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Optimal acyclic edge colouring of grid like graphs

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ABSTRACT

We determine the values of the acyclic chromatic index of a class of graphs referred to as *d-dimensional partial tori*. These are graphs which can be expressed as the cartesian product of d graphs each of which is an induced path or cycle. This class includes some known classes of graphs like d-dimensional meshes, hypercubes, tori, etc. Our estimates are exact except when the graph is a product of a path and a number of odd cycles, in which case the estimates differ by an additive factor of at most 1. Our results are also constructive and provide an optimal (or almost optimal) acyclic edge colouring in polynomial time.

1. Introduction

All graphs we consider are simple and finite. Throughout the paper, we use $\Delta(G)$ to denote the maximum degree of a graph G. A colouring of the edges of a graph is *proper* if no pair of incident edges receives the same colour. A proper colouring $\mathcal{C}$ of the edges of a graph G is *acyclic* if there is no two-coloured (bichromatic) cycle in G with respect to $\mathcal{C}$. In other words, the subgraph induced by the union of any two colour classes in $\mathcal{C}$ is a forest. The minimum number of colours required to edge-colour a graph G acyclically is termed the *acyclic chromatic index* of G and is denoted by $a'(G)$. The notion of acyclic colouring was introduced by Grünbaum in [6].

Determining $a'(G)$ either theoretically or algorithmically has been a very difficult problem. Even for the highly structured and simple class of complete graphs, the value of $a'(G)$ is not yet determined. Determining the exact values of $a'(G)$ even for very special classes of graphs is still open.

It is easy to see that $a'(G) \geq \chi'(G) \geq \Delta(G)$ for any graph G. Here, $\chi'(G)$ denotes the chromatic index of G (the minimum number of colours used in any proper edge colouring of G). Using probabilistic arguments, Alon, McDiarmid, and Reed [1] obtained an upper bound of $64\Delta(G)$ on $a'(G)$. Using the same analysis but with more careful calculations, Molloy and Reed [8] obtained an improvement of $a'(G) \leq 16\Delta(G)$.

Recently, Muthu, Narayanan, and Subramanian [9] obtained a better bound of $a'(G) \leq 4.52\Delta(G)$ for graphs G with girth (the length of the shortest cycle) at least 220. Concerning constructive bounds, Subramanian [16] presents an $O(\Delta(G) \log \Delta(G))$ upper bound which is valid for any graph G.

It follows from the work of Burnštein [5] that $a'(G) \leq \Delta(G) + 2$ for all graphs with $\Delta(G) \leq 3$. It was conjectured by Alon, Sudakov, and Zaks [2] that always $a'(G) \leq \Delta(G) + 2$; they also demonstrated the tightness of the conjecture by providing examples of graphs requiring $\Delta(G) + 2$ colours in any acyclic edge colouring. The conjecture was shown in [2] to be true for almost every d-regular (d fixed) graph. Recently, Nešetřil and Wormald [13] strengthened the latter result by showing that $a'(G) \leq d + 1$ for almost every d-regular graph.

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The proofs of the above mentioned bounds are existential in nature and are not constructive. In this work, we look at the class of those graphs that can be expressed as a finite Cartesian product of graphs each of which is an induced path or cycle. We show that (see Theorem 3) $a'(G) \in \{\Delta(G), \Delta(G)+1\}$ for each member of this class and also obtain the exact value of $a'(G)$ for all G except when G is a product of a path and a number of odd cycles. Thus we verify the above conjecture for these graphs, which we refer to as *partial tori*. As special cases, this class includes other well-known classes like hypercubes, d-dimensional meshes, etc. All these definitions are given below. Our results are proved by an explicit constructive colouring scheme, and the colouring can be constructed in polynomial time in the size of the graph (see Theorem 5). Hence our results are both exact and constructive. There are very few non-trivial graph classes where $a'(G)$ has been determined so closely.

