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Mann and Ishikawa Type Perturbed Iterative Algorithms for Generalized Quasivariational Inclusions

K. R. Kazmi

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In this paper, we introduce a class of generalized quasivariational inclusions and show its equivalence with a class of fixed point problems by making use of the properties of proximal maps. Using this equivalence, we develop the Mann and Ishikawa type perturbed iterative algorithms for this class of generalized quasivariational inclusions. Further, using fixed point techniques, we prove the existence of solutions for the class of generalized quasivariational inclusions and discuss the convergence criteria for the perturbed algorithms. Our algorithms and results improve and generalize many known corresponding algorithms and results.

1. INTRODUCTION

Variational inequality theory has emerged as an elegant and fascinating branch of applicable mathematics in recent years. Variational inequalities arise in models for a large number of mathematical, physical, engineering, and other problems. The theory of variational inequalities has led to exciting and important contributions to pure and applied sciences which include work on differential equations, contact problems in elasticity, fluid flow through porous media, control problems, general equilibrium problems in economics and transportation, and unilateral, obstacle, moving, and free boundary problems; see for instance [1–3, 6, 8]. Inspired and motivated by the recent work of Hassouni and Moudafi [7], we introduce a class of generalized quasivariational inclusions. We remark that one of the most important and difficult problems in variational inequality theory is the development of an efficient and implementable iterative algorithm for

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solving various classes of variational inequalities. There is a substantial number of iterative methods for solving variational inequalities in the literature. Among the most efficient methods is the proximal method introduced by Hassouni and Moudafi [7]. We also remark that the projection method and auxiliary principle technique of Glowinski, Lions, and Tremolieres [6] and its variant forms cannot be applied to study the existence of a solution and to develop the iterative algorithm for our considered class of generalized quasivariational inclusions. Therefore, the aim of this paper is to study the existence theory and to develop the Mann and Ishikawa type perturbed iterative algorithms for the class of generalized quasivariational inclusions. The convergence criteria for these algorithms is also discussed.

2. PRELIMINARIES

Let H be a Hilbert space with norm and inner product denoted by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively; let $T, A\colon H \to 2^H$, where 2^H is the power set of H, be two set valued mappings, and $g, m\colon H \to H$ be two single-valued mappings. Assume $\phi\colon H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semicontinuous function and $\partial\phi$ is the subdifferential of ϕ. Then the generalized quasivariational inclusion problem (GQVIP) is to find $u \in H$, $x \in T(u)$, $y \in A(u)$ such that $(g-m)(u) \cap \operatorname{dom} \partial\phi \neq \emptyset$ and

$$\operatorname{Re}\langle x - y, v - (g-m)(u)\rangle \geq \phi((g-m)(u)) - \phi(v), \qquad \forall v \in H, \tag{2.1}$$

where $g - m$ is defined as

$$(g-m)(u) = g(u) - m(u) \qquad \text{for each } u \in H.$$

Some Special Cases

(I) If T and A are single-valued mappings and m is a zero mapping, then GQVIP is equivalent to finding $u \in H$ such that $g(u) \cap \operatorname{dom} \phi \neq \emptyset$ and

$$\operatorname{Re}\langle T(u) - A(u), v - g(u)\rangle \geq \phi(g(u)) - \phi(v), \qquad \forall v \in H,$$

which is called the variational inclusion problem introduced and studied by Hassouni and Moudafi [7].

(II) If $\phi = \delta_K$, the indicator function of the nonempty closed convex set K in H, then GQVIP is equivalent to finding $u \in H$, $x \in T(u)$, $y \in A(u)$ such that $g(u) \in K + m(u)$ such that

$$\operatorname{Re}\langle x - y, v - g(u)\rangle \geq 0 \qquad \forall v \in K(u),$$

which is called the generalized strongly nonlinear quasivariational inequality problem, introduced and studied by Ding [4].

We remark that GQVIP also includes as special cases, the variational and quasivariational inequality problems considered by [4, 5, 7, 9, 11, 12, 14−16].

3. MANN AND ISHIKAWA TYPE PERTURBED ITERATIVE ALGORITHMS

First of all, we prove that GQVIP is equivalent to a fixed point problem.

