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Weighted and regularity estimates for nonlinear equations on Reifenberg flat domains

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ABSTRACT

Global weighted L^p estimates are obtained for the gradient of solutions to nonlinear elliptic Dirichlet boundary value problems over a bounded nonsmooth domain. Morrey and Hölder regularity of solutions are also established, as a consequence. These results generalize various existing estimates for nonlinear equations. The nonlinearities are of at most linear growth and assumed to have a uniform small mean oscillation. The boundary of the domain, on the other hand, may exhibit roughness but assumed to be sufficiently flat in the sense of Reifenberg. Our approach uses maximal function estimates and Vitali covering lemma, and also known regularity results of solutions to nonlinear homogeneous equations.

1. Introduction

In this paper we study solutions to nonlinear boundary value problems of the form

$$\begin{cases} \operatorname{div} \mathbf{a}(\nabla u, x) = \operatorname{div} \mathbf{f} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the "nonlinearity" $\mathbf{a}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in x for all $\xi \in \mathbb{R}^n$ and continuous in ξ for almost all x. We assume that $\mathbf{f}$ is a given vector valued function at least in $L^2(\Omega, \mathbb{R}^n)$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with nonsmooth boundary. We will specify the nature of the boundary shortly.

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The main goal of this work is to obtain global weighted L^p estimates for the gradient of solutions to (1.1) that will hold for all $p > 2$. Regularity results in Morrey and Hölder spaces will also be established as a consequence of these weighted estimates.

To be specific, for any $p > 2$ we give conditions on the nonlinearity $\mathbf{a}(\xi, x)$, on the domain Ω and on the weight w such that for a solution u of (1.1)

$$\int_\Omega |\mathbf{f}|^p w(x)\,dx < +\infty \quad \Rightarrow \quad \int_\Omega |\nabla u|^p w(x)\,dx < +\infty.$$

Solutions to (1.1) are understood in the standard weak sense, i.e., $u \in W_0^{1,2}(\Omega)$ is a weak solution to (1.1) if

$$\int_\Omega \mathbf{a}(\nabla u, x) \cdot \nabla\phi\,dx = \int_\Omega \mathbf{f} \cdot \nabla\phi\,dx \tag{1.2}$$

for all test functions $\phi \in W_0^{1,2}(\Omega)$.

Existence of such solutions can be proved under the following standard growth and monotonicity assumptions on $\mathbf{a}(\xi, x)$: for some positive constants c_0 and c_1,

$$\big[\mathbf{a}(\xi, x) - \mathbf{a}(\eta, x)\big] \cdot (\xi - \eta) \geqslant c_0|\xi - \eta|^2 \tag{1.3}$$

and

$$\big|\mathbf{a}(\xi, x)\big| \leqslant c_1\big(1 + |\xi|\big) \tag{1.4}$$

for every choice of $\xi, \eta \in \mathbb{R}^n$, and a.e. $x \in \mathbb{R}^n$. Indeed, it follows from the nonvariational method of Browder and Minty for monotone operators (see, e.g., [8, Section 9.1]) that a unique weak solution $u \in W_0^{1,2}(\Omega)$ to the Dirichlet problem (1.1) exists and satisfies the global $W^{1,2}$-estimate

$$\|\nabla u\|_{L^2(\Omega)} \leqslant C\big(\|\mathbf{f}\|_{L^2(\Omega)} + 1\big), \quad C = C(c_0, c_1, \Omega). \tag{1.5}$$

Hereafter we assume that (1.3) and (1.4) are satisfied. For our purpose we also assume that $\mathbf{a}(\xi, x)$ is uniformly Lipschitz continuous in ξ and satisfies, for some positive constant c_2,

$$\big|\nabla_\xi \mathbf{a}(\xi, x)\big| \leqslant c_2. \tag{1.6}$$

We emphasize that our intention is to obtain global weighted L^p estimates for gradients of solutions to (1.1) with $\mathbf{f}$ in the same weighted L^p space, that holds for all $p > 2$. The stated structural assumptions on the nonlinearity $\mathbf{a}(\xi, x)$, however, are not enough to accomplish this. In the linear case when $\mathbf{a}(\xi, x) = A(x)\xi$, and $A(x)$ an $n \times n$ matrix function, for example, one cannot expect such strong estimates to hold for general discontinuous uniformly elliptic $A(x)$. See the counterexample in [19]. In the standard L^p theory for linear equations over smooth domains, requiring coefficients to have small mean oscillations in the x-variable is found to be sufficient. One would expect to require the same in the nonlinear case. To make this precise we define, as in [3], a function that measures the oscillation of $\mathbf{a}(\xi, \cdot)$ over measurable sets. For each measurable set $D \subset \mathbb{R}^n$ we let $\beta = \beta(\mathbf{a}, D) : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\beta(\mathbf{a}, D)(x) := \sup_{\xi \in \mathbb{R}^n} \frac{|\mathbf{a}(\xi, x) - \bar{\mathbf{a}}_D(\xi)|}{1 + |\xi|},$$

where

$$\bar{\mathbf{a}}_D(\xi) = \fint_D \mathbf{a}(\xi, x)\,dx = \frac{1}{|D|}\int_D \mathbf{a}(\xi, x)\,dx \tag{1.7}$$

is the integral average of $\mathbf{a}(\xi, \cdot)$ over D. In the linear case, we see that

$$\beta(\mathbf{a}, D)(x) \leqslant \left|A(x) - \bar{A}_D\right|$$

for almost every $x \in \mathbb{R}^n$. Thus one may then think of $\beta(\mathbf{a}, D)$ as a natural extension of the function that measures oscillations to the nonlinear setting. Below $B_\rho(y)$ refers to a ball centered at y with radius ρ. The nonlinear version of the definition of small mean oscillation condition is given below.

Definition 1.1. We say that $\mathbf{a}(\xi, x)$ satisfies (δ, R)-BMO condition for some δ, $R > 0$, if

$$\sup_{0<\rho\leqslant R}\sup_{y\in\mathbb{R}^n} \fint_{B_\rho(y)} \left|\beta\big(\mathbf{a}, B_\rho(y)\big)(x)\right|^2 dx \leqslant \delta^2. \tag{1.8}$$

The (δ, R)-BMO condition which appeared in [3,4] is also called a small BMO condition and has been used in various work as an appropriate substitute for the Sarason VMO (vanishing mean oscillation [27]) condition (see, e.g., [3,4,11,14,20,28,30]).

Notation. Throughout the paper $\mathcal{A}_{\delta,R}$ denotes the set of vector functions $\mathbf{a}(\xi, x)$ that satisfies inequalities (1.3), (1.4), (1.6), and (1.8).

Another obstacle in obtaining global integrability results is the roughness of the boundary of the ground domain. Even for smooth coefficients, global integrability of gradients of solutions to (1.1) over domains with bad boundary may not be true. The counterexample given in [15] comes to mind. For $\pi/2 < \theta_0 < \pi$, consider the nonconvex domain in $\mathbb{R}^2$ given in polar coordinates by $\Omega_{\theta_0} = \{(r, \theta)\colon\ 0 \leqslant r \leqslant 1,\ -\theta_0 \leqslant \theta \leqslant \theta_0\}$. For $\lambda = \frac{\pi}{2\theta_0} < 1$ consider the function

$$v(r, \theta) = u(r, \theta) - r^2 u(r, \theta), \tag{1.9}$$

where $u(r, \theta) = r^\lambda \cos(\lambda\theta)$ is a harmonic function. Then for $\mathbf{f} = \nabla[-r^2 u(r, \theta)]$, we see that v vanishes on $\partial\Omega_{\theta_0}$ and solves the nonhomogeneous linear equation

$$\Delta v = \operatorname{div}\mathbf{f} \quad \text{in } \Omega.$$

From (1.9) we see that $\nabla v = \nabla u + \mathbf{f}$, and $|\mathbf{f}| \leqslant Cr^{\lambda+1}$ for all $(r, \theta) \in \Omega_{\theta_0}$. As a result, for r close to 0, $|\nabla v| \approx |\nabla u| = \lambda r^{\lambda-1}$. It follows then that, on the one hand, for any $p > 4$ we can find a θ_0 such that $|\nabla v| \notin L^p(\Omega_{\theta_0})$. It is also true, on the other hand, that for any p there exists θ_0 sufficiently close to $\pi/2$ such that $|\nabla v| \in L^p(\Omega_{\theta_0})$. Observe that the choices of θ_0 determine whether the boundary of the domain at the origin is sufficiently flat, as it indeed is for the latter choice.

The above example suggests that we need to impose some kind of flatness assumption on the boundary of the domain in order to obtain a global integrability result. Essentially, at each boundary point and every scale, we require the boundary of the domain to be between two hyperplanes separated by a distance that depends on the scale. The following defines the relevant geometry precisely.

Definition 1.2. We say that Ω is a (δ, R)-Reifenberg flat domain if for every $x \in \partial\Omega$ and every $r \in (0, R]$, there exists a system of coordinates $\{y_1, y_2, \ldots, y_n\}$, which may depend on r and x, so that $x = 0$ in this coordinate system and that

$$B_r(0) \cap \{y_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{y_n > -\delta r\}.$$

Reifenberg flat domains appear naturally in the theory of minimal surfaces and free boundary problems. They first appeared in a paper of E. Reifenberg (see [26]) in the context of a Plateau problem. Reifenberg flat domains can be very rough. They include Lipschitz domains with sufficiently small Lipschitz constants (see [29]) and even some domains with fractal boundaries. We refer to [12,16–18, 26,29] for further discussion on Reifenberg flat domains.