1.1. Definitions and notation

We use P_k to denote a simple path on k vertices. Without loss of generality (w.l.o.g.), we may let $V(P_k) = \{0, \ldots, k-1\}$ and $E(P_k) = \{(i,j) : |i-j| = 1\}$. Similarly, we use C_k to denote a cycle $(0, \ldots, k-1, 0)$ on k vertices. We use "PATHS" to denote the set $\{P_3, P_4, \ldots\}$ of all paths on 3 or more vertices. Similarly, we use "CYCLES" to denote the set $\{C_3, C_4, \ldots\}$ of all cycles. The standard notation $[n]$ is used to denote the set $\{1, 2, \ldots, n\}$.

Our definition of the class of partial tori is based on the so-called *Cartesian product* of graphs defined below.

Definition 1. Given two graphs G_1 and G_2, the *Cartesian product* of G_1 and G_2, denoted by $G_1\Box G_2$, is defined to be the graph G with $V(G) = V(G_1) \times V(G_2)$ and $E(G)$ contains the edge joining (u_1, u_2) and (v_1, v_2) if and only if *either* $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$ *or* $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$.

Note that $\Box$ is a binary operation on graphs that is commutative in the sense that $G_1\Box G_2$ and $G_2\Box G_1$ are isomorphic. Similarly, it is also associative. Hence the graph $G_0\Box G_1\Box \cdots \Box G_d$ is unambiguously defined for any d. We use G^d to denote the d-fold Cartesian product of G with itself. It was shown by Sabidussi [14] and Vizing [17] (see also [7]) that any connected graph G can be expressed as a product $G = G_1\Box \cdots \Box G_k$ of prime factors G_i. Here, a graph is said to be *prime* with respect to the $\Box$ operation if it has at least two vertices and if it is not isomorphic to the product of two non-trivial graphs (those having at least two vertices). Also, this factorisation (or decomposition) is unique except for a re-ordering of the factors and will be referred to as the *Unique Prime Factorisation (UPF)* of the graph. Since $a'(G)$ is a graph invariant, we assume, without loss of generality, that any G_i is from $\{K_2\} \cup$ PATHS $\cup$ CYCLES *if* it is either an induced path or an induced cycle.

Definition 2. A *d-dimensional partial torus* is a connected graph G whose unique prime factorisation is of the form $G = G_1\Box \cdots \Box G_d$, where $G_i \in \{K_2\} \cup$ PATHS $\cup$ CYCLES for each $i \le d$. We denote the class of such graphs by $\mathcal{P}_d$.

Definition 3. If each prime factor of a graph $G \in \mathcal{P}_d$ is a K_2, then G is the *d-dimensional hypercube*. This graph is denoted by K_2^d.

Definition 4. If each prime factor of a graph $G \in \mathcal{P}_d$ is from PATHS, then G is a *d-dimensional mesh*. The class of all such graphs is denoted by $\mathcal{M}_d$.

Definition 5. If each prime factor of a graph $G \in \mathcal{P}_d$ is from CYCLES, then G is a *d-dimensional torus*. The class of all such graphs is denoted by $\mathcal{T}_d$.

1.2. Results

The proof of the results mentioned in the abstract is based on the following useful theorem whose proof is given later.

Theorem 1. *If G is a simple graph with $a'(G) = \eta$, then*

1. $a'(G\Box P_2) \le \eta + 1$, *if* $\eta \ge 2$.
2. $a'(G\Box P_l) \le \eta + 2$, *if* $\eta \ge 2$ *and* $l \ge 3$.
3. $a'(G\Box C_l) \le \eta + 2$, *if* $\eta > 2$ *and* $l \ge 3$.

The first two of the three mentioned results are special cases of the following more general result obtained in [12]. The third result however is stronger than what follows from the result in [12]. Hence, we provide only the proof of the third statement. An independent proof of the first two statements also appeared in a preliminary conference version [10] of the current paper.

Theorem 2 ([12]). *If G and H are two connected non-trivial graphs such that* $\max\{a'(G), a'(H)\} > 1$, *then*

$$a'(G\Box H) \le a'(G) + a'(H).$$

As a corollary, we obtain the following results.

