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Two-dimensional current algebras and affine fusion product

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To Ernest Borisovich Vinberg on the occasion of his seventieth birthday

Abstract

In this paper we study a family of commutative algebras generated by two infinite sets of generators. These algebras are parametrized by Young diagrams. We explain a connection of these algebras with the fusion product of integrable irreducible representations of the affine $\mathfrak{sl}_2$ Lie algebra. As an application we derive a fermionic formula for the character of the affine fusion product of two modules. These fusion products can be considered as a simplest example of the double affine Demazure modules.

Keywords: Kac–Moody algebras; Demazure modules

Introduction

The main goal of this paper is to derive a formula for the character of the fusion product of two integrable irreducible representations of the affine Kac–Moody Lie algebra $\widehat{\mathfrak{sl}_2}$. We first briefly recall the definition.

The notion of the fusion product of cyclic representations $V_1, \dots, V_n$ of the Lie algebra $\mathfrak{g}$ was introduced in [FL]. This object is a cyclic graded representation of the current algebra $\mathfrak{g} \otimes \mathbb{C}[u]$.

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One starts with a tensor product of evaluation representations $V_1(z_1) \otimes \cdots \otimes V_n(z_n)$, where z_i are pairwise distinct complex numbers. Introduce a filtration

$$F_l = \operatorname{span}\left\{\left(g_1 \otimes u^{i_1} \cdots g_k \otimes u^{i_k}\right) \cdot v_1 \otimes \cdots \otimes v_n,\ \sum i_\alpha \leqslant l\right\}, \tag{1}$$

where v_i are cyclic vectors of V_i. Then the fusion product is an adjoint graded space

$$V_1(z_1) * \cdots * V_n(z_n) = F_0 \oplus \bigoplus_{l>0} F_l / F_{l-1}.$$

An important property is that in some cases $V_1(z_1) * \cdots * V_n(z_n)$ is independent on $\{z_i\}$ as $\mathfrak{g} \otimes \mathbb{C}[u]$-module. This is always true for

- $n = 2$ and arbitrary cyclic modules (obvious);
- $\mathfrak{g} = \mathfrak{sl}_2$ and n finite-dimensional modules (see [FL,FF1]);
- $\mathfrak{g} = \mathfrak{sl}_n$ and irreducible representations with special highest weights (see [CL,FKL,Ked]);
- simple Lie algebra $\mathfrak{g}$ and special highest weight irreducible representations (see [FoL]).

In these cases we omit numbers z_i and denote the corresponding fusion product simply by $V_1 * \cdots * V_n$.

Now let $\mathfrak{g}$ be an affine Kac–Moody Lie algebra and V_i be integrable irreducible representations. This situation for $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ and $n = 2$ was studied in [FFJMT] in order to derive some results about monomial bases for vertex operators in minimal models. In particular a bosonic formula for the character of fusion product of level 1 and level k representations was obtained. Here we consider two arbitrary level representations and derive a fermionic formula for the corresponding fusion product. We briefly describe our approach here.

Let $\mathfrak{g} = \widehat{\mathfrak{sl}_2} = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$, where K is a central element and d is a degree operator. For $k \in \mathbb{N}$ and $0 \leqslant i \leqslant k$ let $L_{i,k}$ be the corresponding irreducible integrable representation of $\widehat{\mathfrak{sl}_2}$ with highest weight vector $v_{i,k}$ such that $h_0 v_{i,k} = i v_{i,k}$, $K v_{i,k} = k v_{i,k}$, $d v_{i,k} = 0$, where h is a standard generator of the Cartan subalgebra and for $x \in \mathfrak{sl}_2$ we set $x_i = x \otimes t^i$. In what follows we use the notation e, h, f for standard basis of $\mathfrak{sl}_2$. There exist three different gradings on a fusion product $L_{i_1,k_1} * L_{i_2,k_2}$: $\deg_z$ by an operator h_0, $\deg_q$ by an operator d and $\deg_u$ coming from the fusion filtration (1). This defines a character $\operatorname{ch}_{z,u,q} L_{i_1,k_1} * L_{i_2,k_2}$. To find this character we consider a principal subspace $W^N_{i,k} \hookrightarrow L_{i,k}$ which is generated from the extremal vector $v^N_{i,k}$ by the action of operators e_i, $i \in \mathbb{Z}$ (see [FS,FF2]). The weight of $v^N_{i,k}$ is a weight of $v_{i,k}$ shifted by the Nth power of the translation element from the Weyl group of $\widehat{\mathfrak{sl}_2}$. An important thing is that $L_{i,k}$ is a limit of $W^N_{i,k}$ while $N \to \infty$. This gives

$$L_{i_1,k_1} * L_{i_2,k_2} = \lim_{N \to \infty} W^N_{i_1,k_1} * W^N_{i_2,k_2}.$$

So we only need to find the character of $W^N_{i_1,k_1} * W^N_{i_2,k_2}$. We note that this space is a cyclic representation of an abelian Lie algebra with a basis $e_i, e_i \otimes u$, $i \in \mathbb{Z}$ with cyclic vector $v_{i_1,k_1} \otimes v_{i_2,k_2}$. We give a precise description of $W^N_{i_1,k_1} * W^N_{i_2,k_2}$ in terms of generators and relations.

Let $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_s > 0$ be a set of positive integers. We consider a corresponding partition $\lambda = \{(i,j),\ i,j \in \mathbb{Z},\ i,j \geqslant 0,\ i \leqslant \lambda_j\}$ and define an algebra

$$A_\lambda = \mathbb{C}[a_0, a_{-1}, \ldots; b_0, b_{-1}, \ldots] \big/ \big\langle a(z)^i b(z)^j,\ (i,j) \notin \lambda \big\rangle, \tag{2}$$

where $a(z) = \sum_{k \geqslant 0} a_{-k} z^k$, $b(z) = \sum_{l \geqslant 0} b_{-l} z^l$. This algebra is two-dimensional generalization of an algebra $\mathbb{C}[e_i]_{i \leqslant 0}/e(z)^{k+1}$ which plays an important role in the representation theory of $\widehat{\mathfrak{sl}_2}$ (see for example [FS]). Recall that one of the most useful tools of study of A_k and similar algebras is a vertex operator realizations (see for example [FK,FF3,FLT,FST]). We also use this technique in our situation. Namely for λ such that $\lambda_{i-1} - \lambda_i \leqslant \lambda_i - \lambda_{i+1}$ we embed A_λ into multi-dimensional lattice vertex operator algebra and compute the character of A_λ. In addition we show that there exists λ (depending on k_1 and k_2) such that up to a certain identification of generators e_i, $e_i \otimes u$ and a_j, b_j the space $W^N_{i_1,k_1} * W^N_{i_2,k_2}$ is isomorphic to the quotient of A_λ by some ideal. This ideal depends on i_1 and i_2 and is generated by certain coefficients of the series $a(z)^i$ and $b(z)^j$. Using the realization above we derive a fermionic (Gordon type) formula for the character of the fusion product of principal subspaces and therefore (as a limit) the character of the fusion product of two integrable irreducible representations. For fermionic formulas in the case of finite-dimensional algebras see for example [FF1,FJKLM1,FJKLM2,AK]. We also note that two-dimensional current algebras of the type A_λ can be used for the study of principal subspaces of integrable representations of $\widehat{\mathfrak{sl}_3}$.

We also expect the existence of the bosonic (alternating sign) formula for the fusion product of integrable representations similar to one given in [FFJMT] in the $\widehat{\mathfrak{sl}_2}$ case with one of the representations of the level 1. Let us briefly explain the importance of such formula. Let $\mathfrak{g}$ be an affine Kac–Moody Lie algebra, L_μ, L_ν be its integrable highest weight representations. Then one has a decomposition of the tensor product

$$L_\mu \otimes L_\nu = \bigoplus_\pi C^\pi_{\mu,\nu} \otimes L_\pi$$

into the direct sum of integrable highest weight modules L_π, where $C^\pi_{\mu,\nu}$ are the spaces of multiplicities (see [K1]). The fusion filtration (1) defines a filtration on $C^\pi_{\mu,\nu}$, because all F_l in (1) are representations of $\mathfrak{g} = \mathfrak{g} \otimes 1 \hookrightarrow \mathfrak{g} \otimes \mathbb{C}[t]$. We note that spaces of multiplicities appear in the coset conformal field theories (see [DMS]). In [FFJMT] the character of the filtered space $C^\pi_{\mu,\nu}$ in some particular case was given in the bosonic form and the resulting formula was applied for the study of vertex operators in Virasoro minimal models. We hope that in the general case (at least for $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$) it is also possible to write an analogous bosonic formula and establish connection with the corresponding coset conformal field theory.

In the end of the introduction we comment about the connection of the affine fusion product with two-dimensional affine Demazure modules. Recall that in some special cases affine Demazure modules (see for example [Kum]) can be identified with the fusion product of finite-dimensional irreducible representations of the corresponding simple algebra (see [FF2,CL,FKL, FoL]). In addition some integrable irreducible representations (in particular the basic one) can be realized as an inductive limit of these fusion products (see [FF2,FoL]). All these statements allow to consider affine fusion products of two representations as a simplest example of double affine Demazure modules. To construct another examples one needs to “fuse” N irreducible representations for arbitrary N. In particular it seems to be important to prove the independence of the corresponding fusion products on the evaluation parameters. We hope to return to these questions elsewhere.

Our paper is organized in the following way.

In Section 1 we recall the definition and collect main properties of the lattice vertex operator algebras and principal subspaces.

In Section 2 we study the family of commutative algebras labeled by Young diagram and construct their vertex operator realization.

In Section 3 we apply the results of Section 2 to the computation of the character of the fusion product of two irreducible representations.

1. Lattice vertex operator algebras

In this section we recall main properties of lattice vertex operator algebras (VOA for short) and derive some statements about principal subspaces. The main references are [K2,BF,D].

