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Parabolic Perturbation of a Nonlinear Hyperbolic Problem Arising in Physiology

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We study a transport-diffusion initial value problem where the diffusion coefficient is "small" and the transport coefficient is a time function depending on the solution in a nonlinear and nonlocal way. We show the existence and the uniqueness of a weak solution of this problem. Moreover we discuss its asymptotic behaviour as the diffusion coefficient goes to zero, obtaining a well-posed first-order nonlinear hyperbolic problem. These problems arise from mathematical models of muscle contraction in the framework of the sliding filament theory. © 1993 Academic Press, Inc.

1. INTRODUCTION

The aim of this paper is the study of a nonlinear and nonlocal *transport-diffusion* initial value problem with a *small* diffusion coefficient $\varepsilon > 0$ and the analysis of the *asymptotic behaviour* as ε goes to 0.

This research originated from the following first-order hyperbolic problem. Find $u: \mathbf{R} \times [0, T] \to \mathbf{R}$ and $z: [0, T] \to \mathbf{R}$ satisfying

$$u_t(x, t) + z'(t)\, u_x(x, t) + G(x, t, z(t))\, u(x, t) = F(x, t, z(t)) \qquad \text{for a.e.} \quad (x, t) \in \mathbf{R} \times]0, T[, \tag{1.1}$$

$$u(x, 0) = u_0(x) \qquad \text{for a.e.} \quad x \in \mathbf{R}, \tag{1.2}$$

$$z(t) = L\left(\int_{\mathbf{R}} w(x)\, u(x, t)\, dx\right) \qquad \forall t \in [0, T], \tag{1.3}$$

where $T > 0$, F, G, u_0 are given non-negative *smooth* functions, w is a non-decreasing *weight* function, and L is strictly decreasing. Other suitable assumptions on L will be specified later.

Various applications are concerned with problems of type (1.1)–(1.3).

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For instance, in physiology we refer to rheological models simulating the *cross-bridge dynamics* in *muscle contraction* (see [5, 12, 14, 15, 20, 22, 23] and references therein) according to the *sliding filament theory* due to Huxley [16, 17]. In fact, muscle can be seen as an array of *sarcomeres* (repeating unit in a fibre), where each sarcomere is composed by particles of *myosin* (thick filament) and *actin* (thin filament) [20, 22]. The muscular force is a consequence of the interactions between these filaments. Such interactions are named *cross-bridges* [8–12, 14, 15]. Once a cross-bridge is activated, it links myosin and actin and the force generated depends on the distance of attachment x. The quantity $u(x, t)$ essentially represents a density of cross-bridges attached at distance x and time t. The function z is the *contractile movement* of filaments and it is related to the *contractile force* $\int_{\mathbf{R}} w(x)\, u(x, t)\, dx$ (cf. (1.3)). The data F and G are proportional to the *attachment* and *detachment rate functions* (cf. [16, 8]).

As far as the mathematical viewpoint is concerned, Problem (1.1)–(1.3) has been studied by various authors (see, e.g., [3, 8, 10, 13, 4, 6]), who prove existence and/or uniqueness results using different approaches. In the quoted papers a common assumption states that the functions u_0 and F have a compact support in $\mathbf{R}$ for any time. Therefore the solution u has a compact support too. Actually, in the mentioned physiological application cross-bridges cannot attach at large distances and consequently u must vanish outside a bounded set.

However, the model problem (1.1)–(1.3) does not take into account other physiological phenomena involved in the cross-bridge mechanism. In particular some *slipping* effects (see, e.g., [15, 21]) are neglected. An attempt to include these phenomena in the model has been made in [7] introducing a *viscosity* term $-\varepsilon u_{xx}$ in Eq. (1.1) and studying it in a bounded domain with homogeneous Dirichlet boundary conditions. Nevertheless the investigation of this problem involves some difficulties concerning the global existence of a solution for a given $T > 0$ and its behaviour as $\varepsilon \to 0$ (see [7]). Such difficulties seem to be essentially related to the fact that in general the sign of $z'(t)$ may change. Hence, when ε is small, boundary layers occur, but it is not easy to know where and when. As a consequence, when ε goes to 0, we do not know which part of the boundary conditions is preserved.

The last considerations, along with the necessity of finding a satisfactory connection between the hyperbolic problem and its perturbation, addressed us to study the latter in the whole strip. More precisely, we consider the following transport-diffusion problem. Look for $u\colon \mathbf{R} \times [0, T] \to \mathbf{R}$ and $z\colon [0, T] \to \mathbf{R}$ satisfying (1.2)–(1.3) and

$$
\begin{aligned}
&u_t(x, t) - \varepsilon u_{xx}(x, t) + z'(t)\, u_x(x, t) + G(x, t, z(t))\, u(x, t) \\
&\quad = F(x, t, z(t)) \qquad \text{for a.e.} \quad (x, t) \in \mathbf{R} \times \left]0, T\right[. \qquad (1.4)
\end{aligned}
$$

Due to the presence of the diffusion term, any possible solution u of this problem does not maintain a compact support. Therefore, we study Problem (1.2)–(1.4) only assuming suitable decay conditions on u_0 and F. Moreover we are allowed to take the data u_0, F, G less regular than in [3, 8, 10, 13, 4, 6], though some restrictions on the growth of G are required.

Here we show the existence and the uniqueness of a weak solution of (1.2)–(1.4) using the Contraction Mapping Principle step by step in time. Then we pass to the limit in this problem as ε goes to 0. Provided u_0, F, G satisfy some Lipschitz continuity assumptions, we obtain a solution of (1.1)–(1.3). We also prove the uniqueness of this solution by means of the Gronwall Lemma. Therefore we show a new existence and uniqueness result for Problem (1.1)–(1.3), different from the one presented in, e.g., [4], since here we only prescribe both decay and growth conditions on u_0, F, G (no compact support is assumed). On the other hand, we also show that Problem (1.1)–(1.3) in the formulation of, e.g., [4], can be approximated by a transport-diffusion problem of type (1.2)–(1.4) choosing suitably the function G_ε approximating G.

The outline of the paper is as follows. Section 2 contains a precise formulation of the transport-diffusion problem and the statement of the existence and uniqueness result. In Section 3 we introduce equivalent formulations and we prove estimates on the solution of (1.2)–(1.4) when z is known. In Section 4 we show the *a priori* boundedness of z and we prepare some estimates on the contracting behaviour in time. In Section 5 we prove the existence of a unique global solution of Problem (1.2)–(1.4) using the Contraction Mapping Principle step by step in time. Section 6 is devoted to the study of asymptotic behaviour of the transport-diffusion problem as $\varepsilon \to 0$ and, consequently, to get an existence and uniqueness result for Problem (1.1)–(1.3). Finally, in Section 7 we show that the hyperbolic problem (1.1)–(1.3) in the formulation of [4] (the *muscle contraction problem*) can be approximated by a suitable transport-diffusion problem of type (1.2)–(1.4).

2. Setting of the Transport-Diffusion Problem and of the Existence-Uniqueness Result

Here we give a precise formulation of the transport-diffusion problem (1.2)–(1.4) and we state an existence and uniqueness result. First let us introduce the following assumptions on the data.

$$T > 0, \qquad a, b \in \mathbf{R} \cup \{\pm\infty\}, \qquad a < 0 < b; \tag{2.1}$$

$L\colon]a, b[\to \mathbf{R}$ is a continuous strictly decreasing mapping with inverse function $\lambda\colon \mathbf{R} \to]a, b[$ satisfying $\lambda(0) = 0$, $\lambda(\eta) \searrow a$ (resp. $\nearrow b$) as $\eta \nearrow +\infty$ (resp. $\searrow -\infty$); (2.2) for any $R>0$ there is a constant $M_1(R)>0$ such that for any $\eta_1, \eta_2 \in [-R, R]$ one has $|\eta_1-\eta_2| \leqslant M_1(R)\,|\lambda(\eta_1)-\lambda(\eta_2)|$; (2.3)

$w \in C^1(\mathbf{R})$ is an increasing function such that

$$w(0)=0 \quad \text{and} \quad \frac{dw}{dx} \in W^{1,\infty}(\mathbf{R}); \tag{2.4}$$

$F, G \in L^{\infty}_{loc}(\mathbf{R}\times]0, T[\times \mathbf{R})$ satisfy $0 \leqslant F \leqslant G$ a.e. in $\mathbf{R}\times]0, T[\times\mathbf{R}$ and $F(\cdot, t, \cdot)$, $G(\cdot, t, \cdot) \in C^0(\mathbf{R}\times\mathbf{R})$ for a.e. $t \in]0, T[$; (2.5)

for any $R>0$ there is a constant $M_2(R)>0$ such that for a.e. $(x, t, \xi) \in \mathbf{R}\times]0, T[\times]-R, R[$ one has

$$\int_{\mathbf{R}} (1+|y|)\, F(y, t, \xi)\, dy \leqslant M_2(R),\quad G(x, t, \xi) \leqslant M_2(R)(1+|x|); \tag{2.6}$$

for any $R>0$ there are a number $q>1$ and a function $h \in L^q(0, T)$ such that for a.e. (x_1, t, ξ_1), $(x_2, t, \xi_2) \in \mathbf{R}\times]0, T[\times]-R, R[$ one has

$$\int_{\mathbf{R}} (1+|y|)\, |F(y+\xi_1, t, \xi_1) - F(y+\xi_2, t, \xi_2)|\, dy \leqslant h(t)\, |\xi_1-\xi_2|,$$
$$|G(x_1, t, \xi_1) - G(x_2, t, \xi_2)| \leqslant h(t)\{|x_1-x_2| + |\xi_1-\xi_2|\}; \tag{2.7}$$

there exists a function $\mathscr{F} \in L^1(0, T; L^\infty(\mathbf{R}))$ such that $0 \leqslant F(x, t, \xi) \leqslant \mathscr{F}(x, t)$ for a.e. $(x, t, \xi) \in \mathbf{R}\times]0, T[\times\mathbf{R}$ and $\int_0^T \int_{\mathbf{R}} x^2 \mathscr{F}(x, t)\, dx\, dt < +\infty$; (2.8)

$$u_0 \in L^\infty(\mathbf{R}),\quad 0 \leqslant u_0 \leqslant 1 \quad \text{a.e. in } \mathbf{R},\quad \int_{\mathbf{R}} x^2 u_0(x)\, dx < +\infty; \tag{2.9}$$

$$a < \int_{\mathbf{R}} w(x)\, u_0(x)\, dx < b. \tag{2.10}$$

Remark 2.1. Observe that (2.3) is equivalent to the local Lipschitz continuity of L in $]a, b[$, as it can be easily checked. Thus the assumptions concerning L are the same of [4].

