PART ONE

0. INTRODUCTION

1. We study the orbits of subspaces $F$ in finite-dimensional $\mathcal{E}$-hermitean vector spaces $E$ under the action of the unitary group $U(E)$. More precisely, we shall reduce the classification of pairs $(E, F)$ of spaces $E \equiv F$ to the classification of spaces (up to isometry). The case in which the characteristic of the base field is not 2 is classical and is taken care of by Witt’s theorem. The case of characteristic 2 on the contrary is settled for the first time here; it is plenteous in structure and it is made entirely perspicuous. We wish to stress the fact that the classical case appears naturally as a very special situation within the characteristic 2 setting. In fact, in the general setting the characteristic doesn’t even play any particular role.

2. Consider then an antiautomorphism $\alpha \mapsto \bar{\alpha}$ on the division ring $k$, of any characteristic, and an element $e$ of the group $\{\xi \in k \mid \xi^e - 1\}$. In this...
paper we always consider the data \((k, -, \varepsilon)\) as being fixed throughout (cf. Remark 1.6). Let \(\Phi: E \times E \to k\) be an \(\varepsilon\)-hermitean form on the \(k\)-left vector space \(E\), linear in the first argument and \(\Phi(y, x) = \varepsilon \overline{\Phi(x, y)}\); assume that \(\Phi\) is non-degenerate, i.e., \(E^\perp = \{0\}\). All lengths \(\Phi(x, x)\) of vectors \(x \in E\) are elements of the additive subgroup \(S := \{\xi \in k \mid \xi = \varepsilon \overline{\xi}\}\) of "symmetric" elements in \(k\). Its subgroup \(T := \{\xi + \varepsilon \overline{\xi} \mid \xi \in k\} \subseteq S\) of "traces" occasions us to associate with each linear subspace \(X \subseteq E\) the subspace \(X^* := \{x \in X \mid \Phi(x, x) \in T\}\). The importance of \(X^*\) derives from the fact that each operation of \(O(E)\) leaves the subspace \(E^* \perp\) pointwise fixed (Corollary 1.5).

In the classical situation (\(\text{char}(k) \neq 2\)) we have \(E^* \perp = \{0\}\) because \(S = T\); in the general situation, particularly in the case of symmetric forms over commutative fields, the subspace \(E^* \perp\) obstructs cancellation of isometric summands of \(E\) and it obstructs extension of partial isometries (cf. Examples in IV.3). It is thus inevitable to pay heed to the operator

\[X \mapsto X^*\]

and we therefore put down

3. Definition. Let \(E\) be a finite-dimensional, non-degenerate hermitean space and \(F\) a linear subspace. The pair \((E, F)\) is called decomposable if \(E\) is the orthogonal sum of finitely many non-zero subspaces \(E_1, \ldots, E_m, m \geq 2\), such that \(E^* = \sum_1^m E^* \cap E_i, F = \sum_1^m F \cap E_i\). If this takes place we set \(F_i := F \cap E_i\) and write \((E, F) = \sum_1^m (E_i, F_i)\). We shall say that two pairs are isometric, \((E, F) \cong (\tilde{E}, \tilde{F})\), if there is an isometry \((k\text{-vector space isomorphism that preserves forms})\) between \(E\) and \(\tilde{E}\) which maps the subspaces \(F, \tilde{F}\) onto each other. Any pair that is isometric to a pair \((E_0, F \cap E_0)\) occurring in some decomposition of \((E, F)\) is called an orthogonal summand of \((E, F)\).

4. In this paper we shall give a canonical description of all indecomposable pairs \((E, F)\) as follows. We prove that the class of all indecomposable pairs \((E, F)\) over fixed \((k, -, \varepsilon)\) with \(E\) non-degenerate splits into thirteen (fourteen if the trivial pair \(((0), (0))\) is included) equivalence classes of "isotypic" pairs. These classes correspond to the (subdirectly) irreducible lattices in a certain class of 2-generated modular polarity lattices. Each isotype class is made up of full isometry classes of pairs. The isotype attached to an indecomposable pair \((E, F)\) is described by a diagram; this diagram is a crucial invariant of the pair since together with the isometry type \(\tilde{E}\) it forms a complete set of invariants for the isometry class of the pair but for one case where a further (independent) invariant has to be added and another case, where \(\tilde{F}\) is needed instead of \(\tilde{E}\). Section II.5 presents this information in a table.
5. A major result that flows from our classification is formulated in Theorem III.3: it gives a complete set of fourteen invariants of a pair \((E, F)\) modulo isometry; these invariants are essentially independent (a complete list of relations is appended).

6. There are, of course, many applications to our classification. As an illustration we shall derive in a concise and uniform manner the Extension Theorem and the Congruence Theorem (H. Lenz, G. E. Wall, V. Pless, and W. Bäni); the results on cancellation (R. Wagner) can be obtained in the same style. See Sections IV, X, and XI.

7. We conclude this introduction with a remark on the style of the paper. Each of the authors has found a different proof for the existence of a decomposition

\[
(E, F) = \sum_i (E_i, F_i)
\]

of an arbitrary pair into indecomposable pairs each of which has one of the thirteen isotypes mentioned above: There are relatively short proofs by direct calculation either in the hermitean space or in its associated polarity lattice. The version using polarity lattices works under the sole assumption of modularity and complementation, so that it is valid for regular rings of characteristic 2, also. Even substantial parts of the analysis carry through without complementation. Details shall appear elsewhere. The proof presented here makes use of the classification of vector spaces with a quadruple of subspaces; thereby we exploit an idea of Gabriel [Ga]. Nevertheless, we stress that it is relations in the associated polarity lattice which make this reduction work.

It seems that in finite dimensions the explicit structure of the associated polarity lattice does not have the import it has in the infinite-dimensional case. All the more clearly emerges here the role of lattice theory as a transfer principle.

8. Contents. Part One of the paper contains hardly any proofs. Apart from the Introduction it consists of the following sections:

I. The lattice pertaining naturally to the classification.

II. Table of isotypic pairs and their lattices.

III. Statement of the principal results: Existence and uniqueness of decompositions.

IV. Extension of partial isometries, congruence, and cancellation.

Part Two (Sections V–XII) give proofs that are not routine. Theorems from Section IV are not used in any of these proofs. Facts that are assumed to be known or are referred to frequently are compiled in Section VI.
I. THE LATTICE PERTAINING NATURALLY TO THE CLASSIFICATION

1. In the following, if $E$ is a vector space then $(\mathcal{L}(E), +, \cap)$, or $\mathcal{L}(E)$ for short, is the modular lattice of all linear subspaces in $E$. With a non-degenerate hermitean space $E$ we may associate the polarity lattice $\mathcal{Y} = (\mathcal{P}(E), +, \cap, \perp)$. To a pair $(E, F), F \in \mathcal{L}(E)$, we attach the polarity sublattice $\mathcal{Y} = \mathcal{V}_+(F, E^*)$ of $\mathcal{L}$ generated by the elements $F, E^*, E$. The structure $\mathcal{Y}$ is stable under the operation $\ast$ because $X^* = X \cap E^*$. Also, notice an evident but crucial property: $X \cap X^\perp \subseteq E^*$ for every subspace $X$ of $E$. This means, in particular, that for every term $t(x, y)$ in the language of polarity lattices the relation $(t \cap t')(E, F^*) \subseteq E^*$ is satisfied in the lattice $\mathcal{V}_+(F, E^*)$. The most general such lattice is isomorphic to the modular polarity lattice $\mathcal{Y}(a, b)$ freely generated by $a, b$ subject to the relations $(t \cap t')(a, b) \subseteq b$; its cardinality is 13,080.

2. **Definition.** A pair $(E, F)$ is called *isotypic* if for any two orthogonal summands $(E_1, F_1), (E_2, F_2), E_i \neq (0)$ (Definition 0.3) we have polarity lattice isomorphism $\mathcal{Y}_{E_1}(F_1, E_1^*) \cong \mathcal{Y}_{E_2}(F_2, E_2^*)$ that maps $F_1, E_1^*$ onto $F_2, E_2^*$, respectively.

Since every orthogonal decomposition of the pair induces a subdirect decomposition of the associated polarity lattice (cf. Lemma 1.2 in [P]) a pair $(E, F)$ is isotypic if $\mathcal{V}_+(F, E^*)$ is subdirectly irreducible. Our main result (III.1) implies that the converse is true, too.

3. **Example.** Let $E$ be $m$-dimensional, $m \geq 2$, and $E^* = (0)$. (Such spaces are called rigid [M, p. 105] since $\mathcal{O}(E) = \{1\}$; if there were an operation $\varphi \in \mathcal{O}(E)$ that moved some vector $a$ then $0 \neq a - \varphi a \in E^*$.) The pair $(E, E)$ is, in infinitely many ways, a sum $(E, E) = \sum_i (H_i, H_i)$, dim $H_i = 1$; no two orthogonal summands $(E_1, F_1), (E_2, F_2)$ of $(E, E)$ with $E_1, E_2 \subseteq E$ are ever isometric unless $E_1$ and $E_2$ are the same subset of $E$ (rigidity!). Yet $(E, E)$ is isotypic as each orthogonal summand $\neq ((0), (0))$ has a polarity lattice

$$\cong \begin{pmatrix} E, F \\ E^*, F^\perp \end{pmatrix}$$

4. The interval $[E^*, E]$ of the lattice $\mathcal{V}_+(F, E^*)$ plays an important arithmetic role which we now explain. The additive groups $S$ and $T$ (cf. Introduction) attached to our structure $(k, -, \varepsilon)$ are invariant under each automorphism $\varphi_{\lambda}: \xi \to \lambda \xi \varepsilon^\ast (\lambda \in \hat{k})$ of the group $(k, +)$. Therefore the quotient group $S/T$ can be turned into a $k$-vector space by the definition

$$\lambda \cdot (\sigma + T) := \lambda \sigma \varepsilon^\ast + T \quad (\sigma \in S).$$

Henceforth we shall simply speak of the vector space $S/T$ ("value space")
attached to \((k, -, \varepsilon)\). If \(\text{char } k \neq 2\) or if the center of \(k\) is not left pointwise fixed under the involution we have \(S = T\) so \(S/T\) is trivial in these cases. If \(\dim S/T\) is non-zero and finite then it is a power of 2 \([G3, pp. 372–373]\). (Examples with prescribed dimension 2\(^{r}\) are listed in \([G3, p. 373\) (skew) and p. 87 (commutative)].)

