolating subdivision scheme with C^2 continuity

Jun Pan [a,b,∗], Shujin Lin [a,c], Xiaonan Luo [a,∗∗]

[a] National Engineering Research Center of Digital Life, State-Province Joint Laboratory of Digital Home Interactive Applications, School of Information Science and Technology, Sun Yat-sen University, Guangzhou, 510006, China

[b] College of Computer and Information Engineering, Henan University of Economics and Law, Zhengzhou, 450000, China

[c] School of Communication and Design, Sun Yat-sen University, Guangzhou, 510006, China

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ABSTRACT

In this paper a combined approximating and interpolating subdivision scheme is presented. The relationship between approximating subdivision and interpolating subdivision is derived by directly performing operations on geometric rules. The behavior of the limit curve produced by our combined subdivision scheme is analyzed by the Laurent polynomial and attains C^2 degree of smoothness. Furthermore, a non-uniform combined subdivision with shape control parameters is introduced, which allows a different tension value for every edge of the original control polygon.

1. Introduction

Subdivision is a very popular geometric modeling tool. In general, subdivision can be distinguished into two categories: interpolating schemes and approximating schemes. Although approximating schemes yield smoother curves with higher order continuity [1–3], interpolating schemes are more useful for engineering applications, especially the schemes with the shape control [4,5]. The approximating schemes and interpolating schemes are related by a deep connection. Maillot et al. [6] and Zhang [7] introduced a push-back method to progressively interpolate the control vertices. To obtain interpolating refinement rules from approximating ones, Rossignac [8] used a weak operation and Li et al. [9] used a sequence of weighted averaging operations. Lin et al. [10] proceeded with relationship between the cubic B-spline and four-point curve subdivision scheme. Beccari et al. [11] employed generating functions to extend a unified univariate subdivision. All the considered methods only produced C^1-continuous limit function.

In this paper, we focus on constructing combined subdivision scheme with higher order continuity by directly performing some simple operations on geometric rules. We also offer a method of intuitively modifying the approximating or interpolating tension of the limit shape. The new combined approximating and interpolating four-point C^2 subdivision scheme is defined as follows:

$$\begin{cases} P_{3i}^{k+1} = a_0 P_{i-1}^k + a_1 P_i^k + a_0 P_{i+1}^k, \\ P_{3i+1}^{k+1} = b_0 P_{i-1}^k + b_1 P_i^k + b_2 P_{i+1}^k + b_3 P_{i+2}^k, \\ P_{3i+2}^{k+1} = b_3 P_{i-1}^k + b_2 P_i^k + b_1 P_{i+1}^k + b_0 P_{i+2}^k, \end{cases} \tag{1}$$

∗ Corresponding author at: College of Computer and Information Engineering, Henan University of Economics and Law, Zhengzhou, 450000, China.
∗∗ Corresponding author.
E-mail addresses: panjun@mail2.sysu.edu.cn (J. Pan), lnslxn@mail.sysu.edu.cn (X. Luo).

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where

$$\begin{cases} a_0 = \dfrac{4}{27}\alpha, \\ a_1 = 1 - \dfrac{8}{27}\alpha, \\ b_0 = \dfrac{1}{27} - \dfrac{1}{27}(1-\alpha)(\omega+1), \\ b_1 = \dfrac{16}{27} + \dfrac{1}{27}(1-\alpha)(2+2\omega-\nu), \\ b_2 = \dfrac{10}{27} + \dfrac{1}{27}(1-\alpha)(-1-\omega+2\nu), \\ b_3 = -\dfrac{1}{27}(1-\alpha)\nu, \end{cases}$$

with $\omega = \frac{3}{2}(1+\mu)$, $\nu = \frac{3}{2}(1-\mu)$, μ is free parameter. If $\alpha = 0$, (1) generates the interpolating subdivision; if $\alpha = 1$, (1) generates approximating subdivision; and if $0 < \alpha < 1$, (1) generates the subdivision that produces the limit curves intervening between approximating subdivision and interpolating subdivision. It is also proved that the proposed combined subdivision generates curves with C^2 continuity when $1/5 < \mu < 1/3$ and $0 \le \alpha \le 1$.