Remark 2.2. The physical meaning of the weight function w is related to the elastic law of the cross-bridges. Usually w is assumed to be linear (cf., e.g., [3, 8, 11, 16, 22]), while here we allow w to be more general. Note that the functions λ and w have to vanish at some point, but it is not restrictive to assume $\lambda(0) = w(0) = 0$.

Remark 2.3. Since the assumptions (2.6)–(2.8) on F and G look rather involved, even if it seems to us that there is not a simpler way to write them, we give below an example of possible F and G satisfying (2.5)–(2.8):

$$F(x, t, \xi) = \frac{\tanh(t^{1/2} |\xi|)}{t^{1/2}(1 + x^4)}, \qquad G(x, t, \xi) = 1 + |x| + \xi^2 + \xi \sin^2 \frac{x}{t^{1/2}}.$$

However, for some explicit choices of F, G in the applications we refer to [16, 17, 8–10].

Remark 2.4. Concerning the attachment and detachment rate functions F and G (see [16]) and the initial datum u_0, our assumptions (2.5)–(2.9) are somewhat different from those of [4]. Indeed here F, G, u_0 are not requested to be continuous anywhere and $\mathscr{F}$ (as well as u_0) has no more a compact support: we just require that $x^2 \mathscr{F}$ and $x^2 u_0$ are summable in $\mathbf{R} \times]0, T[$ and $\mathbf{R}$, respectively. On the other hand, we ask for a global Lipschitz continuity of G with respect to x (see (2.7)). These hypotheses are due to the fact that a solution u of the transport-diffusion problem does not maintain a compact support as time increases.

Remark 2.5. It is straightforward to show that from (2.8) and (2.9) it follows that $\mathscr{F} \in L^1(0, T; L^p(\mathbf{R}))$ and $u_0 \in L^p(\mathbf{R})$ for any $p \geq 1$. Besides, condition (2.10) makes sense since w is Lipschitz continuous and $\int_{\mathbf{R}} (1 + |x|) u_0(x)\, dx < +\infty$ (cf. (2.4) and (2.9)).

We now state a weak formulation of the transport-diffusion problem. Let us set $V := H^1(\mathbf{R})$, $H := L^2(\mathbf{R})$, so that $V \subset H \subset V' \equiv H^{-1}(\mathbf{R})$ with dense and continuous inclusions. Let us denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V' and V and by $(\cdot, \cdot)$ the scalar product in H.

PROBLEM (P_1). *Find* $u \in C^0([0, T]; H) \cap L^2(0, T; V)$, $z \in C^{0,1}([0, T])$ *such that* $u_t \in L^2(0, T; V') + L^1(0, T; H)$, $wu \in C^0([0, T]; L^1(\mathbf{R}))$, $G_z u \in L^2(0, T; H)$ *and satisfying*

$$a < \int_{\mathbf{R}} w(x)\, u(x, t)\, dx < b \qquad \forall t \in [0, T], \tag{2.11}$$

$$z(t) = L\left(\int_{\mathbf{R}} w(x)\, u(x, t)\, dx \right) \qquad \forall t \in [0, T], \tag{2.12}$$

$$\begin{aligned} &\langle u_t(\cdot, t), \varphi \rangle + \varepsilon(u_x(\cdot, t), \varphi_x) - z'(t)(u(\cdot, t), \varphi_x) + (G_z(\cdot, t)\, u(\cdot, t) \\ &\quad - F_z(\cdot, t), \varphi) = 0 \qquad \forall \varphi \in V, \ \textit{for a.e.} \ \ t \in]0, T[, \end{aligned} \tag{2.13}$$

$$u(x, 0) = u_0(x) \qquad \textit{for a.e.} \ \ x \in \mathbf{R}, \tag{2.14}$$

where $z' = dz/dt$ *and*

$$F_z(x, t) := F(x, t, z(t)), \qquad G_z(x, t) := G(x, t, z(t))$$
$$\textit{for a.e.} \quad (x, t) \in \mathbf{R} \times]0, T[. \tag{2.15}$$

Remark 2.6. One can easily check that (2.11)–(2.14) have a meaning. For (2.14) note that both $u(\cdot, 0)$ and u_0 belong to $L^2(\mathbf{R})$. Since $u_t \in L^2(0, T; V') + L^1(0, T; H)$, then it can be expressed as $u_t = \bar{u}_1 + \bar{u}_2$, where $\bar{u}_1 \in L^2(0, T; V')$ and $\bar{u}_2 \in L^1(0, T; H)$. Besides, the duality pairing in (2.13) has to be understood as

$$\langle u_t(\cdot, t), \varphi \rangle = \langle \bar{u}_1(\cdot, t), \varphi \rangle + (\bar{u}_2(\cdot, t), \varphi)$$

for a.e. $t \in]0, T[$. Finally, observe that (2.13) yields

$$u_t - \varepsilon u_{xx} + z' u_x + G_z u - F_z = 0 \qquad \text{in} \quad \mathscr{D}'(\mathbf{R} \times]0, T[). \tag{2.16}$$

Indeed it suffices to take in (2.13) any $\varphi = \varphi(x, t) \in \mathscr{D}(\mathbf{R} \times]0, T[)$, integrate by parts in space, and integrate in time from 0 to T in order to get (2.16).

Our main result for the transport-diffusion problem is the following.

THEOREM 2.1. *Under the assumptions* (2.1)–(2.10), *there exists one and only one solution* (u, z) *of Problem* (P_1). *Moreover*

$$0 \leqslant u \leqslant 1 \qquad \textit{a.e. in} \quad \mathbf{R} \times]0, T[. \tag{2.17}$$

We prove this theorem in Sections 3, 4, and 5. Finally, since it will be useful in the sequel, let us introduce the notations

$$\|\zeta\|_t := \|\zeta\|_{C^0([0, t])} \qquad \forall t \in [0, T], \quad \zeta \in C^0([0, T]), \tag{2.18}$$

$$z_0 := L\left(\int_{\mathbf{R}} w(x)\, u_0(x)\, dx\right). \tag{2.19}$$

Note that z_0 is well defined owing to (2.10) and (2.2).

3. EQUIVALENT PROBLEMS

In this section we study equivalent formulations of Problem (P_1) and we prove some preliminary results. The following problem in an equivalent version of (P_1) where the time derivative of z does not appear into the equation corresponding to (2.13).

PROBLEM (P_2). *Find* $v \in C^0([0, T]; H) \cap L^2(0, T; V)$, $z \in C^{0,1}([0, T])$ *such that* $v_t \in L^2(0, T; V') + L^1(0, T; H)$, $w_z^* v \in C^0([0, T]; L^1(\mathbf{R}))$, $G_z^* v \in L^2(0, T; H)$ *and satisfying*

$$a < \int_{\mathbf{R}} w(x + z(t))\, v(x, t)\, dx < b \qquad \forall t \in [0, T], \tag{3.1}$$

$$z(t) = L\left(\int_{\mathbf{R}} w(x + z(t))\, v(x, t)\, dx\right) \qquad \forall t \in [0, T], \tag{3.2}$$

$$\begin{aligned} &\langle v_t(\cdot, t), \psi \rangle + \varepsilon(v_x(\cdot, t), \psi_x) + (G_z^*(\cdot, t)\, v(\cdot, t) \\ &\quad - F_z^*(\cdot, t), \psi) = 0 \qquad \forall \psi \in V, \qquad \textit{for a.e.} \quad t \in]0, T[, \end{aligned} \tag{3.3}$$

$$v(x, 0) = u_0(x + z_0) \qquad \textit{for a.e.} \quad x \in \mathbf{R}, \tag{3.4}$$

where

$$\begin{aligned} &w_z^*(x, t) := w(x + z(t)), \qquad F_z^*(x, t) := F(x + z(t), t, z(t)), \\ &G_z^*(x, t) := G(x + z(t), t, z(t)) \qquad \textit{for a.e.} \quad (x, t) \in \mathbf{R} \times]0, T[. \end{aligned} \tag{3.5}$$

PROPOSITION 3.1. *Problems* (P_1) *and* (P_2) *are equivalent in the following sense: if the pair* (u, z) *solves* (P_1), *then the pair* (v, z), *with* $v(x, t) := u(x + z(t), t)$ *for a.e.* $(x, t) \in \mathbf{R} \times]0, T[$, *solves* ($P_2$); *conversely if* (v, z) *solves* (P_2), *then* (u, z), *with* $u(y, t) := v(y - z(t), t)$ *for a.e.* $(y, t) \in \mathbf{R} \times]0, T[$, *solves* ($P_1$).

Proof. We only show the first part of the statement since the converse is quite analogous. Let (u, z) be a solution of (P_1). We set

$$v(x, t) := u(x + z(t), t) \qquad \text{for a.e.} \quad (x, t) \in \mathbf{R} \times]0, T[.$$

By changing variables in the integrals, it is easy to see that $w_z^* v \in C^0([0, T]; L^1(\mathbf{R}))$, $G_z^* v \in L^2(0, T; H)$, $\varepsilon v_{xx} \in L^2(0, T; V')$, and (3.1)–(3.2) hold. Let now ψ be an arbitrary function of $\mathscr{D}(\mathbf{R} \times]0, T[)$: then we can take

$$\varphi(x, t) = \psi(x - z(t), t) \qquad \forall (x, t) \in \mathbf{R} \times [0, T]$$

as a test function in (2.13) since $\varphi \in C^{0,1}([0, T]; V)$. Besides, φ is such that $\varphi(\cdot, 0) = \varphi(\cdot, T) = 0$. Integrating (2.13) with respect to time from 0 to T and taking into account that

$$\varphi_t(x, t) = -\psi_x(x - z(t), t)\, z'(t) + \psi_t(x - z(t), t)$$

for a.e. $(x, t) \in \mathbf{R} \times]0, T[$, the integration by parts of the first term gives

$$-\int_0^T \int_{\mathbf{R}} u(x, t)\, \psi_t(x - z(t), t)\, dx\, dt + \varepsilon \int_0^T \int_{\mathbf{R}} u_x(x, t)\, \psi_x(x - z(t), t)\, dx\, dt$$

$$+ \int_0^T \int_{\mathbf{R}} (G_z(x, t)\, u(x, t) - F_z(x, t))\, \psi(x - z(t), t)\, dx\, dt = 0$$

where F_z, G_z are defined by (2.15). Replacing x by $x + z(t)$ in the integrals above, a further integration by parts yields

$$v_t = \varepsilon v_{xx} - G_z^* v + F_z^* \qquad \text{in} \quad \mathscr{D}'(\mathbf{R} \times]0, T[),$$

as ψ is arbitrary in $\mathscr{D}(\mathbf{R} \times]0, T[)$. But, since the right hand side belongs to $L^2(0, T; V') + L^1(0, T; H)$, v_t has the required regularity and (3.3) is satisfied. Next, by using (2.12), (2.14), and (2.19), it is easy to see that v fulfils (3.4).