**Definition.** Let \((k, -, \varepsilon)\) be fixed and \(\Phi: E \times E \rightarrow k\) an \(\varepsilon\)-hermitean form. The \(k\)-vector space homomorphism \(E \rightarrow S/T: x \mapsto \|x\| := \Phi(x, x) + T\) is called the value map of \(\Phi\). If \(X \in \mathcal{L}(E)\) set \(\|X\| := \{\|x\| \mid x \in X\}\); the induced map \(\mathcal{L}(E) \rightarrow \mathcal{L}(S/T): X \mapsto \|X\|\) is likewise called “value map of \(\Phi\)” Following Dieudonné we call \(X\) trace-valued iff \(\|X\| = (0)\) (such spaces are sometimes called “even” and non-trace-valued spaces “odd”).

**Example.** As each non-degenerate \(\varepsilon\)-hermitean space that is not alternate admits an orthogonal basis \([G3, p. 65]\) we see that \(\|E\| = (0)\) iff \(E\) is either an orthogonal sum of hyperbolic planes or else an orthogonal sum of straight lines \(k(a)\) with \(\Phi(a, a) \in \{0\}\).

Notice that the value map \(\mathcal{L}(E) \rightarrow \mathcal{L}(S/T)\) has the following properties: for all \(A, B \in \mathcal{L}(E)\)

\[
\begin{align*}
(i) \quad & \|A\| + \|B\| = \|A + B\|, \\
(ii) \quad & \|A\| \cap \|B\| = \|(A + E^*) \cap B\|, \\
(iii) \quad & \|A\| = \|B\| \leftrightarrow A + E^* = B + E^* \leftrightarrow A^\perp \cap E^\perp = B^\perp \cap E^\perp.
\end{align*}
\]

The role of the interval \([E^*, E] \subset \mathcal{V}(F, E^*)\) may conveniently be summed up by saying that

\[X \mapsto \|X\|\] is a lattice monomorphism of \([E^*, E]\) into \(\mathcal{L}(S/T)\).

\([E^*, E]\) is a distributive sublattice of \(\mathcal{V}(F, E^*)\); it has 74 elements. See Remark III.3.

5. **Pointwise Fixedness of \(E^\perp\) under \(\mathcal{C}(E)\).** As an application of the foregoing paragraph we prove the

**Lemma.** Let \(E\) be a non-degenerate hermitean space and \(F, G\) subspaces with \(\|F\| \cap \|G\| = \|G\| \cap \|F^\perp\|\). An isometry \(\varphi: F \rightarrow G\) maps the subspace \((F \cap F^\perp) + (F \cap E^\perp)\) onto the corresponding space \((G \cap G^\perp) + (G \cap E^\perp)\); further, if \(\|F^\perp\| = \|G^\perp\|\) then \(\varphi z - z \in G \cap G^\perp\) for all \(z \in F \cap E^\perp\).

**Proof.** Let \(F_0, G_0\) be the full inverse images (relative to the value map) of \(\|F\| \cap \|F^\perp\|\) in \(F\) and \(G\), respectively. We have \(\varphi F_0 - G_0\) as \(\varphi\) is an isometry. Now \(F_0 = F \cap (F^\perp + E^*)\) ("\(\subseteq\)" because \(\|F_0\| \subseteq \|F^\perp\|\) and "\(\supseteq\)" is trivial) and likewise \(G_0 = G \cap (G^\perp + E^*)\). Since \(\varphi F_0 = G_0\) the space \(F_0 \cap F_0^\perp\) is mapped onto \(G_0 \cap G_0^\perp\). But \(F \cap F_0^\perp = (F \cap F^\perp) + (F \cap E^\perp)\), etc.
Let \( \| F \|^2 = \| G \|^2 \) and \( z \in F \cap E^* \perp = G \cap E^* \perp \) by 4(iii)) and \( g = \varphi f \) \( (f \in F) \) be a typical vector of \( G \). Since \( f - \varphi f \in E^* \) we have \( 0 = \Phi(z, f - \varphi f) = \Phi(\varphi z, \varphi f) - \Phi(z, \varphi f) = \Phi(\varphi z - z, g) \). Thus \( \varphi z - z \in G^\perp \) and so \( \varphi z - z \in G \cap G^\perp \).

**Corollary.** If \( E \) is non-degenerate and \( F + F^\perp = E = G + G^\perp \) with \( \| F \|^2 = \| G \|^2 \) then \( F \cap E^* \perp = G \cap E^* \perp \) and each isometry \( \varphi: F \to G \) leaves \( F \cap E^* \perp \) pointwise fixed. In particular \( (F := E) \) the space \( E^* \perp \) is left pointwise fixed under any operation \( \varphi \in \mathcal{O}(E) \).

6. **Remark on Scaling.** If \( \Phi \) is an \( \varepsilon \)-hermitean form on the \( k \)-vector space \( E \) with respect to \( (k, -, \varepsilon) \) then the right multiple \( \Phi_\mu := \Phi \mu \) \( (\mu \in \mathbb{K}) \) is \( \varepsilon_1 \)-hermitean with respect to \( (k, v, \varepsilon_1) \) where \( v(\xi) = \mu^{-1} \varepsilon_1 \mu, \varepsilon_1 = \varepsilon \cdot (\mu^{-1}) \cdot \mu \).

The symmetric elements and traces are related by \( S_\varepsilon = S_{\mu}, T_\varepsilon = T_\mu; \) furthermore, the map \( \sigma + T \to \sigma \cdot \mu + T_1 \) is a \( k \)-vector space isomorphism \( S/T \to S_\mu/T_\mu \). Thus, the operators \( * \) and \( ^\perp \) on the lattice \( \mathcal{L}(E) \) are not affected if the underlying form \( \Phi \) is replaced by a multiple \( \Phi \cdot \mu; \) in particular, the lattice \( \mathcal{V}_p(F, E^*) \) of a pair \( (E, F), F \in \mathcal{L}(E) \), is an invariant attached to the similarity class of the structure \( (k, -, \varepsilon) \). However, scaling forms in order to have to deal with special types only (symmetric, hermitean,...) does not simplify matters in our classification problem.

**II. Table of Isotypic Pairs and Their Lattices**

1. The first column in the table below gives a numbering of all isotypes for the purpose of reference. There is no strong intrinsic meaning in the numbering chosen.

2. The second column gives the diagrams of all subdirectly irreducible polarity lattices \( \mathcal{V}^\perp(F, E^*) \). The universal bounds in the diagrams represent (by definition of \( \mathcal{V}^\perp(F, E^*) \)) invariably the null space and the entire space \( E \).

3. In the third column a normal form is given for the pair \( (E, F) \) of given isotype: the first row describes \( E \), the second row specifies \( F \). We have used the following **notations**: \( R \oplus R' \) is an orthogonal sum of \( \dim R = \dim R' \) hyperbolic planes; \( R \) and \( R' \) are maximal totally isotropic subspaces of \( R \oplus R' \). \( D \oplus C \) is an orthogonal sum of \( \dim D = \dim C \) metabolic planes (i.e., planes \( k(d) \oplus k(c) \approx \binom{0}{1} \)) with \( \gamma = \Phi(c, c) \in S \setminus T \) such that \( C \) is rigid, i.e., \( C^* = (0) \); hence \( D \) is the unique maximal totally isotropic subspace of \( D \oplus C \); \( B \) is a rigid space in the table and \( A \) is a trace-valued space. If \( (u_i)_{i \in I} \) and \( (v_i)_{i \in I} \) are orthogonal bases of spaces \( U \) and \( V \), respectively, then the span in \( U \oplus V \) of the family \( (u_i + v_i)_{i \in I} \) is ambiguously denoted by \( U \bigoplus V \); our considerations are not affected by this ambiguity.
4. In the fourth column we give invariants that have to be added to the diagram in column 2 in order to obtain a complete set of invariants for the pair \((E, F)\). \(X\) is the isometry class of \(X\) and \(\mathcal{O}(X)\) is the group of metric automorphisms of \(X\). In all cases except for isotypes 12 and 13 this "arithmetic" invariant may be chosen to be \(\hat{E}\). In case 12 we need \(\hat{F}\) instead of \(\hat{E}\); in case 13, \(\hat{E}\) has to be supplemented by a "matrix" that can be picked arbitrarily from certain cosets. Case 13 does not arise for algebraically closed fields whereas all other twelve cases do.

5. Table

<table>
<thead>
<tr>
<th>Number of isotype</th>
<th>Diagram of (\gamma(F, E^*))</th>
<th>Normal form for (E) and (F)</th>
<th>Complete set of invariants for a pair ((E, F)) of given isotype</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(F \cdot F^{-1} \cdot E^*E'^{-1})</td>
<td>(D \oplus C) (\hat{E}) (determined by (|E|))</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(D)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>(F \cdot F^{-1}) (E^*E'^{-1})</td>
<td>(R \oplus R') (\hat{E}) (determined by (\dim E))</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(R)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(E^* \cdot E'^{-1}) (F)</td>
<td>(D \oplus C) (D \oplus C) (\hat{E}) (determined by (|E|))</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(D)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(E^* \cdot E'^{-1}) (F)</td>
<td>(D \oplus C) ({0}) (\hat{E}) (determined by (|E|))</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>({0})</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(\hat{F} \cdot F^{-1}) (E^*E'^{-1})</td>
<td>((R \oplus R') \oplus (D \oplus C)) (R \oplus (R' \Delta C)) (\hat{E}) (determined by (|E|))</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>((R \oplus R')) ((D \oplus C)) (R \oplus (R' \Delta C))</td>
<td></td>
</tr>
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### Table—Continued