LEMMA 3.1. *GQVIP has a solution if and only if, for some given* $\eta > 0$, *the mapping* $F\colon H \to 2^H$ *defined by*

$$F(u) = \bigcup_{x \in T(u)} \bigcup_{y \in A(u)} \Big[u - (g-m)(u) + \mathsf{J}_\eta^\phi((g-m)(u) - \eta(x-y)) \Big], \quad (3.1)$$

where $\eta > 0$ *is a constant and* $\mathsf{J}_\eta^\phi = (I + \eta\,\partial\phi)^{-1}$ *is the so-called proximal mapping, I stands for the identity on H, has a fixed point.*

Proof. Let (u^*, x^*, y^*) be a solution of GQVIP. Then we have $u^* \in H$, $x^* \in T(u^*)$, $y^* \in A(u^*)$ such that $(g-m)(u^*) \cap \operatorname{dom} \partial\phi \neq \emptyset$ and

$$\operatorname{Re}\langle x^* - y^*, v - (g-m)(u^*)\rangle \geq \phi(g-m)(u^*) - \phi(v), \qquad \forall v \in H. \quad (3.2)$$

Using the definition of $\partial\phi$, (3.2) can be written as

$$y^* - x^* \in \partial\phi((g-m)(u^*)),$$

and hence for any given $\eta > 0$,

$$\begin{aligned}(g-m)(u^*) - \eta(x^* - y^*) &\in (g-m)(u^*) + \eta\,\partial\phi((g-m)(u^*)) \\ &= (I + \eta\,\partial\phi)((g-m)(u^*)).\end{aligned}$$

From the definition of J_η^ϕ, one has

$$(g-m)(u^*) = \mathsf{J}_\eta^\phi((g-m)(u^*) - \eta(x^* - y^*)),$$

and hence

$$\begin{aligned} u^* &= u^* - (g-m)(u^*) + J_\eta^\phi((g-m)(u^*) - \eta(x^* - y^*)) \\ &\in \bigcup_{x^* \in T(u^*)} \bigcup_{y^* \in A(u^*)} \big[u^* - (g-m)(u^*) \\ &\qquad + J_\eta^\phi((g-m)(u^*) - \eta(x^* - y^*))\big] \\ &= F(u), \end{aligned}$$

$u^* \in H$ is a fixed point of F.

Conversely, if $u^* \in H$ is a fixed point of F, by definition of F, there exist $x^* \in T(u^*)$ and $y^* \in A(u^*)$ such that

$$u^* = u^* - (g-m)(u^*) + J_\eta^\phi((g-m)(u^*) - \eta(x^* - y^*)).$$

Hence, from the definition of J_η^ϕ, we have

$$(g-m)(u^*) - \eta(x^* - y^*) \in (g-m)(u^*) + \eta\,\partial\phi((g-m)(u^*)).$$

Note $\eta > 0$, and we have

$$y^* - x^* \in \partial\phi((g-m)(u^*)).$$

The definition of $\partial\phi$ yields

$$\mathrm{Re}\langle x^* - y^*, v - (g-m)(u^*)\rangle \geq \phi((g-m)(u^*)) - \phi(v), \quad \forall v \in H$$

and $\mathrm{Im}(g-m) \cap \mathrm{dom}\,\partial\phi \neq \emptyset$.

Thus (u^*, x^*, y^*) is a solution of GQVIP. ∎

The transformation of GQVIP to the fixed point problem (3.1) is very useful in the approximation analysis of GQVIP. One of the consequences of this transformation is that we can obtain an approximate solution by an iterative algorithm.

Based on the above transformation and observations, we now suggest and analyze the following new general and unified algorithms for GQVIP:

MANN TYPE PERTURBED ITERATIVE ALGORITHM (MTPIA). Let $T, A\colon \to 2^H$ and $g, m\colon H \to H$. Given $u_0 \in H$, the iterative sequences $\{u_n\}$, $\{x_n\}$, and $\{y_n\}$ are defined by

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n \\ &\qquad + \alpha_n\big[u_n - (g-m)(u_n) + J_\eta^{\phi_n}((g-m)(u_n) - \eta(x_n - y_n))\big] \\ &\qquad + e_n, x_n \in T(u_n) \qquad \text{and} \qquad y_n \in A(u_n), \qquad n \geq 0, \end{aligned}$$

where $\{\alpha_n\}$ is a real sequence satisfying $\alpha_0 = 1, 0 \le \alpha_n \le 1$ for $n > 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$; $e_n \in H$ for all n is an error which is taken into account for a possible inexact computation of the proximal point; $\{\phi_n\}$ is the sequence approximating ϕ and $\eta > 0$ is a constant.