Theorem 3. *The following is true for each $d \geq 1$.*

- $a'(K_2^d) = \Delta(K_2^d) + 1 = d + 1$ *if* $d \geq 2$; $a'(K_2) = 1$.
- $a'(G) = \Delta(G) = 2d$ *for each* $G \in \mathcal{M}_d$.
- $a'(G) = \Delta(G) + 1 = 2d + 1$ *for each* $G \in \mathcal{T}_d$.
- *Let* $G \in \mathcal{P}_d$ *be any graph. If* e *(respectively* p *and* c*) denote the number of prime factors of* G *which are* K_2*'s (respectively from* PATHS *and* CYCLES*), then*
  - $a'(G) = \Delta(G) + 1 = e + 2c + 1$ *if* $p = 0$.
  - $a'(G) = \Delta(G) = e + 2p + 2c$ *if* either $p \geq 2$, or $p = 1$ *and* $e \geq 1$.
  - $a'(G) = \Delta(G) = 2 + 2c$ *if* $p = 1$, $e = 0$ *and if at least one prime factor of* G *an even cycle.*
  - $a'(G) \in \{\Delta(G) = 2 + 2c, \Delta(G) + 1 = 2 + 2c + 1\}$ *if* $p = 1$, $e = 0$ *and if all prime factors of* G *(except the one path) are odd cycles. There are examples for both values of* $a'(G)$.

2. Proofs

The following useful fact about acyclic edge colouring can be easily verified.

Fact 4. *If a graph* G *is regular with* $\Delta(G) \geq 2$, *then* $a'(G) \geq \Delta(G) + 1$.

This is because in any proper edge-colouring of G with $\Delta(G)$ colours, each colour is used on an edge incident at every vertex. Hence, for each pair of distinct colours a and b and for each vertex u, there is a unique cycle in G going through u that is coloured with a and b.

We first present the proof of Theorem 3.

Proof (*Of* Theorem 3)**.** *Case* $G = K_2^d$: Clearly, $a'(K_2) = 1$ and $a'(K_2^2) = a'(C_4) = 3$. For $d > 2$, we start with $G = K_2^2$ and repeatedly and inductively apply Statement (1) of Theorem 1 to deduce that $a'(K_2^d) \leq d + 1$. Combining this with Fact 4, we get $a'(K_2^d) = d + 1$ for $d \geq 2$.

Case $G \in \mathcal{M}_d$: Again, we use induction on d. If $d = 1$, then $G \in$ PATHS and hence $a'(G) = 2 = \Delta(G)$. or $d > 1$, repeatedly and inductively apply Statement (2) of Theorem 1 to deduce that $a'(G) \leq 2(d-1)+2 = 2d$. Combining this with the trivial lower bound $a'(G) \geq \Delta(G)$, we get $a'(G) = 2d$ for each $G \in \mathcal{M}_d$ and each $d \geq 1$.

Case $G \in \mathcal{T}_d$: We use induction on d. If $d = 1$, then $G \in$ CYCLES and hence $a'(G) = 3 = \Delta(G) + 1$. For $d > 1$, repeatedly and inductively apply Statement (3) of Theorem 1 to deduce that $a'(G) \leq 2(d-1) + 1 + 2 = 2d + 1$. Combining this with Fact 4, we get $a'(G) = 2d + 1$ for each $G \in \mathcal{T}_d$ and each $d \geq 1$.

Case $G \in \mathcal{P}_d$: Let e, p, and c be as defined in the statement of the theorem. If $p = 0$, then G is the product of edges and cycles, and hence G is regular and $a'(G) \geq \Delta(G) + 1$ by Fact 4. Also, we can assume that $c > 0$. Otherwise, $G = K_2^d$, and this case has already been established. Again, without loss of generality, we can assume that the first factor G_1 of G is from CYCLES and $a'(G_1) = 3$. Now, as in the previous cases, we apply induction on d and also repeatedly apply one of the Statements (1) and (3) of Theorem 1 to deduce that $a'(G) \leq \Delta(G) + 1$. This settles the case $p = 0$.