Let L be a lattice of finite rank equipped with a symmetric bilinear form $(\cdot,\cdot): L \times L \to \mathbb{Z}$ such that $(\alpha,\alpha) > 0$ for all $\alpha \in L \setminus \{0\}$. Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$. The form $(\cdot,\cdot)$ induces a bilinear form on $\mathfrak{h}$, for which we use the same notation. Let

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

be the corresponding multi-dimensional Heisenberg algebra with the bracket

$$[\alpha \otimes t^i, \beta \otimes t^j] = i\delta_{i,-j}(\alpha,\beta)K, \quad [K, \alpha \otimes t^i] = 0, \quad \alpha, \beta \in \mathfrak{h}.$$

For $\alpha \in \mathfrak{h}$ define the Fock representation π_α generated by a vector $|\alpha\rangle$ such that

$$(\beta \otimes t^n)|\alpha\rangle = 0, \quad n > 0; \qquad (\beta \otimes 1)|\alpha\rangle = (\beta,\alpha)|\alpha\rangle; \qquad K|\alpha\rangle = |\alpha\rangle.$$

We now define a VOA V_L associated with L. We deal only with an even case, i.e. $(\alpha,\beta) \in 2\mathbb{Z}$ for all $\alpha, \beta \in L$ (in the general case the construction leads to the so called super VOA). As a vector space

$$V_L \simeq \bigoplus_{\alpha \in L} \pi_\alpha.$$

The q-degree on V_L is defined by

$$\deg_q |\alpha\rangle = \frac{(\alpha,\alpha)}{2}, \qquad \deg_q (\beta \otimes t^n) = -n. \tag{3}$$

The main ingredient of the VOA structure on V_L are bosonic vertex operators $\Gamma_\alpha(z)$ which correspond to highest weight vectors $|\alpha\rangle$. One sets

$$\Gamma_\alpha(z) = S_\alpha z^{\alpha \otimes 1} \exp\left(-\sum_{n<0} \frac{\alpha \otimes t^n}{n} z^{-n}\right) \exp\left(-\sum_{n>0} \frac{\alpha \otimes t^n}{n} z^{-n}\right), \tag{4}$$

where $z^{\alpha \otimes 1}$ acts on π_β by $z^{(\alpha,\beta)}$ and the operator S_α is defined by

$$S_\alpha|\beta\rangle = c_{\alpha,\beta}|\alpha+\beta\rangle; \quad [S_\alpha, \beta \otimes t^n] = 0, \quad \alpha, \beta \in \mathfrak{h},$$

where $c_{\alpha,\beta}$ are some nonvanishing constants. The Fourier decomposition is given by

$$\Gamma_\alpha(z) = \sum_{n \in \mathbb{Z}} \Gamma_\alpha(n) z^{-n-(\alpha,\alpha)/2}.$$

In particular,

$$\Gamma_\alpha\bigl(-(\alpha,\alpha)/2 - (\alpha,\beta)\bigr)|\beta\rangle = c_{\alpha,\beta}|\alpha+\beta\rangle. \tag{5}$$

One of the main properties of vertex operators is the following commutation relation:

$$\bigl[\alpha \otimes t^n, \Gamma_\beta(z)\bigr] = (\alpha,\beta) z^n \Gamma_\beta(z). \tag{6}$$

Another important formula describes the product of two vertex operators

$$\begin{aligned} \Gamma_\alpha(z)\Gamma_\beta(w) = (z-w)^{(\alpha,\beta)} S_\alpha S_\beta z^{(\alpha+\beta)\otimes 1} \\ \times \exp\biggl(-\biggl(\sum_{n<0} \frac{\alpha \otimes t^n}{n} z^{-n} + \frac{\beta \otimes t^n}{n} w^{-n}\biggr)\biggr) \\ \times \exp\biggl(-\biggl(\sum_{n>0} \frac{\alpha \otimes t^n}{n} z^{-n} + \frac{\beta \otimes t^n}{n} w^{-n}\biggr)\biggr). \end{aligned} \tag{7}$$

This leads to the proposition:

Proposition 1.1.

$$\bigl(\Gamma_\alpha(z)\bigr)^{(k)}\bigl(\Gamma_\beta(z)\bigr)^{(l)} = 0 \quad \text{if } k + l < (\alpha,\beta), \tag{8}$$

where the superscript (k) *denotes the kth derivative of the corresponding series. In addition if* $(\alpha,\beta) = 0$ *then*

$$\Gamma_\alpha(z)\Gamma_\beta(z) \text{ is proportional to } \Gamma_{\alpha+\beta}(z).$$

We now recall some basic facts about the representation theory of V_L. This VOA is known to be rational, i.e. every V_L-module is completely reducible. The number of irreducible representations is finite. These representations are labeled by the elements of L'/L, where L' is a dual lattice

$$L' = \bigl\{\beta \in L \otimes_{\mathbb{Z}} \mathbb{Q}\colon\ (\alpha,\beta) \in \mathbb{Z}\ \forall \alpha \in L\bigr\}. \tag{9}$$

Namely for $\gamma \in L'/L$ define

$$V_L^\gamma = \bigoplus_{\beta \in L+\gamma} \pi_\beta.$$

For example vertex operators $\Gamma_\alpha(z)$ act on each V_L^γ via the formula (4) (because $(\alpha, \beta+\gamma) \in \mathbb{Z}$ for all $\beta \in L$). The q-degree on V_L^γ is defined as in (3).

In what follows we fix a set $\alpha_1, \dots, \alpha_N$ of linearly independent vectors generating the lattice L. We denote the nondegenerate matrix of scalar products by M: $m_{ij} = (\alpha_i, \alpha_j)$ and assume that $m_{ij} \in \mathbb{N}$.

Lemma 1.1. *For any vector* $\mathbf{v} \in \mathbb{Z}^N_{\geqslant 0}$ *there exist* $\gamma_\mathbf{v} \in L'/L$ *and* $\beta\mathbf{v} \in L + \gamma_\mathbf{v}$ *such that*

$$\Gamma_{\alpha_i}(-n)|\beta_\mathbf{v}\rangle = 0 \quad \text{for } n < v_i,\ 1 \leqslant i \leqslant N; \qquad \Gamma_{\alpha_i}(-v_i)|\beta_\mathbf{v}\rangle = c_i|\beta_\mathbf{v} + \alpha_i\rangle, \tag{10}$$

with some nontrivial constants c_i.

Proof. We only need to find $\beta_\mathbf{v}$ such that

$$\deg_q|\beta_\mathbf{v} + \alpha_i\rangle - \deg_q|\beta_\mathbf{v}\rangle = v_i, \quad 1 \leqslant i \leqslant N. \tag{11}$$

Note that $\beta \in L \otimes_\mathbb{Z} \mathbb{Q}$. So $\beta_\mathbf{v}$ is a rational linear combination of α_i and we can consider $\beta_\mathbf{v}$ as a vector in $\mathbb{Q}^N$. Then (11) is equivalent to

$$\frac{m_{ii}}{2} + (M\beta_\mathbf{v})_i = v_i. \tag{12}$$

In view of $m_{ii}/2 \in \mathbb{Z}$ we obtain that $\beta_\mathbf{v} \in L'$ satisfying (12) really exists, because (9) can be rewritten as

$$L' = \{\beta \in L \otimes_\mathbb{Z} \mathbb{Q}\colon\ M\beta \in \mathbb{Z}\}.$$

Then $\gamma_\mathbf{v}$ is defined as a class of $\beta_\mathbf{v}$. $\square$

We now define principal subspaces. For $\mathbf{v} \in Z^N_{\geqslant 0}$ consider the subspace $W_L(\mathbf{v}) \hookrightarrow V_L^{\gamma_\mathbf{v}}$ generated from the vector $\beta_\mathbf{v}$ by an action of operators $\Gamma_{\alpha_i}(-n_i)$ with $n_i \geqslant v_i$ $(1 \leqslant i \leqslant N)$. Our goal is to describe $W_L(\mathbf{v})$ (in particular we want to find its character). We first realize this subspace as a quotient of a polynomial algebra. Namely define $W'_L(\mathbf{v})$ as a quotient of $\mathbb{C}[a_i(-n)]_{\substack{1\leqslant i\leqslant N\\ n\geqslant v_i}}$ by relations

$$a_i(z)^{(k)} a_j(z)^{(l)}, \quad k + l < m_{ij},$$

where $a_i(z) = \sum_{n \geqslant v_i} z^n a_i(-n)$. $W'_L(\mathbf{v})$ is generated by coefficients

$$a_i(-n), \quad 1 \leqslant i \leqslant N,\ n \geqslant v_i.$$

We note that $W'_L(\mathbf{v}) = \bigoplus_{\mathbf{n} \in \mathbb{Z}^N_{\geqslant 0}} W'_{L,\mathbf{n}}(\mathbf{v})$, where $W'_{L,\mathbf{n}}(\mathbf{v})$ is a subspace spanned by monomials in $a_i(k)$ such that the number of factors of the type $a_{i_0}(k)$ with fixed i_0 is exactly n_{i_0}. The character of $W'_{L,\mathbf{n}}(\mathbf{v})$ is naturally defined by $\deg_q a_i(k) = -k$.

Lemma 1.2.

$$\operatorname{ch}_q W'_{L,\mathbf{n}}(\mathbf{v}) = \frac{q^{\mathbf{n}M\mathbf{n}/2 + \sum_{i=1}^N n_i(v_i - m_{ii}/2)}}{(q)_\mathbf{n}}, \tag{13}$$

where $(q)_\mathbf{n} = \prod_{j=1}^N (q)_{n_j}$, $(q)_n = \prod_{j=1}^n (1 - q^j)$.

Proof. We use the dual space approach (see [FS,FJKLM1]). For $\theta \in (W'_{L,\mathbf{n}}(\mathbf{v}))^*$ define a polynomial $f_\theta \in \mathbb{C}[z_{i,n}]_{\substack{1\leqslant i\leqslant N\\ 1\leqslant n\leqslant n_i}}$ by

$$f_\theta = \theta\left(\prod_{i=1}^{N}\prod_{n=1}^{n_i} a_i(z_{i,n})\right).$$

From the exact form of the relations in $W'_L(\mathbf{v})$ one can see that the space

$$\left\{f_\theta\colon\ \theta \in \left(W'_{L,\mathbf{n}}(\mathbf{v})\right)^*\right\}$$

coincides with the space of polynomials which are divisible by

$$\prod_{i=1}^{N}\prod_{n=1}^{n_i} z_{i,n}^{v_i}\prod_{i=1}^{N}\prod_{1\leqslant n<m\leqslant n_i}(z_{i,n}-z_{i,m})^{m_{ii}}\prod_{1\leqslant i<j\leqslant N}\prod_{\substack{1\leqslant n\leqslant n_i\\ 1\leqslant m\leqslant n_j}}\bigl(z(i,n)-z(j,m)\bigr)^{m_{ij}}. \tag{14}$$

The character of such polynomials coincides with the right-hand side of (13). $\square$

In the next proposition we show that spaces $W_L(\mathbf{v})$ and $W'_L(\mathbf{v})$ are isomorphic.