2. Preliminaries

In this section some fundamental definitions and results are recalled as the basis of the theory developed in the remainder of this paper. For their derivation the reader is referred to [12,13]. Start from a set of initial control vertices $P^0 = \{p_i^0 \in \mathbb{R},\ i \in \mathbb{Z}\}$, the set of control vertices $P^{k+1} = \{p_i^{k+1},\ i \in \mathbb{Z}\}$ at the $(k+1)$th level generated by a ternary subdivision scheme is defined by

$$P_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-3j} P_j^k, \quad i \in \mathbb{Z}. \tag{2}$$

The set of coefficients $\mathbf{a} = \{a_i \in \mathbb{R},\ i \in \mathbb{Z}\}$ determines the subdivision rule. Let S be a convergent subdivision scheme with a mask a. Then

$$\sum_{i\in\mathbb{Z}} a_{3i+j} = 1, \quad j = 0, 1, 2. \tag{3}$$

The generating function associated with the set of coefficients can defined as the following Laurent polynomial:

$$\mathbf{a}(z) = \sum_{-\infty<i<\infty} a_i z^i. \tag{4}$$

A necessary C^m-continuity condition for the subdivision scheme is

$$\mathbf{a}(z) = \frac{1}{3^m}\left(\frac{1-z^3}{1-z}\right)^{m+1} \mathbf{a}_{m+1}(z), \tag{5}$$

with $\mathbf{a}_{m+1}(z)$ denoting the $(m+1)$th divided difference of $\mathbf{a}(z)$. In particular, if $\mathbf{a}^{\upsilon+1}(z)$ satisfies relation (3) for all $\upsilon \in \mathbb{Z} \cap [0, m]$ and there exists an integer $L > 0$ such that $\|S^L_{a^{\upsilon+1}}\|_\infty < 1$, then $S^\infty P^0 \in C^m$.

Let S^k be k-th refinement of the subdivision and

$$\left\|\frac{1}{3}S^{k+1}\right\|_\infty = \frac{1}{3}\max\left(\sum_{i\in Z}|a_{3i}^{k+1}|, \sum_{i\in Z}|a_{3i+1}^{k+1}|, \sum_{i\in Z}|a_{3i+2}^{k+1}|\right), \tag{6}$$

where

$$\mathbf{a}_{m+1}^{k+1}(z) = \prod_{j=0}^{k} \mathbf{a}_{m+1}(z^{3^j}). \tag{7}$$

If $\|\frac{1}{3}S^{k+1}\|_\infty < 1$ for all given initial control vertices P^0, then $S^\infty P^0 \in C^k$.

3. Combined approximating and interpolating four-point ternary subdivision scheme

In this section, we deduce our combined four-point ternary subdivision scheme by directly performing operations on geometric rules. Given a set of initial control vertices $\{P_i^0 \in \mathbb{R},\ i \in \mathbb{Z}\}$, as shown in Fig. 1(a). First, insert two vertices $P_{3i+1}^{\frac{1}{2}}$

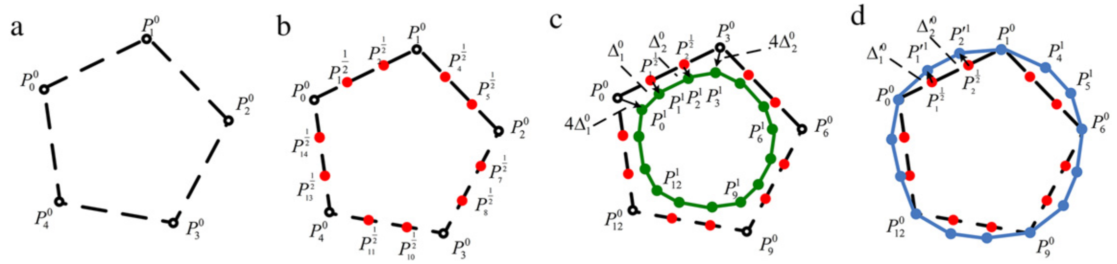


Fig. 1. Relationship between ternary approximating and interpolating subdivision.