The remark given below will be used later in the paper. It follows from the geometry and definition of Reifenberg flat domains.

Remark 1.3. If Ω is (δ, R)-Reifenberg flat domain with $\delta < 1$, then for any point x on the boundary of Ω and $0 < \rho < R(1-\delta)$ there exists a coordinate system $\{z_1, z_2, \ldots, z_n\}$ with the origin at some point in the interior of Ω such that in this coordinate system $x = -\delta'\rho z_n$ and

$$B_\rho^+ \subset \Omega_\rho \subset B_\rho \cap \bigl\{z = (z_1, z_2, \ldots, z_n)\colon z_n > -2\rho\delta'\bigr\}, \quad \text{where } \delta' = \frac{\delta}{1-\delta}.$$

Notation. Here and in what follows we adopt the notation $\Omega_r = B_r \cap \Omega$ and $B_r^+ = B_r \cap \{z = (z_1, z_2, \ldots, z_n)\colon z_n > 0\}$ in a given coordinate system.

2. Main results and applications

2.1. Main results

To state one of the main results we first recall that a nonnegative function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is called an A_s weight, $1 < s < \infty$, if the quantity

$$[w]_s := \sup\biggl(\fint_B w(x)\,dx\biggr)\biggl(\fint_B w(x)^{\frac{-1}{s-1}}\,dx\biggr)^{s-1} < +\infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. In this case $[w]_s$ is called the A_s constant of w. Related to A_s is the weighted Lebesgue space $L^s_w(\Omega)$, $1 < s < \infty$, which is the set of measurable functions g on Ω such that

$$\|g\|_{L^s_w(\Omega)} := \biggl(\int_\Omega |g|^s w\,dx\biggr)^{1/s} < +\infty.$$

Properties of A_s weights and relevant weighted estimates that are used in this paper are given in the next section. Unless otherwise stated we assume that $2 < p < \infty$, and w is an $A_{p/2}$ weight throughout the paper.

Theorem 2.1. *There exist positive constants C and δ such that if $u \in W_0^{1,2}(\Omega)$ is a weak solution to* (1.1) *in a (δ, R)-Reifenberg flat domain Ω with $\mathbf{a} \in \mathcal{A}_{\delta,R}$ for some $R > 0$, and $|\mathbf{f}| \in L^p_w(\Omega)$, then $|\nabla u| \in L^p_w(\Omega)$ and*

$$\int_\Omega |\nabla u|^p w\,dx \leqslant C \int_\Omega \bigl(|\mathbf{f}|^p + 1\bigr) w\,dx.$$

The constants C and δ depend only on $c_0, c_1, c_2, n, p, \Omega, R$, and $[w]_{p/2}$.

In the standard linear case where $\operatorname{div}\mathbf{a}(\nabla u, x) = \Delta u$ and $\Omega = \mathbb{R}^n$, if u solves

$$\Delta u = \operatorname{div}\mathbf{f} \quad \text{in } \mathbb{R}^n,$$

then ∇u can be realized as the second order Riesz transform of the vector field $\mathbf{f}$:

$$\nabla u = \{-R_i R_j\}(\mathbf{f}).$$

Here R_i, $i = 1, \ldots, n$, is the i-th Riesz transform. Thus Theorem 2.1 can be viewed as a nonlinear version (on irregular domains) of the celebrated weighted norm inequalities for singular integrals (see [13,6]).

By applying Theorem 2.1 with an appropriately chosen weight we obtain the following regularity result in Morrey spaces. Recall that a function $g \in L^s(\Omega)$, $s > 1$, is said to belong to the Morrey space $\mathcal{L}^{s,\theta}(\Omega)$, $0 < \theta \leqslant n$, if

$$\|g\|_{\mathcal{L}^{s,\theta}(\Omega)} := \sup_{z\in\Omega,\, 0<r\leqslant \operatorname{diam}(\Omega)} \left(r^{\theta-n} \int_{B_r(z)\cap\Omega} \left|g(x)\right|^s dx \right)^{1/s} < +\infty.$$

Theorem 2.2. *There exist positive constants C and δ such that if $u \in W_0^{1,2}(\Omega)$ is a weak solution to* (1.1) *in a (δ, R)-Reifenberg flat domain Ω with $\mathbf{a} \in \mathcal{A}_{\delta,R}$ for some $R > 0$, and $|\mathbf{f}| \in \mathcal{L}^{p,\theta}(\Omega)$ where $0 < \theta \leqslant n$, then $|\nabla u| \in \mathcal{L}^{p,\theta}(\Omega)$ and*

$$\|\nabla u\|_{\mathcal{L}^{p,\theta}(\Omega)} \leqslant C \left\| \left(|\mathbf{f}| + 1\right) \right\|_{\mathcal{L}^{p,\theta}(\Omega)}.$$

Again the constants C and δ depend only on $c_0, c_1, c_2, n, p, \theta$, R, and Ω.

Some remarks are now in order. In the unweighted and linear case where $w \equiv 1$ and $\mathbf{a}(\xi, x) = A(x)\xi$ with a uniformly elliptic matrix $A(x)$, the regularity estimate in Theorem 2.1 was obtained in [7] for VMO coefficients A_{ij} in $C^{1,1}$ domains, and later extended in [2] for C^1 domains. The same result was also obtained in [4] under the weak hypothesis that the coefficients A_{ij} have small BMO seminorms and the ground domain Ω is sufficiently flat in the Reifenberg sense. For general nonlinearities $\mathbf{a}(\cdot,\cdot)$ that belong to the class $\mathcal{A}_{\delta,R}$ with sufficiently small $\delta > 0$, the global unweighted estimate has been obtained recently in [3] also on sufficiently flat domains. We follow the argument used in [3] to establish the weighted estimate. To our knowledge the estimates obtained in Theorems 2.1 and 2.2 are new and generalize the L^p estimate in [3].

2.2. Other applications

The Morrey space estimate obtained in Theorem 2.2 together with the Sobolev–Morrey Embedding Theorem (see, e.g., [9, Theorem 7.19]) can be applied to yield the following global Hölder regularity of solutions. The significance of the Hölder regularity result is when $p \leqslant n$, since by Sobolev embedding $W_0^{1,p}(\Omega) \subset C^{0,1-n/p}(\bar{\Omega})$ whenever $p > n$.

Corollary 2.3. *There exist positive constants C and δ such that if $u \in W_0^{1,2}(\Omega)$ is a weak solution of* (1.1) *in a (δ, R)-Reifenberg flat domain Ω with $\mathbf{a} \in \mathcal{A}_{\delta,R}$ for some $R > 0$, and $|\mathbf{f}| \in \mathcal{L}^{p,\theta}(\Omega)$ where $0 < \theta \leqslant n$, and $\theta < p$, then $u \in C^{0,1-\theta/p}(\bar{\Omega})$ and for any ball B_r*

$$\operatorname{osc}_{B_r\cap\bar{\Omega}} u \leqslant C \left\| \left(|\mathbf{f}| + 1\right) \right\|_{\mathcal{L}^{p,\theta}(\Omega)} r^{1-\theta/p}.$$

The constants C and δ depend only on $c_0, c_1, c_2, n, p, \theta$, R, and Ω.

We should mention that local Hölder regularity of solutions with the same exponent as in the corollary is obtained in [25] for linear equations with VMO coefficients. In contrast Corollary 2.3 gives a global Hölder regularity for solutions to nonlinear equations over a rough domain.

To discuss another regularity result that also follows from Theorem 2.2, we make the following notes. For a real valued function $f \in \mathcal{L}^{\gamma,\theta}(\Omega)$, $1 < \gamma < \theta$, one can write f in the form $f = \operatorname{div} \mathbf{f}$, where the vector field $\mathbf{f} = -\nabla G(f\chi_\Omega)$. Here $G = G(x, y)$ is the Green function associated to the Laplacian in a ball B containing Ω, and $G(f\chi_\Omega)$ is the Green potential of $f\chi_\Omega$ defined by

$$G(f\chi_\Omega)(x) = \int_B G(x, y) f(y) \chi_\Omega(y)\, dy.$$

From standard estimates for the gradient of G (see, e.g., [32]) and by a result in [1] on embedding properties of Riesz potentials we find that $\mathbf{f} \in \mathcal{L}^{\frac{\theta\gamma}{\theta-\gamma},\theta}(\Omega, \mathbb{R}^n)$ with the estimate

$$\|\mathbf{f}\|_{\mathcal{L}^{\frac{\theta\gamma}{\theta-\gamma},\theta}(\Omega,\mathbb{R}^n)} \leqslant C\|f\|_{\mathcal{L}^{\gamma,\theta}(\Omega)}.$$

Now if $\frac{\theta\gamma}{\theta-\gamma} > 2$, which forces $\frac{2\theta}{\theta+2} < \gamma$, we obtain from Theorem 2.2 the following global regularity result.

Corollary 2.4. *There exist positive constants C and δ such that if $u \in W_0^{1,2}(\Omega)$ is a weak solution of the nonhomogeneous equation*

$$\begin{cases} \operatorname{div} \mathbf{a}(\nabla u, x) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in a (δ, R)-Reifenberg flat domain Ω with $\mathbf{a} \in \mathcal{A}_{\delta,R}$ for some $R > 0$, and $f \in \mathcal{L}^{\gamma,\theta}(\Omega)$ where $0 < \frac{2\theta}{\theta+2} < \gamma < \theta \leqslant n$, then

$$\|\nabla u\|_{\mathcal{L}^{\frac{\theta\gamma}{\theta-\gamma},\theta}(\Omega)} \leqslant C\big\|\big(|f| + 1\big)\big\|_{\mathcal{L}^{\gamma,\theta}(\Omega)},$$

with C and δ depending only on $c_0, c_1, c_2, n, \gamma, \theta$, and Ω.