Proposition 1.2. *The map $\beta_{\mathbf{v}} \mapsto 1$, $\Gamma_{\lambda_i}(n) \mapsto a_i(n)$ induces the isomorphism*

$$W_L(\mathbf{v}) \simeq W'_L(\mathbf{v}).$$

In particular for any $\mathbf{n} = (n_1, \ldots, n_N) \in \mathbb{Z}^N_{\geqslant 0}$

$$\mathrm{ch}_q\bigl(W_L(\mathbf{v}) \cap \pi_{\beta_{\mathbf{v}}+n_1\alpha_1+\cdots+n_N\alpha_N}\bigr) = \frac{q^{\mathbf{n}M\mathbf{n}/2+\sum_{i=1}^N n_i(v_i-m_{ii}/2)}}{(q)_{\mathbf{n}}} q^{\deg_q \beta_{\mathbf{v}}}. \tag{15}$$

Proof. Because of (8) it suffices to prove the equality (15).

We note that

$$|\beta_{\mathbf{v}} + n_1\alpha_1 + \cdots + n_N\alpha_N\rangle \in W_L(\mathbf{v})$$

(see (5)). In addition

$$\deg_q\left|\beta_{\mathbf{v}} + \sum_{i=1}^N n_i\alpha_i\right\rangle - \deg_q|\beta_{\mathbf{v}}\rangle = \frac{1}{2}\bigl((\beta_{\mathbf{v}}+\mathbf{n})M(\beta_{\mathbf{v}}+\mathbf{n}) - \beta_{\mathbf{v}}M\beta_{\mathbf{v}}\bigr)$$

$$= \frac{1}{2}(\mathbf{n}M\mathbf{n} + 2\mathbf{n}M\beta_{\mathbf{v}}) = \frac{1}{2}\mathbf{n}M\mathbf{n} + \sum_{i=1}^N n_i\left(v_i - \frac{m_{ii}}{2}\right),$$

where the last equality is true because of (12). Therefore the minimal power of q in the left-hand side of (15) is equal to

$$\mathbf{n}M\mathbf{n}/2 + \sum_{i=1}^{N} n_i(v_i - m_{ii}/2).$$

For $\theta \in (W_L(\mathbf{v}) \cap \pi_{\beta_\mathbf{v}+n_1\alpha_1+\cdots+n_N\alpha_N})^*$ set

$$f_\theta(z_{i,n})_{\substack{1\leqslant i\leqslant N\\ 1\leqslant n\leqslant n_i}} = \theta\left(\prod_{i=1}^{N}\prod_{n=1}^{n_i} \Gamma_{\alpha_i}(z_{i,n})\right).$$

In view of (6) we obtain that $W_L(\mathbf{v}) \cap \pi_{\beta_\mathbf{v}+n_1\alpha_1+\cdots+n_N\alpha_N}$ is invariant under the action of operators $\alpha_i \otimes t^k$, $k \geqslant 0$, $1 \leqslant i \leqslant N$. For the action on the dual space one has

$$(\alpha_i \otimes t^k) f_\theta = \left(\sum_{j=1}^{N} m_{ij} \sum_{n=1}^{n_j} z_{j,n}^k\right) f_\theta.$$

Using the nondegeneracy of M and a fact that polynomials $\sum_{n=1}^{n_j} z_{j,n}^k$, $k \geqslant 0$ generate the ring of symmetric polynomials in variables $z_{j,n}$ with fixed j we obtain that the character of $W_L(\mathbf{v}) \cap \pi_{\beta_\mathbf{v}+n_1\alpha_1+\cdots+n_N\alpha_N}$ is greater than or equal to the right-hand side of (15). Now using (8) and Lemma 1.2 we obtain our proposition. □

2. Commutative algebras

Let $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_s > 0$ be a set of positive integers. We consider a corresponding partition $\lambda = \{(i, j),\ i, j \in \mathbb{Z}_{\geqslant 0}, i \leqslant \lambda_j\}$ and define an algebra

$$A_\lambda = \mathbb{C}[a_0, a_{-1}, \ldots; b_0, b_{-1}, \ldots] \big/ \big\langle a(z)^i b(z)^j, (i, j) \notin \lambda\big\rangle, \tag{16}$$

where $a(z) = \sum_{k\geqslant 0} a_{-k} z^k$, $b(z) = \sum_{l\geqslant 0} b_{-l} z^l$. This means that our algebra is generated by two sets of variables a_k and b_l and the ideal of relations is generated by coefficients of series $a(z)^i b(z)^j$, $(i, j) \notin \lambda$. For example, $a(z)^{\lambda_0+1} = b(z)^{s+1} = 0$. A_λ is graded by

$$\deg_z a_n = \deg_z b_n = 1, \qquad \deg_u a_n = 0, \qquad \deg_u b_n = 1, \qquad \deg_q a_n = \deg_q b_n = -n.$$

We want to find the character of A_λ, which is given by

$$\mathrm{ch}_{z,u,q} A_\lambda = \sum_{i,j,k\geqslant 0} z^i u^j q^k \dim\{x \in A_\lambda\colon \deg_z x = i,\ \deg_u x = j,\ \deg_q x = k\}. \tag{17}$$

We first recall the corresponding result in one-dimensional case (see [FS,FJKLM1]).

Proposition 2.1. *Let $A_k = \mathbb{C}[a_0, a_{-1}, \ldots]/a(z)^{k+1}$. Then there exists a Gordon filtration F_μ of A_k (labeled by Young diagrams μ) such that the adjoint graded algebra is generated by coefficients of series $a^{[i]}(z)$, which are images of powers $a(z)^i$, $1 \leqslant i \leqslant k$. In addition defining relations in the adjoint graded algebra are of the form*

$$a^{[i]}(z)^{(l)} a^{[j]}(z)^{(r)} = 0 \quad \text{if } s + r < 2\min(i, j). \tag{18}$$

Here the superscript (l) is used for the lth derivative of the corresponding series. Relations (18) *gives the following Gordon type formula*:

$$\mathrm{ch}_{z,q} A_k = \sum_{\mathbf{n} \in \mathbb{Z}^k_{\geqslant 0}} \left(zq^{-1}\right)^{|\mathbf{n}|} \frac{q^{\mathbf{n}T\mathbf{n}/2}}{(q)_\mathbf{n}}, \tag{19}$$

where $T_{i,j} = 2\min(i, j)$, $|\mathbf{n}| = \sum_{i=1}^k i n_i$ and

$$(q)_\mathbf{n} = \prod_{i=1}^k (q)_{n_i}, \qquad (q)_j = \prod_{s=1}^j \left(1 - q^s\right). \tag{20}$$

The following theorem gives two-dimensional generalization of (19) for special λ.

Theorem 2.1. *We consider $\lambda_0 \geqslant \cdots \geqslant \lambda_s > 0$ with a condition*

$$\lambda_{i-1} - \lambda_i \leqslant \lambda_i - \lambda_{i+1}, \quad i = 1, \ldots, s-1. \tag{21}$$

Then

$$\mathrm{ch}_{z,u,q} A_\lambda = \sum_{\mathbf{n} \in \mathbb{Z}^{\lambda_0}_{\geqslant 0},\ \mathbf{m} \in \mathbb{Z}^s_{\geqslant 0}} u^{|\mathbf{m}|} \left(zq^{-1}\right)^{|\mathbf{n}|+|\mathbf{m}|} \frac{q^{\mathbf{n}T\mathbf{n}/2 + \mathbf{n}B\mathbf{m} + \mathbf{m}T\mathbf{m}/2}}{(q)_\mathbf{n}(q)_\mathbf{m}}, \tag{22}$$

where $T_{i,j} = 2\min(i, j)$ and $B_{i,j} = \max(0, i - \lambda_j)$.

We first construct an algebra with the character given exactly by the right-hand side of (22). Let $\mathfrak{A}_\lambda$ be an algebra generated by coefficients of abelian currents

$$a^{[i]}(z) = \sum_{n \geqslant 0} a^{[i]}_{-n} z^n, \quad 1 \leqslant i \leqslant \lambda_0, \qquad b^{[j]}(z) = \sum_{n \geqslant 0} b^{[j]}_{-n} z^n, \quad 1 \leqslant j \leqslant s.$$

These currents are subject to the only relations

$$a^{[i]}(z)^{(l)} a^{[j]}(z)^{(r)} = 0 \quad \text{if } l + r < 2\min(i, j), \tag{23}$$

$$b^{[i]}(z)^{(l)} b^{[j]}(z)^{(r)} = 0 \quad \text{if } l + r < 2\min(i, j), \tag{24}$$

$$a^{[i]}(z)^{(l)} b^{[j]}(z)^{(r)} = 0 \quad \text{if } l + r < i - \lambda_j. \tag{25}$$

We define three gradings on $\mathfrak{A}_\lambda$:

$$\deg_z a_n^{[i]} = \deg_z b_n^{[i]} = i,$$
$$\deg_u a_n^{[i]} = 0, \quad \deg_u b_n^{[i]} = i,$$
$$\deg_q a_n^{[i]} = \deg_q b_n^{[i]} = -n.$$

This defines the character $\mathrm{ch}_{z,u,q}\,\mathfrak{A}_\lambda$ of $\mathfrak{A}_\lambda$.

Lemma 2.1. $\mathrm{ch}_{z,u,q}\,\mathfrak{A}_\lambda$ *coincides with the right-hand side of* (22).

Proof. Follows from Lemma 1.2. □

We now prove our theorem in two steps comparing left- and right-hand sides of (22).