and $P_{3i+2}^{\frac{1}{2}}$ into the control polygon respectively at 1/3 and 2/3 parametric positions between P_i^0 and P_{i+1}^0, as shown in Fig. 1(b). Let

$$\Delta_i^0 = -\frac{1}{27}P_{i-1}^0 + \frac{2}{27}P_i^0 - \frac{1}{27}P_{i+1}^0, \tag{8}$$

then move $P_{3i+1}^{\frac{1}{2}}$ and $P_{3i+2}^{\frac{1}{2}}$ to the new position P_{3i+1}^1 and P_{3i+2}^1 with displacement of Δ_i^0 and Δ_{i+1}^0, while move P_{3i}^0 to the new position P_{3i}^1 with displacement of $4\Delta_i^0$, as shown in Fig. 1(c). After k-th refinement iterations, P^{k+1} can be deduced by the ternary approximating scheme as follows:

$$\begin{cases} P_{3i}^{k+1} = P_i^k - 4\Delta_i^k, \\ P_{3i+1}^{k+1} = \left(\dfrac{2}{3}P_i^k + \dfrac{1}{3}P_{i+1}^k\right) - \Delta_i^k, \\ P_{3i+2}^{k+1} = \left(\dfrac{1}{3}P_i^k + \dfrac{2}{3}P_{i+1}^k\right) - \Delta_{i+1}^k. \end{cases} \tag{9}$$

Based on (9), the corresponding interpolating subdivision curves can be deduced in this way: keep all the vertices P_i^0 fixed and move the two inserted vertices $P_{3i+1}^{\frac{1}{2}}$ and $P_{3i+2}^{\frac{1}{2}}$ to a new position P'^1_{3i+1} and P'^1_{3i+2} with Δ'^0_{3i+1} and Δ'^0_{3i+2}, as shown in Fig. 1(d). Δ'^0_{3i+1} and Δ'^0_{3i+2} are defined by

$$\Delta'^0_{3i+d} = \omega\Delta_{i+d-1}^0 + \nu\Delta_{i+2-d}^0, \quad d = 1, 2, \tag{10}$$

where $\omega = \frac{3}{2}(1+\mu)$, $\nu = \frac{3}{2}(1-\mu)$, μ is free parameter. Thus the new interpolating four-point ternary subdivision can be written in the form

$$\begin{cases} P'^{k+1}_{3i} = P_i^k, \\ P'^{k+1}_{3i+1} = \left(\dfrac{2}{3}P_i^k + \dfrac{1}{3}P_{i+1}^k\right) + \omega\Delta_i^k + \nu\Delta_{i+1}^k, \\ P'^{k+1}_{3i+2} = \left(\dfrac{1}{3}P_i^k + \dfrac{2}{3}P_{i+1}^k\right) + \nu\Delta_i^k + \omega\Delta_{i+1}^k. \end{cases} \tag{11}$$

By adding a parameter α, we can formalize the connection between (9) and (11) as

$$\begin{cases} P_{3i}^{k+1} = P_i^k - 4\alpha\Delta_i^k, \\ P_{3i+1}^{k+1} = \left(\dfrac{2}{3}P_i^k + \dfrac{1}{3}P_{i+1}^k\right) - \alpha\Delta_i^k + (1-\alpha)(\omega\Delta_i^k + \nu\Delta_{i+1}^k), \\ P_{3i+2}^{k+1} = \left(\dfrac{1}{3}P_i^k + \dfrac{2}{3}P_{i+1}^k\right) - \alpha\Delta_{i+1}^k + (1-\alpha)(\nu\Delta_i^k + \omega\Delta_{i+1}^k). \end{cases} \tag{12}$$

(12) is called the combined approximating and interpolating ternary subdivision scheme. It is obvious that both the approximating subdivision defined by (9) and the interpolating subdivision defined by (11) are the special cases of the combined subdivision scheme defined by (12). If $\alpha = 1$, (12) generates the approximating subdivision and if $\alpha = 0$, (12) generates the interpolating subdivision scheme. From (8) and (12), we can get the combined ternary approximating and interpolating subdivision in the form of (1). Fig. 2 illustrates a family of curves produced by our combined ternary subdivision scheme with different α. In Section 4, we will prove that our combined approximating and interpolating subdivision scheme yields C^2 limit curves when $1/5 < \mu < 1/3$ and $0 \le \alpha \le 1$.