ISHIKAWA TYPE PERTURBED ITERATIVE ALGORITHM (ITPIA). Let $T, A\colon H \to 2^H$ and $g, m\colon H \to H$. Given $u_0 \in H$, the iterative sequences $\{u_n\}$, $\{x_n\}$, and $\{y_n\}$ are defined by

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n \\ &\quad + \alpha_n\left[v_n - (g-m)(v_n) + J_\eta^{\phi_n}((g-m)(v_n) - \eta(\bar{x}_n - \bar{y}_n))\right] + e_n, \\ v_n &= (1-\beta_n)u_n \\ &\quad + \beta_n\left[u_n - (g-m)(u_n) + J_\eta^{\phi_n}((g-m)(u_n) - \eta(x_n - y_n))\right] + \beta_n r_n, \end{aligned}$$

for $n \ge 0$, where $\bar{x}_n \in T(v_n)$, $\bar{y}_n \in A(v_n)$, $x_n \in T(u_n)$, $y_n \in A(u_n)$; e_n and r_n in H for all $n \ge 0$ are errors; $\{\phi_n\}$ is the sequence approximating ϕ; $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying $\alpha_0 = 1$, $0 \le \alpha_n, \beta_n \le 1$ for $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\eta > 0$ is a constant.

Next, we review some definitions and results which are needed in the sequel.

DEFINITION 3.1. A mapping $f\colon H \to H$ is said to be

(i) ν-strongly monotone if there exists a constant $\nu > 0$ such that

$$\operatorname{Re}\langle f(u) - f(v), u - v\rangle \ge \nu \|u - v\|^2, \qquad \forall u, v \in H;$$

(ii) σ-Lipschitz continuous if there exists a constant $\sigma > 0$ such that

$$\|f(u) - f(v)\| \le \sigma \|u - v\|, \qquad \forall u, v \in H.$$

DEFINITION 3.2. A set-valued mapping $T\colon H \to 2^H$ is said to be

(i) α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\operatorname{Re}\langle x - y, v - u\rangle \ge \alpha \|u - v\|^2, \quad \forall u, v \in H,\ x \in T(u) \text{ and } y \in T(v);$$

(ii) β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\delta(T(u), T(v)) \le \beta\|u - v\|, \qquad \forall u, v \in H,\ x \in T(u), \text{ and } y \in T(v),$$

where $\delta(A, B) = \sup\{\|a - b\|\colon a \in A,\ b \in B\}$ for any $A, B \in 2^H$.

LEMMA 3.2 [10]. *Let ϕ be a proper convex lower semi-continuous function. Then $\mathrm{J}_\eta^\phi = (I + \eta\,\partial\phi)^{-1}$ is nonexpansive, i.e.,*

$$\left\| \mathrm{J}_\eta^\phi(u) - \mathrm{J}_\eta^\phi(v) \right\| \le \|u - v\|, \qquad \forall u, v \in H.$$

We remark that if $\phi = \delta_K$, the indicator function of a nonempty closed convex set K in H, then $\mathrm{J}_\eta^\phi(u) = P_K(u)$ for each $u \in H$ and $\eta > 0$, where P_K is the projection mapping of H onto K.

Several special cases of ITPIA are listed below.

(i) If $\beta_n = 0$ for all $n \ge 0$, ITPIA reduces to MTPIA.

(ii) If m is a zero mapping, T and A are single-valued, and $\beta_n = 0$ and $\alpha_n = 1$ for all $n \ge 0$, ITPIA reduces to the Algorithm (2.4) of Hassouni and Moudafi [7].

(iii) If for all $n \ge 0$, $\phi_n = \delta_K$, the indicator function of a nonempty closed convex set K in H, $e_n = 0$, $r_n = 0$ for all $n \ge 0$, the ITPIA reduces to the Algorithm 3.2 of Ding [4].

(iv) If $\phi_n = \delta_K$ for all $n \ge 0$, $g = I$, the identity mapping, $e_n = r_n = 0$ for all $n \ge 0$ and $\beta_n = 0$ for all $n \ge 0$ then ITPIA reduces to the Algorithm in Theorem 3.5 of Ding [5].

4. EXISTENCE AND CONVERGENCE RESULT

In this section, we prove the existence of a solution of GQVIP and discuss the convergence criteria for ITPIA and MTPIA.