When viewed locally the above integrability result is a special case of Theorem 4 of [21] when $2 \leqslant \theta \leqslant n$. While Corollary 2.4 gives a global result over rough domains with an explicit quantification of the degree of integrability, Theorem 4 of [21] (when $p = 2$) gives a local result under a weak assumption on the nonlinearity, namely $\mathbf{a}(\xi, x)$ is just a measurable function in x. Corollary 2.4 also provides the missing piece of the range of γ in Theorem 1 of [21] where it is shown that, for $1 < \gamma \leqslant \frac{2\theta}{\theta+2}$ and $2 < \theta \leqslant n$,

$$f \in L^\gamma(\Omega) \quad \Rightarrow \quad |\nabla u| \in \mathcal{L}_{\mathrm{loc}}^{\frac{\theta\gamma}{\theta-\gamma},\theta}(\Omega). \tag{2.1}$$

Here we deduce that the implication (2.1) is still valid globally for all $\frac{2\theta}{\theta+2} < \gamma < \theta \leqslant n$.

To mention yet another application, we notice that for quasilinear elliptic operators of p-Laplacian type

$$\mathcal{L}_p[u] := \operatorname{div}\big[(A\nabla u \cdot \nabla u)^{\frac{p-2}{2}} A\nabla u\big], \quad p > 1,$$

with VMO coefficients $A = A(x)$, a weighted estimate similar to that of Theorem 2.1 is obtained in [24] on C^1 domains. The result obtained in [24] was then applied in [23] to capacitary inequalities and Riccati type equations of the form

$$-\mathcal{L}_p[u] = |\nabla u|^q + \omega,$$

where $1 < p < q < \infty$ on C^1 domains. Here ω is a measure with bounded total variation in Ω. Thus it is reasonable to expect that Theorem 2.1 can be applied to capacitary inequalities and Riccati type equations with BMO coefficients on Reifenberg flat domains but we choose to pursue this application elsewhere.

Finally we would like to comment that the proof of the weighted estimate in this paper is different from that of [24]. The approach in [24] relies on a local version of Fefferman–Stein sharp maximal functions and $C^{1,\alpha}$ regularity of homogeneous equations. In this paper we rather follow the ideas implemented in [3,4] to overcome the difficulty arising from the nonlinearity **a** and the complexity of Reifenberg flat domains. Specifically, we make use of weak compactness, $W^{1,\infty}$ regularity of reference homogeneous equations, the Vitali covering lemma, and the Hardy–Littlewood maximal function.

3. Preliminaries

3.1. Invariance

We start by collecting some useful facts concerning the class of nonlinearities $\mathcal{A}_{\delta,R}$. The first observation is that $\mathcal{A}_{\delta,R}$ is closed under normalization in the first variable ξ, that is, if $\mathbf{a} \in \mathcal{A}_{\delta,R}$ then so is

$$\mathbf{a}_\lambda(\xi, x) := \frac{\mathbf{a}(\lambda\xi, x)}{\lambda} \tag{3.1}$$

for all $\lambda \geqslant 1$. In addition, we see that for $\lambda \geqslant 1$ and $\mathbf{a} \in \mathcal{A}_{\delta,R}$ if $u \in W_0^{1,2}(\Omega)$ is a weak solution of (1.1), then $u_\lambda(x) = u(x)/\lambda$ solves

$$\begin{cases} \operatorname{div} \mathbf{a}_\lambda(\nabla u_\lambda, x) = \operatorname{div} \mathbf{f}_\lambda & \text{in } \Omega, \\ u_\lambda = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $\mathbf{f}_\lambda = \mathbf{f}/\lambda$, and $\mathbf{a}_\lambda$ is as defined in (3.1). That is, (1.1) is invariant under normalization.

The class $\mathcal{A}_{\delta,R}$ is also scaling and translation invariant in the second variable x in sense that if $\mathbf{a} \in \mathcal{A}_{\delta,R}$, then

$$\mathbf{a}^\tau(\xi, x) := \mathbf{a}(\xi, \tau x + x_0) \tag{3.3}$$

belongs to $\mathcal{A}_{\delta,R/\tau}$ for any $\tau > 0$ and $x_0 \in \mathbb{R}^n$. This follows from the easily verified identity:

$$\beta\big(\mathbf{a}^\tau, B_\rho(y)\big)(x) = \beta\big(\mathbf{a}, B_{\tau\rho}(\tau y + x_0)\big)(\tau x + x_0).$$

Note that if Ω is a (δ, R)-Reifenberg flat domain, then for any $x_0 \in \mathbb{R}^n$ the domain $\Omega^{\tau,x_0} = \{(x - x_0)/\tau\colon\ x \in \Omega\}$ is a $(\delta, R/\tau)$-Reifenberg flat domain for any $\tau > 0$. We will drop the index x_0 and simply write Ω^τ whenever it is clear from the context. The following lemma states that (1.1) is invariant under scaling and translation as well.

Lemma 3.1. *For $x_0 \in \mathbb{R}^n$ and $\tau > 0$ define $\mathbf{a}^\tau(\xi, x)$ as in* (3.3) *for $\mathbf{a} \in \mathcal{A}_{\delta,R}$. Let $u^\tau(x) = u(\tau x + x_0)/\tau$ and $\mathbf{f}^\tau(x) = \mathbf{f}(\tau x + x_0)$. If $u \in W_0^{1,2}(\Omega)$ is a weak solution of* (1.1)*, then $u^\tau \in W_0^{1,2}(\Omega^\tau)$ solves*

$$\begin{cases} \operatorname{div} \mathbf{a}^\tau\big(\nabla u^\tau, x\big) = \operatorname{div} \mathbf{f}^\tau & \text{in } \Omega^\tau, \\ u^\tau = 0 & \text{on } \partial\Omega^\tau. \end{cases}$$

The proof of this lemma and the other statements stated in this subsection can be found in [3].

3.2. Weights

The weights considered in this paper belong to the Muckenhoupt class A_s for some $1 < s < \infty$ as defined at the beginning of Section 2. We would like to mention one particular weight that will be used in this paper. Let

$$w(x) = |x|^a, \quad x \in \mathbb{R}^n.$$

Then w is an A_s weight if and only if $-n < a < n(s-1)$. More A_s weights can also be constructed by applying the following lemma.

Lemma 3.2. *Let w be an A_s weight for some $1 < s < \infty$, and let $k > 0$ and $z \in \mathbb{R}^n$. Then*

(a) *the translation of w, $\tilde{w}(x) = w(x - z)$ is an A_s weight with the same A_s constant,*
(b) *the truncation of w, $\tilde{w}(x) = \min(w, k)$ is an A_s weight and satisfies*

$$\big[\min(w, k)\big]_s \leqslant c_s\big([w]_s + 1\big).$$

We remark that part (b) of Lemma 3.2 does not say that $[\min\{w, k\}]_s$ is independent of k. Rather the A_s constant is bounded from above uniformly in k.

We will also use the *strong doubling property* of A_s weights stated below. We denote by $w(E)$ the integral $\int_E w(x)\,dx$, hereafter.

Lemma 3.3. *Let w be an A_s weight for some $1 < s < \infty$, and let E be a measurable subset of a ball B. Then*

$$w(B) \leqslant [w]_s \left(\frac{|B|}{|E|}\right)^s w(E).$$

Proof. The proof follows from Hölder's inequality and goes as follows. For $E \subset B$ we have

$$\begin{aligned} \left(\frac{|E|}{|B|}\right)^s &= \left(\frac{\int_B \chi_E\,dx}{|B|}\right)^s = \left(\frac{\int_B \chi_E w^{\frac{1}{s}} w^{\frac{-1}{s}}\,dx}{|B|}\right)^s \\ &\leqslant \frac{1}{|B|^s}\left(\int_B \chi_E w\,dx\right)\left(\int_B w^{\frac{-1}{s-1}}\,dx\right)^{s-1} \\ &= \frac{w(E)}{w(B)}\left(\fint_B w\,dx\right)\left(\fint_B w^{\frac{-1}{s-1}}\,dx\right)^{s-1} \leqslant [w]_s \frac{w(E)}{w(B)}. \qquad \square \end{aligned}$$

A broader class of weights is the A_∞ weights which, by definition, is the union of A_s weights for $1 < s < \infty$. The following characterization of A_∞ weights will be needed later.

Lemma 3.4. *A weight $w \in A_\infty$ if and only if there are constants $A, \nu > 0$ such that for every ball $B \subset \mathbb{R}^n$ and every measurable subset E of B*

$$w(E) \leqslant A\left(\frac{|E|}{|B|}\right)^{\nu} w(B). \tag{3.4}$$

Remark 3.5. For weights $w \in A_s$ inequality (3.4) holds true with constants A and ν depending only on n and $[w]_s$. More importantly if $[w]_s \leqslant \omega$, then the constants A and ν can be chosen depending only on n and ω. Applying this remark to the truncated weight in Lemma 3.2, we see that if w is an A_s weight then (3.4) holds for $\tilde{w} = \min\{w, k\}$ with A and ν being independent of k. For convenience, we will use the (non-standard) notation $[w]_\infty$ to denote any pair of constants (A, ν) satisfying (3.4).