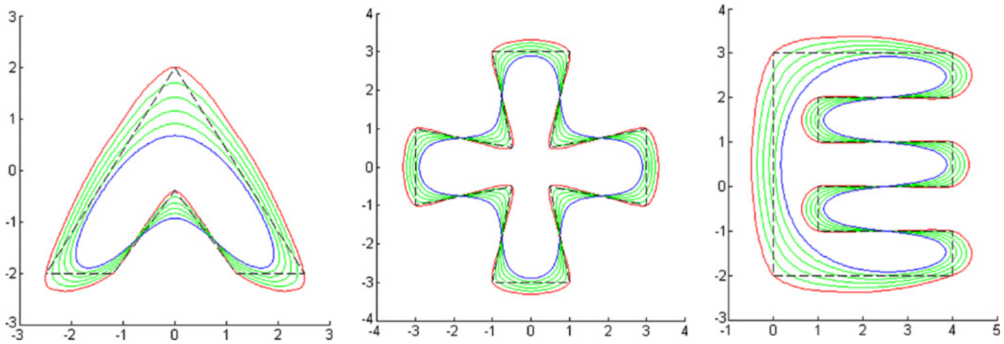


Fig. 2. A family of limit curves generated by our combined subdivision scheme from approximation to interpolation with $\alpha = 1.0, 0.8, 0.6, 0.4, 0.2, 0$ from inside to outside, respectively.

4. Smoothness analysis

Theorem 1. *The scheme defined by (1) converges and has smoothness C^2 when $0 \le \alpha \le 1$ and $1/5 < \mu < 1/3$.*

Proof. The generating function corresponding to the proposed combined ternary subdivision defined by (1) has the following sequence of coefficients:

$$a = (a_i) = (\dots, b_3,\ b_0,\ a_0,\ b_2,\ b_1,\ a_1,\ b_1,\ b_2,\ a_0,\ b_0,\ b_3, \dots). \tag{13}$$

The Laurent polynomial $\mathbf{a}(z)$ here can be written as:

$$\mathbf{a}(z) = b_3 z^{-5} + b_0 z^{-4} + a_0 z^{-3} + b_2 z^{-2} + b_1 z^{-1} + a_1 z^0 + b_1 z^1 + b_2 z^2 + a_0 z^3 + b_0 z^4 + b_3 z^5. \tag{14}$$

In order to prove the smoothness of this scheme to be C^2 according to (5), let

$$b^{(m)}(z) = \frac{1}{3} a_m(z), \quad m \in \mathbb{Z}^+, \tag{15}$$

where

$$a_m(z) = \left(\frac{3z}{1+z}\right) a_{m-1}(z) = \left(\frac{3z}{1+z}\right)^m a(z). \tag{16}$$

From Laurent polynomial, for $m = 1$, we have

$$b^{(1)}(z) = \frac{1}{3} a_1(z) = \frac{1}{3} \sum \xi_i z^i, \quad i = -3, -2, \dots, 5, \tag{17}$$

where

$$\begin{aligned}
\xi_{-3} &= \xi_5 = -\frac{1}{6}(-1+\alpha)(-1+\mu),\\
\xi_{-2} &= \xi_4 = \frac{1}{9}(\alpha - 3\mu + 3\alpha\mu),\\
\xi_{-1} &= \xi_3 = \frac{1}{6}(1 + \alpha + \mu - \alpha\mu),\\
\xi_0 &= \xi_2 = \frac{1}{3}(3 + \alpha(-1+\mu) - \mu),\\
\xi_1 &= \frac{1}{9}(9 + 6\mu - 2\alpha(1+3\mu)).
\end{aligned} \tag{18}$$

When $1/5 < \mu < 1/3$ and $0 \le \alpha \le 1$, the norm of subdivision $\frac{1}{3}S_1$ is

$$\left\| \frac{1}{3} S_1 \right\| = \max\left\{ \sum_\beta |b^{(1)}_{\gamma+3\beta}| : \gamma = 0, 1, 2 \right\} = \max\left\{ \frac{13}{27}, \frac{19}{45}, \frac{19}{45} \right\} = \frac{13}{27} < 1. \tag{19}$$

From Laurent polynomial, for $m = 2$, we have

$$b^{(2)}(z) = \frac{1}{3} a_2(z) = \frac{1}{18} \sum \xi'_i z^i, \quad i = -1, 0, \dots, 5, \tag{20}$$

where

$$\begin{aligned}
&\xi'_{-1} = \xi'_5 = 3(1-\alpha)(-1+\mu),\\
&\xi'_0 = \xi'_4 = 3-\alpha-9(1-\alpha)\mu,\\
&\xi'_1 = \xi'_3 = 3+\alpha+9(1-\alpha)\mu,\\
&\xi'_2 = 12-6\alpha-6(1-\alpha)\mu.
\end{aligned} \tag{21}$$