Remark 3.1. Observe that Problem (P_2) has a meaning even if we only ask for $z \in C^0([0, T])$, while that is no longer true for Problem (P_1).

In order to find other equivalent formulations, first we study the problem (3.3)–(3.4) when z is only continuous and given. The next technical result is concerned with the existence and uniqueness of the solution.

LEMMA 3.1. *For any given function $z \in C^0([0, T])$ there exists a unique $v_z \in C^0([0, T]; H) \cap L^2(0, T; V)$ satisfying*

$$\frac{\partial v_z}{\partial t} \in L^2(0, T; V') + L^1(0, T; H), \qquad G_z^* v_z \in L^2(0, T; H)$$

and such that

$$\left\langle \frac{\partial v_z}{\partial t}(\cdot, t), \psi \right\rangle + \varepsilon \left(\frac{\partial v_z}{\partial x}(\cdot, t), \psi_x \right) + ((G_z^* v_z - F_z^*)(\cdot, t), \psi) = 0$$

$$\forall \psi \in V, \quad \textit{for a.e.} \quad t \in]0, T[, \tag{3.6}$$

$$v_z(x, 0) = u_0(x + z(0)) \qquad \textit{for a.e.} \quad x \in \mathbf{R}, \tag{3.7}$$

where F_z^* and G_z^* are defined in (3.5). *Moreover*

$$0 \leqslant v_z \leqslant 1 \qquad \textit{a.e. in} \quad \mathbf{R} \times]0, T[\tag{3.8}$$

and there exists a constant $C_1 > 0$, independent of z and ε, such that

$$\|v_z\|^2_{C^0([0, T]; H)} + \varepsilon \left\| \frac{\partial v_z}{\partial x} \right\|^2_{L^2(\mathbf{R} \times]0, T[)} + \|G_z^* v_z^2\|_{L^1(\mathbf{R} \times]0, T[)} \leqslant C_1, \tag{3.9}$$

$$\|v_z\|_{L^\infty(0, T; L^1(\mathbf{R}))} + \|G_z^* v_z\|_{L^1(\mathbf{R} \times]0, T[)} \leqslant C_1. \tag{3.10}$$

Proof. Since it is rather long and technical, the proof is split into five steps.

Step 1 (approximation). We regularize the functions F_z^* and G_z^* by

$$F_n := \min\{F_z^*, n\}, \quad G_n := \min\{G_z^*, n\} \text{ a.e. in } \mathbf{R}\times]0, T[, \quad n\in\mathbf{N}, \tag{3.11}$$

and the initial datum $u_0(\cdot + z(0))$ by a sequence $\{v_0^n\}_{n\in\mathbf{N}}$ such that

$$v_0^n \in V, \qquad 0 \leqslant v_0^n \leqslant 1 \qquad \text{a.e. in } \mathbf{R}, \tag{3.12}$$

the sequence $\{v_0^n\}$ is bounded (independently of n and z) in $L^p(\mathbf{R})$ for any $p\in[1, +\infty]$ and

$$\exists C_2 > 0\colon \int_{\mathbf{R}} x^2 v_0^n(x)\, dx \leqslant C_2 \qquad \forall n\in\mathbf{N}, \tag{3.13}$$

$$v_0^n \to u_0(\cdot + z(0)) \qquad \text{strongly in } H. \tag{3.14}$$

Owing to (2.9) (see also Remark 2.5), it is easy to check that such a sequence $\{v_0^n\}$ exists: for instance, one can take $v_0^n(x) = \int_{\mathbf{R}} u_0(y + z(0)) \exp\{-(n+4)(x-y)^2\}\, dy$. Concerning F_n and G_n, since $z\in C^0([0, T])$, from (3.11), (3.5), (2.5), (2.8) (one can see Remark 2.5 too) it follows that

$$F_n \in L^2(\mathbf{R}\times]0, T[), \qquad G_n \in L^\infty(\mathbf{R}\times]0, T[), \tag{3.15}$$

$$0 \leqslant F_n \leqslant G_n \qquad \text{a.e. in} \quad \mathbf{R}\times]0, T[, \tag{3.16}$$

the sequence $\{F_n\}$ is bounded (independently of n and z) in $L^1(0, T; L^p(\mathbf{R}))$ for any $p\in[1, +\infty]$ and

$$\exists C_3 > 0\colon \int_0^T \int_{\mathbf{R}} x^2 F_n(x, t)\, dx\, dt \leqslant C_3 \qquad \forall n\in\mathbf{N}. \tag{3.17}$$

Let now $v_n \in C^0([0, T]; V) \cap L^2(0, T; H^2(\mathbf{R})) \cap H^1(0, T; H)$ be the unique solution of the following problem (cf., e.g., [19])

$$\frac{\partial v_n}{\partial t} - \varepsilon \frac{\partial^2 v_n}{\partial x^2} + G_n v_n = F_n \qquad \text{a.e. in} \quad \mathbf{R}\times]0, T[, \tag{3.18}$$

$$v_n(x, 0) = v_0^n(x) \qquad \forall x\in\mathbf{R}. \tag{3.19}$$

Step 2 (*a priori* estimates). Multiplying (3.18) by v_n, then integrating by parts in space and time, we get

$$\tfrac{1}{2}\|v_n(\cdot, t)\|_H^2 + \varepsilon \left\| \frac{\partial v_n}{\partial x} \right\|^2_{L^2(\mathbf{R}\times]0, t[)} + \|G_n v_n^2\|_{L^1(\mathbf{R}\times]0, t[)}$$

$$\leqslant \tfrac{1}{2}\|v_0^n\|_H^2 + \int_0^t \|F_n(\cdot, \tau)\|_H \|v_n(\cdot, \tau)\|_H\, d\tau \qquad \forall t\in[0, T].$$

Using now (3.13), (3.17) and applying the Gronwall Lemma, we have that there exists a constant C_4, independent of n, z, ε, such that

$$\|v_n\|^2_{C^0([0,T];H)} + \varepsilon \left\| \frac{\partial v_n}{\partial x} \right\|^2_{L^2(\mathbf{R}\times]0,T[)} + \|G_n v_n^2\|_{L^1(\mathbf{R}\times]0,T[)} \leqslant C_4 \qquad \forall n \in \mathbf{N}. \tag{3.20}$$

Next, we show that

$$0 \leqslant v_n \leqslant 1 \qquad \forall n \in \mathbf{N}. \tag{3.21}$$

These maximum principle estimates are obtained multiplying (3.18) by $(v_n)^-$ and $(v_n - 1)^+$ (where $(\cdot)^-$ and $(\cdot)^+$ denote the negative and positive part, respectively): note that $(v_n)^-$, $(v_n-1)^+ \in C^0([0,T];V) \cap H^1(0,T;H)$. An integration by parts in space and time and the use of (3.19), (3.12), (3.16) yield $(v_n)^- = (v_n-1)^+ = 0$ a.e. in $\mathbf{R}\times]0,T[$, that is, (3.21). For instance, we detail the proof of the right inequality of (3.21). After an integration in time, we have

$$\tfrac{1}{2}\|(v_n-1)^+(\cdot,t)\|^2_H + \varepsilon \left\| \frac{\partial}{\partial x}(v_n-1)^+ \right\|^2_{L^2(\mathbf{R}\times]0,t[)} + \int_0^t \int_{\mathbf{R}} (F_n v_n - F_n)(v_n-1)^+ \leqslant 0 \qquad \forall t \in [0,T],$$

since $(v_0^n - 1)^+ = 0$, $F_n \leqslant G_n$, and we already know that $v_n \geqslant 0$. As

$$(F_n v_n - F_n)(v_n-1)^+ \geqslant 0 \qquad \text{a.e. in} \quad \mathbf{R}\times]0,T[,$$

it follows that $(v_n-1)^+ = 0$.

The following *a priori* estimate concerns the boundedness of $G_n v_n$ in $L^2(\mathbf{R}\times]0,T[)$. We start to observe that, thanks to (3.11), (3.5), and (2.6), there is a positive constant C_5, depending only on $\|z\|_T$, such that

$$0 \leqslant G_n(x,t) \leqslant \min\{n, C_5(1+|x|)\} \qquad \text{for a.e.} \quad (x,t)\in \mathbf{R}\times]0,T[.$$

So, owing to (3.20), if we show that

$$\int_0^T \int_{\mathbf{R}} |x|\, G_n(x,t)\, v_n^2(x,t)\, dx\, dt \quad \text{is bounded independently of } n, \tag{3.22}$$

then we can find a constant C_6 such that

$$\|G_n v_n\|^2_{L^2(\mathbf{R}\times]0,T[)} \leqslant C_6 \qquad \forall n \in \mathbf{N}, \tag{3.23}$$

where C_6 depends on $\|z\|_T$. In order to prove (3.22), we multiply (3.18) by $v_n(x, t)\,\varphi_m(x)$, where $\varphi_m(x) := \min\{|x|, m\} \in W^{1,\infty}(\mathbf{R})$ for $m \in \mathbf{N}$, and integrate by parts in space and time using (3.19). Thus we get

$$\begin{aligned} &\tfrac{1}{2}\|\varphi_m v_n^2(\cdot, T)\|_{L^1(\mathbf{R})} + \varepsilon \left\| \varphi_m \left| \frac{\partial v_n}{\partial x} \right|^2 \right\|_{L^1(\mathbf{R}\times]0,T[)} + \int_0^T \int_{\mathbf{R}} \varphi_m G_n v_n^2 \\ &\quad \leqslant \tfrac{1}{2} \int_{\mathbf{R}} |x|\, |v_0^n(x)|^2\, dx + \varepsilon \int_0^T \int_{-m}^{m} |v_n| \left| \frac{\partial v_n}{\partial x} \right| \\ &\qquad + \tfrac{1}{2} \int_0^T \int_{\mathbf{R}} F_n\, |v_n|^2 + \tfrac{1}{2} \int_0^T \int_{\mathbf{R}} x^2 F_n(x, t)\, dx\, dt. \end{aligned} \tag{3.24}$$

Since

$$\int_{\mathbf{R}} |x|\, |v_0^n(x)|^2\, dx \leqslant \left\{ \int_{\mathbf{R}} x^2 v_0^n(x)\, dx \right\}^{1/2} \|v_0^n\|_{L^3(\mathbf{R})}^{3/2}$$

and, by (3.16),

$$\int_0^T \int_{\mathbf{R}} F_n\, |v_n|^2 \leqslant \|G_n v_n^2\|_{L^1(\mathbf{R}\times]0,T[)},$$

from (3.13), (3.17), (3.20), and (3.24) it follows that $\int_0^T \int_{\mathbf{R}} \varphi_m G_n v_n^2$ is bounded independently of m, n, ε. Then, taking the limit as $m \nearrow +\infty$ and using the Beppo Levi Monotone Convergence Theorem, we obtain (3.22).