We next state a fundamental result on weighted norm inequalities for maximal functions. Recall that for a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the Hardy–Littlewood maximal function of f is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \fint_{B_r(x)} \left|f(y)\right| dy.$$

For a function f that is defined only on Ω, we define $\mathcal{M}f := \mathcal{M}(f\chi_\Omega)$. Note that for $f \in L^s_w(\mathbb{R}^n)$, $s > 1$, $\mathcal{M}f$ is meaningful since, by Hölder's inequality, $L^s_w(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Lemma 3.6. *(See Muckenhoupt [22].) Let w be an A_s weight for some $1 < s < \infty$. There exists a constant $C = C(n, s, [w]_s)$ such that*

$$\|\mathcal{M}f\|_{L^s_w(\mathbb{R}^n)} \leqslant C\|f\|_{L^s_w(\mathbb{R}^n)} \tag{3.5}$$

for all $f \in L^s_w(\mathbb{R}^n)$. Conversely, if (3.5) holds for all $f \in L^s_w(\mathbb{R}^n)$, then w must be an A_s weight.

The following result comes from standard measure theory.

Lemma 3.7. *Assume that $g \geqslant 0$ is a measurable function in a bounded subset $U \subset \mathbb{R}^n$. Let $\theta > 0$, $\Lambda > 1$ be constants, and let w be a weight in $\mathbb{R}^n$. Then for $0 < s < \infty$, we have*

$$g \in L^s_w(U) \quad \Leftrightarrow \quad S := \sum_{k \geqslant 1} \Lambda^{ks} w\left(\left\{x \in U\colon\ g(x) > \theta\Lambda^k\right\}\right) < \infty$$

and moreover,

$$C^{-1}S \leqslant \|g\|^s_{L^s_w(U)} \leqslant C\left(w(U) + S\right),$$

where $C > 0$ is a constant depending only on θ, Λ, and s.

The results stated in this subsection and their proof can be found in [10, Chapter 9].

3.3. Technical lemma

One of the ingredients in the proof of the weighted estimate is the following technical lemma whose proof is a consequence of Lebesgue Differentiation Theorem and the standard Vitali covering lemma. In the unweighted case various versions of this lemma have been obtained (see, e.g., [31,4]). A very similar lemma was also obtained in [5] based on the Calderón–Zygmund decomposition. We give a proof to the weighted version that is similar to the one given in [4] with obvious modifications to fit our setting.

Lemma 3.8. *Let Ω be a (δ, R)-Reifenberg flat domain with $\delta < 1/4$, and let w be an A_s weight for some $s > 1$. Suppose that the sequence of balls $\{B_r(y_i)\}_{i=1}^L$ with centers $y_i \in \bar{\Omega}$ and a common radius $r \leqslant R/2000$ covers Ω. Let $C \subset D \subset \Omega$ be measurable sets for which there exists $0 < \epsilon < 1$ such that*

(1) *$w(C) < \epsilon w(B_r(y_i))$ for all $i = 1, \dots, L$, and*
(2) *for all $x \in \Omega$ and $\rho \in (0, 2r]$, if $w(C \cap B_\rho(x)) \geqslant \epsilon w(B_\rho(x))$, then $B_\rho(x) \cap \Omega \subset D$.*

Then we have the estimate

$$w(C) \leqslant \epsilon \left(\frac{10}{1-4\delta}\right)^{ns} [w]_s^2 w(D).$$

Proof. We first observe that for almost all $x \in C$, the function

$$\phi(\rho) = \frac{w(C \cap B_\rho(x))}{w(B_\rho(x))}, \quad \rho > 0,$$

is continuous with $\phi(0) = \lim_{\rho \to 0} \phi(\rho) = 1$ (by Lebesgue Differentiation Theorem). Moreover $\phi(2r) < \epsilon$ since $w(C) < \epsilon w(B_r(y_i))$ and $B_r(y_i) \subset B_{2r}(x)$ whenever $x \in B_r(y_i)$. Therefore, for almost all $x \in C$ there exists $0 < \rho_x < 2r$ such that

$$w\big(C \cap B_{\rho_x}(x)\big) = \epsilon w\big(B_{\rho_x}(x)\big) \quad \text{and} \tag{3.6}$$

$$w\big(C \cap B_\rho(x)\big) < \epsilon w\big(B_\rho(x)\big) \quad \text{for all } \rho > \rho_x. \tag{3.7}$$

The family of balls $\{B_{\rho_x}(x)\}$ for almost all $x \in C$ covers C. By Vitali's covering lemma there exists a countable $\{x_i\}_{i=1}^\infty$ such that the balls $B_{\rho_i}(x_i)$ are mutually disjoint and

$$C \subset \bigcup B_{5\rho_i}(x_i) \quad (\rho_i = \rho_{x_i}).$$

Thus it follows from (3.7) and Lemma 3.3 that

$$w\big(C \cap B_{5\rho_i}(x_i)\big) < \epsilon w\big(B_{5\rho_i}(x_i)\big) \leqslant \epsilon [w]_s 5^{sn} w\big(B_{\rho_i}(x_i)\big).$$

Next we claim that

$$w\big(B_{\rho_i}(x_i)\big) \leqslant [w]_s \left(\frac{2}{1-4\delta}\right)^{sn} w\big(B_{\rho_i}(x_i) \cap \Omega\big). \tag{3.8}$$

To verify this we first show that

$$\sup_{0<\rho\leqslant 2r} \sup_{x\in\Omega} \frac{|B_\rho(x)|}{|B_\rho(x) \cap \Omega|} \leqslant \left(\frac{2}{1-4\delta}\right)^n. \tag{3.9}$$

Indeed, choose any $\rho \in (0, 2r)$ and any $x \in \Omega$. If $\operatorname{dist}(x, \partial\Omega) \geqslant \rho$, then

$$\frac{|B_\rho(x)|}{|B_\rho(x) \cap \Omega|} = 1$$

since $B_\rho(x) \subset \Omega$. If $\operatorname{dist}(x, \partial\Omega) < \rho$, we argue as follows. There exists a $z \in \partial\Omega$ such that

$$\operatorname{dist}(x, \partial\Omega) = |x - z| < \rho \quad \text{and} \quad B_\rho(x) \subset B_{2\rho}(z).$$

Since Ω is a (δ, R)-Reifenberg flat domain, there exists a coordinate system with $z = 0$ such that

$$B_{2\rho}(z) \cap \{x_n > 2\rho\delta\} \subset B_{2\rho}(z) \cap \Omega \subset B_{2\rho}(z) \cap \{x_n > -2\rho\delta\}.$$

It follows then that

$$B_\rho(x) \cap \{x_n > 2\delta\rho\} \subset B_{2\rho}(z) \cap \{x_n > 2\rho\delta\} \subset \Omega$$

and so

$$B_\rho(x) \cap \{x_n > 2\delta\rho\} \subset B_\rho(x) \cap \Omega.$$

Therefore after some calculations and noting that $\delta < 1/4$,

$$\frac{|B_\rho(x)|}{|B_\rho(x) \cap \Omega|} \leqslant \frac{|B_\rho(x)|}{|B_\rho(x) \cap \{x_n > 2\rho\delta\}|} \leqslant \left(\frac{2}{1 - 4\delta}\right)^n.$$

This gives (3.9) and hence by Lemma 3.3 we obtain inequality (3.8). We are now ready to finish the proof of the lemma. From (3.8) we have

$$\begin{aligned} w(C) &\leqslant w\left(\bigcup_{i \geqslant 1} B_{5\rho_i}(x_i) \cap C\right) \\ &\leqslant \sum_{i \geqslant 1} \epsilon [w]_s 5^{sn} w\big(B_{\rho_i}(x_i)\big) \\ &\leqslant \sum_{i \geqslant 1} \epsilon [w]_s^2 \left(\frac{10}{1 - 4\delta}\right)^{sn} w\big(B_{\rho_i}(x_i) \cap \Omega\big). \end{aligned}$$

Since the balls $B_{\rho_i}(x_i)$ are mutually disjoint we can continue to estimate

$$\begin{aligned} w(C) &\leqslant \epsilon [w]_s^2 \left(\frac{10}{1 - 4\delta}\right)^{sn} w\left(\bigcup_{i \geqslant 1} B_{\rho_i}(x_i) \cap \Omega\right) \\ &\leqslant \epsilon [w]_s^2 \left(\frac{10}{1 - 4\delta}\right)^{sn} w(D), \end{aligned}$$

where the last inequality follows from equality (3.6) and the second hypothesis. This completes the proof of the lemma. $\square$

4. Weighted local interior and boundary estimates

In this section we obtain certain weighted local interior and boundary estimates for weak solutions u of

$$\begin{cases} \operatorname{div} \mathbf{a}(\nabla u, x) = \operatorname{div} \mathbf{f} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

These are good λ type estimates and will be used in the proof of Theorem 2.1. They are motivated by and obtained from the unweighted local interior and boundary estimates that are established in [3].

4.1. Review of unweighted estimates

We review first the local interior and boundary estimates established in [3]. To make our exposition relatively complete, we sketch the proof of the interior estimate. A similar local boundary estimate will be stated later.