Lemma 2.2. *For any* λ *the left-hand side of* (22) *is less than or equal to the right-hand side.*

Proof. Consider a filtration $F_\mu(A_{\lambda_0}) \cdot \mathbb{C}[b_i]_{i=0}^\infty$ on A_λ (this filtration comes from the filtration from Proposition 2.1 on the subalgebra of A_λ generated by coefficients of $a(z)$). The adjoint graded algebra A'_λ is generated by coefficients of series $a^{[i]}(z)$ (images of $a(z)^i$) and $b(z)$. We now consider a filtration on A'_λ coming from the filtration from Proposition 2.1 on the subalgebra of A'_λ generated by b_j. We denote the adjoint graded algebra by A_λ^{gr}. This algebra is generated by coefficients of currents

$$a^{[i]}(z),\ b^{[j]}(z), \quad i \leqslant \lambda_0,\ j \leqslant s. \tag{26}$$

First note that in view of (18) currents (26) are subject to the relations

$$a^{[i]}(z)^{(l)} a^{[j]}(z)^{(r)} = 0 \quad \text{if } l + r < 2\min(i, j). \tag{27}$$
$$b^{[i]}(z)^{(l)} b^{[j]}(z)^{(r)} = 0 \quad \text{if } l + r < 2\min(i, j). \tag{28}$$

We show that in addition

$$a^{[i]}(z)^{(l)} b^{[j]}(z)^{(r)} = 0 \quad \text{if } l + r < \max(0, i - \lambda_j). \tag{29}$$

Note that it is enough to show that relations

$$\bigl(a(z)^i\bigr)^{(l)} \bigl(b(z)^j\bigr)^{(r)} = 0 \quad \text{if } l + r < \max(0, i - \lambda_j) \tag{30}$$

hold in A_λ. We use the induction on i. For $i = \lambda_j + 1$ we need to show that $a(z)^{\lambda_j+1} b(z)^j = 0$. But this is a relation in A_λ. Now suppose that (30) is proved for all $i \leqslant i_0$. We want to show that

$$\bigl(a(z)^{i_0+1}\bigr)^{(l)} \bigl(b(z)^j\bigr)^{(r)} = 0 \quad \text{if } l + r < i_0 - \lambda_j + 1. \tag{31}$$

If $l+r<i_0-\lambda_j$ then (31) holds by induction assumption, because

$$\bigl(a(z)^{i_0}a(z)\bigr)^{(l)}\bigl(b(z)^j\bigr)^{(r)}=\sum_{\gamma=0}^{l}\Bigl(a(z)^{i_0}\Bigr)^{(\gamma)}\bigl(b(z)^j\bigr)^{(r)}x_\gamma(z) \tag{32}$$

for some x_γ. Suppose $l+r=i_0-\lambda_j$.

Case 1. $l\neq 0$. Then for some $x(z)$

$$\bigl(a(z)^{i_0+1}\bigr)^{(l)}\bigl(b(z)^j\bigr)^{(r)}=a(z)^{i_0+1-l}\bigl(b(z)^j\bigr)^{(r)}x(z).$$

In view of $i_0+1-l\leqslant i_0$ we can use the induction assumption, which gives

$$a(z)^{i_0+1-l}\bigl(b(z)^j\bigr)^{(r)}=0,$$

because $l+r=i_0-\lambda_j$ and so $r<(i_0+1-l)-\lambda_j$.

Case 2. $l=0$. We need to show that $a(z)^{i_0+1}(b(z)^j)^{(i_0-\lambda_j)}=0$. Note that $a(z)^{i_0+1}b(z)^j=0$ for $i_0\geqslant\lambda_j$. Therefore

$$\bigl(a(z)^{i_0+1}b(z)^j\bigr)^{(i_0-\lambda_j)}=0.$$

But the following equality holds in A_λ:

$$\bigl(a(z)^{i_0+1}b(z)^j\bigr)^{(i_0-\lambda_j)}=a(z)^{i_0+1}\bigl(b(z)^j\bigr)^{(i_0-\lambda_j)}.$$

In fact

$$\bigl(a(z)^{i_0+1}b(z)^j\bigr)^{(i_0-\lambda_j)}=\sum_{l=0}^{i_0-\lambda_j}\binom{i_0-\lambda_j}{l}\bigl(a(z)^{i_0+1}\bigr)^{(l)}\bigl(b(z)^j\bigr)^{(i_0-\lambda_j-l)}.$$

But if $l\neq 0$ then

$$\bigl(a(z)^{i_0+1}\bigr)^{(l)}\bigl(b(z)^j\bigr)^{(i_0-\lambda_j-l)}=0$$

because of Case 1. We thus obtain

$$a(z)^{i_0+1}\bigl(b(z)^j\bigr)^{(i_0-\lambda_j)}=0.$$

Equalities (31) and (30) are proved.

We now consider an algebra generated by currents (26) with only relations (27), (28), (29). Because of Lemma 2.1 the character of this algebra is given by the right-hand side of (22). This finishes the proof of our lemma. □

To prove an equality in (22) we construct a vertex operator realization of an algebra A_λ. Consider a vector space $\mathfrak{h}\simeq\mathbb{R}^N$ equipped with a standard scalar product $(\cdot,\cdot)$. We fix N such that there exists a set of linearly independent vectors $p_1, \ldots, p_{\lambda_0}, q_1, \ldots, q_s \in \mathbb{R}^N$ with the scalar products $(p_i, p_j) = 2\delta_{i,j}$, $(q_i, q_j) = 2\delta_{i,j}$ and

$$(p_i, q_j) = 1 \quad \text{if } \lambda_{j-1} \geqslant i > \lambda_j; \qquad (p_i, q_j) = 0 \quad \text{otherwise}. \tag{33}$$

(For example, take $N = \lambda_0 + s$ and let $e_1, \ldots, e_N$ be some orthonormal basis. Put

$$q_j = \sqrt{2} e_j, \quad 1 \leqslant j \leqslant s; \qquad p_i = \frac{1}{\sqrt{2}} e_j + \sqrt{\frac{3}{2}} e_{s+i}, \quad j \text{ such that } \lambda_{j-1} \geqslant i > \lambda_j.$$

Then these vectors obviously satisfy (33).) In what follows we fix a lattice L generated by vectors p_i, q_j. Let $\Gamma_{p_i}(z)$, $\Gamma_{q_j}(z)$ be corresponding bosonic vertex operators. Set

$$\tilde{a}(z) = \sum_{i=1}^{\lambda_0} \Gamma_{p_i}(z), \qquad \tilde{b}(z) = \sum_{j=1}^{s} \Gamma_{q_j}(z). \tag{34}$$

Lemma 2.3. *Suppose that*

$$\lambda_{i-1} - \lambda_i \leqslant \lambda_i - \lambda_{i+1}, \quad i = 1, \ldots, s-1. \tag{35}$$

Then $\tilde{a}(z)^i \tilde{b}(z)^j = 0$ *for* $(i, j) \notin \lambda$.

Proof. It suffices to check that

$$\Gamma_{p_{l_1}}(z) \cdots \Gamma_{p_{l_{\lambda_i+1}}}(z) \Gamma_{q_{r_1}}(z) \cdots \Gamma_{q_{r_i}}(z) = 0 \tag{36}$$

for any $l_1, \ldots, l_{\lambda_i+1}, r_1, \ldots, r_i$. Note that we can assume that

$$l_1 < \cdots < l_{\lambda_i+1}, \quad r_1 < \cdots < r_i,$$

because $(p_l, p_l) = 2 = (q_r, q_r)$ and therefore $\Gamma_{p_l}(z)^2 = \Gamma_{q_r}(z)^2 = 0$. In view of (33) we have

$$\Gamma_{p_l}(z)\Gamma_{q_r}(z) = 0 \quad \text{for } \lambda_{r-1} \geqslant l > \lambda_r.$$

So (36) holds if there exists $k \leqslant \lambda_r + 1$ such that $\lambda_{\beta_t - 1} \geqslant l_k > \lambda_{\beta_t}$ for some $t \leqslant r$. The number of such k equals to

$$\sum_{t=1}^{r} (\lambda_{\beta_t} - \lambda_{\beta_t - 1}).$$

Because of the condition (35) this sum is greater than or equal to $\lambda_0 - \lambda_r$. So we obtain (36), because the number of factors of the type $\Gamma_{p_l}(z)$ is equal to $\lambda_r + 1$. The lemma is proved. □

This lemma can be used for the proof of (22). But in the last section we will need a modification of Theorem 2.1. So we formulate and prove its slight generalization.

Let

$$\mathbf{c}=(c_1,\dots,c_{\lambda_0})\in\mathbb{Z}^{\lambda_0}_{\geqslant 0},\qquad \mathbf{d}=(d_1,\dots,d_s)\in\mathbb{Z}^{s}_{\geqslant 0}.$$

We consider an ideal $I_{\lambda;\mathbf{c},\mathbf{d}}\hookrightarrow A_\lambda$ generated by conditions

$$a(z)^i\div z^{c_i+2c_{i-1}+\dots+ic_1},\quad 1\leqslant i\leqslant\lambda_0, \tag{37}$$

$$b(z)^j\div z^{d_j+2d_{j-1}+\dots+jd_1},\quad 1\leqslant j\leqslant s. \tag{38}$$

This means $I_{\lambda;\mathbf{c},\mathbf{d}}$ is generated by coefficients of $a(z)^i$ in front of the powers z^r, $0\leqslant r<\sum_{l=1}^{i} lc_{i+1-l}$ and by coefficients of $b(z)^j$ in front of the powers z^r, $0\leqslant r<\sum_{l=1}^{j} ld_{j+1-l}$. We define

$$A_{\lambda;\mathbf{c},\mathbf{d}}=A_\lambda/I_{\lambda;\mathbf{c},\mathbf{d}}.$$

Theorem 2.2. *Let*

$$\lambda_{i-1}-\lambda_i\leqslant\lambda_i-\lambda_{i+1},\quad i=1,\dots,s-1. \tag{39}$$

Then

$$\operatorname{ch}_{z,u,q}A_{\lambda;\mathbf{c},\mathbf{d}}=\sum_{\mathbf{n}\in\mathbb{Z}^{\lambda_0}_{\geqslant 0},\mathbf{m}\in\mathbb{Z}^{s}_{\geqslant 0}}u^{|\mathbf{m}|}\bigl(zq^{-1}\bigr)^{|\mathbf{n}|+|\mathbf{m}|}\times\frac{q^{\mathbf{n}T\mathbf{n}/2+\mathbf{n}B\mathbf{m}+\mathbf{m}T\mathbf{m}/2}}{(q)_{\mathbf{n}}(q)_{\mathbf{m}}}q^{\sum_{1\leqslant i\leqslant j\leqslant\lambda_0}(j-i+1)n_jc_i+\sum_{1\leqslant i\leqslant j\leqslant s}(j-i+1)m_jd_i}, \tag{40}$$

where $T_{i,j}=2\min(i,j)$ *and* $B_{i,j}=\max(0,i-\lambda_j)$.