Step 3 (passage to the limit). Thanks to (3.20)–(3.21) and (3.23), there exist $v \in L^\infty(0, T; H \cap L^\infty(\mathbf{R})) \cap L^2(0, T; V)$, $\rho \in L^2(\mathbf{R} \times]0, T[)$ such that, possibly taking subsequences,

$$v_n \to v \qquad \text{weakly star in } L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^\infty(\mathbf{R} \times]0, T[), \tag{3.25}$$

$$G_n v_n \to \rho \qquad \text{weakly in } L^2(\mathbf{R} \times]0, T[), \tag{3.26}$$

and v satisfies (3.8). To prove that v is a solution of (3.6)–(3.7) and that $\rho = G_z^* v$ a.e. in $\mathbf{R} \times]0, T[$, we multiply (3.18) by a smooth function ψ having a compact support contained in $\mathbf{R} \times]0, T[$. Integrating by parts in space and time and using (3.19), we get

$$-\int_0^T \int_{\mathbf{R}} v_n \frac{\partial \psi}{\partial t} + \varepsilon \int_0^T \int_{\mathbf{R}} \frac{\partial v_n}{\partial x} \frac{\partial \psi}{\partial x} + \int_0^T \int_{\mathbf{R}} G_n v_n \psi = \int_{\mathbf{R}} v_0^n \psi(\cdot, 0) + \int_0^T \int_{\mathbf{R}} F_n \psi.$$

As (cf. (2.6), (2.8), (3.5), (3.11)) $F_n \to F_z^*$ strongly in $L^1(0, T; H)$ and $G_n \psi \to G_z^* \psi$ strongly in $L^1(\mathbf{R} \times]0, T[)$ (indeed $G_n \psi$ converges to $G_z^* \psi$ on the set where $\psi \neq 0$), taking the limit in the last equality, by (3.14) and (3.25) we obtain

$$-\int_0^T \int_{\mathbf{R}} v\psi_t + \varepsilon \int_0^T \int_{\mathbf{R}} v_x \psi_x + \int_0^T \int_{\mathbf{R}} G_z^* v\psi$$

$$= \int_{\mathbf{R}} u(\cdot + z_0)\, \psi(\cdot, 0) + \int_0^T \int_{\mathbf{R}} F_z^* \psi. \tag{3.27}$$

Then $\rho = G_z^* v$ (see (3.26)), besides, using (3.27), it is not difficult to infer that

$$v_t = \varepsilon v_{xx} - G_z^* v + F_z^* \qquad \text{in} \quad \mathscr{D}'(\mathbf{R} \times]0, T[).$$

Thus $v_t \in L^2(0, T; V') + L^1(0, T; H)$ and v satisfies (3.6): moreover (see [18, Sect. 3]) $v \in C^0([0, T]; H)$, so that, from (3.6) and (3.27), (3.7) follows.

Step 4 (uniqueness). The uniqueness of the solutions of (3.6)–(3.7) is simple to prove: since the problem is linear, it suffices to show that $F_z^* = 0$, $u_0 = 0$ imply $v_z = 0$. But this follows easily taking $\psi = v_z$ in (3.6), integrating (3.6) from 0 to $t \in]0, T[$, and using the fact that G_z is non-negative.

Step 5 (estimate (3.10)). We choose as test function in (3.6), $\psi_m(x) = \exp(-x^2/m) \in V$ for $m \in \mathbf{N}$ and $x \in \mathbf{R}$, and integrate in time. With the help of (3.7), we obtain

$$\int_{\mathbf{R}} \psi_m v_z(\cdot, t) + \int_0^t \int_{\mathbf{R}} G_z^* v_z \psi_m \leqslant \int_{\mathbf{R}} u_0(x + z(0))\, dx$$

$$+ \int_0^t \int_{\mathbf{R}} F_z^*(x, \tau)\, dx\, d\tau + \varepsilon \left\| \frac{\partial v_z}{\partial x} \right\|_{L^2(\mathbf{R} \times]0, t[)}$$

$$\times \left(t \int_{\mathbf{R}} 4x^2 m^{-2} \exp(-2x^2/m)\, dx \right)^{1/2} \qquad \forall t \in [0, T]. \tag{3.28}$$

Since

$$\int_{\mathbf{R}} 4x^2 m^{-2} \exp(-2x^2/m)\, dx$$

$$= (4m)^{-1/2} \int_{\mathbf{R}} x^2 \exp(-x^2)\, dx \to 0 \qquad \text{as} \quad m \nearrow +\infty,$$

by (2.8)–(2.9) and (3.9) we have that the left hand side of (3.28) is bounded independently of m, z, ε. Owing to (3.8) and (2.5), letting $m \nearrow +\infty$, and using the Beppo Levi Theorem, we deduce (3.10).

The next two lemmas give an integral representation of v_z and other properties of this function.

LEMMA 3.2. *Given* $z \in C^0([0, T])$, *the solution* v_z of (3.6)–(3.7) *solves the integral equation*

$$v_z(x, t) = \int_{\mathbf{R}} u_0(y + z(0)) K_\varepsilon(x - y, t)\, dy$$
$$+ \int_0^t \int_{\mathbf{R}} (F_z^*(y, \tau) - G_z^*(y, \tau)\, v_z(y, \tau))\, K_\varepsilon(x - y, t - \tau)\, dy\, d\tau$$
$$\textit{for a.e.} \quad (x, t) \in \mathbf{R} \times]0, T[, \tag{3.29}$$

where

$$K_\varepsilon(x, t) = (4\pi\varepsilon t)^{-1/2} \exp(-|x|^2/4\varepsilon t) \qquad \forall (x, t) \in \mathbf{R} \times [0, T].$$

Moreover for any $R > 0$ *there is a constant* $M_3(R) > 0$ *such that for any* $z \in C^0([0, T])$ *satisfying* $\|z\|_T \leqslant R$ *and for any* $\alpha \in [0, 2]$ *we have*

$$\int_{\mathbf{R}} |x|^\alpha\, v_z(x, t)\, dx \leqslant M_3(R) \qquad \textit{for a.e.} \quad t \in]0, T[. \tag{3.30}$$

Proof. The first part of the statement follows easily from Lemma 3.1 (cf., e.g., [2]): indeed observe that K_ε is the fundamental solution related to the parabolic operator $\{\partial/\partial t - \varepsilon \partial^2/\partial x^2\}$. In order to complete the proof, note that, owing to (3.10) and to the inequality

$$|x|^\alpha \leqslant 1 + x^2 \qquad \forall x \in \mathbf{R}, \quad \forall \alpha \in [0, 2],$$

it suffices to show that $\int_{\mathbf{R}} x^2 v_z(x, t)\, dx$ is bounded in $L^\infty(0, T)$ by a constant depending on $\|z\|_T$. Using (2.5), (3.5), (3.8), and (3.29), it is easy to see that

$$\int_{\mathbf{R}} x^2 v_z(x, t)\, dx \leqslant \int_{\mathbf{R}} x^2 \int_{\mathbf{R}} u_0(y + z(0))\, K_\varepsilon(x - y, t)\, dy\, dx$$
$$+ \int_{\mathbf{R}} x^2 \int_0^t \int_{\mathbf{R}} F_z^*(y, \tau)\, K_\varepsilon(x - y, t - \tau)\, dy\, d\tau\, dx$$
$$= \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}} u_0(y + z(0)) \int_{\mathbf{R}} (4\varepsilon t \xi^2 + y^2) \exp(-\xi^2)\, d\xi\, dy$$
$$+ \frac{1}{\sqrt{\pi}} \int_0^t \int_{\mathbf{R}} F(y + z(\tau), \tau, z(\tau))$$
$$\times \int_{\mathbf{R}} (4\varepsilon(t - \tau) \xi^2 + y^2) \exp(-\xi^2)\, d\xi\, dy\, d\tau$$

for a.e. $t \in]0, T[$. Then (3.30) is a consequence of (2.8)–(2.9) (see also Remark 2.5) and of the Tonelli Integration Theorem.

LEMMA 3.3. *Let* $z \in C^0([0, T])$. *Then* $w_z^* v_z \in C^0([0, T]; L^1(\mathbf{R}))$, *where* w_z^* *is defined in* (3.5).

Proof. Since $w_z^* \in C^0([0, T]; C^{0,1}(\mathbf{R}))$ (see (3.5) and (2.4)), then, by (3.10) and (3.30) with $\alpha = 1$, $w_z^* v_z \in L^\infty(0, T; L^1(\mathbf{R}))$. Moreover, using (3.29) and the Tonelli Theorem, the equality

$$\begin{aligned}
&\int_{\mathbf{R}} w(x+z(t))\, v_z(x, t)\, dx \\
&\quad = \int_{\mathbf{R}} \int_{\mathbf{R}} w(x+z(t))\, u_0(y+z(0))\, K_\varepsilon(x-y, t)\, dx\, dy \\
&\qquad + \int_0^t \int_{\mathbf{R}} (F_z^*(y, \tau) - G_z^*(y, \tau)\, v_z(y, \tau)) \\
&\qquad \times \int_{\mathbf{R}} w(x+z(t))\, K_\varepsilon(x-y, t-\tau)\, dx\, dy\, d\tau
\end{aligned} \tag{3.31}$$

holds for a.e. $t \in]0, T[$. It is now a standard matter to prove (for instance by changing variables in the integrals) that the right hand side of (3.31) is continuous in time.

Then, when z is given in $C^0([0, T])$ and $z(0) = z_0$, the function v_z solves (3.3)–(3.4). We now introduce a function μ_z solving (2.13)–(2.14) provided that z is Lipschitz continuous.

LEMMA 3.4. *For any* $z \in C^0([0, T])$, *set*

$$\mu_z(x, t) := v_z(x - z(t), t) \qquad \textit{for a.e.} \quad (x, t) \in \mathbf{R} \times]0, T[, \tag{3.32}$$

where v_z *is defined by Lemma* 3.1. *Then* $G_z \mu_z \in L^2(0, T; H)$ (*see* (2.15) *for the definition of* G_z), $w\mu_z \in C^0([0, T]; L^1(\mathbf{R}))$,

$$0 \leqslant \mu_z \leqslant 1 \qquad \textit{a.e. in} \quad \mathbf{R} \times]0, T[, \tag{3.33}$$

and there exists a constant $C_7 > 0$, *independent of* z *and* ε, *such that*

$$\begin{aligned}
&\|\mu_z\|_{C^0([0, T]; L^2(\mathbf{R}) \cap L^1(\mathbf{R}))} + \varepsilon \left\| \frac{\partial \mu_z}{\partial x} \right\|_{L^2(\mathbf{R} \times]0, T[)} \\
&\quad + \|G_z(\mu_z^2 + \mu_z)\|_{L^1(\mathbf{R} \times]0, T[)} \leqslant C_7.
\end{aligned} \tag{3.34}$$

Moreover for any $R>0$ *there is a constant* $M_4(R)>0$ *such that for any* $z \in C^0([0,T])$ *satisfying* $\|z\|_T \leq R$ *and for any* $\alpha \in [0,2]$ *we have*

$$\int_{\mathbf{R}} |x|^\alpha \mu_z(x,t)\,dx \leq M_4(R) \qquad \text{for a.e.} \quad t \in]0,T[. \tag{3.35}$$

Proof. It is an easy consequence of (3.32) and Lemmas 3.1–3.3.