Lemma 4.1. *There exists a constant $\lambda > 0$ so that the following statement holds: for any $\epsilon > 0$ there exists a small $\delta = \delta(\epsilon) > 0$ such that, if $u \in W^{1,2}(\Omega)$ is a weak solution of* (4.1) *with $B_8 \subset \Omega$, $\mathbf{a}$ satisfying inequalities* (1.3), (1.4), (1.6), *along with*

$$\fint_{B_6} \left|\beta(\mathbf{a}, B_6)\right|^2 dx \leqslant \delta^2, \tag{4.2}$$

and

$$B_1 \cap \left\{x \in \Omega \colon \mathcal{M}\left(|\nabla u|^2\right) \leqslant 1\right\} \cap \left\{x \in \Omega \colon \mathcal{M}\left(|\mathbf{f}|^2\right) \leqslant \delta^2\right\} \neq \emptyset, \tag{4.3}$$

then

$$\left|\left\{x \in \Omega \colon \mathcal{M}\left(|\nabla u|^2\right) > \lambda^2\right\} \cap B_1\right| < \epsilon |B_1|.$$

The proof of this lemma employs a perturbation argument, where the basic nonlinear equation (4.1) is considered as a perturbation of a reference equation whose solutions are locally Lipschitz continuous. The reference equation is taken to be

$$\operatorname{div} \bar{\mathbf{a}}_{B_6}(\nabla v) = 0 \quad \text{in } B_6, \tag{4.4}$$

where $\bar{\mathbf{a}}_{B_6}(\xi)$ is the average of $\mathbf{a}$ over B_6 in the second variable, as defined in (1.7). A function $v \in W^{1,2}(B_6)$ is a weak solution of (4.4) if

$$\int_{B_6} \bar{\mathbf{a}}_{B_6}(\nabla v) \cdot \nabla \phi \, dx = 0$$

for all $\phi \in W_0^{1,2}(B_6)$. One of the main ingredients of the argument is the following local $W^{1,\infty}$ estimate which we simply state.

Lemma 4.2. *Suppose that* **a** *satisfies* (1.3), (1.4), *and* (1.6). *Then for any weak solution* $v \in W^{1,2}(B_6)$ *of* (4.4) *we have*

$$\|\nabla v\|^2_{L^\infty(B_3)} \leqslant C \fint_{B_5} |\nabla v|^2\,dx$$

for some universal constant $C > 0$.

The link between the solutions of the reference and the main equations is established in the following comparison estimate which says that any solution of (4.1) can be made arbitrarily close to a solution of (4.4) whenever the nonlinearity **a** has a uniform small mean oscillation.

Lemma 4.3. *Given* $\epsilon > 0$ *there exists a small* $\delta = \delta(\epsilon) > 0$ *such that if* $u \in W^{1,2}(\Omega)$ *is a weak solution of* (4.1) *with* $B_8 \subset \Omega$, **a** *satisfying* (1.3), (1.4), (1.6), *and the following normalization*

$$\fint_{B_6} |\nabla u|^2\,dx \leqslant 1, \quad \fint_{B_6} \big|\beta(\mathbf{a}, B_6)\big|^2\,dx \leqslant \delta^2, \quad \textit{and} \quad \fint_{B_6} |\mathbf{f}|^2\,dx \leqslant \delta^2,$$

then there exists a weak solution $v \in W^{1,2}(B_6)$ *of* (4.4) *such that*

$$\fint_{B_6} |u - \bar{u}_{B_6} - v|^2\,dx \leqslant \epsilon^2. \tag{4.5}$$

Remark 4.4. For any $v \in W^{1,2}(B_6)$ as in Lemma 4.3 (solving (4.4) and satisfying (4.5)) we have that

$$\|\nabla v\|^2_{L^\infty(B_3)} \leqslant C$$

for some universal constant C independent of v. Indeed by applying Lemma 4.2 and a local L^2 estimate we obtain first

$$\|\nabla v\|_{L^\infty(B_3)} \leqslant C \fint_{B_5} |\nabla v|^2\,dx \leqslant C \fint_{B_6} |v|^2\,dx.$$

And then we estimate the last term using (4.5) and Poincaré's inequality as follows:

$$\begin{aligned}\fint_{B_6} |v|^2\,dx &\leqslant \fint_{B_6} |v - u - \bar{u}_{B_6}|^2\,dx + \fint_{B_6} |u - \bar{u}_{B_6}|^2\,dx \\ &\leqslant \epsilon^2 + C \fint_{B_6} |\nabla u|^2\,dx \leqslant \epsilon^2 + C \leqslant C + 1,\end{aligned}$$

as we may assume that $0 < \epsilon < 1$.

We now apply Remark 4.4 and Lemma 4.3 to obtain an L^2 approximation of the gradient of the solution to (4.1) by a gradient of a solution to (4.4).

Lemma 4.5. *Given $\epsilon > 0$ there exists a small $\delta = \delta(\epsilon) > 0$ such that if $u \in W^{1,2}(\Omega)$ is a weak solution of* (4.1) *with $B_8 \subset \Omega$, $\mathbf{a}$ satisfying* (1.3), (1.4), (1.6), *and the following normalization*

$$\fint_{B_6} |\nabla u|^2\,dx \leqslant 1, \quad \fint_{B_6} \left|\beta(\mathbf{a}, B_6)\right|^2 dx \leqslant \delta^2, \quad \textit{and} \quad \fint_{B_6} |\mathbf{f}|^2\,dx \leqslant \delta^2,$$

then there exists a weak solution $v \in W^{1,2}(B_6)$ of (4.4) *such that for some universal constant N_0,*

$$\|\nabla v\|_{L^\infty(B_3)} \leqslant N_0 \quad \textit{and} \quad \fint_{B_2} |\nabla u - \nabla v|^2\,dx \leqslant \epsilon^2.$$

Sketch of the proof of Lemma 4.1. Let $\epsilon > 0$ be given. From condition (4.3) and for a $\delta > 0$ to be determined, there exists a point $x_0 \in B_1$ such that for any $\rho > 0$

$$\fint_{B_\rho(x_0)} |\nabla u|^2\,dx \leqslant 1 \quad \text{and} \quad \fint_{B_\rho(x_0)} |\mathbf{f}|^2\,dx \leqslant \delta^2.$$

Then since $B_6 \subset B_7(x_0) \subset B_8 \subset \Omega$, we find that

$$\fint_{B_6} |\nabla u|^2\,dx \leqslant (7/6)^n \quad \text{and} \quad \fint_{B_6} |\mathbf{f}|^2\,dx \leqslant (7/6)^n\delta^2. \tag{4.6}$$

In view of the above inequalities we set $\kappa = \sqrt{(7/6)^n}$ and normalize $\mathbf{a}$ to $\mathbf{a}_\kappa$ as in (3.1). We now recall that $u_\kappa = u/\kappa$ solves the equation

$$\begin{cases} \operatorname{div}\mathbf{a}_\kappa(\nabla u_\kappa, x) = \operatorname{div}\mathbf{f}_\kappa & \text{in } \Omega, \\ u_\kappa = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

with $\mathbf{f}_\kappa = \mathbf{f}/\kappa$. As discussed in Section 3, $\mathbf{a}_\kappa$ satisfies all the hypotheses of Lemma 4.5. Applying Lemma 4.5 we deduce that for any $\eta > 0$, there exist a small $\delta = \delta(\eta) > 0$ and a weak solution $v_\kappa \in W^{1,2}(B_6)$ of

$$\operatorname{div}\bar{\mathbf{a}}_{\kappa B_6}(\nabla v_\kappa) = 0 \quad \text{in } B_6,$$

such that for some universal constant N_0,

$$\|\nabla v_\kappa\|_{L^\infty} \leqslant N_0 \quad \text{and} \quad \fint_{B_2} |\nabla u_\kappa - \nabla v_\kappa|^2\,dx \leqslant \eta^2$$

provided

$$\fint_{B_6} |\nabla u_\kappa|^2\,dx \leqslant 1, \quad \fint_{B_6} \left|\beta(\mathbf{a}_\kappa, B_6)\right|^2 dx \leqslant \delta^2, \quad \text{and} \quad \fint_{B_6} |\mathbf{f}_\kappa|^2\,dx \leqslant \delta^2,$$

which are indeed true by (4.6) and (4.2) for this choice of δ.

Now taking $\Lambda^2 = \max\{2^n, (2N_0)^2\}$, it can be shown that

$$\left\{x \in B_1 \colon \mathcal{M}\left(|\nabla u_\kappa|^2\right) > \Lambda^2\right\} \subset \left\{x \in B_1 \colon \mathcal{M}\left(|\nabla u_\kappa - \nabla v_\kappa|^2 \chi_{B_6}\right) > N_0^2\right\}. \tag{4.8}$$

Finally set $\lambda = \kappa \Lambda$ to obtain the estimate

$$\begin{aligned}
\left|\left\{x \in B_1 \colon \mathcal{M}\left(|\nabla u|^2\right) > \lambda^2\right\}\right| &= \left|\left\{x \in B_1 \colon \mathcal{M}\left(|\nabla u_\kappa|^2\right) > \Lambda^2\right\}\right| \\
&\leqslant \left|\left\{x \in B_1 \colon \mathcal{M}\left(|\nabla u_\kappa - \nabla v_\kappa|^2\right) > N_0^2\right\}\right| \\
&\leqslant C \int_{B_2} |\nabla u_\kappa - \nabla v_\kappa|^2 \, dx \leqslant C\eta^2
\end{aligned}$$

for some universal constant $C > 0$. The first inequality follows from (4.8), whereas the second follows from the weak-type $(1,1)$ inequality for maximal functions (see, e.g., [10, Theorem 2.1.6]). Now we select a small $\eta > 0$, thereby $\delta = \delta(\eta) > 0$, so that $C\eta^2 \leqslant \epsilon |B_1|$ to complete the proof of the lemma. □

We will state the following boundary estimate whose proof follows a similar procedure as above but with a careful analysis to deal with the roughness of the boundary of Ω. We would like to emphasize that Lemmas 4.2, 4.3, 4.5 and their boundary counterparts are nontrivial results obtained in [3].