Now, suppose *either* $p \geq 2$, *or* $p = 1$ and $e \geq 1$. Order the d prime factors of G so that $G = G_1 \square \cdots \square G_d$ and the first p factors are from PATHS and the next e factors are copies of K_2. By the previously established cases and from Theorem 1, it follows that

$$a'(G_1 \square \cdots \square G_{p+e}) = \Delta(G_1 \square \cdots \square G_{p+e}) = 2p + e \geq 3.$$

As before, applying (3) of Theorem 1 inductively, it follows that

$$a'(G) = a'(G_1 \square \cdots \square G_{p+e+c}) \leq \Delta(G) = 2p + e + 2c.$$

Combining this with the trivial lower bound establishes this case also.

Suppose $p = 1$, $e = 0$, and at least one prime factor of G is an even cycle. Let $G_1 = P_k$ for some $k \geq 3$ and $G_2 = C_{2l}$ for some $l \geq 2$. We note that it is enough to show that $G' = G_2 \square G_1$ is acyclically colourable with $\Delta(G')$ colours, where $\Delta(G') = 4$. Extending this colouring to an optimal colouring of G can be achieved by repeated applications of Statement (3) of Theorem 1 as before. Hence we focus on showing $a'(G') = 4$.

First, colour the cycle $G_2 = C_{2l} = (0, 1, \ldots, 2l-1, 0)$ acyclically as follows. For each i, $0 \leq i \leq 2l-2$, colour the edge $(i, i+1)$ with 1 if i is even and with 2 if i is odd. Colour the edge $(2l-1, 0)$ with 3. Now, use the same colouring on each of the k isomorphic copies (numbered with $0, \ldots, k-1$) of G_2. For each j with $0 \leq j < k-1$, the jth and $(j+1)$th copies of G_2 are joined by edges which constitute a perfect matching between similar vertices in the two copies. These edges are coloured as follows. For every i and j, the edge joining (i, j) and $(i, j+1)$ is coloured as follows: If $(i+j)$ is even, the edge is coloured with 4. Otherwise, it is coloured with the unique colour from $\{1, 2, 3\}$ which is missing at this vertex i in both copies. See Fig. 1 for an illustration.

The colouring is such that in each perfect matching joining two adjacent copies of G_2, the edges which are part of this matching are alternately coloured with 4 and a colour from $\{1, 2, 3\}$. Note that there can be no bichromatic cycle within each copy of G_2. Hence any bichromatic cycle (if it exists) should use edges of this perfect matching.

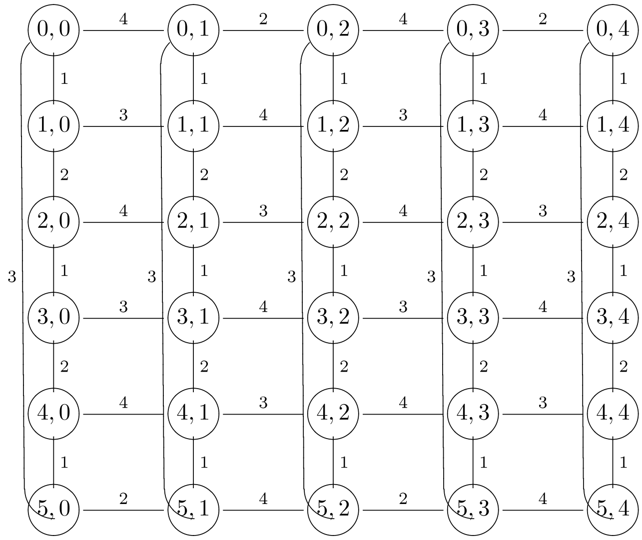

Fig. 1. Colouring of $C_6 \square P_5$.