THEOREM 4.1. *Let T: $H \to 2^H$ be α-strongly monotone and β-Lipschitz continuous; A: $H \to 2^H$ be μ-Lipschitz continuous; $(g - m)$: $H \to H$ be ν-strongly monotone; and g, m: $H \to H$ be σ-Lipschitz continuous and ξ-Lipschitz continuous, respectively. Assume that*

$$\mathrm{Re}\langle m(v) - m(u), g(u) - g(v)\rangle \le \lambda \|u - v\|^2, \qquad \forall u, v \in H \quad (4.1)$$

for some constant λ such that $\lambda_0 \le \lambda \le \sigma\xi$, where

$$\lambda_0 = \inf\{M: \mathrm{Re}\langle m(v) - m(u), g(u) - g(v)\rangle \le M \|u - v\|^2, \forall u, v \in H\}.$$

If there exists a constant $\eta > 0$ *such that*

$$\left|\eta - \frac{\alpha + \mu(k-1)}{\beta^2 - \mu^2}\right| < \frac{\sqrt{(\alpha + \mu(k-1))^2 - (\beta^2 - \mu^2)k(2-k)}}{\beta^2 - \mu^2}$$

$$\alpha > \mu(1-k) + \sqrt{(\beta^2 - \mu^2)k(2-k)}, \qquad \mu(1-k) < \beta$$

and (4.2)

$$k = 2\sqrt{1 - 2\nu + \xi^2 + \sigma^2 + 2\lambda} < 1,$$

then (u^*, x^*, y^*) *is a solution of GQVIP. Moreover, if*

$$\lim_{n\to\infty} \left\| \mathrm{J}_\eta^{\phi_n}(v) - \mathrm{J}_\eta^{\phi}(v) \right\| = 0, \qquad \forall v \in H$$

and $\{u_n\}$, $\{\bar{x}_n\}$, *and* $\{\bar{y}_n\}$ *are defined by ITPIA with conditions*

(i) $\lim_{n\to\infty} \|e_n\| = 0 = \lim_{n\to\infty} \|r_n\|$ *and*

(ii) $\sum_{i=0}^{n} \prod_{j=i+1}^{n} (1 - \alpha_j(1-c))$ *converges*, $0 \le c < 1$,

then $\{u_n\}$, $\{\bar{x}_n\}$, *and* $\{\bar{y}_n\}$ *strongly converge to* u^*, x^*, *and* y^*, *respectively*.

Proof. First we prove that the GQVIP has a solution (u^*, x^*, y^*). By Lemma 3.1, it is enough to show that the mapping $F: H \to 2^H$ defined by (3.1) has a fixed point u^*. For any $u, v \in H$, $p \in F(u)$, and $q \in F(v)$, there exist $x_1 \in T(u)$, $x_2 \in T(v)$, $y_1 \in A(u)$, and $y_2 \in A(v)$ such that

$$p = u - (g-m)(u) + \mathrm{J}_\eta^{\phi}((g-m)(u) - \eta(x_1 - y_1))$$

and

$$q = v - (g-m)(v) + \mathrm{J}_\eta^{\phi}((g-m)(v) - \eta(x_2 - y_2)).$$

By Lemma 3.2, we have

$$\begin{aligned}\|p - q\| \le{}& \|u - v - ((g-m)(u) - (g-m)(v))\| \\ &+ \|(g-m)(u) - (g-m)(v) - \eta(x_1 - x_2) + \eta(y_1 - y_2)\| \\ \le{}& 2\|u - v - ((g-m)(u) - (g-m)(v))\| \\ &+ \|u - v - \eta(x_1 - x_2)\| + \eta\delta(A(u), A(v)).\end{aligned}$$

By using the technique of Noor [12], the Lipschitz continuity of T, A, g, and m, the strong monotonicity of T and $(g - m)$, and (4.1), we obtain

$$\begin{aligned}\|u - v - ((g - m)(u) - (g - m)(v))\|^2 &= \|u - v\|^2 - 2\,\mathrm{Re}\langle u - v, (g - m)(u) - (g - m)(v)\rangle \\ &\quad + \|m(u) - m(v)\|^2 + \|g(u) - g(v)\|^2 \\ &\quad + 2\,\mathrm{Re}\langle m(v) - m(u), g(u) - g(v)\rangle \\ &\leq \left(1 - 2\nu + \xi^2 + \sigma^2 + 2\lambda\right)\|u - v\|^2, \end{aligned} \tag{4.3}$$