Lemma 4.6. *There exists a constant $\lambda > 0$ so that the following statement holds: for any $\epsilon > 0$ there exists a small $\delta = \delta(\epsilon) > 0$ such that, if $u \in W_0^{1,2}(\Omega)$ is a weak solution of (4.1) with*

$$B_6^+ \subset \Omega_6 \subset B_6 \cap \{x_n > -12\delta\},$$

a *satisfying inequalities (1.3), (1.4), (1.6), along with*

$$\fint_{\Omega_6} \left|\beta(\mathbf{a}, \Omega_6)\right|^2 dx \leqslant \delta^2,$$

and

$$B_1 \cap \left\{x \in \Omega \colon \mathcal{M}\left(|\nabla u|^2\right) \leqslant 1\right\} \cap \left\{x \in \Omega \colon \mathcal{M}\left(|\mathbf{f}|^2\right) \leqslant \delta^2\right\} \neq \emptyset,$$

then

$$\left|\left\{x \in \Omega \colon \mathcal{M}\left(|\nabla u|^2\right) > \lambda^2\right\} \cap B_1\right| < \epsilon |B_1|.$$

Remark 4.7. In the above two lemmas, if the statements are true for some $\delta_0 > 0$, then they are true for all other $0 < \delta < \delta_0$.

4.2. Weighted estimates

Next we will use properties of A_s weights to give weighted versions of the local interior and boundary estimates reviewed in the previous subsection. We begin with the following translated and scaled versions.

Lemma 4.8. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty$ and let λ be as in Lemma 4.1. For any $\epsilon > 0$ there exists $\delta = \delta(\epsilon, [w]_s) > 0$ such that the following holds. Suppose that $u \in W^{1,2}(\Omega)$ is a weak solution of (4.1) with $\mathbf{a} \in \mathcal{A}_{\delta,6r}$ for some $r > 0$. Then for any $y \in \Omega$ such that $B_{8r}(y) \subset \Omega$ and*

$$B_r(y) \cap \big\{x \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big) \leqslant 1\big\} \cap \big\{x \in \Omega\colon \mathcal{M}\big(|\mathbf{f}|^2\big) \leqslant \delta^2\big\} \neq \emptyset,$$

we have

$$w\big(\big\{x \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big) > \lambda^2\big\} \cap B_r(y)\big) < \epsilon\, w\big(B_r(y)\big).$$

The proof of this lemma will be omitted as it is similar to the next.

Lemma 4.9. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty$ and let λ be as in Lemma 4.6. For any $\epsilon > 0$ there exists $\delta = \delta(\epsilon, [w]_s) > 0$ such that the following holds. Suppose that $u \in W_0^{1,2}(\Omega)$ is a weak solution of (4.1) with $\mathbf{a} \in \mathcal{A}_{\delta,6r}$ for some $r > 0$. Then for any $y = (y', y_n) \in \Omega$ such that*

$$B_{6r}^+(y) \subset \Omega_{6r}(y) \subset B_{6r}(y) \cap \{x_n > y_n - 12r\delta\}$$

and

$$B_r(y) \cap \big\{x \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big) \leqslant 1\big\} \cap \big\{x \in \Omega\colon \mathcal{M}\big(|\mathbf{f}|^2\big) \leqslant \delta^2\big\} \neq \emptyset, \tag{4.9}$$

we have

$$w\big(\big\{x \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big) > \lambda^2\big\} \cap B_r(y)\big) < \epsilon\, w\big(B_r(y)\big).$$

Proof. Let $\Omega^r = \Omega^{r,y} = \{\frac{x-y}{r}\colon x \in \Omega\}$, $\mathbf{a}^r(x) = \mathbf{a}(rx + y)$, $u^r(x) = u(rx + y)/r$, and $\mathbf{f}^r(x) = \mathbf{f}(rx + y)$, where y is as in the lemma. By Lemma 3.1 we find

$$\begin{cases} \operatorname{div} \mathbf{a}^r\big(\nabla u^r, x\big) = \operatorname{div} \mathbf{f}^r & \text{in } \Omega^r, \\ u^r = 0 & \text{on } \partial\Omega^r. \end{cases}$$

Let $\epsilon > 0$ be given and choose $\delta = \delta(\epsilon, [w]_s)$ as in Lemma 4.6 with $(2\epsilon/A)^{1/\nu}$ replacing ϵ where (A, ν) is an A_∞ constant of w. By hypothesis there exists $x_0 \in B_r(y)$ such that (4.9) holds. Then the point $z_0 = \frac{x_0-y}{r}$ belongs to the set

$$B_1 \cap \big\{z \in \Omega^r\colon \mathcal{M}\big(\big|\nabla u^r\big|^2\big) \leqslant 1\big\} \cap \big\{z \in \Omega^r\colon \mathcal{M}\big(\big|\mathbf{f}^r\big|^2\big) \leqslant \delta^2\big\}.$$

This follows from the identities

$$\mathcal{M}\big(\big|\nabla u^r\big|^2\big)\big((x - y)/r\big) = \mathcal{M}\big(|\nabla u|^2\big)(x) \tag{4.10}$$

and

$$\mathcal{M}\big(\big|\mathbf{f}^r\big|^2\big)\big((x - y)/r\big) = \mathcal{M}\big(|\mathbf{f}|^2\big)(x).$$

Moreover, it can easily be seen that

$$B_6^+ \subset \Omega_6^r \subset B_6 \cap \{x_n > -12\delta\}.$$

Now all the hypotheses of Lemma 4.6 are satisfied and so we have

$$\left|\left\{z \in \Omega^r \colon \mathcal{M}\left(\left|\nabla u^r\right|^2\right) > \lambda^2\right\} \cap B_1\right| \leqslant (2\epsilon/A)^{1/\nu}|B_1|.$$

Since Lebesgue measure is scale and translation invariant it follows that

$$\left|\left\{rz + y \in \Omega \colon \mathcal{M}\left(\left|\nabla u^r\right|^2\right)(z) > \lambda^2\right\} \cap B_r(y)\right| \leqslant (2\epsilon/A)^{1/\nu}\left|B_r(y)\right|.$$

By (4.10) we have $\mathcal{M}(|\nabla u^r|^2)(z) = \mathcal{M}(|\nabla u|^2)(rz + y)$, which implies that

$$\left|\left\{x \in \Omega \colon \mathcal{M}\left(|\nabla u|^2\right) > \lambda^2\right\} \cap B_r(y)\right| \leqslant (2\epsilon/A)^{1/\nu}\left|B_r(y)\right|. \tag{4.11}$$

Finally, using the A_∞ characterization of w (Lemma 3.4), we get from (4.11) that

$$\begin{aligned} w\left(\left\{x \in \Omega \colon \mathcal{M}\left(|\nabla u|^2\right) > \lambda^2\right\} \cap B_r(y)\right) &\leqslant A\left[\frac{|\{x \in \Omega \colon \mathcal{M}(|\nabla u|^2) > \lambda^2\} \cap B_r(y)|}{|B_r(y)|}\right]^\nu w\left(B_r(y)\right) \\ &\leqslant 2\epsilon\, w\left(B_r(y)\right) < \epsilon\, w\left(B_r(y)\right) \end{aligned}$$

as desired. This completes the proof of the lemma. □

Theorem 4.10. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty$ and let $\lambda > 1$ validate both Lemmas 4.8 and 4.9. For any $\epsilon > 0$ there exists $\delta = \delta(\epsilon, [w]_s)$, $0 < \delta < 1/8$, that satisfies the following: suppose that $u \in W_0^{1,2}(\Omega)$ is a weak solution of (4.1) with $\mathbf{a} \in \mathcal{A}_{\delta,R}$ and Ω a (δ, R)-Reifenberg flat domain for some $R > 0$. Then if $y \in \bar{\Omega}$, $0 < r \leqslant R/1000$, and*

$$w\left(\left\{x \in \Omega \colon \mathcal{M}\left(|\nabla u|^2\right) > \lambda^2\right\} \cap B_r(y)\right) \geqslant \epsilon\, w\left(B_r(y)\right), \tag{4.12}$$

then

$$\Omega \cap B_r(y) \subset \left\{x \in \Omega \colon \mathcal{M}\left(|\nabla u|^2\right) > 1\right\} \cup \left\{x \in \Omega \colon \mathcal{M}\left(|\mathbf{f}|^2\right) > \delta^2\right\}. \tag{4.13}$$

Proof. The case where $B_{8r}(y) \subset \Omega$ is the contrapositive of the local interior weighted estimate obtained in Lemma 4.8. Thus we prove the theorem for the case where $B_{8r}(y)$ intersects $\partial\Omega$. We argue by contradiction. Suppose there exists an $\epsilon = \epsilon_0 > 0$ such that, for any $\delta > 0$, we can find $u \in W_0^{1,2}(\Omega)$ a weak solution to (4.1) with Ω a (δ, R)-Reifenberg flat domain and $\mathbf{a} \in \mathcal{A}_{\delta,R}$ with the property that

(1) there exist $y \in \bar{\Omega}$, $0 < r \leqslant R/1000$ satisfying (4.12), and $\partial\Omega \cap B_{8r}(y) \neq \emptyset$; but
(2) there exists also $x_0 \in \Omega \cap B_r(y)$ such that

$$\mathcal{M}\left(|\nabla u|^2\right)(x_0) \leqslant 1 \quad \text{and} \quad \mathcal{M}\left(|\mathbf{f}|^2\right)(x_0) \leqslant \delta^2.$$

Choose $\delta' = \delta(\epsilon, [w]_s) > 0$ as in Lemma 4.9 with $\epsilon = \frac{\epsilon_0}{[w]_s 144^{ns}}$. We will assume, as we may, that $\delta' < 1/7$. Then take $\delta = \frac{\delta'}{1+\delta'}$, and let $y_0 \in \partial\Omega \cap B_{8r}(y)$. We note that

$$x_0 \in B_r(y) \cap \Omega \subset B_{9r}(y_0) \cap \Omega. \tag{4.14}$$

The assumption that Ω is a (δ, R)-Reifenberg flat domain, $y_0 \in \partial\Omega$, and Remark 1.3 imply that for $M < R$ to be determined there exists a coordinate system $\{z_1, z_2, \ldots, z_n\}$ in which

$$0 = y_0 + \delta M z_n \in \Omega, \qquad y = \hat{z}, \qquad x_0 = z_0,$$

and

$$B^+_{6\rho}(0) \subset \Omega_{6\rho} \subset B_{6\rho} \cap \{z_n > -12\rho\delta'\},$$

where we have used the fact $\delta' = \frac{\delta}{1-\delta}$ and let $\rho = \frac{M(1-\delta)}{6}$. We now choose M large enough so that $z_0 \in B_\rho(0)$. $M = 432r$ will do, since $\delta < 1/8$, $|\hat{z}| \leqslant 8r + \delta M$, and $|z_0| \leqslant 9r + \delta M$ in this coordinate system.