When $1/5 < \mu < 1/3$ and $0 \le \alpha \le 1$, the norm of subdivision $\frac{1}{3}S_2$ is

$$\left\| \frac{1}{3}S_2 \right\| = \max\left\{ \sum_\beta |b^{(2)}_{\gamma+3\beta}| : \gamma = 0, 1, 2 \right\} = \max\left\{ \frac{13}{15}, \frac{1}{3}, \frac{1}{3} \right\} < 1. \tag{22}$$

From Laurent polynomial, for $m = 3$, we have

$$b^{(3)}(z) = \frac{1}{3}a_2(z) = \frac{1}{3}\sum \xi''_i z^i, \quad i = 1, 2, 3, 4, 5, \tag{23}$$

where

$$\begin{aligned}
&\xi''_1 = \xi''_5 = \frac{3}{2}(1-\alpha)(-1+\mu),\\
&\xi''_2 = \xi''_4 = 3-6\mu+\alpha(-2+6\mu),\\
&\xi''_3 = \alpha+9\mu-9\alpha\mu.
\end{aligned} \tag{24}$$

When $1/5 < \mu < 1/3$ and $0 \le \alpha \le 1$, the norm of subdivision $\frac{1}{3}S_3$ is

$$\left\| \frac{1}{3}S_3 \right\| = \max\left\{ \sum_\beta |b^{(3)}_{\gamma+3\beta}| : \gamma = 0, 1, 2 \right\} = \max\left\{ \frac{11}{15}, \frac{1}{3}, \frac{1}{3} \right\} < 1. \tag{25}$$

The calculation of $b^{(4)}$ does not give us a Laurent polynomial, i.e., the Laurent polynomial is disconverge everywhere except at the origin. So, for $1/5 < \mu < 1/3$, we have $\|\frac{1}{3}S_1\|_\infty, \|\frac{1}{3}S_2\|_\infty, \|\frac{1}{3}S_3\|_\infty < 1$. It is easy to verify that $a^{(0)}, a^{(1)}, a^{(2)}, a^{(3)}$ all satisfy $\sum_{i\in Z} a_{3i+j} = 1$. Hence, the limit curve produced by our ternary combined subdivision is C^2. □

5. Non-uniform combined subdivision with shape control parameters

Let α_i and β_i be a substitution of α in (12), the non-uniform combined subdivision scheme can be defined by

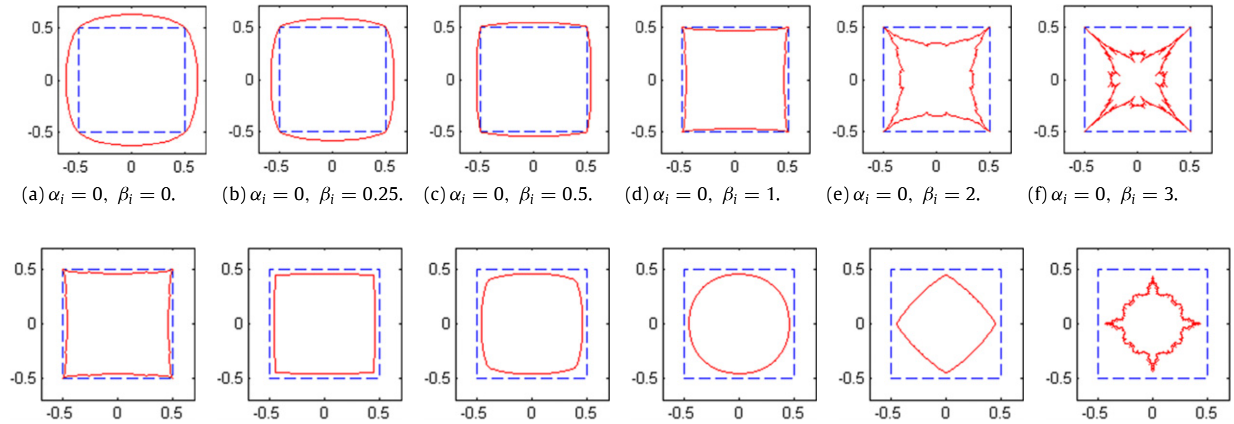
$$\begin{cases}
P^{k+1}_{3i} = a_{0,i-1}P^k_{i-1} + a_{1,i}P^k_i + a_{0,i+1}P^k_{i+1},\\
P^{k+1}_{3i+1} = b_{0,i-1}P^k_{i-1} + b_{1,i}P^k_i + b_{2,i+1}P^k_{i+1} + b_{3,i+2}P^k_{i+2},\\
P^{k+1}_{3i+2} = b_{3,i-1}P^k_{i-1} + b_{2,i}P^k_i + b_{1,i+1}P^k_{i+1} + b_{0,i+2}P^k_{i+2},
\end{cases} \tag{26}$$