Proof. Recall that L is a lattice generated by vectors p_i,q_j. Consider a vector $\mathbf{v}\in L$

$$\mathbf{v}=c_1p_1+(c_1+c_2)p_2+\dots+(c_1+\dots+c_{\lambda_0})p_{\lambda_0}+d_1q_1+\dots+(d_1+\dots+d_s)q_s$$

and the corresponding principal subspace $W_L(\mathbf{v})$. Denote

$$\tilde{\Gamma}_\alpha(z)=\sum_{n\geqslant 0}\Gamma_{p_i}(-n)z^n,$$

which differs from usual vertex operators by a certain power of z. Then for

$$\tilde a(z)=\sum_{i=1}^{\lambda_0}\tilde\Gamma_{p_i}(z),\qquad \tilde b(z)=\sum_{j=1}^{s}\tilde\Gamma_{q_j}(z) \tag{41}$$

one gets

$$\tilde a(z)^i\cdot|\beta_{\mathbf{v}}\rangle\div z^{ic_1+\dots+c_i},\qquad \tilde b(z)^j\cdot|\beta_{\mathbf{v}}\rangle\div z^{id_1+\dots+d_i}.$$

Combining these relations with Lemma 2.3 we obtain a surjection $A_{\lambda;\mathbf{c},\mathbf{d}} \to W_L(\mathbf{v})$. We now construct the degeneration of $W_L(\mathbf{v})$ in order to show that this surjection is an isomorphism.

Consider a family of spaces $W_L(\mathbf{v},\varepsilon)$ ($\varepsilon \in \mathbb{R}$, $\varepsilon > 0$) generated from the vector $|\beta_{\mathbf{v}}\rangle$ by coefficients of series

$$\tilde{a}_\varepsilon(z) = \sum_{i=1}^{\lambda_0} \varepsilon^i \tilde{\Gamma}_{p_i}(z), \qquad \tilde{b}_\varepsilon(z) = \sum_{j=1}^{s} \varepsilon^j \tilde{\Gamma}_{q_j}(z). \tag{42}$$

For any positive ε a space $W_L(\mathbf{v}, ve)$ is isomorphic to $W_L(\mathbf{v})$. Denote the limit $\lim_{\varepsilon\to 0} W_L(\mathbf{v},\varepsilon)$ by $W_L(\mathbf{v},0)$. Then the q-characters of $W_L(\mathbf{v},0)$ and $W_L(\mathbf{v})$ coincide. We note that

$$\lim_{\varepsilon\to 0} \tilde{a}_\varepsilon(z)^i \varepsilon^{-i(i+1)/2} = \prod_{l=1}^{i} \tilde{\Gamma}_{p_l}(z), \tag{43}$$

$$\lim_{\varepsilon\to 0} \tilde{b}_\varepsilon(z)^j \varepsilon^{-j(j+1)/2} = \prod_{r=1}^{j} \tilde{\Gamma}_{q_r}(z), \tag{44}$$

where $1 \leqslant i \leqslant \lambda_0$ and $1 \leqslant j \leqslant s$. Denote the right-hand side of (43) by $\tilde{a}^{[i]}(z)$ and the right-hand side of (44) by $\tilde{b}^{[j]}(z)$. We have

$$\tilde{a}^{[i]}(z) = \tilde{\Gamma}_{p_1+\cdots+p_i}(z), \qquad \tilde{b}^{[j]}(z) = \tilde{\Gamma}_{q_1+\cdots+q_j}(z).$$

Because of

$$(p_1+\cdots+p_i, p_1+\cdots+p_j) = (q_1+\cdots+q_i, q_1+\cdots+q_j) = 2\min(i,j), \tag{45}$$

$$(p_1+\cdots+p_i, q_1+\cdots+q_j) = \max(0, i-\lambda_j) \tag{46}$$

we obtain that the character of the algebra generated by $\tilde{a}^{[i]}(z)$, $\tilde{b}^{[j]}(z)$ is equal to the right-hand side of (40) (see Proposition 1.2). This gives that $\mathrm{ch}_{z,u,q} A_{\lambda;\mathbf{c},\mathbf{d}}$ is greater than or equal to the right-hand side of (40). To finish the proof we use Lemma 1.2 which gives the upper bound for the character of $A_{\lambda;\mathbf{c},\mathbf{d}}$, which coincides with the right-hand side of (40). The theorem is proved. □

Corollary 2.1. *The statement of Theorem* 2.1 *is true.*

We will need the following corollary from the proof of the previous theorem.

Corollary 2.2. *Vertex operators* (41) *provides a vertex operator realization of the algebra* $A_{\lambda;\mathbf{c},\mathbf{d}}$.

3. Fusion products

3.1. Principal subspaces

In this section we apply results of the previous section to the study of fusion products of integrable irreducible representations of $\widehat{\mathfrak{sl}_2}$. We first recall main definitions.

Let $\mathfrak{g}$ be some Lie algebra, $V_1, \ldots, V_n$ be its cyclic representations with cyclic vectors $v_1, \ldots, v_n$, $Z = (z_1, \ldots, z_n)$ be the set of pairwise distinct complex numbers. Denote by $V_i(z_i)$ the corresponding evaluation representations of $\mathfrak{g} \otimes \mathbb{C}[u]$. We consider a filtration F_l on the tensor product $\bigotimes_{j=1}^n V_j(z_j)$ defined by

$$F_l = \operatorname{span}\{x_1 \otimes u^{i_1} \cdots x_p \otimes u^{i_p}(v_1 \otimes \cdots \otimes v_n)\colon i_1 + \cdots + i_p \leqslant l,\ x_j \in \mathfrak{g}\}.$$

The adjoint graded representation of $\mathfrak{g} \otimes \mathbb{C}[u]$ is called the fusion product of V_i and is denoted by $V_1(z_1) * \cdots * V_n(z_n)$. Note that fusion products possess natural u-grading defined by $\deg v = l$ for $v \in F_l/F_{l-1}$.

Conjecture 3.1. *Let $\mathfrak{g}$ be an affine Kac–Moody algebra, V_i be its integrable representations. Then the corresponding fusion product does not depend on Z as a representation of $\mathfrak{g} \otimes \mathbb{C}[u]$.*

This conjecture is obvious in the case $n = 2$ (for an arbitrary Lie algebra $\mathfrak{g}$). In what follows we study the case $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$ and $n = 2$.

We fix some notations first. Let $\widehat{\mathfrak{sl}_2} = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$, where K is the central element, and d is the degree element. Let e, h, f be the standard basis of $\mathfrak{sl}_2$. For $x \in \mathfrak{sl}_2$ we set $x_i = x \otimes t^i \in \widehat{\mathfrak{sl}_2}$. Let $V_1 = L_{i_1,k_1}$ and $V_2 = L_{i_2,k_2}$ be two integrable irreducible $\widehat{\mathfrak{sl}_2}$-modules with $0 \leqslant i_1 \leqslant k_1$, $0 \leqslant i_2 \leqslant k_2$. We fix a vector $v_{i,k} \in L_{i,k}$ which satisfies

$$f_0 v_{i,k} = e_1 v_{i,k} = 0, \qquad U(\widehat{\mathfrak{sl}_2}) v_{i,k} = L_{i,k},$$
$$h_0 v_{i,k} = -i v_{i,k}, \qquad K v_{i,k} = k v_{i,k}, \qquad d v_{i,k} = 0,$$

($v_{i,k}$ is a highest weight vector with respect to the nilpotent algebra of annihilation operators generated by f_0 and e_1). The principal subspace $W_{i,k} \hookrightarrow L_{i,k}$ is defined by

$$W_{i,k} = \mathbb{C}[e_0, e_{-1}, \ldots] \cdot v_{i,k}.$$

This space is z, q bi-graded by

$$\deg_z e_i = 1, \qquad \deg_q e_i = -i, \qquad \deg_z v_{i,k} = \deg_q v_{i,k} = 0.$$

$W_{i,k}$ is a representation of an abelian algebra $\mathfrak{A}$ spanned by e_i, $i \leqslant 0$. So the fusion product $W_{i_1,k_1} * W_{i_2,k_2}$ is a representation of $\mathfrak{A} \oplus \mathfrak{A} \otimes u$. Let $w_{i_1,k_1;i_2,k_2} \in W_{i_1,k_1} * W_{i_2,k_2}$ be the image of $v_{i_1,k_1} \otimes v_{i_2,k_2}$. Then the action of the operators e_i and $e_i \otimes u$ ($i \leqslant 0$) on $w_{i_1,k_1;i_2,k_2}$ generates the fusion product. Our goal is to describe the ideal of relations in $\mathbb{C}[e_i, e_i \otimes u]_{i \leqslant 0}$. We will show that this ideal is isomorphic to $I_{\lambda;\mathbf{c},\mathbf{d}}$ for special values of parameters.

Namely let $I_{i_1,k_1;i_2,k_2} \hookrightarrow \mathbb{C}[e_j, e_j \otimes u]_{j \leqslant 0}$ be the ideal of relations, i.e.

$$W_{i_1,k_1} * W_{i_2,k_2} \simeq \mathbb{C}[e_i, e_i \otimes u]_{i \leqslant 0} / I_{i_1,k_1;i_2,k_2}.$$

Proposition 3.1. *An isomorphism of algebras*

$$\mathbb{C}[e_j, e_j \otimes u]_{j \leqslant 0} \to \mathbb{C}[a_j, b_j]_{i \leqslant 0}, \quad e_i \mapsto a_i,\ e_i \otimes u \mapsto b_i \tag{47}$$

induces the isomorphism of ideals

$$I_{i_1,k_1;i_2,k_2} \to I_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}},$$

where $\lambda^{(k_1,k_2)} = (k_1+k_2, k_1+k_2-2, \ldots, |k_1-k_2|)$ *and* $(\delta^{(i)})_j = \delta_{i,j}$.