<table>
<thead>
<tr>
<th>Number of isotype</th>
<th>Diagram of isotype $\mathcal{V}(F, E^*)$</th>
<th>Normal form for $E$ and $F$</th>
<th>Complete set of invariants for a pair $(E, F)$ of given isotype</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$E^* \cdot F^\perp$</td>
<td>$E = R \oplus R'$ (alternate case) or else orthogonal sum of lines $k(a)$ with $\Phi(a, a) \in T \setminus {0}$</td>
<td>$\hat{E}$ (determined by the bijective value map $E^* \rightarrow |E^*|$)</td>
</tr>
<tr>
<td>7</td>
<td>$F \cdot E^\perp \cdot F^\perp$</td>
<td>$(R \oplus R') \oplus B$</td>
<td>$F = E$, $\hat{E}$ (determined by the bijective value map $E^* \rightarrow |E^*|$)</td>
</tr>
<tr>
<td>8</td>
<td>$E^* \cdot F^\perp$</td>
<td>$E$ as in No. 6 $F = E$</td>
<td>$\hat{E}$</td>
</tr>
<tr>
<td>9</td>
<td>$F^\perp \cdot E^\perp \cdot F$</td>
<td>$(R \oplus R') \oplus B$</td>
<td>$E \hat{E}$ (determined by the bijective value map $E^* \rightarrow |E^*|$)</td>
</tr>
<tr>
<td>10</td>
<td>$E^\perp \cdot F^\perp$</td>
<td>$B$</td>
<td>$\hat{E}$ (determined by the bijective value map $E \rightarrow |E|$)</td>
</tr>
<tr>
<td>11</td>
<td>$E^* \cdot F^\perp$</td>
<td>$B$ (0)</td>
<td>$\hat{E}$ (determined by the bijective value map $E \rightarrow |E|$)</td>
</tr>
<tr>
<td>Number of isotype</td>
<td>Diagram of isotype $\mathcal{F}(F, E^*)$</td>
<td>Normal form for $E$ and $F$</td>
<td>Complete set of invariants for a pair $(E, F)$ of given isotype</td>
</tr>
<tr>
<td>------------------</td>
<td>------------------------------------------</td>
<td>-----------------------------</td>
<td>-------------------------------------------------------------</td>
</tr>
<tr>
<td>12</td>
<td>$F \otimes E^* \otimes E^{*\perp}$</td>
<td>$D \oplus C$</td>
<td>$\mathcal{F}$ (the set is determined by $|F|$; $\bar{E}$ does not fix $\mathcal{F}$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$C$</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>$F \otimes F^{\perp} \otimes E^* \otimes E^{*\perp}$</td>
<td>$A \oplus B$</td>
<td>$\bar{E}$ and an element of the factor set $GL_r(k) \otimes (E^<em>)$ ($r := \dim E^</em>$). The orbits of pairs $(E, F)$ with $E$ fixed are in 1-1 correspondence with the elements of the factor set indicated.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$A \oplus B$</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>$F \otimes F^{\perp} \otimes E^* \otimes E^{*\perp}$</td>
<td>$E = F = (0)$</td>
<td></td>
</tr>
</tbody>
</table>

6. Remarks Concerning the Table. From the fourth column we see that the classification of pairs $(E, F)$ up to isometry is reduced to the classification up to isometry of spaces in the cases of numbers 6 and 8, i.e., in the classical situation where $E = E^*$ and a classical problem of linear algebra where $E = E^*$ in case 13.

In the case of symmetric bilinear forms the rigid spaces $E^{*\perp}$ in 7, 9, 10, and 11 may equally be characterized by the Clifford determinant $\Delta(E^{*\perp})$ in lieu of the value map $E^{*\perp} \to \|E^{*\perp}\|$ [Mi, p. 69].

It is remarkable that already in the case of “perfect” fields, $\dim S/T = 1$, all isotypes do occur [G2, p. 66]; in type 13 the field is skew if perfect.

For pairs of the same isotype (excluding 13) and dimension there is a linear isomorphism matching the four distinguished subspaces. Indeed, as vector spaces with a quadruple of subspaces they arise from any minimal dimension as direct powers.

The orthogonally indecomposable pairs have dimension equal to the length of the lattice except in isotypes 6, 8, and 13. For 6 and 8 the dimension is 1 or 2. In isotype 13 there are indecomposables of arbitrary dimension containing arbitrary numbers of linear indecomposables. The proof that there are no other indecomposables is given in Section V. All other assertions made by the table are routinely verified (make use of Section VI); no proofs will be written out.
III. STATEMENT OF THE PRINCIPAL RESULTS:
EXISTENCE AND UNIQUENESS OF DECOMPOSITIONS

1. \( E \) is a finite-dimensional non-degenerate \( \varepsilon \)-hermitean space and \( F, \bar{F}, \ldots \) are linear subspaces. Our main result is

**Theorem.** Each pair \( (E, F) \) has a decomposition

\[
(E, F) = \sum_{i} (E_i, F_i), \quad I \subseteq \{1, 2, \ldots, 14\}
\]  

with \( (E_i, F_i) \) of isotype \( i \) as given in Table II.5.

Sometimes it is convenient to formally extend summation in (1) over the entire interval \([1, 14]\); we then interpret \( (E_i, F_i) \) with \( i \in [1, 14]\setminus I \) as \( (E_{14}, F_{14}) \). The proof of the Theorem is given in Section V.

2. **Lemma.** The following isometry types read off from any decomposition (1) are invariants of the pair \( (E, F) \) and hence of the isometry class of the pair \( (E, F) \): \( \tilde{E}_j, \tilde{F}_j^{-1} \) \( (j = 7, 9, 12, 13) \), \( \tilde{E}_i \) where \( i \in \{2, 6, 7, 8, 9, 10, 11, 12, 13\} \).

The isometry classes \( \tilde{E}_1, \tilde{E}_3, \tilde{E}_4, \tilde{E}_5 \) are not invariants; they do depend on the decomposition (1). In order to give more precise information we shall introduce certain frequently occurring combinations of objects defined via (1). They are all invariants of the isometry class of the pair \( (E, F) \). Here is their list:

\[
\begin{align*}
\| F \| + \| F^\perp \| &= \bigoplus \{ \| E_i \| \mid i \in I \cap \{3, 4, 5, 7, 9, 10, 11, 12, 13\}\} \\
\| F \| &= \bigoplus \{ \| E_i \| \mid i \in I \cap \{3, 5, 7, 9, 10, 12, 13\}\} \\
\| F^\perp \| &= \bigoplus \{ \| E_i \| \mid i \in I \cap \{4, 5, 7, 9, 11, 12, 13\}\} \\
\| F \| \cap \| F^\perp \| &= \bigoplus \{ \| E_i \| \mid i \in I \cap \{5, 7, 9, 12, 13\}\} \\
\| E \| &= \bigoplus \{ \| E_i \| \mid i \in I \cap \{1, 3, 4, 5, 7, 9, 10, 11, 12, 13\}\} \\
\| F^* \cap F \| &= \bigoplus \{ \| E_i \| \mid i \in I \cap \{7, 10, 12, 13\}\} \\
\| F^\perp \cap F^* \| &= \bigoplus \{ \| E_i \| \mid i \in I \cap \{9, 11, 12, 13\}\} \\
W_1 &= := \| F \| + \| F^\perp \|, \quad W_3 := (\| F \| \cap \| F^\perp \|) + \| F^* \cap F \| \\
W_4 &= := (\| F \| \cap \| F^\perp \|) + \| F^\perp \cap F^* \| \\
W_5 &= := (\| F \| \cap \| F^\perp \|) \cap (\| F^* \cap F \| + \| F^\perp \cap F^* \|)
\end{align*}
\]
We have $Y_i = \| E_i \|$ ($i = 7, 9, 10, 11, 12, 13$).

3. Theorem. A complete set of invariants for the pair $(E, F)$ up to isometry is formed by the following collection of fourteen objects:

(i) $\tilde{E}_6, \tilde{E}_8, \tilde{E}_{13}^*$: isometry classes of arbitrary non-degenerate trace-valued spaces over $(k, -, \varepsilon)$;

(ii) $\tilde{E}^*_{10}, \tilde{E}^*_{11}, \tilde{E}^*_{12}, \tilde{E}^*_{13}:$ isometry classes of arbitrary rigid spaces over $(k, -, \varepsilon)$;

(iii) $\dim E_7$;

(iv) $M_{13}:$ element of the factor set $GL_r(k)/O(E_{13}^*)$ where $r := \dim E_{13}^*$;

(v) $\| E \|, \| F \|, \| F^\perp \|$; linear subspaces in the value space $S/T$.

The complete list of relations among these invariants is made up by the following obvious ones:

(i) the sum $\| E_7^\perp \| + \| E_8^\perp \| + \| E_{10} \| + \| E_{11} \| + \| F_{12} \| + \| E_{13}^* \|$ is a direct sum;

(jj) $\dim E_2 \equiv 0 \pmod{2}$, $\dim E_{13}^* = \dim E_{13}^\perp$;

(jjj) $\| E_7^\perp \| + \| E_8^\perp \| + \| F_{12} \| + \| E_{13}^* \| \subseteq \| F \| \cap \| F^\perp \|$, $\| E_{10} \| \leq \| F \| \leq \| E \|$, $\| E_{11} \| \leq \| F^\perp \| \leq \| E \|$, $\| E_{10} \| \cap \| F^\perp \| = (0)$, $\| E_{11} \| \cap \| F \| = (0)$.

Thus (i) and (jjj) may conveniently be replaced by the diagram in Fig. 1.
Remark. We explain how we can obtain, from the diagram in Fig. 1, a diagram for the lattice \([E^*, E] \subseteq \mathcal{V}_E(F, E^*)\) (recall that the interval \([E^*, E]\) is mapped monomorphically into the lattice \(\mathcal{L}(S/T)\) under the value map): Replace the edge between \((0)\) and \(Y\) by the distributive lattice \(2^4\) generated by the atoms \(\|E^*_1\|, \|E^*_3\|, \|F_{12}\|, \|F^*_3\|\) thereby transforming the lowermost "cube" of the diagram in Fig. 1 into the lattice \(2^6\) in Fig. 2. The rest of the original diagram in Fig. 1 with its ten elements is left unchanged: we have \(\|F\| \cap \|F^*\| = Y \oplus \|E_5\|\). Thus the interval \([E^*, E] \subseteq \mathcal{V}_E(F, E^*)\) has 74 elements in general (cf. the definitions of \(Y_i\) in (2)).