LEMMA 3.5. *For any* $z \in C^{0,1}([0,T])$ *the function* μ_z *defined by* (3.32) *satisfies*

$$\frac{\partial \mu_z}{\partial t} \in L^2(0,T;V') + L^1(0,T;H), \tag{3.36}$$

$$\left\langle \frac{\partial \mu_z}{\partial t}(\cdot,t), \varphi \right\rangle + \varepsilon \left(\frac{\partial \mu_z}{\partial x}(\cdot,t), \varphi_x \right) - z'(t)(\mu_z(\cdot,t), \varphi_x)$$

$$+ ((G_z \mu_z - F_z)(\cdot,t), \varphi) = 0 \qquad \forall \varphi \in V, \quad \text{for a.e.} \quad t \in]0,T[, \tag{3.37}$$

$$\mu_z(x,0) = u_0(x) \qquad \text{for a.e.} \quad x \in \mathbf{R}, \tag{3.38}$$

where F_z *and* G_z *are defined by* (2.15).

The proof of this lemma is quite similar to that of Proposition 3.1: using Lemma 3.1, it is not difficult to recover (3.36)–(3.38). The last result allows us to write another equivalent formulation of Problems (P_1) and (P_2).

PROBLEM (P_3). *Find* $z \in C^{0,1}([0,T])$ *such that*

$$a < \int_{\mathbf{R}} w(x)\,\mu_z(x,t)\,dx < b \qquad \forall t \in [0,T], \tag{3.39}$$

$$z(t) = L\left(\int_{\mathbf{R}} w(x)\,\mu_z(x,t)\,dx \right) \qquad \forall t \in [0,T]. \tag{3.40}$$

PROPOSITION 3.2. *Problems* (P_1) and (P_3) *are equivalent in the following sense: if* (u,z) *solves* (P_1), *then* z *solves* (P_3) *and* $\mu_z \equiv u$; *conversely if* z *solves* (P_3), *then* (μ_z, z) *solves* (P_1).

Remark 3.2. Like Problem (P_2), also (P_3) has a meaning even when $z \in C^0([0,T])$, since the definition of μ_z is given for $z \in C^0([0,T])$ (cf. Lemma 3.4).

4. A PRIORI ESTIMATES

In this section we deal with the *a priori* boundedness of the solution of Problem (P_1) (or (P_3)) and we prepare some estimates related to the contracting behaviour of possible solutions, in order to apply the Banach fixed point theorem step by step in time.

LEMMA 4.1. *There exists a positive constant K, independent of ε, such that any solution (u, z) of Problem* (P_1) *satisfies*

$$|z(t)| \leqslant K \qquad \forall t \in [0, T], \tag{4.1}$$

$$a < \lambda(K) \leqslant \int_{\mathbf{R}} w(x)\, u(x, t)\, dx \leqslant \lambda(-K) < b \qquad \forall t \in [0, T], \tag{4.2}$$

where λ is the inverse function of L (see (2.2)).

Proof. Let us set

$$W(x) := \int_0^x w(\xi)\, d\xi \qquad \forall x \in \mathbf{R}, \tag{4.3}$$

$$\Lambda(\eta) := \int_0^\eta \lambda(\xi)\, d\xi \qquad \forall \eta \in \mathbf{R}. \tag{4.4}$$

By means of (2.4) and (2.2), it is easy to see that

$$\begin{gathered} W \text{ is a convex function satisfying} \\ \exists C_8 > 0\colon \quad 0 \leqslant W(x) \leqslant C_8 x^2 \qquad \forall x \in \mathbf{R}, \end{gathered} \tag{4.5}$$

$$\begin{gathered} -\Lambda \text{ is a convex function such that} \\ \Lambda(\eta) \leqslant 0 \quad \forall \eta \in \mathbf{R} \qquad \text{and} \qquad -\Lambda(\eta) \nearrow +\infty \text{ as } |\eta| \nearrow +\infty. \end{gathered} \tag{4.6}$$

Let (u, z) be a solution to Problem (P_1). Then z solves (P_3) and $\mu_z \equiv u$ (cf. Proposition 3.2) satisfies (3.37)–(3.38). As test function in (3.37) we can take $\varphi_m = W\psi_m$, where $\psi_m(x) = \exp(-x^2/m)$: note that $\varphi_m \in H^2(\mathbf{R})$ owing to (4.3), (2.4). Integrating by parts in space and time and using (3.38), we have

$$\begin{aligned} &\int_{\mathbf{R}} \mu_z(\cdot, t)\, W\psi_m + \int_0^t \int_{\mathbf{R}} G_z \mu_z W\psi_m \\ &\quad = \int_{\mathbf{R}} u_0 W\psi_m + \int_0^t \int_{\mathbf{R}} F_z W\psi_m + \varepsilon \int_0^t \int_{\mathbf{R}} \mu_z \frac{d}{dx}\left(w\psi_m + W \frac{d\psi_m}{dx}\right) \\ &\qquad + \int_0^t z'(\tau) \int_{\mathbf{R}} \mu_z(\cdot, \tau) \left(w\psi_m + W \frac{d\psi_m}{dx}\right) d\tau \qquad \forall t \in [0, T], \quad \forall m \in \mathbf{N}. \end{aligned} \tag{4.7}$$

Noting that $0 \leqslant \psi_m \leqslant 1$ in $\mathbf{R}$, from (4.5), (2.9), (2.15), (2.8) it follows that there is a constant C_9 such that

$$\left| \int_{\mathbf{R}} u_0 W \psi_m + \int_0^t \int_{\mathbf{R}} F_z W \psi_m \right| \leqslant C_9 \qquad \forall t \in [0, T], \quad \forall m \in \mathbf{N}. \tag{4.8}$$

On account of (2.4), (4.5), (3.35), we have

$$\begin{aligned} &\left| \varepsilon \int_0^t \int_{\mathbf{R}} \mu_z \frac{d}{dx} \left(w \psi_m + W \frac{d\psi_m}{dx} \right) \right| \\ &\quad \leqslant \varepsilon \, \|\mu_z\|_{L^1(\mathbf{R} \times]0, T[)} \left\| \frac{dw}{dx} \right\|_{L^\infty(\mathbf{R})} + 2\varepsilon T \, \|\mu_z w\|_{L^\infty(0, T; L^1(\mathbf{R}))} \left\| \frac{d\psi_m}{dx} \right\|_{L^\infty(\mathbf{R})} \\ &\qquad + \varepsilon T \, \|\mu_z W\|_{L^\infty(0, T; L^1(\mathbf{R}))} \left\| \frac{d^2\psi_m}{dx^2} \right\|_{L^\infty(\mathbf{R})} \qquad \forall t \in [0, T], \quad \forall m \in \mathbf{N}, \end{aligned} \tag{4.9}$$

and it is easy to check that

$$\left\| \frac{d\psi_m}{dx} \right\|_{L^\infty(\mathbf{R})} + \left\| \frac{d^2\psi_m}{dx^2} \right\|_{L^\infty(\mathbf{R})} \to 0 \qquad \text{as} \quad m \nearrow +\infty. \tag{4.10}$$

Besides

$$\begin{aligned} &\left| \int_0^t z'(\tau) \int_{\mathbf{R}} \mu_z(\cdot, \tau) \left(w \psi_m + W \frac{d\psi_m}{dx} \right) d\tau \right| \leqslant \|z'\|_{L^1(0, T)} \, \|\mu_z w\|_{L^\infty(0, t; L^1(\mathbf{R}))} \\ &\qquad + \|z'\|_{L^1(0, T)} \, \|\mu_z W\|_{L^\infty(0, T; L^1(\mathbf{R}))} \left\| \frac{d\psi_m}{dx} \right\|_{L^\infty(\mathbf{R})} \qquad \forall t \in [0, T], \quad \forall m \in \mathbf{N}. \end{aligned} \tag{4.11}$$

By (4.8)–(4.11), the right hand side of (4.7) is bounded independently of m. Then, as μ_z, W, ψ_m, G_z are non-negative, we can pass to the limit in (4.7) using the Beppo Levi Monotone Convergence Theorem, obtaining (see (4.9)–(4.11))

$$\begin{aligned} &\int_{\mathbf{R}} \mu_z(\cdot, t) W - \int_0^t z'(\tau) \int_{\mathbf{R}} \mu_z(\cdot, \tau) w \, d\tau + \int_0^t \int_{\mathbf{R}} G_z \mu_z W \\ &\qquad = \int_{\mathbf{R}} u_0 W + \int_0^t \int_{\mathbf{R}} \left(F_z W + \varepsilon \mu_z \frac{dw}{dx} \right) \qquad \forall t \in [0, T]. \end{aligned} \tag{4.12}$$

But, as λ is the inverse function of L, using (3.40), (3.38), (2.19), and (4.4), it is easy to see that

$$-\int_0^t z'(\tau) \int_{\mathbf{R}} \mu_z(\cdot, \tau) w \, d\tau = -\Lambda(z(t)) + \Lambda(z_0) \qquad \forall t \in [0, T].$$

Then, from (4.8), (3.34), (2.4), (4.6), (4.12) it follows that there exists a constant C_{10}, independent of z and ε, such that

$$0 \leqslant -A(z(t)) \leqslant \int_{\mathbf{R}} \mu_z(\cdot, t)\, W - A(z(t)) + \int_0^t \int_{\mathbf{R}} G_z \mu_z W$$
$$\leqslant C_{10} \qquad \forall t \in [0, T],$$

and, owing to (4.6), the last inequality yields (4.1), whose (4.2) is an immediate consequence (cf. (2.2)).

Remark 4.1. The method used to prove the *a priori* estimate (4.1) can be applied with minor effort to show the *a priori* boundedness of z in the muscle contraction problem (see, e.g., [9, 10, 13, 4]). Moreover, this method allows us to drop the technical assumption (2.16) of [4].