We start with the following lemma.

Lemma 3.1. *We have an embedding of ideals* (*with respect to the isomorphism* (47))

$$I_{i_1,k_1;i_2,k_2} \hookrightarrow I_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}},$$

or the surjection

$$A_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}} \to W_{i_1,k_1} * W_{i_2,k_2}.$$

Proof. Let $I_{i,k} \hookrightarrow \mathbb{C}[e_j]_{j\leqslant 0}$ be the ideal of relations in $W_{i,k}$, i.e.

$$I_{i,k} = \{p(e_0, e_{-1}, \ldots)\colon\ pv_{i,k} = 0\}, \qquad W_{i,k} \simeq \mathbb{C}[e_j]_{j\leqslant 0}/I_{i,k}.$$

Denote $e(z) = \sum_{i\leqslant 0} e_i z^{-i}$. Then $I_{i,k}$ is generated by coefficients of series $e(z)^{k+1}$ and first $l-i$ coefficients of the series $e(z)^l$, $l > i$ (the defining relations in $W_{i,k}$ are $e(z)^{k+1} = 0$ and $e(z)^l \div z^{l-i}$, $l > i$). We recall that $W_{i_1,k_1} * W_{i_2,k_2}$ is an adjoint graded space with respect to the filtration on the tensor product of evaluation representations $W_{i_1,k_2}(z_1) \otimes W_{i_2,k_2}(z_2)$, $z_1 \neq z_2$. We set

$$e(z) = \sum_{j\geqslant 0} e_{-j} z^j, \qquad e^{(1)}(z) = e(z)\otimes \mathrm{Id}, \qquad e^{(2)}(z) = \mathrm{Id}\otimes e(z).$$

Then obviously

$$\bigl(e^{(1)}(z) + e^{(2)}(z)\bigr)^l (v_{i_1,k_1} \otimes v_{i_2,k_2}) \div z^{l-i_1-i_2} \quad \text{for } l > i_1 + i_2 \tag{48}$$

in the tensor product $W_{i_1,k_1} \otimes W_{i_2,k_2}$. This gives (37) for $\mathbf{c} = \delta^{(i_1+i_2+1)}$. Now let $z_1 = 1$, $z_2 = 0$. Then $e_j \otimes u$ acts on the tensor product $W_{i_1,k_1}(1) \otimes W_{i_2,k_2}(0)$ as $e_j \otimes \mathrm{Id}$. We state that in $W_{i_1,k_2} * W_{i_2,k_2}$ the following equation is true:

$$e^{(1)}(z)^j \div z^{j-\min(i_1,i_2)} \quad \text{for } j > \min(i_1, i_2). \tag{49}$$

In fact, $e^{(1)}(z)^l \div z^{l-i_1}$. We also have

$$\begin{aligned} e^{(2)}(z)^l &= \bigl(e^{(1)}(z) + e^{(2)}(z) - e^{(1)}(z)\bigr)^l \\ &= \bigl(-e^{(1)}(z)\bigr)^l + \sum_{j=1}^{l} \binom{l}{j} \bigl(-e^{(1)}(z)\bigr)^{l-j} \bigl(e^{(1)}(z) + e^{(2)}(z)\bigr)^j. \end{aligned}$$

Therefore in $W_{i_1,k_2} * W_{i_2,k_2}$ we have

$$e^{(1)}(z)^l = \left(-e^{(2)}(z)\right)^j \div z^{l-i_2}.$$

This gives (38) for $\mathbf{d} = \delta^{(\min(i_1,i_2)+1)}$.

We now check that

$$\left(e^{(1)}(z) + e^{(2)}(z)\right)^{k_1+k_2-2i+1} e^{(1)}(z)^i = 0, \quad i = 0, 1, \ldots, \min(k_1, k_2). \tag{50}$$

Consider one-dimensional fusion product $(\mathbb{C}[e]/e^{k_1+1}) * (\mathbb{C}[e]/e^{k_2+1})$. As shown in [FF1] the relations in this fusion product are of the form

$$\left(e^{(1)} + e^{(2)}\right)^{k_1+k_2-2i+1} \left(e^{(1)}\right)^i = 0, \quad i = 0, 1, \ldots, \min(k_1, k_2),$$

where $e^{(1)} = e \otimes \mathrm{Id}$, $e^{(2)} = \mathrm{Id} \otimes e$. This gives (50). Combining together (48), (49) and (50) we obtain our lemma. $\square$

Now our goal is to prove that

$$W_{i_1,k_1} * W_{i_2,k_2} \simeq A_{\lambda(k_1,k_2);\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}}.$$

We use the fermionic realization of $W_{i,k}$ (see [FF2]). Let us briefly recall the construction.

Let ϕ_i, ψ_i, $i \geqslant 0$ be anticommuting variables,

$$\phi(z) = \sum_{i \geqslant 0} \phi_{-i} z^i, \qquad \psi(z) = \sum_{i \geqslant 0} \psi_{-i} z^i.$$

We denote by $\Lambda = \Lambda(\phi_i, \psi_i)_{i \leqslant 0}$ an exterior algebra in variables ϕ_i, ψ_i. Then there exists an embedding $\imath_{i,k} : W_{i,k} \hookrightarrow \Lambda^{\otimes k}$ such that

$$v_{i,k} \mapsto \tilde{v}_{i,k} = \underbrace{1 \otimes \cdots \otimes 1}_{i} \otimes \phi_0 \otimes \cdots \otimes \phi_0. \tag{51}$$

This embedding is defined via the identification $e(z) = \sum_{n=1}^k \phi^{(n)}(z)\psi^{(n)}(z)$, where

$$\phi^{(n)}(z) = \underbrace{\mathrm{Id} \otimes \cdots \otimes \mathrm{Id}}_{n-1} \otimes \phi(z) \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}$$

and similarly for $\psi^{(n)}(z)$. Note that

$$\imath_{i,k}(W_{i,k}) \hookrightarrow \Lambda_{ev} \cdot \tilde{v}_{i,k},$$

where Λ_{ev} is an even part of Λ generated by the products $\psi_i\psi_j$, $\psi_i\phi_j$, $\phi_i\phi_j$. The algebra Λ_{ev} is naturally (z, q) bi-graded with

$$\deg_z \psi_i\psi_j = 1, \quad \deg_q \psi_i\psi_j = i + j$$

and similarly for other generators. Fixing $\deg_z \tilde{v}_{i,k} = \deg_q \tilde{v}_{i,k} = 0$ we obtain a bi-grading on $\Lambda_{ev} \cdot \tilde{v}_{i,k}$ such that $\iota_{i,k}$ is an embedding of (z,q) bi-graded spaces.

Note that our construction gives an embedding

$$\iota_{i_1,k_1} \otimes \iota_{i_2,k_2} : W_{i_1,k_1} \otimes W_{i_2,k_2} \hookrightarrow \Lambda^{\otimes(k_1+k_2)}.$$

Our goal is to construct a continuous family of algebras $B(\varepsilon) \hookrightarrow \Lambda^{\otimes(k_1+k_2)}$, $0 \leqslant \varepsilon \leqslant 1$ which "connects" $W_{i_1,k_1} \otimes W_{i_2,k_2}$ and $A_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}}$ inside $\Lambda^{\otimes(k_1+k_2)}$. So we need a fermionic realization of an algebra

$$A_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}}.$$

Lemma 3.2. *The map $\jmath$ defined by*

$$1 \mapsto \tilde{v}_{i_1,k_2} \otimes \tilde{v}_{i_2,k_2}, \tag{52}$$

$$a(z) \mapsto \tilde{a}(z) = \sum_{n=1}^{k_1+k_2} \phi^{(n)}(z)\psi^{(n)}(z), \tag{53}$$

$$b(z) \mapsto \tilde{b}(z) = \begin{cases} \sum_{n=1}^{k_1} \phi^{(n)}(z)\psi^{(n+k_1)}(z), & i_1 \leqslant i_2, \\ \sum_{n=1}^{k_1} \phi^{(z)}(z)\psi^{(n+k_1)}(z), & i_1 > i_2, \end{cases} \tag{54}$$

provides an embedding

$$A_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}} \hookrightarrow \Lambda^{\otimes(k_1+k_2)}.$$

Proof. Our lemma is an immediate consequence of vertex operator realization of

$$A_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}}$$

(see Corollary 2.2) and a version of the boson-fermion correspondence (see for example [BF]). We give a sketch of the proof here.

Recall that the boson-fermion correspondence allows to realize lattice VOAs inside the space build up from the fermionic particles of the types ϕ_i and ψ_i. In particular the space of states of corresponding VOA (the direct sum of Fock modules) is replaced by the space of semi-infinite forms or by the tensor products of such spaces. In this realization fields $\phi(z)$ and $\phi(z)\psi(z)$ correspond to the one-dimensional vertex operators $\Gamma_\alpha(z)$ with the length of α equal 1 or 2 respectively. To proceed to multi-dimensional even case one must consider independent fields of the type $\phi^{(n)}(z)\psi^{(m)}(z)$. The independence means that these fields represent vertex operators $\Gamma_{\beta_{n,m}}(z)$ and the scalar products are given by $(\beta_{n,m}, \beta_{k,l}) = \delta_{n,k} + \delta_{m,l}$.