$$\|u - v - \eta(x_1 - x_2)\|^2 \leq \left(1 - 2\eta\alpha + \eta^2\beta^2\right)\|u - v\|^2$$

and

$$\delta(A(u), A(v)) \leq \|u - v\|. \tag{4.4}$$

Therefore, it follows that

$$\begin{aligned}\delta(F(u), F(v)) &\leq \left\{2\sqrt{1 - 2\nu + \xi^2 + \sigma^2 + 2\lambda} + \sqrt{1 - 2\eta\alpha + \eta^2\beta^2} + \eta\mu\right\}\|u - v\| \\ &= \{k + t(\eta) + \eta\mu\}\|u - v\|, \\ &= \theta\|u - v\|, \end{aligned} \tag{4.5}$$

where $k = 2\sqrt{1 - 2\nu + \xi^2 + \sigma^2 + 2\lambda}$, $t(\eta) = \sqrt{1 - 2\eta\alpha + \eta^2\beta^2}$, and $\theta = k + t(\eta) + \eta\mu$. By condition (4.2), we see that $0 < \theta < 1$. It follows from (4.5) and Theorem 3.1 of Siddiqi and Ansari [14] that F has a fixed point $u^* \in H$. Hence by Lemma 3.1, there exist $x^* \in T(u^*)$ and $y^* \in A(u^*)$ such that (u^*, x^*, y^*) is a solution of GQVIP.

Next we prove that the iterative sequences $\{u_n\}$, $\{\bar{x}_n\}$, and $\{\bar{y}_n\}$ defined by ITPIA strongly converge to u^*, x^*, and y^*, respectively.

Since GQVIP has a solution (u^*, x^*, y^*) then, by Lemma 3.1, we have

$$u^* = u^* - (g - m)(u^*) + J_\eta^\phi((g - m)(u^*) - \eta(x^* - y^*)).$$

By making use of the same arguments used for obtaining (4.3) and (4.4), we get

$$\|u_n - u^* - ((g - m)(u_n) - (g - m)(u^*))\| \leq 2\sqrt{1 - 2\nu + \xi^2 + \sigma^2 + 2\lambda}\,\|u_n - u^*\|,$$

$$\|u_n - u^* - \eta(x_n - x^*)\| \leq \sqrt{1 - 2\eta\alpha + \eta^2\beta^2}\,\|u_n - u^*\|,$$

$$\|v_n - u^* - ((g - m)(v_n) - (g - m)(u^*))\| \leq \sqrt{1 - 2\nu + \xi^2 + \sigma^2 + 2\lambda}\,\|v_n - u^*\|,$$

and

$$\|v_n - u^* - \eta(\bar{x}_n - x^*)\| \le \sqrt{1 - 2\eta\alpha + \eta^2\beta^2}\,\|v_n - u^*\|.$$

By setting

$$h(u^*) := (g - m)(u^*) - \eta(x^* - y^*)$$

and

$$h(v_n) := (g - m)(v_n) - \eta(\bar{x}_n - \bar{y}_n),$$

we have

$$\begin{aligned}\|u_{n+1} - u^*\| = \Big\|&(1 - \alpha_n)u_n - \alpha_n\big[v_n - (g - m)(v_n) + \mathsf{J}_\eta^{\phi_n}(h(v_n))\big] + e_n \\ &+ (1 - \alpha_n)u^* + \alpha_n\big[u^* - (g - m)(u^*) + \mathsf{J}_\eta^{\phi}(h(u^*))\big]\Big\| \\ \le{}& (1 - \alpha_n)\|u_n - u^*\| \\ &+ \alpha_n\|v_n - u^* - ((g - m)(v_n) - (g - m)(u^*))\| \\ &+ \alpha_n\left\|\mathsf{J}_\eta^{\phi_n}(h(v_n)) - \mathsf{J}_\eta^{\phi}(h(u^*))\right\| + \|e_n\|. \end{aligned} \tag{4.6}$$