LEMMA 4.2. *Let $\theta, \zeta \in C^0([0, T])$ be such that*

$$\max\{\|\theta\|_T, \|\zeta\|_T\} \leqslant 2K, \tag{4.13}$$

where K is defined in Lemma 4.1. Assume also that for some $r \in [0, T[$ we have

$$\theta(t) = \zeta(t) \qquad \forall t \in [0, r]. \tag{4.14}$$

Then there exist two constants $\beta \in]0, 1[$ and $C_{11} > 0$, depending on K but independent of ε, r, θ, ζ, such that

$$\int_r^t \int_{\mathbf{R}} G_\zeta^*(x, \tau)\, |v_\zeta(x, \tau) - v_\theta(x, \tau)|\, (1 + |x|)\, dx\, d\tau$$
$$\leqslant C_{11}(t - r)^\beta \|\zeta - \theta\|_t \qquad \forall t \in [r, T], \tag{4.15}$$

where G_ζ^* is defined in (3.5) and v_ζ, v_θ are defined by Lemma 3.1.

Proof. Setting $v = v_\zeta - v_\theta$, by (3.6)–(3.7) it is not difficult to see that v satisfies

$$\left\langle \frac{\partial v}{\partial t}(\cdot, t), \psi \right\rangle + \varepsilon \left(\frac{\partial v}{\partial x}(\cdot, t), \psi_x \right) + ((G_\zeta^* v)(\cdot, t), \psi)$$
$$= (\{(G_\theta^* - G_\zeta^*) v_\theta + F_\zeta^* - F_\theta^*\}(\cdot, t), \psi) \qquad \forall \psi \in V, \text{ for a.e. } t \in]0, T[, \tag{4.16}$$

and homogeneous initial conditions because of (4.14). We now choose, as a test function in (4.16), $\psi = \tanh(mv)\, \varphi \psi_m \in V$ for $m \in \mathbf{N}$, where $\tanh(m\,\cdot)$

is a smooth approximation of the sign function, $\psi_m(x) = \exp(-x^2/m)$, and $\varphi(x) = 2 + x \tanh x$ satisfies

$$1 + |x| \leqslant \varphi(x) \leqslant 2(1 + |x|) \qquad \forall x \in \mathbf{R}. \tag{4.17}$$

Note also that $\varphi\psi_m \in H^2(\mathbf{R})$. Setting then

$$\Phi_m(\xi) := \int_0^\xi \tanh(m\eta)\, d\eta \in [0, |\xi|] \qquad \forall \xi \in \mathbf{R}, \quad \forall m \in \mathbf{N},$$

integrating by parts in space and time in (4.16), and using (4.14), we get

$$\int_{\mathbf{R}} \Phi_m(v(\cdot, t))\, \varphi\psi_m + \varepsilon \int_r^t \int_{\mathbf{R}} \frac{m\varphi\psi_m}{\cosh^2(mv)} \left| \frac{\partial v}{\partial x} \right|^2 + \int_r^t \int_{\mathbf{R}} G_\zeta^* v \tanh(mv)\, \varphi\psi_m$$
$$\leqslant \varepsilon \int_r^t \int_{\mathbf{R}} \Phi_m(v) \frac{d^2}{dx^2} (\varphi\psi_m) + \int_r^t \int_{\mathbf{R}} |G_\theta^* - G_\zeta^*|\, v_0 \varphi$$
$$+ \int_r^t \int_{\mathbf{R}} |F_\zeta^* - F_\theta^*|\, \varphi \qquad \forall t \in [r, T]. \tag{4.18}$$

From (3.5), (2.7), (3.30), (4.13), (4.17) it follows that there are a positive constant C_{12} and a function h such that

$$\int_r^t \int_{\mathbf{R}} |G_\theta^* - G_\zeta^*|\, v_0 \varphi + \int_r^t \int_{\mathbf{R}} |F_\zeta^* - F_\theta^*|\, \varphi$$
$$\leqslant C_{12}\, \|\zeta - \theta\|_t \int_r^t h(\tau)\, d\tau \qquad \forall t \in [r, T], \tag{4.19}$$

where, for instance, $C_{12} = 8M_3(2K) + 2$, and $h \in L^q(0, T)$ and q depends on K. Applying the Hölder inequality we obtain

$$\int_r^t h(\tau)\, d\tau \leqslant C_{13}(t - r)^\beta \qquad \forall t \in [r, T], \tag{4.20}$$

where $C_{13} = \|h\|_{L^q(0, T)}$ and $\beta = (q-1)/q \in]0, 1[$. Next, it is a standard matter to check that

$$\varepsilon \int_r^t \int_{\mathbf{R}} \Phi_m(v) \frac{d^2}{dx^2} (\varphi\psi_m) \leqslant 2\varepsilon \int_r^t \int_{\mathbf{R}} \Phi_m(v) \psi_m$$
$$+ \varepsilon \int_r^t \int_{\mathbf{R}} v(x, \tau)\, \varphi(x)\, 4x^2 m^{-2} \exp(-x^2/m)\, dx\, d\tau \qquad \forall t \in [r, T].$$

Hence, thanks to (4.13), (4.17), (3.30), there is a positive constant C_{14} such that

$$\varepsilon \int_r^t \int_{\mathbf{R}} \Phi_m(v) \frac{d^2}{dx^2} (\varphi\psi_m) \leqslant \varepsilon \int_r^t \int_{\mathbf{R}} \Phi_m(v)\, \varphi\psi_m + C_{14} m^{-1} \qquad \forall t \in [r, T], \tag{4.21}$$

where, for instance, $C_{14} = 8\varepsilon M_3(2K)\, T \max_{x \in \mathbf{R}} |4x^2 \exp(-x^2)|$. From (4.18)–(4.21) it follows that

$$\int_{\mathbf{R}} \Phi_m(v(\cdot, t))\, \varphi\psi_m + \int_r^t \int_{\mathbf{R}} G_\zeta^* v \tanh(mv)\, \varphi\psi_m \leqslant \varepsilon \int_r^t \int_{\mathbf{R}} \Phi_m(v)\, \varphi\psi_m + C_{14} m^{-1} + C_{15}(t-r)^\beta \|\zeta - \theta\|_t \qquad \forall t \in [r, T], \tag{4.22}$$

where $C_{15} = C_{12} C_{13}$. Applying then the Gronwall Lemma (cf., e.g., [1, Appendice]) to the function $\int_{\mathbf{R}} \Phi_m(v(\cdot, t))\, \varphi\psi_m$ and using the obtained estimate in the right hand side of (4.22), we get

$$\int_r^t \int_{\mathbf{R}} G_\zeta^* v \tanh(mv)\, \varphi\psi_m \leqslant \{1 + \exp(\varepsilon T)\}\{C_{14} m^{-1} + C_{15}(t-r)^\beta \|\zeta - \theta\|_t\}$$

for any $t \in [r, T]$. Finally, taking the limit as $m \nearrow +\infty$, using the Beppo Levi Theorem, recalling (4.17), and observing that $v \tanh(mv) \nearrow |v|$ a.e. in $\mathbf{R} \times]0, T[$, we easily get (4.15).

The statement of Lemma 4.2 will be used in the next lemma.

For any $\zeta \in C^0([0, T])$, we define a function Γ_ζ: $]0, T[\times]0, T[\to \mathbf{R}$ by setting

$$\Gamma_\zeta(t, \tau) := \int_{\mathbf{R}} w(x + \zeta(t)) \int_{\mathbf{R}} K_\varepsilon(x - y, t - \tau)\{F_\zeta^*(y, \tau) - G_\zeta^*(y, \tau)\, v_\zeta(y, \tau)\}\, dy\, dx \qquad \text{for a.e.} \quad (t, \tau) \in]0, T[\times]0, T[. \tag{4.23}$$

Owing to (2.4), (3.29)–(3.30) it is easy to see that (4.23) makes sense. The following lemma states some contracting properties of Γ_ζ.

LEMMA 4.3. *Let* $\theta, \zeta \in C^0([0, T])$ *satisfy* (4.13)–(4.14). *Then there exist* *two constants* $\beta \in]0, 1[$ *and* $C_{16} > 0$, *depending on* K *but independent of* ε, r, θ, ζ, *such that for any* $t \in [r, T]$ *we have*

$$\left| \int_r^t \Gamma_\zeta(t, \tau) \, d\tau \right| \leqslant C_{16}(t-r), \tag{4.24}$$

$$\left| \int_r^t (\Gamma_\zeta(t, \tau) - \Gamma_\theta(t, \tau)) \, d\tau \right| \leqslant C_{16}(t-r)^\beta \, \|\zeta - \theta\|_t. \tag{4.25}$$

Proof. By (4.23), applying the Tonelli Integration Theorem, changing variables in the integrals, and recalling (2.15), (3.5), (3.32), we obtain

$$\Gamma_\zeta(t, \tau) = \int_{\mathbf{R}} \int_{\mathbf{R}} \frac{\exp(-\eta^2)}{\sqrt{\pi}} w(y + 2\eta(\varepsilon(t-\tau))^{1/2} - \zeta(\tau) + \zeta(t))$$
$$\times \{F_\zeta(y, \tau) - G_\zeta(y, \tau) \, \mu_\zeta(y, \tau)\} \, d\eta \, dy$$
$$\text{for a.e.} \quad \tau \in]0, t[\quad \text{and} \quad t \in]0, T[. \tag{4.26}$$

Using now (2.15), (2.6), (2.4), (4.13) and denoting by C_w the Lipschitz constant of w, we get

$$|\Gamma_\zeta(t, \tau)| \leqslant 2C_w \int_{\mathbf{R}} \int_{\mathbf{R}} \frac{\exp(-\eta^2)}{\sqrt{\pi}} \{|y| + (\varepsilon T)^{1/2} |\eta| + 2K\}$$
$$\times \{F(y, \tau, \zeta(\tau)) + M_2(2K)(1 + |y|) \, \mu_\zeta(y, \tau)\} \, d\eta \, dy$$
$$\text{for a.e.} \quad \tau \in]0, t[\quad \text{and} \quad t \in]0, T[.$$

Next, thanks to (3.35) and again to (2.6), we find a constant C_{17} such that

$$|\Gamma_\zeta(t, \tau)| \leqslant C_{17} \qquad \text{for a.e.} \quad \tau \in]0, t[\quad \text{and} \quad t \in]0, T[, \tag{4.27}$$

where, for instance,

$$C_{17} = 2C_w(1 + 2K + (\varepsilon T)^{1/2}) \, M_2(2K)$$
$$\times \{1 + M_4(2K)\} \int_{\mathbf{R}} \frac{(1 + |\eta|) \exp(-\eta^2)}{\sqrt{\pi}} \, d\eta.$$