4. Lemma. The spaces \(E_1, E_3, E_4, E_5\) in any decomposition (1) satisfy the following relations: \(\|E_1\| \oplus W_1 = \|E\|, \|E_3\| \oplus W_3 = \|F\|, \|E_4\| \oplus W_4 = \|F^*\|, \|E_5\| \oplus W_5 = \|F\| \cap \|F^*\|\). Conversely, if we pick in \(S/T\) arbitrary linear supplements \(S_1, S_3, S_4, S_5\) of the invariant subspaces \(W_1, W_3, W_4, W_5\) in \(\|E\|, \|F\|, \|F^*\|, \|F\| \cap \|F^*\|\), respectively, then there exists a decomposition (1) of the pair \((E, F)\) such that

\[
\|E_i\| = S_i \quad (i = 1, 3, 4, 5).
\]

Definition. The elements \(S_1, S_3, S_4, S_5 \in \mathcal{L}(S/T)\) described in the lemma are called value-complements of \((E, F)\). Thus if (3) holds we say that (1) is a decomposition with value complements \(S_1, S_3, S_4, S_5\).

Remark. Assume that (3) holds for a decomposition (1). It would be erroneous to think that for arbitrary subspaces \(X_i \subseteq E\) with \(\|X_i\| = S_i\), there would exist a decomposition with \(X_i \subseteq E_i\) \((i = 1, 3, 4, 5)\).

5. Uniqueness of Decomposition. We start with an obvious remark: If two pairs \((E, F), (\tilde{E}, \tilde{F})\) are isometric then there is an isometry that maps all spaces in \(\mathcal{V}_E(F, E^*)\) onto the corresponding elements of the lattice \(\mathcal{V}_{\tilde{E}}(\tilde{F}, \tilde{E}^*)\). From this one may extract various sets of conditions that are necessary in order for two pairs to be isometric. Here our selection of such conditions is motivated by Lemma III.4:
From (4) we obtain $W_i = \tilde{W}_i$, $i = 1, 3, 4, 5$ (cf. (2) for the definition of the $W_i$) and thus by Lemmata III.2 and III.4 follows directly:

**Theorem.** (A) If $(E, F) \preceq (\tilde{E}, \tilde{F})$ then (4) holds. Conversely, assume that (4) is satisfied. Then, if $S_1, S_3, S_4, S_5$ are any value-complements for $(E, F)$ they serve as value-complements for $(\tilde{E}, \tilde{F})$ as well, $\tilde{S}_i = S_i$ ($i = 1, 3, 4, 5$).

(B) If (1) and (1) are decompositions for $(E, F)$ and $(\tilde{E}, \tilde{F})$, respectively, and with value-complements $S_i = S_i$, then

$$(E, F) \simeq (\tilde{E}, \tilde{F}) \iff (E_i, F_i) \simeq (\tilde{E}_i, \tilde{F}_i) \quad (1 \leq i \leq 14).$$

**Remark.** It is possible to give another version of uniqueness of our decomposition. It avoids the concept of value space in the formulation of the result as well as in its proof. It is, by necessity, more involved than the account given here. However, it allows for effective construction of isometries from decomposition (1). We shall not discuss it in this paper.

IV. Extension of Partial Isometries, Congruence, and Cancellation

As an example of how the decomposition theorem can be applied we treat in this section some classical questions. Our treatment rounds off the list of results and, more important, it replaces a variety of different and in cases rather cumbersome proofs by short and perspicuous arguments. We start off with

1 **Extension Theorem.** Let $F, \tilde{F}$ be subspaces in the non-degenerate hermitean space $E$. An isometry $\varphi_0 : F \rightarrow \tilde{F}$ can be extended to all of $E$ iff

$$F \cap E^* \perp = \tilde{F} \cap E^* \perp \quad \text{and} \quad \varphi_0 \upharpoonright F \cap E^* \perp = \text{identity.} \quad (5)$$

**Remark.** If $S = T$, hence in particular when $\text{char } k \neq 2$, we have $E^* \perp = (0)$ so the requirement (5) is empty and the statement reduces to the classical Witt theorem. Also, it is very easy to formulate the Extension Theorem for degenerate spaces $E$.

The history of the Extension Theorem is somewhat curious. G. E. Wall stated a very special result by requiring $F, \bar{F} \subseteq E^*$ [Wa, Theorem 1.2.1, p. 93], yet he adduced a complete proof for the general result without being aware of it. V. Pless has reported on this fact in [P1]. Independently, W. Bäni discovered the same result (see [G3, p. 382] for the story); in
2. **Congruence Theorem** [P1, Theorem 2]. Let \( F, \tilde{F} \) be subspaces in the non-degenerate hermitian space \( E \). There exists an isometry \( \varphi: E \to E \) with \( \varphi F = \tilde{F} \) iff \( F \cong \tilde{F} \) and
\[
F \cap E^{*\perp} = \tilde{F} \cap E^{*\perp} \quad \text{(i.e., } \| F^{\perp} \| = \| \tilde{F}^{\perp} \| \text{ by I.4(iii)).} \tag{6}
\]
In other words, a complete set of invariants for the orbit of a subspace \( F \subseteq E \) under the action of \( O(E) \) is made up by \( \tilde{F} \) and \( \| F^{\perp} \| \).

From the Congruence Theorem and Decomposition Theorem III.1 one obtains the **Corollary.** Assume that the subspaces \( F \) and \( \tilde{F} \) are congruent, i.e., belong to the same orbit under \( O(E) \). Then each isometry \( \varphi_0: F \to \tilde{F} \) can be extended to all of \( E \) if and only if we are in one of the following cases: (1) \( F \oplus F^{\perp} = E = \tilde{F} \oplus \tilde{F}^{\perp} \), (2) \( F \cap E^{*\perp} = (0) = \tilde{F} \cap E^{*\perp} \), (3) the base field is \( \mathbb{Z}_2 \) and \((E, F) = \sum (E_i, F_i) \mid i = 1, 6, 8\) with \( E_1 = \langle 1, 1 \rangle \) and \( F \cap F^{\perp} = F_1 = \tilde{F}_1 \) the unique isotropic line in \( E_1 \).

3. **On Cancellation.** The following result [W, Theorem 23(i)] is a special case of the Congruence Theorem.

**Cancellation Theorem.** Let \( F \oplus F^{\perp} = E = \tilde{F} \oplus \tilde{F}^{\perp} \) and \( F \cong \tilde{F} \). Then the following are equivalent: (1) \( F^{\perp} \cong \tilde{F}^{\perp} \), (2) \( F \cap E^{*\perp} = \tilde{F} \cap E^{*\perp} \), (3) \( F^{\perp} + E^{*} = \tilde{F}^{\perp} + E^{*} \), (4) \( \| F^{\perp} \| = \| \tilde{F}^{\perp} \| \).

By the Decomposition Theorem it is easy to see through the cases “where cancellation fails”: Since \( F \cap E^{*\perp} = F_3 + F_{10} \) in terms of (1) when \( F \) is non-degenerate we see that “cancellation fails” if and only if \( F \cong \tilde{F} \) and the following inequality on sets holds:
\[
F_3 \oplus F_{10} \neq \tilde{F}_3 \oplus \tilde{F}_{10}. \tag{7}
\]
We shall now give some typical examples. In the first two cases we have \( F \cong \tilde{F} \) but no lattice isomorphism \( \nu(F, E^{*}) \cong \nu(\tilde{F}, E^{*}) \) and hence a fortiori no cancellation. In the last two cases we have \( F \cong \tilde{F} \) and also a lattice isomorphism \( \nu(F, E^{*}) \cong \nu(\tilde{F}, E^{*}) \) yet no cancellation because (7) holds.

**Example 1.** \( E := k(e_1) \oplus k(e_2) \oplus k(e_3) = \langle \alpha, \alpha, \alpha \rangle, \) \( F := k(e_4), \) \( \tilde{F} := k(e_1 + e_2 + e_3). \) \((E, E) = (E_7, F_7), (E, \tilde{F}) = (\tilde{E}_6, \tilde{F}_6) + (\tilde{E}_{10}, \tilde{F}_{10}), F \cong \tilde{F}. \)

---

1. **Note added in proof.** M. Kneser (in a letter to H. Gross dated September 8, 1985) kindly drew our attention to yet another source: The Extension Theorem is also proved in full generality in Hanfried Lenz, Grundlagen der Elementarmathematik, erste Auflage 1961, VEB Verlag der Wissenschaften, Berlin (Satz 7, p. 335). This material is not contained in later editions of Lenz' book!
EXAMPLE 2. \( E := k(e_1) \oplus k(e_2) \oplus k(e_3) \oplus k(e_4) \cong \langle \alpha, \alpha, \alpha, \alpha \rangle \),
\( F := k(e_1 + e_2 + e_3, e_4), \quad \tilde{F} := k(e_1 + e_2, e_1 + e_3 + e_4). \)
\((E, F) = (E_6, F_6) + (E_3, F_3), \quad (E, \tilde{F}) = (\tilde{E}_5, \tilde{F}_5), \quad F \cong \tilde{F}.

EXAMPLE 3. \( E := k(e_1) \oplus k(e_2) \oplus k(e_3) \cong \langle \alpha, \alpha, \beta \rangle \) with \( \alpha, \beta \) linearly independent in the value space \( S/T \), \( F := k(e_1 + e_2 + e_3). \)
\((E, F) = (E_4, F_4) + (E_{10}, F_{10}), \quad (E, \tilde{F}) = (\tilde{E}_4, \tilde{F}_4) + (\tilde{E}_{10}, \tilde{F}_{10}), \quad F \cong \tilde{F}, \quad F_{10} = F \neq \tilde{F} = \tilde{F}_{10} \) so (7) holds.

EXAMPLE 4. \( E := k(e_1) \oplus k(e_2) \oplus k(e_3) \oplus k(e_4) \cong \langle \alpha, \alpha, \beta, \beta \rangle \) with \( \alpha, \beta \) independent in \( S/T \), \( F := k(e_1, e_2), \quad \tilde{F} := k(e_1 + e_3 + e_4, e_2). \)
\((E, F) := (E_3, F_3) + (E_4, F_4), \quad (E, \tilde{F}) = (\tilde{E}_3, \tilde{F}_3) + (\tilde{E}_4, \tilde{F}_4), \quad F \cong \tilde{F}, \quad F_3 = F \neq \tilde{F} = \tilde{F}_3 \) so (7) holds.