Now, since $\mathsf{J}_\eta^{\phi_n}$ is nonexpansive, we have

$$\begin{aligned}&\left\|\mathsf{J}_\eta^{\phi_n}(h(v_n)) - \mathsf{J}_\eta^{\phi}(h(u^*))\right\| \\ &\quad\le \|h(v_n) - h(u^*)\| + \left\|\mathsf{J}_\eta^{\phi_n}(h(u^*)) - \mathsf{J}_\eta^{\phi}(h(u^*))\right\| \\ &\quad\le \|v_n - u^* - ((g - m)(v_n) - (g - m)(u^*))\| \\ &\qquad + \|v_n - u^* - \eta(\bar{x}_n - x^*)\| + \left\|\mathsf{J}_\eta^{\phi_n}(h(u^*)) - \mathsf{J}_\eta^{\phi}(h(u^*))\right\| \\ &\qquad + \eta\|\bar{y}_n - y^*\| \\ &\quad\le \sqrt{1 - 2\nu + \xi^2 + \sigma^2 + 2\lambda}\,\|v_n - u^*\| \\ &\qquad + \sqrt{1 - 2\eta\alpha + \eta^2\beta^2}\,\|v_n - u^*\| \\ &\qquad + \eta\delta(A(v_n), A(u^*)) + \left\|\mathsf{J}_\eta^{\phi_n}(h(u^*)) - \mathsf{J}_\eta^{\phi}(h(u^*))\right\|. \end{aligned} \tag{4.7}$$

On combining (4.6) and (4.7) and using the μ-Lipschitz continuity of A, we get

$$\begin{aligned}\|u_{n+1} - u^*\| &\le (1-\alpha_n)\|u_n - u^*\| \\ &\quad + \alpha_n\Big[2\sqrt{1-2\nu+\xi^2+\sigma^2+2\lambda} \\ &\quad + \sqrt{1-2\eta\alpha+\eta^2\beta^2} + \eta\mu\Big]\|v_n - u^*\| \\ &\quad + \alpha_n\left\|J_\eta^{\phi_n}(h(u^*)) - J_\eta^{\phi}(h(u^*))\right\| + \|e_n\| \\ &= (1-\alpha_n)\|u_n - u^*\| + \alpha_n\theta\|v_n - u^*\| + \alpha_n\varepsilon_n + \|e_n\|, \end{aligned} \tag{4.8}$$

where $\theta = 2\sqrt{1-2\nu+\xi^2+\sigma^2+2\lambda} + \sqrt{1-2\eta\alpha+\eta^2\beta^2} + \eta\mu$ and

$$\varepsilon_n = \left\|J_\eta^{\phi_n}(h(u^*)) - J_\eta^{\phi}(h(u^*))\right\|.$$

Next

$$\begin{aligned}\|v_n - u^*\| &= \Big\|(1-\beta_n)u_n + \beta_n\big[u_n - (g-m)(u_n) + J_\eta^{\phi_n}(h(u_n))\big] + \beta_n r_n \\ &\qquad + (1-\beta_n)u^* + \beta_n\big[u^* - (g-m)(u^*) + J_\eta^{\phi}(h(u^*))\big]\Big\| \\ &\le (1-\beta_n)\|u_n - u^*\| \\ &\quad + \beta_n\|u_n - u^* - ((g-m)(u_n) - (g-m)(u^*))\| \\ &\quad + \beta_n\left\|J_\eta^{\phi_n}(h(u_n)) - J_\eta^{\phi}(h(u^*))\right\| + \beta_n\|r_n\|. \end{aligned} \tag{4.9}$$

By making use of the same arguments used for obtaining (4.7), we get

$$\begin{aligned}&\left\|J_\eta^{\phi_n}(h(u_n)) - J_\eta^{\phi}(h(u^*))\right\| \\ &\quad \le \left[\sqrt{1-2\nu+\xi^2+\sigma^2+2\lambda} + \sqrt{1-2\eta\alpha+\eta^2\beta^2}\right]\|u_n - u^*\| \\ &\qquad + \eta\delta(A(u_n), A(u^*)) + \varepsilon_n. \end{aligned} \tag{4.10}$$

On combining (4.9) and (4.10) and using the μ-Lipschitz continuity of A, we get

$$\begin{aligned}\|v_n - u^*\| &\le (1-\beta_n)\,\|u_n - u^*\| + \beta_n\theta\|u_n - u^*\| + \beta_n\varepsilon_n + \beta_n\|r_n\| \\ &\le \big(1-\beta_n(1-\theta)\big)\,\|u_n - u^*\| + \beta_n\big(\varepsilon_n + \|r_n\|\big) \\ &\le \|u_n - u^*\| + \beta_n\big(\varepsilon_n + \|r_n\|\big), \qquad (4.11)\end{aligned}$$

since $(1-\beta_n(1-\theta)) \le 1$.