In summary, for the choice $M = 432r$ and up to a change of coordinate system we have

(1) $u \in W^{1,2}_0(\Omega)$ a weak solution to (4.1) with $\mathbf{a} \in \mathcal{A}_{\delta,R} \subset \mathcal{A}_{\delta',R}$;
(2) $B^+_{6\rho}(0) \subset \Omega_{6\rho} \subset B_{6\rho} \cap \{z_n > -12\rho\delta'\}$;
(3) $z_0 \in B_\rho(0) \cap \{\mathcal{M}(|\nabla u|^2) \leqslant 1\} \cap \{\mathcal{M}(|\mathbf{f}|^2) \leqslant \delta'^2\}$.

Now all the hypotheses of Lemma 4.9 are satisfied with $B_\rho(0)$ replacing $B_r(y)$. Thus we conclude that

$$w\big(\big\{z \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big) > \lambda^2\big\} \cap B_\rho(0)\big) < \frac{\epsilon_0}{[w]_s 144^{ns}} w\big(B_\rho(0)\big). \tag{4.15}$$

A simple calculation yields $B_r(y) \subset B_\rho(0) \subset B_{2\rho}(y) \subset B_{144r}(y)$. Then it follows from (4.15) that

$$\begin{aligned} w\big(\big\{x \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big) > \lambda^2\big\} \cap B_r(y)\big) &\leqslant w\big(\big\{x \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big) > \lambda^2\big\} \cap B_\rho(0)\big) \\ &< \frac{\epsilon_0}{[w]_s 144^{ns}} w\big(B_\rho(0)\big) \leqslant \frac{\epsilon_0}{[w]_s 144^{ns}} w\big(B_{144r}(y)\big) \\ &\leqslant \epsilon_0 w\big(B_r(y)\big), \end{aligned}$$

where we have used Lemma 3.3. The last chain of inequalities contradict hypothesis (4.12) of the theorem, and thus the proof is complete. □

Corollary 4.11. *Let w be an A_s weight in $\mathbb{R}^n$ for some $1 < s < \infty$ and let $\lambda > 1$ be as in Theorem 4.10. For any $\epsilon > 0$ there exists $\delta = \delta(\epsilon, [w]_s)$, $0 < \delta < 1/8$, such that the following holds. Suppose that $u \in W^{1,2}_0(\Omega)$ is a weak solution of (4.1) with $\mathbf{a} \in \mathcal{A}_{\delta,R}$ and Ω a (δ, R)-Reifenberg flat domain. Let $\{B_r(y_i)\}_{i=1}^L$ be a sequence of balls with centers $y_i \in \bar{\Omega}$ and a common radius $0 < r \leqslant R/2000$ that covers Ω. Set $\epsilon_1 = \epsilon(\frac{10}{1-4\delta})^{ns}[w]_s^2$ and let k be a positive integer. If for all $i = 1, \ldots, L$*

$$w\big(\big\{x \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big) > \lambda^2\big\}\big) < \epsilon w\big(B_r(y_i)\big), \tag{4.16}$$

then we have

$$\begin{aligned} w\big(\big\{x \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big) > \lambda^{2k}\big\}\big) \leqslant \sum_{i=1}^{k} \epsilon_1^i w\big(\big\{x \in \Omega\colon \mathcal{M}\big(|\mathbf{f}|^2\big) > \delta^2 \lambda^{2(k-i)}\big\}\big) \\ + \epsilon_1^k w\big(\big\{x \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big) > 1\big\}\big). \end{aligned}$$

Proof. Given $\epsilon > 0$, we take δ as in Theorem 4.10. We now prove this corollary by induction. The case $k = 1$ follows from Theorem 4.10 and the technical lemma, Lemma 3.8. Indeed, let

$$C = \big\{x \in \Omega\colon \mathcal{M}\big(|\nabla u|^2\big) > \lambda^2\big\}$$

and

$$D = \{x \in \Omega\colon \mathcal{M}(|\nabla u|^2) > 1\} \cup \{x \in \Omega\colon \mathcal{M}(|\mathbf{f}|^2) > \delta^2\}.$$

Then from assumption (4.16) it follows that $w(C) < \epsilon w(B_r(y_i))$ for all $i = 1, \dots, L$. Moreover, if $x \in \Omega$ and $\rho \in (0, 2r)$ such that $w(C \cap B_\rho(x)) \geqslant \epsilon w(B_\rho(x))$, then $0 < \rho \leqslant R/1000$ and all the hypotheses of Theorem 4.10 are satisfied with ρ replacing r. We then have $B_\rho(x) \cap \Omega \subset D$. Applying now Lemma 3.8 yields

$$w(C) \leqslant \epsilon_1 w(D),$$

from which the desired inequality follows.

Suppose now that the conclusion is true for some $k > 1$. Normalizing u to $u_\lambda = u/\lambda$ and $\mathbf{f}_\lambda = \mathbf{f}/\lambda$, we see that

$$\begin{aligned} w(\{x \in \Omega\colon \mathcal{M}(|\nabla u_\lambda|^2) > \lambda^2\}) &= w(\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2) > \lambda^4\}) \\ &\leqslant w(\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2) > \lambda^2\}) \\ &< \epsilon w(B_r(y_i)), \end{aligned}$$

which holds for all $i = 1, \dots, L$. Here we have used the fact that $\lambda > 1$. Now by induction assumption it follows that

$$\begin{aligned} w(\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2) > \lambda^{2(k+1)}\}) &= w(\{x \in \Omega\colon \mathcal{M}(|\nabla u_\lambda|^2) > \lambda^{2k}\}) \\ &\leqslant \sum_{i=1}^{k} \epsilon_1^i w(\{x \in \Omega\colon \mathcal{M}(|\mathbf{f}_\lambda|^2) > \delta^2 \lambda^{2(k-i)}\}) \\ &\quad + \epsilon_1^k w(\{x \in \Omega\colon \mathcal{M}(|\nabla u_\lambda|^2) > 1\}). \end{aligned}$$

Rewriting the right-hand side we obtain

$$\begin{aligned} w(\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2) > \lambda^{2(k+1)}\}) &= \sum_{i=1}^{k} \epsilon_1^i w(\{x \in \Omega\colon \mathcal{M}(|\mathbf{f}|^2) > \delta^2 \lambda^{2(k+1-i)}\}) \\ &\quad + \epsilon_1^k w(\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2) > \lambda^2\}). \end{aligned} \tag{4.17}$$

Finally, applying the case $k = 1$ to the last term in (4.17) we conclude that

$$\begin{aligned} w(\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2) > \lambda^{2(k+1)}\}) &\leqslant \sum_{i=1}^{k+1} \epsilon_1^i w(\{x \in \Omega\colon \mathcal{M}(|\mathbf{f}|^2) > \delta^2 \lambda^{2(k+1-i)}\}) \\ &\quad + \epsilon_1^{k+1} w(\{x \in \Omega\colon \mathcal{M}(|\nabla u|^2) > 1\}), \end{aligned}$$

which completes the proof of the theorem. $\square$

5. Global weighted $W^{1,p}$ and Morrey estimates

In this section we present the proof of Theorems 2.1 and 2.2.

Proof of Theorem 2.1. Let w be an $A_{p/2}$ weight with $2 < p < \infty$. Let $|\mathbf{f}| \in L^p_w(\Omega)$. Then $|\mathbf{f}| \in L^2(\Omega)$. Indeed, by Hölder's inequality

$$\|\mathbf{f}\|^2_{L^2(\Omega)} \leqslant \int_\Omega |\mathbf{f}|^2 w^{2/p} w^{-2/p}\, dx \leqslant \|\mathbf{f}\|^2_{L^p_w(\Omega)} \bigl(w^{\frac{-2}{p-2}}(\Omega)\bigr)^{\frac{p-2}{p}}. \tag{5.1}$$

Then a unique weak solution $u \in W^{1,2}_0(\Omega)$ to (1.1) exists satisfying the global estimate (1.5). We take λ as in Corollary 4.11. For a given $\epsilon > 0$ we also take $\delta = \delta(\epsilon, [w]_{p/2})$, $0 < \delta < 1/8$, as in Corollary 4.11. We aim to show that $\mathcal{M}(|\nabla u|^2) \in L^{p/2}_w(\Omega)$ with an appropriate estimate. We fix $R > 0$ and begin by selecting a finite collection of points $\{y_i\}_{i=1}^L \subset \bar{\Omega}$, and a ball B such that

$$\bar{\Omega} \subset \bigcup_{i=1}^L B_r(y_i) \subset B,$$

where $r = R/2000$. We claim that we can choose N large (depending on $\|\nabla u\|_{L^2(\Omega)}$) such that for $u_N = u/N$ and for all $i = 1, \ldots, L$

$$w\bigl(\bigl\{x \in \Omega\colon \mathcal{M}\bigl(|\nabla u_N|^2\bigr)(x) > \lambda^2\bigr\}\bigr) < \epsilon\, w\bigl(B_r(y_i)\bigr). \tag{5.2}$$