Remark. In view of (7) it is now possible, in principle, to invent cancellation results at will: Just find conditions sufficient to rule out (7). The main results in \([W, \text{Theorem 2.3(ii), (iii)}]\) give rather useful conditions that are sufficient for cancellation. Let \( E = F \oplus F^\perp = G \oplus G^\perp \) and \( F \cong G \).
Then \( F^\perp \cong G^\perp \) if one of the following conditions is satisfied:

(j) \( F^\perp \) or \( G^\perp \) is anisotropic,

(jj) both \( F^\perp \) and \( G^\perp \) have the property \((W)\) that every non-zero isotropic vector belongs to a hyperbolic plane.

The proofs given in \([W]\) can be shortened considerably by using the Decomposition Theorem in the style of Sections X and XI. Also, the two enigmatic assumptions on \( F^\perp, G^\perp \) become perspicuous: From any decomposition (1) one reads off: (A) \( F^\perp \) is anisotropic iff \( \text{rad}(F^\perp *) = (0) \); (B) \( F^\perp \) has property \((W)\) iff \( \text{rad}(F^\perp *) = (0) \) and \( E_6 \) is anisotropic. Thus

(jj) \( F^\perp \) is anisotropic \( \Leftrightarrow F^\perp \) has property \((W)\) and \( E_6 \) is anisotropic.

In particular, for symmetric forms in characteristic 2 the two assumptions (j) and (jj) are identical since \( E_6 = (0) \) if anisotropic! A proof following the Decomposition Theorem also shows just why it suffices to postulate anisotropy for one among \( F^\perp, G^\perp \) only. (Example 1 above shows, as pointed out in \([W]\), that \((W)\) must be postulated for both \( F^\perp \) and \( G^\perp \).)

PART TWO: PROOFS

V. PROOF OF THE DECOMPOSITION THEOREM

I. To employ the classification of quadruples we need some terminology first. By a quadruple \( \mathcal{U} \) we mean a \( k \)-vector space \( U \) together with four
linear subspaces $U_1, U_2, U_3, U_4$. Its polar $U^\circ$ is the dual space $U^*$ of $U$ together with the subspaces $U_2^\circ, U_3^\circ, U_4^\circ$ where $U_i^\circ$ is the polar subspace $\{ f \in U^* \mid f(U_i) = 0 \}$ of $U_i$.

Subspaces $A$ and $B$ of $U$ provide a (linear) decomposition of the quadruple $\mathcal{U} = (U_1, U_2, U_3, U_4)$ if and only if $U = A \oplus B$ and $U_i = (U_i \cap A) \oplus (U_i \cap B)$ for all $i$. Then, $A$ with the $U_i \cap A$ is a summand of $\mathcal{U}$.

Associating with a pair $(E, F)$ the quadruple $\mathcal{S} = (E; F, F^*, E^*, E^*\perp)$ we have that a linear decomposition $A, B$ of $\mathcal{S}$ is an orthogonal decomposition of $(E, F)$ if and only if $B = A^\perp$. Also, if $A, B$ is a linear decomposition of $\mathcal{S}$ then so is $A^\perp, B^\perp$ and the summand $B^\perp$ is the polar of the summand $A$. Thus, following Gabriel [Ga] we obtain an orthogonal decomposition $A, A^\perp$ of $(E, F)$ if we choose a set $\Sigma$ of isomorphism types of indecomposable summands of $\mathcal{S}$, closed under polarity, and a maximal summand $A$ built up from summands whose types are in $\Sigma$. This can be proved by applying the Krull-Remak-Schmidt Exchange Theorem to conclude $A = B^\perp$ and $A^\perp = B$.

Consequently, for every indecomposable pair $(E, F)$ there are an indecomposable quadruple $\mathcal{U}$, a multiplicity $m$, and linear quadruple isomorphism

$$\mathcal{S} \cong m \mathcal{U} \quad \text{and} \quad \mathcal{U} \cong \mathcal{U}^\circ$$

or

$$\mathcal{S} \cong m(\mathcal{U} \oplus \mathcal{U}^\circ).$$

Compare Bäni [B] and Quebbemann, Scharlau, and Schulte [Q] for the case of characteristic $\neq 2$.

2. The classification of indecomposable quadruples has been established by uncountably many authors beginning with Kronecker at various levels of generality. For our purpose the following information will suffice.

Any indecomposable quadruple is (up to isomorphism) of one of the following types (see Brenner [Br2, pp. 597–599]):

operator quadruple: $U = U_i \oplus U_j$ for all $i \neq j$ \hfill (10)

uniserial, zero defect, and self-polar \hfill (11)

$\dim U$ even: $U = fQ \oplus gQ, U_1 = fQ, U_2 = gQ, U_3 = (f + g)Q, U_4 = (f + g(j + 1))Q$.

$U_4 = (f + g(j + 1))Q$

$\dim U$ odd: $U = fQ \oplus gQ \oplus hX, U_1 = (f + g)Q, U_2 = gQ \oplus hX, U_3 = fQ \oplus hX, U_4 = (f + g(j + 1) + b)Q$

or any permutation thereof. Here, $Q$ and $X$ are $k$-vector spaces, $\dim X = 1$, ...
$f$ and $g$ monomorphisms of $Q$ into $U$, $h$ a monomorphism of $X$ into $U$, $j$ an indecomposable nilpotent endomorphism of $Q$, $1$ the identity map, and $b$ a homomorphism of $Q$ into $U$ such that $b(\ker j) = hX$. Moreover, the permutations (cycles) $(12)$ and $(34)$, $(14)$ and $(23)$, respectively, are induced by quadruple isomorphisms.

negative defect, preprojective:

\[ U_i \cap U_j = (0) \text{ for all } i \neq j; \quad \dim U > 2 \text{ implies } U_i \neq (0) \text{ for all } i, \]
\[ \dim U = 2 \text{ implies } U_i = (0) \text{ for an } i \text{ and } \dim U_j = 1 \text{ for all } j \neq i, \]
\[ \dim U = 1 \text{ implies } U_i = U \text{ for an } i \text{ and } U_j = (0) \text{ for all } j \neq i \quad (12) \]

positive defect, preinjective: dual of $(12)$. (13)

Moreover, if $\mathcal{U}$ is of type $(12)$ then there is a lattice term $q(x, y, z, w)$—namely a perfect one in the sense of Gelfand and Ponomarev $[GP]$—such that

\[ q(\mathcal{U}) = (0) \quad \text{and} \quad q(\mathcal{U}^\vee) = \hat{U} \quad (14) \]

where $q(\mathcal{U})$ means $q(U_1, U_2, U_3, U_4)$ evaluated in the subspace lattice of $U$. Such terms can be defined inductively: $q^0 = 1$, and $q^{m+1} = ((x \cap q^m) + (y \cap q^m)) \cap ((z \cap q^m) + (w \cap q^m))$.

Then, $q^m(\mathcal{U}) = (0)$ if and only if $\defect \mathcal{U} = -1$ and $\dim U \leq m$ or $\defect \mathcal{U} = -2$ and $\dim U \leq 2m - 1$. Indeed, using Brenner's list one has that $q^i(\mathcal{U})$ with $U_i \cap q^i(\mathcal{U})$ is again an indecomposable quadruple of the same defect and one can apply induction.

3. For a lattice term $t(x, y, z, w)$ define its polar $t^\vee$ as $t^0(y, x, w, z)$ where $t^0$ is the term dual to $t$. Then, for the quadruple $\mathcal{E}$ associated with the pair $(E, F)$ we have $t^\vee(\mathcal{E}) = t(\mathcal{E})^\perp$ whence by I.1

\[ t(\mathcal{E}) \cap t^\vee(\mathcal{E}) \subseteq E^* \quad \text{and} \quad t(\mathcal{E}) + t^\vee(\mathcal{E}) \supseteq E^\perp. \quad (15) \]

Observe that the lattice $\mathcal{Y}(F, E^*)$ is isomorphic to the sublattice $\mathcal{L}(\mathcal{U})$ of $\mathcal{L}(U)$ generated by $U_1, U_2, U_3, U_4$ in case (8) and a subdirect product of $\mathcal{L}(\mathcal{U})$ and $\mathcal{L}(\mathcal{U}^\vee)$ in case (9). Thus, (15) provides relations for the quadruples $\mathcal{U}$ and $\mathcal{U}^\vee$ involved in $\mathcal{E}$. Using these we can single out the pairs listed in Section II. For case (9) with $\mathcal{U} \neq \mathcal{U}^\vee$ this is done easily: we may assume that $\mathcal{U}$ is of type $(12)$ and choose $t = q$ as in $(14)$. Then we have

\[ q(\mathcal{E}) = m(q(\mathcal{U}) \oplus q(\mathcal{U}^\vee)) = m((0) + \hat{U}) = m(q^*(\mathcal{U}) \oplus q^*(\mathcal{U}^\vee)) = q^*(\mathcal{E}). \]

By (15) it follows $E^\perp \subseteq E^*$ whence $U_4 \subseteq U_3$ and $U_4 = (0)$. This leaves dimensions 1 and 2, only, and the isotypes 1 to 5. For $\mathcal{U}$ of type $(10)$ we have isotype 13.
4. Now, consider \( \mathcal{U} \) of type (11) and the self-polar case (8). The dimensions up to three are worked through, easily, yielding isotypes 6 to 12. Excluding these we have by Brenner [Br1, Lemma 2]

\[
U_i \cap U_j \subseteq U_k + U_i \quad \text{and} \quad U_i \cap U_j \cap U_k = 0
\]

for any listing of the index set \( \{1, 2, 3, 4\} \). Having

\[
(F + E^*) \cap (F^\perp + E^{*\perp}) \cap ((F^\perp \cap E^{*\perp}) + (F \cap E^*)) \subseteq E^*
\]

by (15), we derive \( F^\perp \cap E^{*\perp} \subseteq E^* \) and \( F^\perp \cap E^{*\perp} = 0 \). Similarly, we get \( F \cap E^{*\perp} = F \cap F^\perp = 0 \). This leaves us to deal with the quadruples as listed in (11)—the remaining cases follow by interchanging \( F \) and \( F^\perp \). Let

\[
t(x, y, z, w) = (x \cap (y + (z \cap w))) + (y \cap (x + (z \cap w))).
\]

For even dimension one has, evidently,

\[
t(\mathcal{U}) = f(\ker j) \oplus g(\ker j) \quad \text{and} \quad t^*(\mathcal{U}) = f(\text{im } j) \oplus g(\text{im } j),
\]

dually. Since \( j \) is an indecomposable nilpotent endomorphism of the vector space \( Q \) of dimension at least two the kernel of \( j \) must be contained in the image of \( j \). This implies \( t(\mathcal{U}) \subseteq t^*(\mathcal{U}) \) and, by (15), the obvious contradiction \( t(\mathcal{U}) \subseteq U_3 \). For odd dimension one has, similarly, by interchanging \( y \) and \( w \) in \( t \),

\[
t(\mathcal{U}) = (f + g)(\ker j) \oplus b(\ker j),
\]

\[
t^*(\mathcal{U}) = (f + g)(\text{im } j) \oplus b(\text{im } j)
\]

and \( t(\mathcal{U}) = t(\mathcal{U}) \cap t^*(\mathcal{U}) \subseteq U_3 \), a contradiction.