We now apply this boson-fermion correspondence to the proof of our lemma. Let $p_1, \ldots, p_{k_1+k_2}, q_1, \ldots, q_{k_1}$ be vectors such that

$$\phi^{(n)}(z)\psi^{(n)}(z) = \Gamma_{p_{k_1+k_2+1-n}}(z), \quad 1 \leqslant n \leqslant k_1 + k_2 \tag{55}$$

$$\phi^{(m)}(z)\psi^{(m+k_1)}(z) = \Gamma_{q_m}(z), \quad i_1 \leqslant i_2, \tag{56}$$

$$\phi^{(m+k_1)}(z)\psi^{(m)}(z) = \Gamma_{q_m}(z), \quad i_1 > i_2, \tag{57}$$

where $1 \leqslant m \leqslant k_1$. Then $(p_n, q_m) = 0$ unless $k_1 + k_2 - 2(m-1) \geqslant n > k_1 + k_2 - 2m$. In the latter case the corresponding scalar product equals 1. Therefore according to Corollary 2.2 to finish the proof of our lemma we only need to check the initial conditions

$$\tilde{a}(z)^{i_1+i_2+l} \tilde{v}_{i_1,k_1} \otimes \tilde{v}_{i_2,k_2} \div z^l, \qquad \tilde{b}(z)^{\min(i_1,i_2)+l} \tilde{v}_{i_1,k_1} \otimes \tilde{v}_{i_2,k_2} \div z^l.$$

But this just follows from $\phi^{(n)}(z)\tilde{v}_{i,k} \div z^l$ for $n > i$ (see (51)). □

Proof of Proposition 3.1.

$$W_{i_1,k_1} * W_{i_2,k_2} \simeq A_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}}. \tag{58}$$

Because of Lemma 3.1 it suffices to check that the character of the left-hand side of (58) coincides with the character of the right-hand side.

We construct a continuous family of algebras $B(\varepsilon) \hookrightarrow \Lambda^{\otimes(k_1+k_2)}$, $0 \leqslant \varepsilon < 1$ which "connects" $W_{i_1,k_1} \otimes W_{i_2,k_2}$ and $A_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}}$. We want $B(\varepsilon)$ to satisfy

(A) $B(0) = (\imath_{i_1,k_1} \otimes \imath_{i_2,k_2})(W_{i_1,k_1} \otimes W_{i_2,k_2})$,
(B) $B(\varepsilon) \simeq B(0)$ as bi-graded vector spaces,
(C) $\lim_{\varepsilon\to 1} B(\varepsilon) = \jmath A_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}}$.

Note that the existence of such deformation proves our proposition.

Set

$$\tilde{b}^{\varepsilon}(z) = \begin{cases} \sum_{n=1}^{k_1} \phi^{(n)}(z)(\varepsilon\psi^{(n)}(z) + (1-\varepsilon)\psi^{(n+k_1)}(z)), & i_1 \leqslant i_2, \\ \sum_{n=1}^{k_1} (\varepsilon\phi^{(n)}(z) + (1-\varepsilon)\phi^{(n+k_1)}(z))\psi^{(n)}(z), & i_1 > i_2. \end{cases} \tag{59}$$

We denote by $B(\varepsilon) \hookrightarrow \Lambda^{\otimes(k_1+k_2)}$ the subspace generated by the coefficients of $\tilde{a}(z)$ and $b^{\varepsilon}(z)$ from the vector $\tilde{v}_{i_1,k_1} \otimes \tilde{v}_{i_2,k_2}$. We now check (A), (B) and (C).

(A) is obvious because $\tilde{b}^0(z)$ reduces to the current $\tilde{b}(z)$.

We now check (B). First note that for any $0 < \varepsilon_1 \leqslant \varepsilon_2 < 1$ we have $B(\varepsilon_1) \simeq B(\varepsilon_2)$. In fact, fix some $0 < \varepsilon < 1$ and redefine

$$\begin{gathered} 1/2\psi'^{(n)}(z) = (1-\varepsilon)\psi^{(n)}(z), \quad 1 \leqslant n \leqslant k_1, \\ 1/2\psi'^{(m)}(z) = \varepsilon\psi^{(m)}(z), \quad k_1+1 \leqslant m \leqslant k_1+k_2, \\ 2\phi'^{(n)}(z) = (1-\varepsilon)^{-1}\phi^{(n)}(z), \quad 1 \leqslant n \leqslant k_1, \\ 2\phi'^{(m)}(z) = \varepsilon^{-1}\phi^{(m)}(z), \quad k_1+1 \leqslant m \leqslant k_1+k_2, \end{gathered}$$

for $i_1 \leqslant i_2$ and

$$\begin{gathered} 1/2\phi'^{(n)}(z) = (1-\varepsilon)\phi^{(n)}(z), \quad 1 \leqslant n \leqslant k_1, \\ 1/2\phi'^{(m)}(z) = \varepsilon\phi^{(m)}(z), \quad k_1+1 \leqslant m \leqslant k_1+k_2, \\ 2\psi'^{(n)}(z) = (1-\varepsilon)^{-1}\psi^{(n)}(z), \quad 1 \leqslant n \leqslant k_1, \\ 2\psi'^{(m)}(z) = \varepsilon^{-1}\psi^{(m)}(z), \quad k_1+1 \leqslant m \leqslant k_1+k_2, \end{gathered}$$

for $i_1 > i_2$. Then one can easily check that $\tilde{a}(z)$ does not change and $\tilde{b}^{\varepsilon}(z)$ becomes $\tilde{b}^{1/2}(z)$ (up to the nonzero constant). Therefore, $B(\varepsilon) \simeq B(1/2)$ for any $0 < \varepsilon < 1$.

Now note that

$$\big(2\tilde{b}^{1/2}(z)\big)^{k_1+1}\tilde{v}_{i_1,k_1} \otimes \tilde{v}_{i_2,k_2} = \big(a(z) - 2\tilde{b}^{1/2}(z)\big)^{k_2+1}\tilde{v}_{i_1,k_1} \otimes \tilde{v}_{i_2,k_2} = 0 \tag{60}$$

and

$$\big(2\tilde{b}^{1/2}(z)\big)^{l}\tilde{v}_{i_1,k_1} \otimes \tilde{v}_{i_2,k_2} \div z^{l-i_1+1}, \quad l > i_1, \tag{61}$$

$$\big(\tilde{a}(z) - 2\tilde{b}^{1/2}(z)\big)^{l}\tilde{v}_{i_1,k_1} \otimes \tilde{v}_{i_2,k_2} \div z^{l-i_2+1}, \quad l > i_2. \tag{62}$$

In fact, one has

$$2\tilde{b}^{1/2}(z) = \begin{cases} \sum_{n=1}^{k_1} \phi^{(n)}(z)(\psi^{(n)}(z) + \psi^{(n+k_1)}(z)), & i_1 \leqslant i_2, \\ \sum_{n=1}^{k_1} (\phi^{(n)}(z) + \phi^{(n+k_1)}(z))\psi^{(n)}(z), & i_1 > i_2, \end{cases}$$

and

$$\tilde{a}(z) - 2\tilde{b}^{1/2}(z) = \begin{cases} \sum_{n=k_1+1}^{k_1+k_2} (\phi^{(n)}(z) - \phi^{(n-k_1)}(z))\psi^{(n)}(z), & i_1 \leqslant i_2, \\ \sum_{n=k_1+1}^{k_1+k_2} \phi^{(n)}(z)(\psi^{(n)}(z) - \psi^{(n-k_1)}(z)), & i_1 > i_2. \end{cases}$$

Now (60), (61), (62) follow from the above exact expressions of the currents. Relations (60), (61), (62) provide a surjection $B(0) \to B(1/2)$ ($2\tilde{b}^{1/2}(z)$ and $\tilde{a}(z)$ correspond to the currents $e^{(1)}(z)$ and $e^{(2)}(z)$ in $B(0) \simeq W_{i_1,k_1} \otimes W_{i_2,k_2}$). This gives (B) (because all $B(\varepsilon)$ with $0 < \varepsilon < 1$ are isomorphic and our deformation is continuous).

To show (C) we note that in view of formulas (52), (53), (54) and (59) we have an embedding

$$\jmath A_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}} \hookrightarrow \lim_{\varepsilon \to 1} B(\varepsilon).$$

But all $B(\varepsilon)$ (including $B(0)$) are isomorphic and

$$\mathrm{ch}_{z,q} B(0) \leqslant \mathrm{ch}_{z,q} A_{\lambda^{(k_1,k_2)};\delta^{(i_1+i_2+1)},\delta^{(\min(i_1,i_2)+1)}}.$$

Therefore, using Lemma 3.1 we obtain (C). $\square$

Corollary 3.1. *Let $k_1 \leqslant k_2$. Then*

$$\begin{aligned} \mathrm{ch}_{z,u,q}(W_{i_1,k_1} * W_{i_2,k_2}) = &\sum_{\mathbf{n} \in \mathbb{Z}_{\geqslant 0}^{k_1+k_2}, \mathbf{m} \in \mathbb{Z}_{\geqslant 0}^{k_1}} u^{|\mathbf{m}|} z^{|\mathbf{n}|+|\mathbf{m}|} \\ &\times q^{-|\mathbf{n}|-|\mathbf{m}|+\sum_{j=i_1+i_2+1}^{k_1+k_2}(j-i_1-i_2)n_j + \sum_{j=\min(i_1,i_2)+1}^{k_1}(j-\min(i_1,i_2))m_j} \\ &\times \frac{q^{\mathbf{n}T\mathbf{n}/2+\mathbf{n}B\mathbf{m}+\mathbf{m}T\mathbf{m}/2}}{(q)_{\mathbf{n}}(q)_{\mathbf{m}}}, \end{aligned} \tag{63}$$

where $T_{i,j} = 2\min(i,j)$ and $B_{i,j} = \max(0, i - k_1 - k_2 + 2j)$.

3.2. The limit construction

In this subsection we derive a fermionic formula for the character of the fusion product $L_{i_1,k_1} * L_{i_2,k_2}$. Note that $L_{i,k}$ is bi-graded by operators d and h_0. Therefore for the fusion product the z,u,q character is naturally defined.

Let $v_{i,k}^N \in L_{i,k}$, $N \in \mathbb{Z}$ be the set of extremal vectors (the weight of $v_{i,k}^N$ is a weight of $v_{i,k}$ shifted by the Nth power of the translation element from the Weyl group of $\widehat{\mathfrak{sl}_2}$). We fix $h_0 v_{i,k}^N = (-i-2Nk)v_{i,k}^N$. Introduce the Nth principal subspace $W_{i,k}^N \hookrightarrow L_{i,k}$ by

$$W_{i,k}^N = \mathbb{C}[e_{2N}, e_{2N-1}, \ldots] \cdot v_{i,k}^N$$

(note that $e_{2N+1} v_{i,k}^N = 0$). We recall that there exists an isomorphism

$$W_{i,k} \simeq W_{i,k}^N, \quad v_{i,k} \mapsto v_{i,k}^N, \ e_j \to e_{j+2N} \tag{64}$$

and

$$L_{i,k} = W_{i,k}^0 \hookrightarrow W_{i,k}^1 \hookrightarrow W_{i,k}^2 \hookrightarrow \cdots.$$

Using this limit construction we can write a fermionic formula for the character of the fusion product of integrable modules.