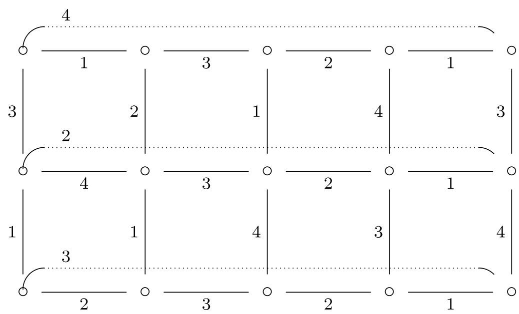

Fig. 2. Colouring of $P_3 \square C_5$.

First, we claim that there can be no $(4, c)$-coloured cycle for any $c \in \{1, 2, 3\}$. To see this, note that no two successive edges of any such cycle can be from the same copy of G_2 since there is no edge coloured 4 in any copy of G_2. In addition, to complete a cycle it is necessary that there must be two adjacent copies, say the jth and the $(j+1)$th, such that the cycle passes from the jth to the $(j+1)$th and back to jth copy using exactly 3 edges. This contradicts the fact that the edges between adjacent copies are alternately coloured with 4 and a colour from $\{1, 2, 3\}$.

In addition, there can be no (c, c')-coloured cycle for any $c, c' \in \{1, 2, 3\}$. To see this, we first note that any maximal (c, c')-coloured path in the jth (for any j) copy of G_2 is of odd length (= number of edges) and hence the first and last edge of such a path are coloured the same, say with c. This means the c'-coloured edges incident at the two end points u and v connect them to the different, namely the $(j-1)$th and $(j+1)$th, copies (because of the way these edges are coloured). Extending this further, we see that any (c, c')-coloured maximal path starts at some $(u, 0)$ and ends at some $(v, k-1)$ and does not complete to a cycle. This shows that $a'(G') = 4$ as desired.

Finally, suppose $p = 1$, $e = 0$, and all prime factors of G (except the one path) are odd cycles. In this case, $a'(G)$ can take both values as the following examples show. If $G = P_3 \square C_3$, then it can be easily verified that $a'(G) = 5 = \Delta(G) + 1$. Also, if $G = P_3 \square C_5$, then $a'(G) = 4 = \Delta(G)$ as shown by the colouring in Fig. 2. □

We now present the proof of Theorem 1. A restricted class of bijections (defined below) will play an important role in this proof.

Definition. A bijection σ from a set $\mathcal{A}$ to a set $\mathcal{B}$ of the same cardinality is a *non-fixing* bijection, if $\sigma(i) \neq i$ for each i.

Proof (*Of* Theorem 1)**.** Throughout this proof, by the term *cross edge*, we mean an edge in the perfect matching joining two consecutive copies of G in $G \square H$ where $H \in \{K_2, \text{PATHS}, \text{CYCLES}\}$.

Since $a'(G) = \eta$, we can edge-colour G acyclically using colours from $[\eta]$. Fix one such colouring $\mathcal{C}_0 : E(G) \to [\eta]$.

Define $\mathcal{C}_1$ to be the colouring defined by $\mathcal{C}_1(e) = \sigma(\mathcal{C}_0(e))$ where $\sigma : [\eta] \to [\eta]$ is any bijection which is *non-fixing*. For concreteness, define $\sigma(i) = (i \bmod \eta) + 1$.

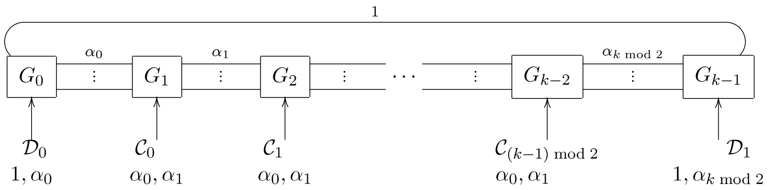

Fig. 3. Colouring of $G \Box C_k$; Note: the two colours indicated under G_i represent colours unused in that copy.

The first two statements of Theorem 1 relating to bounds on $a'(G \Box P_2)$ (with $\eta \geq 2$) and on $a'(G \Box P_l)$ (with $\eta \geq 2$ and $l \geq 3$) follow directly from Theorem 2. See also an independent proof in [10].