VI. Basic Facts on Hermitean Forms That Are Presupposed in Proofs

1. Each non-degenerate \( \varepsilon \)-hermitean space \((E, \Phi)\) of finite dimension admits certain well known canonical decompositions [D, p. 61]. It is appropriate, in this connection, to use terminology from Table II.5 in order to formulate them. They are as follows (\( F := (0) \)):

\[
E = E_6 \oplus (D_4 \oplus C_4) \oplus E_{11}
\]

\[
E^* = E_6 \oplus D_4
\]

\[
E^{*\perp} = D_4 \oplus E_{11}.
\]
It is not difficult to establish, in a straightforward manner, that $E$ is determined up to isometry by the three invariants (see, e.g., [Wa, Lemma 3.4.2; Gl, Theorem 21]):

$$\|E\|$$ (value set in $S/T$), $\hat{E}^*$, $\hat{E}^\perp$ (isometry classes).

(17)

The elementary proof makes use of certain particularly simple transformations of canonical bases. As these are very useful for practical work we set them down in full (Terminology: $k(r, r') \cong (\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})$, $k(d, c) \cong (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ are hyperbolic and metabolic planes, respectively, $k(e)$ is a straight line with $\Phi(e, e) \in T \setminus \{0\}$; see II.3 for further conventions):

(i) $k(r, r') \oplus^\perp k(d, c)$$$
= k(r, r' - \varepsilon \bar{\alpha}^{-1} d) \oplus^\perp k(d, \alpha c + r); \quad \alpha \in \bar{k}$

$k(e) \oplus^\perp k(d, c)$

$= k(\Phi(e, e) \bar{\alpha}^{-1} e) \oplus^\perp k(d, \alpha c + e)$

(ii) $k(d, c) \oplus^\perp k(b)$

$= k(d, \alpha c + b) \oplus^\perp k(b - \beta \bar{\alpha}^{-1} d); \quad \alpha, \beta := \Phi(b, b) \in \bar{k}$

(iii) $k(r, r') \oplus^\perp k(d, c)$

$= k(\alpha r + d, r') \oplus^\perp k(d, c - \varepsilon \bar{\alpha}^{-1} \bar{r} r'); \quad \alpha \in \bar{k}$

$k(e) \oplus^\perp k(d, c)$

$= k(\alpha e + d) \oplus^\perp k(d, c - \mu e), \quad \mu := \varepsilon \bar{\alpha}^{-1} \bar{\Phi}(e, e) \bar{\varepsilon}$

(iv) $k(d, c) \oplus^\perp k(b)$

$= k(d, c - \varepsilon \bar{\alpha}^{-1} \beta \bar{r} e + d) \oplus^\perp k(\alpha b + d); \quad \alpha, \beta := \Phi(b, b) \in \bar{k}$

(v) $k(d_1, c_1) \oplus^\perp k(d_2, c_2)$

$= k(d_1, c_1 + \bar{c} d_2) \oplus^\perp k(d_2, c_2 - d_1)$

(vi) $k(d_1, c_1) \oplus^\perp k(d_2, c_2)$

$= k(d_1, c_1 + c_2) \oplus^\perp k(-d_1 + d_2, -\gamma_2 d_1 + c_2)$.

We use (i) to change $C_4 \mod E_6$, (ii) to change $C_4 \mod E_{11}$, (iii) to change $E_6 \mod D_3$, (iv) to change $E_{11} \mod D_4$, and (v) to change $C_4 \mod D_4$. Incidentally, in order to see completeness of (17) only (ii) and (vi) are used. Further direct corollaries of these transformations are:

(vii) If the subspace $X$ of $E$ has $X \subset E^*$ and $X \cap E^\perp = (0)$ then there is a decomposition (16) with $X \subset E_6$.

(viii) If the subspace $Y$ of $E$ has $Y \subset E^\perp$ and $Y \cap E^* = (0)$ then there is a decomposition (16) with $Y \subset E_{11}$.

(ix) If $Z$ is a subspace of $E$ with $Z \cap (E^* + E^\perp) = (0)$ then there is a decomposition (16) with $Z \subset C_4$. 
Another immediate consequence of (ii) and (vi) worthy of note is

(x) ("adjusting the value set \(\|C_4\|\) in (16)). If \(X \subseteq C_4\) in (16) is fixed and \(S\) is any fixed complement of \(\|E^* + X\|\) in \(\|E\|\) then there is some other decomposition (16') \(E = E_6 \oplus (D_4 \oplus C_4) \oplus E_{11}\) with \(C_4 = X \oplus Y\) and \(\|Y\| = S\).

2. Lemma. Let \(\bar{C} \subseteq E\) be isometric subspaces, \(C \cong \bar{C}\), with \(C \cap (E^* + E^* {\perp}) = (0) = \bar{C} \cap (E^* + E^* {\perp})\). Then there exists \(D \subseteq E^* \cap E^* {\perp}\) such that \(D \oplus C\) is non-degenerate. Furthermore, for every \(D \subseteq E^* \cap E^* {\perp}\) such that \(D \oplus C\) is non-degenerate there is an isometry \(\varphi: E \to E\) with \(\varphi(D \oplus C) = D \oplus \bar{C}\).

Proof. By (ix) there is a decomposition (16) with \(C \subseteq C_4\) so that we can find a \(D \subseteq D_4\) of the requisite shape (symplectic bases for \(D_4 \oplus C_4\)). There is therefore a decomposition

\[
E = E_6 \oplus (D \oplus C) \oplus (D' \oplus C') \oplus E_{11};
\]

\[
D \oplus D' = E^* \cap E^* {\perp}, \quad C \oplus C' = C_4.
\]

Since \(C \cong \bar{C}\) we have \(\bar{C} \subseteq C + E^*\); as \(C \cap E^* = (0)\) we see that \(D \oplus \bar{C}\) is non-degenerate provided \(D \oplus C\) is non-degenerate. Thus, we may again quote (ix) and obtain a decomposition

\[
E = E_6' \oplus (D \oplus \bar{C}) \oplus (D'' \oplus C'') \oplus E_{11};
\]

\[
D \oplus D'' = E^* \cap E^* {\perp}, \quad \bar{C} \oplus C'' = C_4.
\]

By (x) we may furthermore adapt (19) such that \(\|C''\| = S := \|C\|\). The invariants (17) attached to the spaces \(D \oplus C\), \(D \oplus \bar{C}\) coincide, obviously, so \(D \oplus C \cong D \oplus \bar{C}\) (only \(\|C\| = \|\bar{C}\|\) is needed for this, and not \(C \cong \bar{C}\)). By (18) and (19) we read off that the invariants (17) attached to \((D \oplus C) {\perp}\), \((D \oplus \bar{C}) {\perp}\) coincide as well. Thus there is an isometry \(E \to E\) of the required kind.

3. Lemma. Let \(X, \bar{X} \subseteq E\) be isometric subspaces with \(X, \bar{X} \subseteq E^*\) and \(X \cap E^* {\perp} = \bar{X} \cap E^* {\perp} = (0)\). Then there is \(\varphi \in \mathcal{O}(E)\) with \(\varphi X = \bar{X}\).

Proof. By (vii) there are decompositions (16) \(E = E_6 \oplus E_4 \oplus E_{11}\), \(E = E_6' \oplus E_4' \oplus E_{11}'\) with \(X \subseteq E_6, \bar{X} \subseteq E_6\). We have \(E_6 \cong E_6', E_4 \oplus E_{11} \cong E_4' + E_{11}'\) since the invariants (17) coincide. Thus, by Witt's Theorem an isomorphism \(X \cong \bar{X}\) can be extended to an isomorphism \(E_6 \cong E_6'\) and, joining it with an isomorphism \(E_4 \oplus E_{11} \cong E_4' \oplus E_{11}'\), we are done.

4. Remark. Recall that a hermitean space \(X\) is called rigid if \(X^* = (0)\). An isometry between rigid spaces is unique. From this we obtain a simple
instance of cancellation that is used systematically in a proof of Lemma III.2:
If $X \oplus^\perp Y \cong X' \oplus^\perp Y'$ and $X \cong X'$ and $X \oplus Y$ is rigid then $Y \cong Y'$.

VII. PROOF OF LEMMA III.2

Ad type 2: $\hat{E}_2$ is hyperbolic hence determined by $\dim E_2 = \dim(F \cap F^\perp) - \dim(F \cap F^\perp \cap E^* \perp)$.

Ad type 6: $F^* \perp = \text{rad}(F^* \perp) \oplus E_6$ so $\hat{E}_6$ is an invariant.

Ad type 8: $F^* = \text{rad}(F^*) \oplus E_8$ so $\hat{E}_8$ is an invariant.

Ad type 10: $X := F \cap E^* \perp = \text{rad} X \oplus E_{10}$ hence $\hat{E}_{10}$ is an invariant.

Ad type 7: $X := F \cap F^* \perp \perp \perp = \text{rad} X \oplus (F_7 \oplus^\perp E_{10})$ thus $F_7 \oplus^\perp E_{10}$ is determined up to isometry. Remark VI.4 now yields invariance of $F_7$. Furthermore $F_7 \cong E^* \perp$ and $E^* \perp$ determines $\hat{E}_7$ and $\hat{F}_7^\perp$.