where

$$\begin{cases}
a_{0,i} = \dfrac{4}{27}\alpha_i,\\
a_{1,i} = 1 - \dfrac{8}{27}\alpha_i,\\
b_{0,i} = \dfrac{1}{27} - \dfrac{1}{27}(1-\beta_i)(\omega+1),\\
b_{1,i} = \dfrac{16}{27} + \dfrac{1}{27}(1-\beta_i)(2+2\omega-\nu),\\
b_{2,i} = \dfrac{10}{27} + \dfrac{1}{27}(1-\beta_i)(-1-\omega+2\nu),\\
b_{3,i} = -\dfrac{1}{27}(1-\beta_i)\nu,
\end{cases}$$

with $\omega = \frac{3}{2}(1+\mu),\ \nu = \frac{3}{2}(1-\mu)$, μ is free parameter.

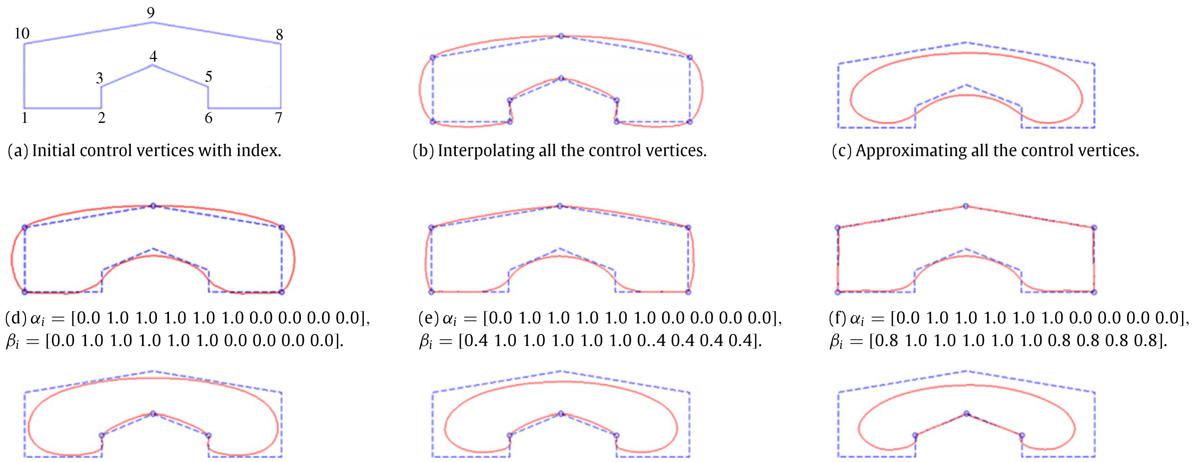
The parameter α_i controls the interpolating property of the subdivision curve and the parameter β_i controls the approximating property of the subdivision curve. The two parameters can intuitively modify the approximating or interpolating tension of the limit shape. If $0 \le \alpha_i \equiv \beta_i \le 1$ and $1/5 < \mu < 1/3$ for the adjacent vertices, it keeps the



(a) $\alpha_i = 0,\ \beta_i = 0.$ (b) $\alpha_i = 0,\ \beta_i = 0.25.$ (c) $\alpha_i = 0,\ \beta_i = 0.5.$ (d) $\alpha_i = 0,\ \beta_i = 1.$ (e) $\alpha_i = 0,\ \beta_i = 2.$ (f) $\alpha_i = 0,\ \beta_i = 3.$