Hence we focus only on the third statement relating to $a'(G \Box C_k)$ (with $\eta > 2$ and $k \geq 3$). Consider $G \Box C_k$, $k \geq 3$. Here we have k isomorphic copies of G numbered $G_0, G_1, \ldots, G_{k-2}, G_{k-1}$ such that there is a perfect matching between successive copies G_i and $G_{(i+1) \bmod k}$ (see Fig. 3). Our colouring is as follows.

For each i, $1 \leq i \leq k-2$, colour the edges of G_i with $\mathcal{C}_{(i+1) \bmod 2}$.

Let α_0, α_1 be two new colours which are not in $[\eta]$. Let $\mathcal{D}_0$ be a colouring of G_0 defined by $\mathcal{D}_0(e) = \tau(\mathcal{C}_0(e))$ where $\tau(i) = i+1, i < \eta, \tau(\eta) = \alpha_1$.

In order to colour G_{k-1}, define the colouring $\mathcal{D}_1(e) = \mu(\mathcal{C}_0(e))$ where $\mu(i) = i+2, i < \eta - 1$ and $\mu(\eta - 1) = \alpha_{(k+1) \bmod 2}, \mu(\eta) = 2$.

Now, colour any edge of the form $((u, i), (u, i+1))$, $0 \leq i < k-1$ with the new colour $\alpha_{i \bmod 2}$. Colour the edges of the form $((u, k-1), (u, 0))$ with the colour 1. Denote this colouring of $G \Box C_k$ by $\mathcal{C}$.

We claim that $\mathcal{C}$ is proper and acyclic. For each i, the colouring $\mathcal{C}$ restricted to G_i is proper and acyclic by definition. Also note that, each edge $((u, i), (u, (i+1) \bmod k))$ is coloured with a colour γ (say) which is not used in either of the copies G_i and G_{i+1}. Hence $\mathcal{C}$ is proper.

Also, in $\mathcal{C}$, any edge $e \in G_i$ and its copy $e' \in G_{(i+1) \bmod k}$ receive different colours (since the colourings on successive copies of G are based on mutually non-fixing bijections). It can be seen from the proof for the Case $G \Box P_2$ (see [10]) that there can be no bichromatic cycle in $\mathcal{C}$ restricted to two successive copies of G. Hence any such bichromatic cycle C should pass through at least 3 consecutive copies of G, again fixing the two colours of C to be those used on two incident cross edges. Also, it is easy to see that there can be no bichromatic cycle involving only cross edges since any such cycle uses the three colours $\{\alpha_0, \alpha_1, 1\}$.

Note that each of $G_1, \ldots, G_{k-2}$ are coloured free of both α_0 and α_1. Hence any (α_0, α_1)-bichromatic cycle C should start from some vertex $(u_1, 0)$ in G_0, then reach $(u_1, k-1)$ using only cross edges, then go to some vertex $(u_2, k-1)$ using an edge of G_{k-1}, then reach $(u_2, 0)$ using only cross edges and then some vertex $(u_3, 0)$ using a α_1-coloured edge of G_0 and continue this until it finally reaches a vertex $(u_k, 0)$ (where k is an even number), and then go to $(u_1, 0)$ using a α_1-coloured edge of G_0. Here the only non-cross edges used in C are either from G_0 (and coloured with α_1) or from G_{k-1} (and coloured with either α_0 or α_1 depending on the parity of k). From the definitions of $\mathcal{D}_0$ and $\mathcal{D}_1$, it follows that for each edge $(u_{2l+1}, k-1) \rightarrow (u_{2l+2}, k-1)$ from G_{k-1} used in C, its isomorphic copy in G_1, namely $(u_{2l+1}, 0) \rightarrow (u_{2l+2}, 0)$, is coloured with η. This implies the existence of a (α_1, η)-coloured bichromatic cycle in G_1, and this is a contradiction.