Ad types 11 and 9: Replace in the two previous discussions the role of $F$ by that of $F^\perp$ in order to obtain invariance of $\hat{E}_{11}$, $\hat{F}_{11}^\perp$, $\hat{E}_{9}$, $\hat{F}_{9}$.

Ad type 13: $X := F^* \perp \perp \perp \perp = \text{rad} X \oplus F_7 \oplus F_{10} \oplus F_{13}$. By the previously discussed invariances we obtain invariance of $\hat{F}_{13}$ by repeated application of Remark VI.4 (because the sum complementing rad $X$ is rigid). $Y := F \cap F^* \perp \perp \perp \perp \perp = \text{rad} Y \oplus F_9 \oplus E_{11} \oplus F_{13}^\perp$ yields invariance of $\hat{F}_{13}$. As $E_{13} = F_{13}^\perp \oplus F_{13}$ we have also invariance of $\hat{E}_{13}$ and hence of $\hat{E}_{13}^\perp$ and $\hat{E}_{13}^* \perp$ used in Theorem III.3.

Ad type 12: $X := F^* \perp \cap F = \text{rad} X \oplus F_7 \oplus F_{10} \oplus F_{12} \oplus F_{13}$. We may again quote Remark VI.4 to obtain invariance of $\hat{F}_{12}$. Again we have $E_{12} = F_{12} \oplus F_{12}^\perp$ and $F_{12} \cong F_{12}^\perp$ so that $\hat{E}_{12}$ is an invariant also.

VIII. PROOF OF LEMMA III.4

That the relations indicated for $\|E_i\|$ ($i = 1, 3, 4, 5$) are indeed satisfied can be read off from decomposition (1). The proof of the rest of the lemma goes by systematic use of the transformations VI.1(v), (iv).

Case $i = 1$. We show how to replace a metabolic plane $k(d, c)$ in $E_1$ by a plane $k(d, \lambda c + x)$, $0 \neq \lambda \in k$ and $x \in E_j$ ($2 \leq j \leq 13$ arbitrarily fixed). To this end we replace the pair $(E_j, F_j)$ by the pair $(E'_j, F'_j)$ where

$$E'_j := \{ e - \Phi(e, x) \bar{\lambda}^{-1} d \mid e \in E_j \}$$

$$F'_j := \{ f - \Phi(f, x) \bar{\lambda}^{-1} d \mid f \in F_j \}.$$
(E', F') is again of isotype \( j \) and another decomposition of the pair \((E, F)\) is obtained by this change of a plane in \( E_1 \). \( E_1' \) is again of isotype 1. By a repetition of the procedure we may adapt \( \| E_1' \| \) as required.

**Case \( i = 3 \).** Here we replace planes \( k(d, c) \subseteq E_3 \) by \( k(d, \lambda c + x) \), \( 0 \neq \lambda \in k, x \in E_3 \) \((j = 5, 7, 9, 10, 12, 13)\). \( E_j, F_j \) are replaced by \( E_j', F_j' \) as in \((20)\).

**Case \( i = 4 \).** This is treated as Case \( "i = 3" \) but with roles of \( F, F^\perp \) interchanged.

**Case \( i = 5 \).** We have \( E_5 = F_5 \oplus F_5^\perp \); let \( F_5 = D \oplus C, F_5^\perp = D' \oplus C' \) be metabolic decompositions. Fix some symplectic basis \((d_i, c_i)_{i \in I}\) in \( D \oplus C \). The procedure is to replace one hyperbolic plane \( k(d, c) \subseteq F_5 \) at a time by \( k(d, \lambda c + x) \), \( x \in F_5 \subseteq E_5 \), where \( j \in \{7, 9, 12, 13\} \). Since \( F_5^\perp + E^* = F_5^\perp + E^* \) (i.e., \( \| F_5 \| = \| F_5^\perp \| \)) there is a corresponding plane \( k(d', c') \subseteq F_5^\perp \cap E_5 \) with \( \| c' \| = \| c \| \) (use a corresponding symplectic basis \((d_i, c_i)_{i \in I}\) with \( \| c' \| = \| c \| \) for all \( i \in I \) in \( D' \oplus C' \)). This plane \( k(d', c') \) has to be replaced, in the same step, by \( k(d', \lambda c' + x') \) where \( x' \in F_5^\perp \cap E_5 \) satisfies \( \| x' \| = \| x \| \). The latter condition can be met because \( \| x \| \in ||F^\perp \cap F^\perp\| \) by assumption. We have \( E_5 = F_5 \oplus F_5^\perp \) \((j = 7, 9, 12, 13)\). The rigid summands \( F_j^\perp, F_j^\perp \) are replaced by the spaces \( F_j^\perp := \{z - \Phi(z, x)\lambda^{-1} d | z \in F_j\}, F_j^\perp := \{z - \Phi(z, x')\lambda^{-1} d | z \in F_j^\perp\} \). The space \( E_j^\perp := F_j^\perp \oplus F_j^\perp \) is again of isotype \( j \). The procedure yields a new decomposition of the pair \((E, F)\): We now have another summand \((E_5, F_5)\) of isotype 5 with varied \( \| E_5 \| \). By a repetition of the steps we may change \( \| E_5 \| \) modulo \( \| E_7 \| + \| E_9 \| + \| E_{12} \| + \| E_{13} \| \) any way we like.

IX. PROOF OF THEOREM III.3

1. The objects listed in the theorem are invariants. \( M_{13} \) is given by a representative \((x_\eta) \in GL_3(k)\) as follows. Fix bases \((e_j)_{j \in J}, (e_j')_{j \in J}\) of \( E_{13}^*, E_{13}^\perp \), respectively. Then \( F_{13} \) has a unique basis of the kind

\[
  f_j := \sum_{j} \alpha_{ij}e_j' + e_j'.
\]

Since \( E_{13}^\perp \) is an invariant subspace under each \( \phi \in \mathcal{O}(E) \) and since \( E_{13}^\perp \) is left pointwise fixed under each \( \phi \in \mathcal{O}(E) \) it is clear that the cosets of \( GL_3(k)/\mathcal{O}(E_{13}^\perp) \) are in one–one correspondence with the isometry classes of \((E, F), E \) fixed. The remaining invariances are by Lemma III.2.

2. The relations listed are obvious.

3. Let \((E, F)\) and \((E', F')\) be pairs with equal invariants. By Lemma III.4 we can choose a decomposition of \((E, F)\) that has \( \| E_i \| = S_i := \| E'_i \| \) for \( i = 1, 3, 4, 5 \) and some previously fixed decomposition of \((E', F')\). We then
have \((E_i, F_i) \cong (E'_i, F'_i), i = 1, 3, 4, 5\). Hence \((E_i, F_i) \cong (E'_i, F'_i)\) for all \(i\) by Lemma III.2. Hence \((E, F) \cong (E', F')\) by joining isometries of the summands.

4. It remains to show that the relations listed are all relations. Let \(A, A_6, A_7, A_{13}\) be arbitrary non-degenerate trace-valued spaces; let \(B_7, B_9, B_{10}, B_{11}, B_{12}, B_{13}\) be arbitrary rigid spaces (all spaces for fixed \((k, - , \epsilon))\). Let furthermore \(e_2 \in \mathbb{N}, M \in GL_r(k), r := \dim B_{13}, \) and, finally, \(G, H, K\) subspaces in the \(k\)-vector space \(S/T\). Assume that the following relations are satisfied:

(i) The sum of the \(\|B_j\|\) is a direct sum in \(S/T;\) (jj) \(e_2 \equiv 0 \mod 2, \dim A_{13} = \dim B_{13};\) (jjj) \(\|B_7\| + \|B_9\| + \|B_{12}\| + \|B_{13}\| \subseteq H \cap K \subseteq G, \|B_{10}\| \subseteq H, \|B_{11}\| \subseteq K, \|B_{10}\| \cap K = (0), \|B_{11}\| \cap H = (0).\)

We shall now define pairs \((E_i, F_i)\) where \(1 \leq i \leq 13\).

\(E_2 \) := orthogonal sum of \(\frac{1}{2}e_2\), hyperbolic planes; \(F_2\) is some fixed maximal totally isotropic subspace in \(E_2\).

\((E_6, F_6) := (A_6, (0)).\)
\((E_8, F_8) := (A_8, A_6).\)

\(E_7 := B_2 \oplus \frac{1}{2} \bigoplus_{i \in J} k(r_i, r'_i), \text{ card } J = \dim B_7, \text{ and } k(r_i, r'_i) \text{ hyperbolic planes}; F_7 := \sum_j k(r'_j + b_j), (b_j)_j \text{ some orthogonal basis of } B_7.\)

\(E_9 := B_9 \oplus \frac{1}{2} \bigoplus_{i \in J} k(r_i, r'_i), \text{ card } J = \dim B_9, \text{ and } k(r_i, r'_i) \text{ hyperbolic planes}; F_9 := \sum_j (k(r_j + k(r'_j + b_j)), (b_j)_j \text{ an orthogonal basis of } B_9.\)

\((E_{10}, F_{10}) := (B_{10}, B_{10}).\)
\((E_{11}, E_{11}) := (B_{11}, (0)).\)

\((E_{12}, B_{12}) := B_{12} \oplus \bigoplus_{i \in J} k(d_i, b_i), (b_i)_i \text{ some orthogonal basis of } B_{12}; F_{12} := B_{12}.\)

\((E_{13}, A_{13} \oplus B_{13} \oplus A_{13}). \text{ Pick some bases } (a_j)_j, (b_j)_j \text{ of } A_{13}, B_{13} \text{ respectively and } F_{13} := \sum_j k(f_j) \text{ where } f_j := \sum_i a_i a_i + b_i.\)

It is routine to verify that the pairs \((E_i, F_i)\) defined here for \(i = 2, 6, 7, 9, 10, 11, 12, 13\) are of isotype \(i\). Define furthermore subspaces in \(S/T\) as follows (cf. (2)):\\
\(W_1 := H + K, \quad W_3 := (H \cap K) + \|B_{10}\|\)
\(W_4 := (H \cap K) + \|B_{11}\|\)
\(W_5 := \|B_7\| + \|B_9\| + \|B_{12}\| + \|B_{13}\|.\)