Corollary 3.2.

$$\begin{aligned}
&\mathrm{ch}_{z,u,q}(L_{i_1,k_1} * L_{i_2,k_2}) \\
&\quad = \lim_{N\to\infty} z^{-i_1-i_2-2N(k_1+k_2)} q^{N^2(k_1+k_2)+N(i_1+i_2)} \\
&\qquad \times \sum_{\mathbf{n}\in\mathbb{Z}_{\geqslant 0}^{k_1+k_2}, \mathbf{m}\in\mathbb{Z}_{\geqslant 0}^{k_1}} u^{|\mathbf{m}|} z^{2(|\mathbf{n}|+|\mathbf{m}|)} q^{(-2N-1)(|\mathbf{n}|+|\mathbf{m}|)} \\
&\qquad \times q^{\sum_{j=i_1+i_2+1}^{k_1+k_2}(j-i_1-i_2)n_j + \sum_{j=\min(i_1,i_2)+1}^{k_1}(j-\min(i_1,i_2))m_j} \frac{q^{\mathbf{n}T\mathbf{n}/2+\mathbf{n}B\mathbf{m}+\mathbf{m}T\mathbf{m}/2}}{(q)_\mathbf{n}(q)_\mathbf{m}}.
\end{aligned} \tag{65}$$

Proof. Follows from Corollary 3.1, isomorphism (64) and equalities

$$h_0 v_{i,k}^N = (-i-2Nk)v_{i,k}^N, \qquad d v_{i,k}^N = \left(kN^2 + Ni\right) v_{i,k}^N.$$

We only note that a factor 2 in $z^{2(|\mathbf{n}|+|\mathbf{m}|)}$ comes from the relation $[h_0, e_i] = 2e_i$ (in $W_{i,k}$ we had $\deg_z e_i = 1$) and the power $q^{(-2N-1)(|\mathbf{n}|+|\mathbf{m}|)}$ comes from the shift $e_j \to e_{j+2N}$ (see (64)). □

Introduce a new set of variables s_i instead of n_i in (65) by

$$n_i = s_i - s_{i+1}, \quad 1 \leqslant i < k_1 + k_2, \qquad n_{k_1+k_2} = s_{k_1+k_2} + N - \frac{|\mathbf{m}|}{k_1+k_2}.$$

Then for the power of z in (65) we have

$$-i_1 - i_2 - 2N(k_1+k_2) + 2\big(|\mathbf{n}| + |\mathbf{m}|\big) = -i_1 - i_2 + 2\sum_{i=1}^{k_1+k_2} s_i.$$

We now rewrite the power of q in (65) in terms of new variables. Note that

$$\mathbf{n}T\mathbf{n}/2 = \sum_{i=1}^{k_1+k_2} (n_i + \cdots + n_{k_1+k_2})^2 = \sum_{i=1}^{k_1+k_2} \left(s_i + N - \frac{|\mathbf{m}|}{k_1+k_2}\right)^2,$$

$$\mathbf{n}B\mathbf{m} = |\mathbf{m}|\left(2N - 2\frac{|\mathbf{m}|}{k_1+k_2}\right) + \sum_{i+2j\geqslant k_1+k_2+1} m_j s_i,$$

$$|\mathbf{n}| + |\mathbf{m}| = N(k_1+k_2) + \sum_{i=1}^{k_1+k_2} s_i.$$

Therefore the power of q in (65) is equal to

$$\sum_{i=1}^{k_1+k_2} s_i^2 - \frac{|\mathbf{m}|}{k_1+k_2}\left(|\mathbf{m}| + k_1 + k_2 - i_1 - i_2 + 2\sum_{i=1}^{k_1+k_2} s_i\right) + \sum_{i+2j\geqslant k_1+k_2+1} m_j s_i + \mathbf{m}T\mathbf{m}/2 - \sum_{i=1}^{i_1+i_2} s_i + \sum_{j=\min(i_1,i_2)+1}^{k_1} \big(j - \min(i_1,i_2)\big) m_j. \quad (66)$$

(Note that the powers which contain N cancel.) We thus obtain the following theorem

Theorem 3.1.

$$\mathrm{ch}_{z,u,q}\, L_{i_1,k_1} * L_{i_2,k_2} = \frac{1}{(q)_\infty} \sum_{\substack{\mathbf{s}\in\mathbb{Z}^{k_1+k_2},\, \mathbf{m}\in\mathbb{Z}_{\geqslant 0}^{k_1} \\ s_1\geqslant\cdots\geqslant s_{k_1+k_2}}} z^{-i_1-i_2+2\sum_{i=1}^{k_1+k_2} s_i} \frac{q^{P(\mathbf{s},\mathbf{m})}}{(q)_\mathbf{m} \prod_{i=1}^{k_1+k_2-1} (q)_{s_i}},$$

where $P(\mathbf{s},\mathbf{m})$ *is given by* (66) *and* $(q)_\infty = \prod_{i=1}^\infty (1-q^i)$.

Proof. We only note that the factor $(q)_\infty$ comes from the limit

$$\lim_{N\to\infty} (q)_{n_{k_1+k_2}} = \lim_{N\to\infty} (q)_{s_{k_1+k_2} + N - \frac{|\mathbf{m}|}{k_1+k_2}}. \qquad \square$$

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References

[AK] E. Ardonne, R. Kedem, Fusion products of Kirillov–Reshetikhin modules and fermionic formulas, math.RT/0602177.

[BF] E. Frenkel, D. Ben-Zvi, Vertex Algebras and Algebraic Curves, Math. Surveys Monogr., vol. 88, Amer. Math. Soc., 2001.

[CL] V. Chari, S. Loktev, Weyl, fusion and Demazure modules for the current algebra of $\mathfrak{sl}_{r+1}$, preprint, math.QA/0502165, 2005.

[D] C. Dong, Vertex algebras associated with even lattices, J. Algebra 161 (1993) 245–265.

[DMS] P. Di Francesco, P. Mathieu, D. Sénéchal, Conformal Field Theory, Springer GTCP, New York, 1997.

[FF1] B. Feigin, E. Feigin, Q-characters of the tensor products in $\mathfrak{sl}_2$ case, Moscow Math. J. 2 (3) (2002) 567–588.

[FF2] B. Feigin, E. Feigin, Integrable $\widehat{\mathfrak{sl}_2}$-modules as infinite tensor products., in: S. Lando, O. Sheinman (Eds.), Fundamental Mathematics Today, Independent University of Moscow, 2003, pp. 304–334 (in honor of the 10th anniversary of the Independent University of Moscow).

[FF3] B. Feigin, E. Feigin, Principal subspace for bosonic vertex operator $\phi_{\sqrt{2m}}$ and Jack polynomials, Adv. Math. 206 (2) (2006) 307–328, math.QA/0407372.

[FFJMT] B. Feigin, E. Feigin, M. Jimbo, T. Miwa, Y. Takeyama, A $\phi_{1,3}$-filtration on the Virasoro minimal series $M(p,p')$ with $1 < p'/p < 2$, preprint, math.QA/0603070, 2006.

[FJKLM1] B. Feigin, M. Jimbo, R. Kedem, S. Loktev, T. Miwa, Spaces of coinvariants and fusion product I. From equivalence theorem to Kostka polynomials, Duke Math. J. 125 (3) (2004) 549–588.

[FJKLM2] B. Feigin, M. Jimbo, R. Kedem, S. Loktev, T. Miwa, Spaces of coinvariants and fusion product II. $\widehat{\mathfrak{sl}_2}$ character formulas in terms of Kostka polynomials, J. Algebra 279 (1) (2004) 147–179.

[FK] I.B. Frenkel, V.G. Kac, Basic representations of affine Lie algebras and dual resonance models, Invent. Math. 62 (1980) 23–66.

[FKL] B. Feigin, A.N. Kirillov, S. Loktev, Combinatorics and geometry of higher level Weyl modules, preprint, math.QA/0503315, 2005.

[FL] B. Feigin, S. Loktev, On generalized Kostka polynomials and quantum Verlinde rule, in: Differential Topology, Infinite-Dimensional Lie Algebras and Applications, in: Amer. Math. Soc. Transl. Ser. 2, vol. 194, Amer. Math. Soc., RI, 1999, pp. 61–79.

[FLT] B. Feigin, S. Loktev, I. Tipunin, Coinvariants for lattice VOAs and q-supernomials, Comm. Math. Phys. 229 (2002) 271–291.

[FoL] G. Fourier, P. Littelmann, Weyl modules, affine Demazure modules, fusion products and limit constructions, preprint, math.RT/0509276, 2005.

[FST] B.L. Feigin, A.M. Semikhatov, I.Yu. Tipunin, Semi-infinite realization of unitary representations of the $N = 2$ algebra and related constructions, Theoret. and Math. Phys. 126 (1) (2001) 1–47.

[FS] B. Feigin, A. Stoyanovsky, Quasi-particle models for the representations of Lie algebras and geometry of flag manifold, hep-th/9308079, RIMS 942;
B. Feigin, A. Stoyanovsky, Functional models for the representations of current algebras and the semi-infinite Schubert cells, Funct. Anal. Appl. 28 (1994) 55–72.

[K1] V. Kac, Infinite Dimensional Lie Algebras, third ed., Cambridge Univ. Press, Cambridge, 1990.

[K2] V. Kac, Vertex Algebras for Beginners, Univ. Lecture Ser., vol. 10, 1997.

[Ked] R. Kedem, Fusion products, cohomology of GL_N flag manifolds and Kostka polynomials, Int. Math. Res. Not. 2004 (25) (2004) 1273–1298.

[Kum] S. Kumar, Kac–Moody Groups, Their Flag Varieties and Representation Theory, Progr. Math., vol. 204, Birkhäuser, Boston, MA, 2002.