(g) $\alpha_i = 0,\ \beta_i = 1.$ (h) $\alpha_i = 0.25,\ \beta_i = 1.$ (i) $\alpha_i = 0.5,\ \beta_i = 1.$ (j) $\alpha_i = 1,\ \beta_i = 1.$ (k) $\alpha_i = 2,\ \beta_i = 1.$ (l) $\alpha_i = 3,\ \beta_i = 1.$

Fig. 3. $\mu = \frac{3}{10}$. Top row: limit curves obtained by the combined subdivision scheme with $\alpha_i \equiv 0$ and $\beta_i = 0, 0.25, 0.5, 1, 2, 3$, respectively. Bottom row: limit curves obtained by the combined subdivision scheme with $\beta_i \equiv 1$ and $\alpha_i = 0, 0.25, 0.5, 1, 2, 3$, respectively.



(a) Initial control vertices with index. (b) Interpolating all the control vertices. (c) Approximating all the control vertices.

(d) $\alpha_i = [0.0\ 1.0\ 1.0\ 1.0\ 1.0\ 1.0\ 0.0\ 0.0\ 0.0\ 0.0]$, $\beta_i = [0.0\ 1.0\ 1.0\ 1.0\ 1.0\ 1.0\ 0.0\ 0.0\ 0.0\ 0.0]$.

(e) $\alpha_i = [0.0\ 1.0\ 1.0\ 1.0\ 1.0\ 1.0\ 0.0\ 0.0\ 0.0\ 0.0]$, $\beta_i = [0.4\ 1.0\ 1.0\ 1.0\ 1.0\ 1.0\ 0..4\ 0.4\ 0.4\ 0.4]$.

(f) $\alpha_i = [0.0\ 1.0\ 1.0\ 1.0\ 1.0\ 1.0\ 0.0\ 0.0\ 0.0\ 0.0]$, $\beta_i = [0.8\ 1.0\ 1.0\ 1.0\ 1.0\ 1.0\ 0.8\ 0.8\ 0.8\ 0.8]$.

(g) $\alpha_i = [1.0\ 1.0\ 0.0\ 0.0\ 0.0\ 1.0\ 1.0\ 1.0\ 1.0\ 1.0]$, $\beta_i = [1.0\ 1.0\ 0.0\ 0.0\ 0.0\ 1.0\ 1.0\ 1.0\ 1.0\ 1.0]$.

(h) $\alpha_i = [1.5\ 1.5\ 0.0\ 0.0\ 0.0\ 1.5\ 1.5\ 1.5\ 1.5\ 1.5]$, $\beta_i = [1.5\ 1.5\ 0.0\ 0.0\ 0.0\ 1.5\ 1.5\ 1.5\ 1.5\ 1.5]$.

(i) $\alpha_i = [2.0\ 2.0\ 0.0\ 0.0\ 0.0\ 2.0\ 2.0\ 2.0\ 2.0\ 2.0]$, $\beta_i = [2.0\ 2.0\ 0.8\ 0.8\ 0.8\ 2.0\ 2.0\ 2.0\ 2.0\ 2.0]$.

Fig. 4. The effect of local interpolation by our combined subdivision with $\mu = \frac{3}{10}$ and different α_i, β_i.

limit curve near these vertices is C^2-continuous. The parameter μ is the free parameter. It is obvious that μ only has the influence to the interpolation situation, because if the scheme is the approximation mode with $\beta_i \equiv 1$, it means μ would have no contribution to the shape of curves. In the interpolation mode, if $1/5 < \mu < 1/3$, the limit subdivision curve is convergent; if $\mu \leq 1/5$, the curvature of curve becomes very large and if $\mu \geq 1/3$ the curvature becomes discontinuous [12]. Fig. 3 shows some subdivision curves of fitting a square produced by (26) with different control parameters α_i and β_i. Fig. 4 shows that one can easily interpolate the control vertices in a local manner and change tension value for every edge of the original control polygon.

6. Conclusion

This paper presents a new combined approximating and interpolating subdivision scheme for designing curves. Our combined subdivision scheme can produce limit curves intervening between approximating subdivision and interpolating subdivision. Our combined subdivision scheme attains C^2 degree of smoothness. Furthermore, a non-uniform combined subdivision with shape control parameters is introduced, which allows a different approximating or interpolating tension value for every edge of the original control polygon.

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