Pick any subspaces \(S_i \subseteq S/T\) such that \(S_1 \oplus W_1 = G, S_3 \oplus W_3 = H, S_4 \oplus W_4 = K, S_5 \oplus W_5 = H \cap K\). Define metabolic spaces

\(E_i := D_1 \oplus C_i \text{ with } \|C_i\| = S_i \text{ (i = 1, 3, 4); } F_1 := D_1, F_3 := E_3, F_4 := (0).\)

\(E_5 := \bigoplus_{i} (k(r_i, r'_i) \oplus k(d_i, c_i)), \text{ card } J = \dim S_5, k(r_i, r'_i) \text{ hyperbolic and } k(d_i, c_i) \text{ metabolic planes with } \|\sum_j k(c_j)\| = S_5, F_5 := \sum_j k(r_j + r'_j + c_i).\)
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It is easy to see that the pairs \((E_i, F_i)\) \((i = 1, 3, 4, 5)\) are of isotype \(i\). Let \(E := \bigoplus_{i=1}^{13} E_i\) (external orthogonal sum) and \(F := \sum_{i=1}^{13} F_i\). Hence \(F_i = E_i \cap F\). Furthermore, it follows from the construction that \(\|E_i\| = \|B_i\|\) \((i = 7, 9, 10, 11, 12, 13)\), \(\|E_j\| = S_j\) \((j = 1, 3, 4, 5)\). By the assumptions (j) and (jjj) on \(\|B_i\|, G, H, K\), and the choice of the \(S_j\) \((j = 1, 3, 4, 5)\) it follows that the sum of all spaces \(\|E_i\|\) \((1 \leq i \leq 13)\) is a direct sum in \(S/T\). Therefore \(E^* = \bigoplus_{i=1}^{13} (E_i \cap E^*)\). Thus we have a decomposition of \((E, F)\): \((E, F) = \bigoplus_{i=1}^{13} (E_i, F_i)\).

If we compute, from this decomposition, the invariants of Theorem III.3 then we find \(E_6 = A_6, E_8 = A_8, E_{13} = A_{13}, E_{13}^\perp = B_7, E_9^\perp = B_9, E_{10} = B_{10}, E_{11} = B_{11}, E_{12} = B_{12}, E_{13}^\perp = B_{13}, \text{dim } E_2 = e_2, M_{13} - M, \|E\| - G, \|F\| - H, \|F^\perp\| = K\).

Hence there exists, for each set of objects satisfying the relations, a pair \((E, F)\) with these objects as the invariants attached to \((E, F)\).

X. PROOF OF THE EXTENSION THEOREM IV.1

1. By Corollary I.5 condition (5) is clearly necessary for an extension of \(\varphi_0 : F \rightarrow \tilde{F}\) to exist. So we now assume (5). This enables us in a first step, to extend \(\varphi_0\) to \(\varphi_1 : F + E_{*}^\perp \rightarrow \tilde{F} + E_{*}^\perp\) simply by setting \(\varphi_0 \upharpoonright E_{*}^\perp = \text{id}\). We see that it suffices to prove the Extension Theorem in the special case

\[
E_{*}^\perp \subset F, \quad E_{*}^\perp \subset \tilde{F}.
\]

Therefore, if we consider decompositions (1) for \((E, F)\), \((E, \tilde{F})\),

\[
(E, F) = \sum_{i \in I} (E_i, F_i), \quad (E, \tilde{F}) = \sum_{i \in \tilde{I}} (\tilde{E}_i, \tilde{F}_i)
\]

we have \(I \cup \tilde{I} \subset \{1, 2, 3, 6, 8, 10\}\). Our aim is to chop off the pairs of isotypes 10, 3, 1 (in this order of succession).

2. Since \(\|F\| = \|\tilde{F}\|\) we have (I.4(iii)) \(F^\perp \cap E_{*}^\perp = \tilde{F}^\perp \cap E_{*}^\perp\); hence by (5), \(F_1 = F \cap F^\perp \cap E_{*}^\perp = \tilde{F} \cap \tilde{F}^\perp \cap E_{*}^\perp\). Let \(D_3\) be a fixed complement of \(F_1\) in \(E_{*}^\perp \times \text{and } A_{10}\) a fixed complement of \(E_{*}^\perp \times \text{in } E_{*}^\perp\). We have \(E = A_{10} \oplus A_{10}^\perp\) and \(F = A_{10} \oplus F \cap A_{10}^\perp = \tilde{F} \oplus \tilde{F} \cap A_{10}^\perp\). As \(\varphi_0 \upharpoonright A_{10} = \text{id}\) we have \(\varphi_0 (F \cap A_{10}^\perp) = \tilde{F} \cap A_{10}^\perp\). In other words, without loss of generality

\[
E_{*}^\perp \subset E^* \quad \text{so } I \cup \tilde{I} \subset \{1, 2, 3, 6, 8\}.
\]

3. Let \(C_3\) be a supplement of \(D_3^\perp \cap F\) in \(F\). The pairs \((D_3 \oplus C_3, D_3 \oplus C_3), (D_3 \oplus \varphi_0 C_3, D_3 \oplus \varphi_0 C_3)\) are isometric, orthogonal summands of \((E, F), (E, \tilde{F})\), respectively, and they may be cancelled by virtue of Lemma VI.2: without loss of generality \(3 \notin I \cup \tilde{I}\).
4. At this stage we have \( F = F \cap E^{\perp} = \tilde{F} \). Let \( F_0 \) be a supplement of \( F \) in \( F \cap E^{\perp} \): Because \( F_1 \cap \varphi_0 F_0 = (0) \) there is a supplement \( C_1 \) of \( F_1 \) in \( E \) with \( C_1 \perp \varphi_0 F \). Chopping off the pairs \((E_1, F_1)\) and \((F_1 \oplus C_1, F_1)\) brings us into the situation where (23) has \( I \subseteq \{2, 6, 8\} \), i.e., \( E = E^* \) and \( \varphi_0 \) extends by Witt's Theorem.

XI. PROOF OF THE CONGRUENCE THEOREM IV.2

1. We assume that the condition
\[
F \cong \tilde{F} \quad \text{and} \quad F \cap E^{\perp} = \tilde{F} \cap E^{\perp}
\] (24)
holds; by Corollary I.5, (24) is clearly necessary for \( F \) and \( \tilde{F} \) to lie in the same orbit under \( \partial(E) \). We reduce our problem to the situation where, in addition to (24), we shall have
\[
F \cap F^\perp = (0) = \tilde{F} \cap F^\perp.
\] (25)
By (25) we shall be in the position to quote Corollary I.5: each isometry \( F \to \tilde{F} \) satisfies the assumptions in the Extension Theorem and may therefore be extended to all of \( E \).

2. From (24) we have \( \|F\| = \|\tilde{F}\| \) so \( F^\perp \cap E^{*\perp} = \tilde{F}^\perp \cap E^{*\perp} \) and hence \( D_1 := F \cap F^\perp \cap E^{*\perp} = \tilde{F} \cap \tilde{F}^\perp \cap E^{*\perp} \). If \( C_1 \) is any complement of \( D_1 \) in \( E \) then the pair \((D_1 \oplus C_1, D_1)\) is an orthogonal summand in both \((E, F)\) and \((E, \tilde{F})\): \( F = D_1 \oplus F \cap (D_1 \oplus C_1)^\perp, \tilde{F} = D_1 \oplus \tilde{F} \cap (D_1 \oplus C_1)^\perp \). Thus without loss of generality \( D_1 = (0) \) in addition to (24), i.e.,
\[
(F \cap F^\perp) \cap E^{*\perp} = (0) = (\tilde{F} \cap \tilde{F}^\perp) \cap E^{*\perp}.
\] (26)
Set \( R := F \cap F^\perp \) and \( \tilde{R} := \tilde{F} \cap \tilde{F}^\perp \). By (26) there are hyperbolic spaces \( H := R \oplus R^\perp, \tilde{H} := \tilde{R} \oplus \tilde{R}^\perp \subset E \); hence the pairs \((H, R), (\tilde{H}, \tilde{R})\) are orthogonal summands of \((E, F), (E, \tilde{F})\), respectively. As \( \dim R = \dim \tilde{R} \) by (24) these two summands are isometric and they may be cancelled by virtue of Lemma VI.3. We now have (25).

XII. PROOF OF COROLLARY IV.2

Assume that \((E, F) \cong (E, \tilde{F})\). To say that each isometry \( F \to \tilde{F} \) can be extended to \( E \) is equivalent to saying that each isometric automorphism \( \varphi_0 : F \to F \) can be extended to all of \( E \). Thus, let us assume that \( \varphi_0 \) admits an extension to \( E \). We shall base our reasonings on a decomposition (1). By the Extension Theorem each \( \varphi_0 \) must be the identity on \( F \cap E^{*\perp} \).
Hence, if $F_1 = F \cap F_1 \perp E_* \perp \neq (0)$ we must have $k = \mathbb{Z}_2$ and $E_1, F_1, F_1^*$ as specified by the corollary (in order to exclude maps $\varphi_0$ with $\varphi_0 \uparrow F_1 = \lambda_1, \lambda \neq 1$, etc.). In particular, $\dim S/T = 1$ so that $(E, F)$ has summands of isotypes 1, 2, 6, 8 only (cf. the relations in Lemma III.4 and Theorem III.3).

Yet we have $E_2 = (0)$ for otherwise there are $\varphi_0$ with $\varphi_0 F_1 \subset (F_1 + F_2) \setminus F_1$, contradicting the Extension Theorem. We have proved that "$F_1 \neq (0)$" implies that we are in Case (3) of the Corollary.

We are left with the possibility $F_1 = (0) = F_1^*$. So $F_3^* \neq (0)$ or $F_{10} \neq (0)$ (or else, we are in Case (2) of the Corollary). If we had $F_2 = (0)$ we could define $\varphi_0$ with $\varphi_0 F_3 \subset (F_3^* + F_2) \setminus F_3^*$ or $\varphi_0 F_{10} \subset (F_{10} + F_2) \setminus F_{10}$. Hence $F_2 = (0)$ and we are in Case (1) of the Corollary.

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