Estimate (4.24) is an easy consequence of (4.27). In order to show (4.25), we first remark that, by the Tonelli Theorem and (2.15), (3.5), (3.32), we obtain

$$\left| \int_r^t (\Gamma_\zeta(t, \tau) - \Gamma_\theta(t, \tau)) \, d\tau \right| \leqslant I_1(t) + I_2(t) + I_3(t), \tag{4.28}$$

where

$$I_1(t) := \int_r^t \int_{\mathbf{R}} \int_{\mathbf{R}} \frac{\exp(-\eta^2)}{\sqrt{\pi}} \, |w(y + 2\eta(\varepsilon(t-\tau))^{1/2} - \zeta(\tau) + \zeta(t))$$
$$- w(y + 2\eta(\varepsilon(t-\tau))^{1/2} - \zeta(\tau) + \theta(t))|$$
$$\times |F_\zeta(y, \tau) - G_\zeta(y, \tau) \, \mu_\zeta(y, \tau)| \, d\eta \, dy \, d\tau,$$
$$I_2(t) := \int_r^t \int_{\mathbf{R}} \int_{\mathbf{R}} \frac{\exp(-\eta^2)}{\sqrt{\pi}} \, |w(y - 2\eta(\varepsilon(t-\tau))^{1/2} + \theta(t))|$$
$$\times |F_\zeta^*(y, \tau) - F_\theta^*(y, \tau) - v_\theta(y, \tau)(G_\zeta^*(y, \tau) - G_\theta^*(y, \tau))| \, d\eta \, dy \, d\tau,$$
$$I_3(t) := \int_r^t \int_{\mathbf{R}} \int_{\mathbf{R}} \frac{\exp(-\eta^2)}{\sqrt{\pi}} \, |w(y - 2\eta(\varepsilon(t-\tau))^{1/2} + \theta(t))|$$
$$\times G_\zeta^*(y, \tau) \, |v_\zeta(y, \tau) - v_\theta(y, \tau)| \, d\eta \, dy \, d\tau,$$

for any $t \in [r, T]$. Following the same procedure adopted to prove (4.24), it is not difficult to show that there is a constant C_{18} such that

$$I_1(t) \leqslant C_{18}(t-r) \, \|\zeta - \theta\|_t \qquad \forall t \in [r, T]. \tag{4.29}$$

Then, by (2.4) and (4.13), we have that

$$I_2(t) \leqslant C_{19} \int_r^t \int_{\mathbf{R}} (1 + |y|) \{ |F_\zeta^*(y, \tau) - F_\theta^*(y, \tau)|$$
$$+ v_\theta(y, \tau) \, |G_\theta^*(y, \tau) - G_\zeta^*(y, \tau)| \} \, dy \, d\tau, \tag{4.30}$$

$$I_3(t) \leqslant C_{19} \int_r^t \int_{\mathbf{R}} (1 + |y|) \, G_\zeta^*(y, \tau) \, |v_\zeta(y, \tau) - v_\theta(y, \tau)| \, dy \, d\tau \tag{4.31}$$

for any $t \in [r, T]$, where, for instance,

$$C_{19} = C_w(1 + 2K + 2(\varepsilon T)^{1/2}) \int_{\mathbf{R}} \frac{(1 + |\eta|) \exp(-\eta^2)}{\sqrt{\pi}} \, d\eta.$$

Next, from (4.17), (4.19), (4.20), (4.30), (4.15), (4.31) it follows that there is a constant $C_{20} > 0$ such that

$$I_2(t) + I_3(t) \leqslant C_{20}(t-r)^\beta \, \|\zeta - \theta\|_t \qquad \forall t \in [r, T]. \tag{4.32}$$

Finally, by (4.28), (4.29), and (4.32), we infer (4.25).

5. Proof of Theorem 2.1

Here we prove the existence and uniqueness result for Problem (P_1) (or, equivalently, (P_2) or (P_3)) by using the Contraction Mapping Principle step by step in time. As each time interval will have the same length $\delta > 0$ (to be specified later), we get existence and uniqueness of the solution on the whole given interval $[0, T]$.

Thus, let $r \in [0, T[$ and assume the solution z (resp. $(u = \mu_z, z)$) of Problem (P_3) (resp. (P_1)) to be known in $[0, r]$. We introduce the Banach space

$$X_r := \{\zeta \in C^0([0, r+\delta]) : \zeta = z \quad \text{in} \quad [0, r], \|\zeta\|_{r+\delta} \leqslant 2K\}, \qquad (5.1)$$

where the constant K is defined by Lemma 4.1 (so that a possible solution of Problem (P_3) belongs to X_r). Observe now that Eq. (3.40) can be equivalently written in $[r, T]$ as

$$\begin{aligned}\lambda(z(t)) = \int_{\mathbf{R}} w(x + z(t)) \int_{\mathbf{R}} v_z(y, r)\, K_\varepsilon(x - y, t - r)\, dy\, dx + \int_{\mathbf{R}} w(x + z(t))\\ \times \int_r^t \int_{\mathbf{R}} \{F_z^*(y, \tau) - G_z^*(y, \tau)\, v_z(y, \tau)\}\, K_\varepsilon(x - y, t - \tau)\, dy\, d\tau\, dx,\end{aligned} \qquad (5.2)$$

owing to (2.2), (3.32), and Lemma 3.2. Recalling (4.23) and applying the Tonelli Integration Theorem, (5.2) yields

$$\lambda_r(z(t), t) = \int_r^t \Gamma_z(t, \tau)\, d\tau \qquad \forall t \in [r, T], \qquad (5.3)$$

where

$$\begin{aligned}\lambda_r(\xi, t) := \lambda(\xi) - \int_{\mathbf{R}} \mu_z(y, r) \int_{\mathbf{R}} \frac{\exp(-\eta^2)}{\sqrt{\pi}}\, w(2\eta(\varepsilon(t - r))^{1/2}\\ + y + \xi - z(r))\, d\eta\, dy \qquad \forall \xi \in \mathbf{R}, \quad \forall t \in [r, T].\end{aligned} \qquad (5.4)$$

Using (2.2)–(2.4) and Lemma 3.4 it is straightforward to check that λ_r is continuous and strictly decreasing in ξ for any t, and moreover that $\lambda_r(\cdot, t)$ satisfies (2.3) for any $t \in [r, T]$. Denoting by L_r the inverse function (in ξ) of λ_r, we define the operator

$$\begin{aligned}&S_r : X_r \to X_r, \qquad S_r(\zeta) := z \quad \text{in} \quad [0, r],\\ &S_r(\zeta)(t) := L_r\left(\int_r^t \Gamma_\zeta(t, \tau)\, d\tau\right) \qquad \forall t \in [r, r+\delta], \quad \zeta \in X_r.\end{aligned} \qquad (5.5)$$

We show that, provided δ is small enough, S_r is well defined and is a contraction mapping in X_r. Then S_r will have a unique fixed point $z \in X_r$. As we shall see, δ does not depend on r, thus the theorem will be completely proved.

First, we check that S_r is well defined. Note that, by (5.2)–(5.4), $\lambda_r(z(r), r) = 0$ and for any $\zeta \in X_r$ we have

$$\begin{aligned}(C_{21})^{-1} &|S_r(\zeta)(t) - z(r)| \\ &\leqslant |\lambda_r(S_r(\zeta)(t), t) - \lambda_r(z(r), t)| \\ &\leqslant \left| \int_r^t \Gamma_\zeta(t, \tau)\, d\tau \right| + |\lambda_r(z(r), r) - \lambda_r(z(r), t)| \qquad \forall t \in [r, r+\delta], \end{aligned} \tag{5.6}$$

where, for instance, $C_{21} = M_1(2K)$. From (5.4) it follows that

$$\begin{aligned}|\lambda_r(z(r), r) &- \lambda_r(z(r), t)| \\ &= \left| \int_{\mathbf{R}} \mu_z(y, r) \int_{\mathbf{R}} \frac{\exp(-\eta^2)}{\sqrt{\pi}} \{w(2\eta(\varepsilon(t-r))^{1/2} + y) - w(y)\}\, d\eta\, dy \right|. \end{aligned} \tag{5.7}$$

By means of (2.4) and Taylor Formula we obtain

$$\begin{aligned} w(2\eta(\varepsilon(t&-r))^{1/2} + y) - w(y) \\ &= w'(y)\, 2\eta(\varepsilon(t-r))^{1/2} + \int_0^{2\eta(\varepsilon(t-r))^{1/2}} (2\eta(\varepsilon(t-r))^{1/2} - \xi)\, w''(y+\xi)\, d\xi \end{aligned} \tag{5.8}$$

for any $y, \eta \in \mathbf{R}$, so that from (2.4), (3.34), (5.7), (5.8) we get

$$|\lambda_r(z(r), r) - \lambda_r(z(r), t)| \leqslant C_{22}\varepsilon(t-r) \qquad \forall t \in [r, r+\delta], \tag{5.9}$$

where, for instance,

$$C_{22} = 2C_7 \left\| \frac{d^2 w}{dx^2} \right\|_{L^\infty(\mathbf{R})} \int_{\mathbf{R}} \frac{\eta^2 \exp(-\eta^2)}{\sqrt{\pi}}\, d\eta.$$

Then, from (5.6), (4.24), and (5.9) it follows that there is a constant C_{23}, independent of ε and r, such that for any $\zeta \in X_r$

$$|S_r(\zeta)(t) - z(r)| \leqslant C_{23}(t-r), \qquad \forall t \in [r, r+\delta]. \tag{5.10}$$

Thus, if

$$\delta \leqslant (C_{23})^{-1} K, \tag{5.11}$$

then $S_r(X_r) \subseteq X_r$, since, by (5.10) and Lemma 4.1,

$$\|S_r(\zeta)\|_{r+\delta} \leqslant |z(r)| + K \leqslant 2K \qquad \forall \zeta \in X_r.$$

Finally, we show that S_r is a contraction mapping. Recalling (5.5)–(5.6), it is easy to see that

$$\begin{aligned} &\|S_r(\zeta) - S_r(\theta)\|_t \\ &\quad \leqslant C_{21} \left| \int_r^t (\Gamma_\zeta(t,\tau) - \Gamma_\theta(t,\tau))\, d\tau \right| \qquad \forall t \in [r, r+\delta], \quad \forall \zeta, \theta \in X_r. \end{aligned} \tag{5.12}$$

Using now (4.25), we get

$$\|S_r(\zeta) - S_r(\theta)\|_{r+\delta} \leqslant C_{21} C_{16} \delta^\beta \, \|\zeta - \theta\|_{r+\delta}, \tag{5.13}$$

so that, by (5.11) and (5.13), if, for instance,

$$\delta = \min\{(C_{23})^{-1} K, (2C_{21} C_{16})^{-1/\beta}\},$$

then S_r is a contraction mapping in X_r. As a conclusion, we remark that the fixed point z found here is a Lipschitz continuous function, since from (5.10) it follows that

$$|z(t) - z(r)| \leqslant C_{23}\, |t - r|, \qquad \forall t \in [r, T], \quad \forall r \in [0, T[. \tag{5.14}$$

Remark 5.1. We stress that in our proof the assumption $dw/dx \in C^{0,1}(\mathbf{R})$ is utilized only in (5.7) to infer (5.9). But, if w is merely Lipschitz continuous and monotone, we are just able to prove that

$$|\lambda_r(z(r), r) - \lambda_r(z(r), t)| \leqslant C(\varepsilon(t - r))^{1/2} \qquad \forall t \in [r, T], \tag{5.15}$$

for some constant $C > 0$ and this does not guarantee the Lipschitz continuity of z. However, letting $\varepsilon \searrow 0$, observe that the term (5.15) disappears, as we could expect from the hyperbolic problem (see, e.g., [4]).