On combining (4.8) and (4.11), we get

$$\begin{aligned}\|u_{n+1} - u^*\| &\le (1-\alpha_n)\,\|u_n - u^*\| + \alpha_n\theta\|u_n - u^*\| \\ &\qquad + \theta\alpha_n\beta_n\big(\varepsilon_n + \|r_n\|\big) + \alpha_n\varepsilon_n + \|e_n\| \\ &= \big(1-\alpha_n(1-\theta)\big)\,\|u_n - u^*\| + \alpha_n\varepsilon_n \\ &\qquad + \theta\alpha_n\beta_n\big(\varepsilon_n + \|r_n\|\big) + \|e_n\| \\ &\le \prod_{i=0}^{n}\big(1+\alpha_i(1-\theta)\big)\,\|u_0 - u^*\| \\ &\qquad + \sum_{j=0}^{n}\alpha_j\prod_{i=j+1}^{n}\big(1-\alpha_i(1-\theta)\big)\varepsilon_j \\ &\qquad + \theta\sum_{j=0}^{n}\alpha_j\beta_j\prod_{i=j+1}^{n}\big(1-\alpha_j(1-\theta)\big)\big(\varepsilon_j + \|r_j\|\big) \\ &\qquad + \sum_{j=0}^{n}\prod_{i=j+1}^{n}\big(1-\alpha_i(1-\theta)\big)\,\|e_j\|, \qquad (4.12)\end{aligned}$$

where $\prod_{i=j+1}^{n}(1-\alpha_i(1-\theta)) = 1$ when $j = n$.

Now, let B denote the lower triangular matrix with entries

$$b_{nj} = \alpha_j\prod_{i=j+1}^{n}\big(1-\alpha_i(1-\theta)\big).$$

Then B is multiplicative, see Rhoades [13], so that

$$\lim_{n\to\infty}\sum_{j=0}^{n}\alpha_j\prod_{i=j+1}^{n}\big(1-\alpha_i(1-\theta)\big)\varepsilon_j = 0$$

$$\lim_{n\to\infty}\theta\sum_{j=0}^{n}\alpha_j\beta_j\prod_{i=j+1}^{n}\big(1-\alpha_i(1-\theta)\big)\big(\varepsilon_j + \|r_j\|\big) = 0,$$

since $\lim_{n\to\infty}\|r_n\| = 0$ and $\lim_{n\to\infty}\varepsilon_n = \|\mathrm{J}_\eta^{\phi_n}(h(u^*)) - \mathrm{J}_\eta^{\phi}(h(u^*))\| = 0$.

Let D be the lower triangular matrix with entries

$$d_{nj} = \prod_{i=j+1}^{n} \left(1 - \alpha_i(1 - \theta)\right).$$

Condition (ii) implies that D is multiplicative, and hence

$$\lim_{n\to\infty} \sum_{j=0}^{n} \prod_{i=j+1}^{n} \left(1 - \alpha_i(1 - \theta)\right) \|e_j\| = 0,$$

since $\lim_{n\to\infty} \|e_n\| = 0$.

Also

$$\lim_{n\to\infty} \prod_{i=0}^{n} \left(1 - \alpha_i(1 - \theta)\right) = 0,$$

since $\sum_{i=0}^{n} \alpha_i = \infty$.

Hence, it follows from inequality (4.12) that $\lim_{n\to\infty} \|u_{n+1} - u^*\| = 0$, i.e., the sequence $\{u_n\}$ strongly converges to u^* in H. The inequality (4.11) implies that the sequence $\{v_n\}$ also converges to u^*. Since $\bar{x}_n \in T(u_n)$, $x^* \in T(u^*)$, and T is β-Lipschitz continuous, we have

$$\begin{aligned}\|\bar{x}_n - x^*\| &\le \delta\left(T(v_n), T(u^*)\right)\\ &\le \|v_n - u^*\| \to 0 \qquad \text{as } n \to \infty,\end{aligned}$$

i.e., $\{\bar{x}_n\}$ strongly converges to x^*. Similarly we can prove that $\{\bar{y}_n\}$ strongly converges to y^*. ∎

We remark that if $\beta_n = 0$ for all $n \ge 0$, Theorem 4.1 gives the conditions under which the sequences $\{u_n\}$, $\{x_n\}$, and $\{y_n\}$ defined by MTPIA strongly converge to u^*, x^*, and y^*, respectively.

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