Similarly, any $(\alpha_0, 1)$-coloured bichromatic cycle should only visit vertices in the copies $G_1, G_0, G_{k-1}, G_{k-2}$ (or G_1, G_0, G_{k-1}) depending on whether k is even (or odd). As argued before, this would imply the existence of a $(1, \eta)$-coloured cycle in G_1 (or a $(1, (\eta - 1))$-coloured cycle in G_1) contradicting our definition of $\mathcal{C}$.

Also, if k is even, then any $(1, \alpha_1)$-coloured cycle should only visit vertices in G_0 and G_{k-1} (which are consecutive) and hence cannot exist. If k is odd, then such a cycle can only visit vertices in G_0, G_{k-1} and G_{k-2} and its existence would imply the existence of a $(2, \alpha_1)$-coloured cycle in G_0, again a contradiction. This shows that $\mathcal{C}$ is acyclic. $\Box$

3. Conclusions

There is very little study of algorithmic aspects of acyclic edge colouring. In [3], Alon and Zaks prove that it is NP-complete to determine if $a'(G) \leq 3$ for an arbitrary graph G. They also describe a deterministic polynomial time algorithm which obtains an acyclic $(\Delta(G) + 2)$-edge-colouring for any graph G whose girth g is at least $c\Delta(G)^3$ for some large absolute constant c. Skulrattanakulchai [15] presents a linear time algorithm to acyclically edge colour any graph with $\Delta \leq 3$ using at most 5 colours. Also, Muthu, Narayanan, and Subramanian [11] present an $O(n \log \Delta(G))$ time algorithm which obtains an acyclic edge colouring of any outerplanar graph using $\Delta(G) + 1$ colours.

The proofs of Theorems 1 and 3 are constructive and readily translate to efficient (polynomial-time) algorithms which find optimal (or almost optimal) acyclic edge colourings of the partial tori. Further, if the input partial tori is given with its prime factorisation the algorithm computes the colouring in linear time. Formally,

Theorem 5. *If $G \in \mathcal{P}_d$ is a graph (on n vertices and m edges) specified by its Unique Prime Factorisation, then an acyclic edge colouring of G using $\Delta(G)$ or $\Delta(G) + 1$ colours can be obtained in $O(n + m)$ time. Also, the colouring is optimal except when G is a product of a path and a number of odd cycles.*

For the sake of completeness, we present a brief and formal description of these Algorithms in the Appendix. Before we finish, we need to say a few words about how the input is presented to the algorithm. It is known from the work of Aurenhammer, Hagauer, and Imrich [4] that the UPF of a connected graph G (on n vertices and m edges) can be obtained in $O(m \log n)$ time. Hence we assume that our connected input $G \in \mathscr{P}_d$ is given by the list of its prime factors $G_1, \ldots, G_d$. Also, without loss of generality, we assume that the list is such that

(i) $G_i \in$ PATHS for $i \in \{1, \ldots, p\}$;
(ii) $G_i = K_2$ for $i \in \{p+1, \ldots, p+e\}$;
(iii) $G_i \in$ CYCLES for $i \in \{p+e+1, \ldots, d = p+c+e\}$ and all even cycles appear before all odd cycles in the order.

Here p, e, c denote respectively the number of prime factors which are from PATHS , $\{K_2\}$, and CYCLES.

If G is isomorphic to the product of a path and a number of odd cycles, the acyclic chromatic index can take either of the values in $\{\Delta(G), \Delta(G)+1\}$. It would be interesting to see if we can classify such graphs for which $a'(G) = \Delta(G)$. It would also be nice to construct an optimal colouring efficiently for such graphs.