Indeed, by the weak-type $(1,1)$ estimate for the maximal function there exists a constant $C > 0$ ($C = 3^n$ would do) such that

$$\begin{aligned}\bigl|\bigl\{x \in \Omega\colon \mathcal{M}\bigl(|\nabla u_N|^2\bigr)(x) > \lambda^2\bigr\}\bigr| &\leqslant \frac{C}{\lambda^2} \int_\Omega |\nabla u_N|^2\, dx \\ &= \frac{C}{(\lambda N)^2} \int_\Omega |\nabla u|^2\, dx. \end{aligned} \tag{5.3}$$

We now choose $N > 0$ so that for a pair $(A, \nu) = [w]_\infty$

$$\frac{C}{(\lambda N)^2} \int_\Omega |\nabla u|^2\, dx = \left(\frac{\epsilon}{A[w]_{p/2}[|B|/|B_r(y_i)|]^{p/2}}\right)^{1/\nu} |B|. \tag{5.4}$$

This is possible as we may assume that $\|\nabla u\|_{L^2(\Omega)} > 0$. Then it follows from (5.3), (5.4), and Lemma 3.4 that

$$w\bigl(\bigl\{x \in \Omega\colon \mathcal{M}\bigl(|\nabla u_N|^2\bigr)(x) > \lambda^2\bigr\}\bigr) < \frac{\epsilon}{[w]_{p/2}[|B|/|B_r(y_i)|]^{p/2}} w(B). \tag{5.5}$$

But applying the strong doubling property of weights (Lemma 3.3) we get

$$w(B) < [w]_{p/2}\bigl[|B|/|B_r(y_i)|\bigr]^{p/2} w\bigl(B_r(y_i)\bigr)$$

for all $i = 1, \ldots, L$. Now we combine this with (5.5) to get (5.2).

We now estimate the sum

$$S=\sum_{k=1}^{\infty}\lambda^{pk}w\big(\big\{x\in\Omega\colon\ \mathcal{M}\big(|\nabla u_N|^2\big)(x)>\lambda^{2k}\big\}\big).$$

To this end, we use (5.2) and apply Corollary 4.11 to calculate that with $\epsilon_1=\epsilon[w]^2_{p/2}(\frac{10}{1-4\delta})^{\frac{np}{2}}$:

$$\begin{aligned}S&\leqslant\sum_{k=1}^{\infty}\lambda^{pk}\Bigg[\sum_{i=1}^{k}\epsilon_1^i w\big(\big\{x\in\Omega\colon\ \mathcal{M}\big(|\mathbf{f}_N|^2\big)(x)>\delta^2\lambda^{2(k-i)}\big\}\big)\Bigg]\\&\quad+\sum_{k=1}^{\infty}\lambda^{pk}\epsilon_1^k w\big(\big\{x\in\Omega\colon\ \mathcal{M}\big(|\nabla u_N|^2\big)(x)>1\big\}\big).\end{aligned}$$

The first term in the right-hand side can be rewritten to obtain

$$\begin{aligned}S&\leqslant\sum_{i=1}^{\infty}\big(\lambda^p\epsilon_1\big)^i\Bigg[\sum_{k=i}^{\infty}\lambda^{p(k-i)}w\big(\big\{x\in\Omega\colon\ \mathcal{M}\big(|\mathbf{f}_N|^2\big)(x)>\delta^2\lambda^{2(k-i)}\big\}\big)\Bigg]\\&\quad+\sum_{k=1}^{\infty}\big(\lambda^p\epsilon_1\big)^k w\big(\big\{x\in\Omega\colon\ \mathcal{M}\big(|\nabla u_N|^2\big)(x)>1\big\}\big)\\&\leqslant C\big[\big\|\mathcal{M}\big(|\mathbf{f}_N|^2\big)\big\|^{p/2}_{L^{p/2}_w(\Omega)}+w(\Omega)\big]\sum_{k=1}^{\infty}\big(\lambda^p\epsilon_1\big)^k,\end{aligned}$$

where we used Lemma 3.7 in the last inequality. Now choosing ϵ small so that $\lambda^p\epsilon_1<1$, we obtain (see Lemma 3.7)

$$\big\|\mathcal{M}\big(|\nabla u_N|^2\big)\big\|^{p/2}_{L^{p/2}_w(\Omega)}\leqslant C\big(w(\Omega)+\big\|\mathcal{M}\big(|\mathbf{f}_N|^2\big)\big\|^{p/2}_{L^{p/2}_w(\Omega)}\big)$$

for a constant C depending only on $c_0, c_1, c_2, n, p, \delta, R, \Omega$, and $[w]_{p/2}$. From the last inequality, Lemmas 3.6 and 3.7 we obtain after multiplying both sides by N^p that

$$\|\nabla u\|^p_{L^p_w(\Omega)}\leqslant C\big(w(\Omega)N^p+\|\mathbf{f}\|^p_{L^p_w(\Omega)}\big).\tag{5.6}$$

On the other hand, in view of (5.4) and estimate (1.5) we find that

$$\begin{aligned}w(\Omega)N^p&\leqslant Cw(\Omega)\|\nabla u\|^p_{L^2(\Omega)}\leqslant Cw(\Omega)\big(\|\mathbf{f}\|^p_{L^2(\Omega)}+1\big)\\&\leqslant Cw(\Omega)\big[\|\mathbf{f}\|^p_{L^p_w(\Omega)}\big(w^{\frac{-2}{p-2}}(\Omega)\big)^{\frac{p-2}{2}}+1\big]\\&\leqslant C\big[[w]_{p/2}\|\mathbf{f}\|^p_{L^p_w(\Omega)}+w(\Omega)\big],\end{aligned}\tag{5.7}$$

where in the next to last inequality we used (5.1). Finally, combining inequalities (5.6) and (5.7) we get

$$\|\nabla u\|^p_{L^p_w(\Omega)}\leqslant C\big(w(\Omega)+\|\mathbf{f}\|^p_{L^p_w(\Omega)}\big)$$

as desired. □

Proof of Theorem 2.2. Suppose that $\mathbf{f} \in \mathcal{L}^{p,\theta}(\Omega, \mathbb{R}^n)$ where $2 < p < \infty$ and $0 < \theta \leqslant n$. For $z \in \Omega$, $0 < r \leqslant \operatorname{diam}(\Omega)$, and $0 < \epsilon < \theta$, we define the function

$$w(x) = \min\{|x-z|^{-n+\theta-\epsilon},\ r^{-n+\theta-\epsilon}\}.$$

By Lemma 3.2, w is an A_s weight for any $1 < s < \infty$, and its A_s constant $[w]_s$ is bounded from above by a constant independent of z and r. Since $w = r^{-n+\theta-\epsilon}$ on $B_r(z)$, we have

$$\begin{aligned} \int_{B_r(z)\cap\Omega} |\nabla u|^p\,dx &= r^{n-\theta+\epsilon} \int_{B_r(z)\cap\Omega} |\nabla u|^p w\,dx \\ &\leqslant C r^{n-\theta+\epsilon} \int_{\Omega} \big(|\mathbf{f}|^p + 1\big) w\,dx \end{aligned} \tag{5.8}$$

where we have used Theorem 2.1 in the inequality. Recalling Remark 3.5 the constant C is independent of z and r. Now by Fubini's theorem and the fact that $w \leqslant r^{-n+\theta-\epsilon}$ in $\mathbb{R}^n$, it follows that

$$\begin{aligned} \int_{\Omega} \big(|\mathbf{f}|^p + 1\big) w\,dx &= \int_0^\infty \Bigg[\int_{\{x\in\Omega:\ w(x)>t\}} \big(|\mathbf{f}|^p + 1\big)\,dx \Bigg] dt \\ &\leqslant \int_0^{r^{-n+\theta-\epsilon}} \int_{B_{t^{\frac{1}{-n+\theta-\epsilon}}}(z)\cap\Omega} \big(|\mathbf{f}|^p + 1\big)\,dx\,dt \\ &\leqslant \big\| \big(|\mathbf{f}| + 1\big) \big\|^p_{\mathcal{L}^{p,\theta}(\Omega)} \int_0^{r^{-n+\theta-\epsilon}} t^{\frac{n-\theta}{-n+\theta-\epsilon}}\,dt \\ &= \frac{n-\theta+\epsilon}{\epsilon} \big\| \big(|\mathbf{f}| + 1\big) \big\|^p_{\mathcal{L}^{p,\theta}(\Omega)} r^{-\epsilon}. \end{aligned} \tag{5.9}$$

We have used the inclusion $\{x \in \mathbb{R}^n\colon\ w(x) > t\} \subset B_{t^{\frac{1}{-n+\theta-\epsilon}}}(z)$ in the first inequality. Combining inequalities in (5.8) and (5.9) we obtain

$$\int_{B_r(z)\cap\Omega} |\nabla u|^p\,dx \leqslant C \big\| \big(|\mathbf{f}| + 1\big) \big\|^p_{\mathcal{L}^{p,\theta}(\Omega)} r^{n-\theta}$$

for a positive universal constant C independent of r and z. As the preceding inequality holds for all $z \in \Omega$ and $0 < r \leqslant \operatorname{diam}(\Omega)$ the desired estimate follows. $\square$

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