Appendix. Algorithms

Algorithm 1 AcycColPCGrid($G_1, \ldots, G_d$)

if $d = 1$, **then** output an optimal acyclic edge-colouring of G_1 using 2 (1 or 3) colours depending on whether $G_1 \in$ PATHS ($G_1 = K_2$ or $G_1 \in$ CYCLES) and exit.
if $d = 2$ **then**
  if both $G_1 = G_2 = K_2$, **then** output an optimal colouring of $G_1 \square G_2$ using 3 colours and exit.
  if *either* $G_1 = K_2$ and $G_2 \in$ CYCLES *or* $G_1 \in$ PATHS and G_2 is an even cycle, **then** interchange G_1 and G_2; Otherwise, let G_1 and G_2 remain the same.
  Let $\mathcal{C}_0$ be an optimal acyclic colouring of G_1 (on l vertices) defined as follows: For each i, $0 \le i < l-1$, colour the edge $(i, i+1)$ with i mod 2. Colour the edge $(l-1, 0)$ (if it exists) with 3.
  Output the optimal colouring obtained by applying Acycol2fac($G_2, G_1, \mathcal{C}_0$) and exit.
end if
if $d > 2$ **then**
  Apply AcycColPCGrid($G_1, \ldots, G_{(d-1)}$) to get an optimal colouring $\mathcal{C}_0$ of $G = G_1 \square \cdots \square G_{d-1}$.
  Obtain an optimal colouring of $G \square G_d$ by applying Acycol2fac($G, G_d, \mathcal{C}_0$).
  Output the optimal colouring of $G_1 \square \cdots \square G_d$ thus obtained and exit.
end if

Algorithm 2 Acycol2fac($G, H, \mathcal{C}_0$)

Let H be a path or cycle on $k \ge 2$ vertices $\{0, \ldots, k-1\}$. Let $G_0, \ldots, G_{k-1}$ be the k isomorphic copies of G induced respectively by the sets $\{(u, i) : u \in V(G)\}$ for each i.
if G is an even cycle C_{2l} and $H = P_k$, **then** colour each of the k isomorphic copies of G by the colouring $\mathcal{C}_0$. For every j ($0 \le j < k-1$) and i ($0 \le i \le 2l-1$), colour the edge joining (i, j) and $(i, j+1)$ with 4 *if* $i+j$ is even and colour it with the unique colour from $\{1, 2, 3\}$ which is missing at both copies of i *if* $i+j$ is odd and exit.
Otherwise, suppose $\mathcal{C}_0$ uses colours from $[\eta] = \{1, \ldots, \eta\}$ for some $\eta > 0$. Let σ, τ, μ be three permutations over $[\eta+2] = \{1, \ldots, \eta+2\}$ defined by
  $\sigma(i) = (i \bmod \eta) + 1$ for $i \in [\eta]$ and $\sigma(i) = i$ for $i > \eta$.
  $\tau(i) = i+1$ for $i < \eta$, $\tau(\eta) = \eta+1$, $\tau(\eta+1) = 1$ and $\tau(\eta+2) = \eta+2$.
  $\mu(i) = i+2$ for $i < \eta-1$, $\mu(\eta-1) = \eta+1+((k+1) \bmod 2)$, $\mu(\eta) = 2$, $\mu(\eta+1+((k+1) \bmod 2)) = 1$ and $\mu(\eta+1+(k \bmod 2)) = \eta+1+(k \bmod 2)$.
Let $\mathcal{C}_1, \mathcal{D}_0, \mathcal{D}_1$ be three new colourings of G obtained respectively by colouring each edge e of G by the colour $\sigma(\mathcal{C}_0(e)), \tau(\mathcal{C}_0(e)), \mu(\mathcal{C}_0(e))$.
if $H = P_k$, **then** colour each copy G_i by the colouring $\mathcal{C}_{i \bmod 2}$. Also, for each $i < k-1$, colour the edges between G_i and G_{i+1} with the common colour missing from both of them. This missing colour is $\eta+1+(i \bmod 2)$.
if $H = C_k$, **then**, for each i, $0 < i < k-1$, colour G_i by the colouring $\mathcal{C}_{(i+1) \bmod 2}$. Also, colour G_0 by $\mathcal{D}_0$ and colour G_{k-1} by $\mathcal{D}_1$. Also, for each $0 \le i < k-1$, colour the edges between G_i and G_{i+1} with the common colour, namely $\eta+1+(i \bmod 2)$, missing from both of them. Colour the edges between G_0 and G_{k-1} with 1.

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