6. The Limit of the Diffusion Problem

This section is devoted to studying the asymptotic behaviour of the diffusion problem when the parameter ε goes to zero. We prove that the solution of Problem (P_2) converges to the unique solution of the following problem.

PROBLEM (P_4). *Find*

$$v \in W^{1,1}(0, T; L^1(\mathbf{R})) \cap L^\infty(\mathbf{R} \times]0, T[), \qquad z \in C^{0,1}([0, T])$$

such that $w_z^* v \in C^0([0, T]; L^1(\mathbf{R}))$, $G_z^* v \in L^1(\mathbf{R} \times]0, T[)$ *and satisfying* (3.1), (3.2), (3.4), *and*

$$v_t + G_z^* v = F_z^* \qquad in \quad L^1(\mathbf{R} \times]0, T[), \tag{6.1}$$

where w_z^*, F_z^*, G_z^* *are defined by* (3.5).

First we show the uniqueness of the solution to Problem (P_4). The existence will be a consequence of the limit procedure in ε.

THEOREM 6.1. *There exists at most one solution of Problem* (P_4). *Moreover*

$$0 \leqslant v \leqslant 1 \qquad a.e.\ in \quad \mathbf{R} \times]0, T[. \tag{6.2}$$

Proof. Since (6.1) is an ordinary differential equation with respect to time, it is easy to check that

$$v(x, t) = u_0(x + z_0) \exp\left(- \int_0^t G_z^*(x, \tau)\, d\tau \right) + \int_0^t F_z^*(x, \tau)$$
$$\times \exp\left(- \int_\tau^t G_z^*(x, s)\, ds \right) d\tau \qquad \text{for a.e.} \quad (x, t) \in \mathbf{R} \times]0, T[, \tag{6.3}$$

for any solution (v, z) of Problem (P_4). Using (2.5), (2.9), and (6.3), we easily obtain (6.2). We prove uniqueness by contradiction. Then let (ζ, v_ζ), (θ, v_θ) be two solutions of Problem (P_4). By means of (2.2), (3.2), (6.3) we have

$$\left| \lambda(\zeta(t)) - \lambda(\theta(t)) - \int_{\mathbf{R}} \{ w(x + \zeta(t)) - w(x + \theta(t)) \}\, v_\zeta(x, t)\, dx \right|$$
$$= \left| \int_{\mathbf{R}} w(x + \theta(t))\, u_0(x + z_0) \left\{ \exp\left(- \int_0^t G_\zeta^*(x, \tau)\, d\tau \right) \right.\right.$$
$$\left. - \exp\left(- \int_0^t G_\theta^*(x, \tau)\, d\tau \right) \right\} dx + \int_0^t \int_{\mathbf{R}} w(x + \theta(t))\, F_\theta^*(x, \tau)$$
$$\times \left\{ \exp\left(- \int_\tau^t G_\zeta^*(x, s)\, ds \right) - \exp\left(- \int_\tau^t G_\theta^*(x, s)\, ds \right) \right\} dx\, d\tau$$
$$+ \int_0^t \int_{\mathbf{R}} w(x + \theta(t)) \{ F_\zeta^*(x, \tau) - F_\theta^*(x, \tau) \}$$
$$\left. \times \exp\left(- \int_\tau^t G_\zeta^*(x, s)\, ds \right) dx\, d\tau \right| \tag{6.4}$$

for any $t \in [0, T]$. Setting

$$\hat{K} = \max\{\|\zeta\|_T, \|\theta\|_T\}$$

and taking into account the properties of function $\exp(-\xi)$, $\xi \geq 0$, from (2.2)–(2.4), (6.2), (3.5) it follows that

$$
\begin{aligned}
&(M_1(\hat{K}))^{-1} |\zeta(t) - \theta(t)| \\
&\leq \int_{\mathbf{R}} C_w(|x| + \hat{K})\, u_0(x + z_0) \\
&\quad \times \int_0^t |G(x + \zeta(\tau), \tau, \zeta(\tau)) - G(x + \theta(\tau), \tau, \theta(\tau))|\, d\tau\, dx \\
&\quad + \int_0^t \int_{\mathbf{R}} C_w(|x| + \hat{K})\, F(x + \theta(\tau), \tau, \theta(\tau)) \\
&\quad \times \int_\tau^t |G(x + \zeta(s), s, \zeta(s)) - G(x + \theta(s), s, \theta(s))|\, ds\, dx\, d\tau \\
&\quad + \int_0^t \int_{\mathbf{R}} C_w(|x| + \hat{K})\, |F(x + \zeta(\tau), \tau, \zeta(\tau)) \\
&\quad - F(x + \theta(\tau), \tau, \theta(\tau))|\, dx\, d\tau
\end{aligned}
\tag{6.5}
$$

for any $t \in [0, T]$, where C_w is the Lipschitz constant of w. With the help of (2.6)–(2.7), (2.9), and (6.5), it is a standard matter to find a constant $C_{24} > 0$ and a function $h \geq 0$ such that

$$\|\zeta - \theta\|_t \leq (C_{24} + T) \int_0^t h(s)\, \|\zeta - \theta\|_s\, ds \qquad \forall t \in [0, T], \tag{6.6}$$

where C_{24}, $h \in L^q(0, T)$ and $q > 1$ depend only on $\hat{K}$. By (6.6), the Gronwall Lemma allows us to conclude that $\zeta = \theta$ and the theorem is proved.

THEOREM 6.2. *For any $\varepsilon > 0$ let $(v_\varepsilon, z_\varepsilon)$ be the solution of Problem* (P_2). *Then we have*

$$v_\varepsilon \to v \qquad \textit{weakly star in } L^\infty(0, T; L^2(\mathbf{R})) \cap L^\infty(\mathbf{R} \times]0, T[), \tag{6.7}$$

$$z_\varepsilon \to z \qquad \textit{weakly star in } W^{1,\infty}(0, T) \tag{6.8}$$

as $\varepsilon \searrow 0$, where (v, z) is the solution of Problem (P_4).

Proof. We start recalling the *a priori* estimates. From Lemmas 3.1 and 4.1 and from (5.14) it follows that there is a positive constant C_{25}, independent of ε, such that

$$\|v_\varepsilon\|_{L^\infty(0,T;L^2(\mathbf{R}))} + \varepsilon^{1/2}\left\|\frac{\partial v_\varepsilon}{\partial x}\right\|_{L^2(\mathbf{R}\times]0,T[)} + \|G^*_{z_\varepsilon} v_\varepsilon\|_{L^1(\mathbf{R}\times]0,T[)} \leqslant C_{25}, \tag{6.9}$$

$$0 \leqslant v_\varepsilon \leqslant 1 \quad \text{a.e. in} \quad \mathbf{R}\times]0,T[, \tag{6.10}$$

$$\|z_\varepsilon\|_{W^{1,\infty}(0,T)} \leqslant C_{25}, \tag{6.11}$$

where $G^*_{z_\varepsilon}$ is defined in (3.5). Then there exists a pair (v, z) satisfying (6.2) and such that (6.7), (6.8), and

$$\varepsilon\frac{\partial v_\varepsilon}{\partial x} \to 0 \qquad \text{strongly in} \quad L^2(\mathbf{R}\times]0,T[) \tag{6.12}$$

hold for some subsequences. Proving that (v, z) solves Problem (P_4), the uniqueness of the solution (cf. Theorem 6.1) yields the convergence of the whole sequences.

Hence let us show that (v, z) is a solution of (P_4). Taking in (3.3) a smooth function ψ with compact support contained in $\mathbf{R}\times[0,T[$ and integrating by parts in time, by (3.4) we get

$$-\int_0^T\int_{\mathbf{R}} v_\varepsilon\psi_t + \int_0^T\int_{\mathbf{R}} \varepsilon\frac{\partial v_\varepsilon}{\partial x}\frac{\partial\psi}{\partial x}$$
$$= \int_0^T\int_{\mathbf{R}} (F^*_{z_\varepsilon} - G^*_{z_\varepsilon} v_\varepsilon)\psi + \int_{\mathbf{R}} u_0(x+z_0)\,\psi(x,0)\,dx. \tag{6.13}$$

By the Ascoli–Arzelà Theorem, from (6.11) and (6.8) it follows that

$$z_\varepsilon \to z \qquad \text{strongly in } C^0([0,T]) \text{ as } \varepsilon \searrow 0. \tag{6.14}$$

Then, by (2.7),

$$\left|\int_0^T\int_{\mathbf{R}} F^*_{z_\varepsilon}\psi - \int_0^T\int_{\mathbf{R}} F^*_z\psi\right| \to 0, \tag{6.15}$$

and, since $G^*_z\psi \in L^1(0,T;L^2(\mathbf{R}))$ (cf. (2.6)), again by (6.7),

$$\left|\int_0^T\int_{\mathbf{R}} G^*_{z_\varepsilon} v_\varepsilon\psi - \int_0^T\int_{\mathbf{R}} G^*_z v\psi\right|$$
$$\leqslant \int_0^T h(t)\,\|z_\varepsilon - z\|_t\,\|v_\varepsilon(\cdot,t)\|_{L^2(\mathbf{R})}\,\|\psi(\cdot,t)\|_{L^2(\mathbf{R})}\,dt$$
$$+\left|\int_0^T\int_{\mathbf{R}} G^*_z\psi(v_\varepsilon - v)\right| \to 0 \tag{6.16}$$

as $\varepsilon \searrow 0$. Using (6.7), (6.12), (6.15)–(6.16), taking the limit in (6.13) we obtain

$$-\int_0^T \int_{\mathbf{R}} v\psi_t + \int_0^T \int_{\mathbf{R}} G_z^* v\psi = \int_0^T \int_{\mathbf{R}} F\psi + \int_{\mathbf{R}} u_0(x+z_0)\, \psi(x,0)\, dx. \tag{6.17}$$

From (6.17) we can easily deduce that

$$v_t + G_z^* v = F_z^* \quad \text{in} \quad \mathscr{D}'(\mathbf{R} \times]0, T[). \tag{6.18}$$

Observe now that $G_z^* v \in L^1(\mathbf{R} \times]0, T[)$ since $G_z^* v \geqslant 0$ (cf. (3.5), (2.5), (6.2)) and, by (6.9) and the Beppo Levi Monotone Convergence Theorem,

$$\int_0^T \int_{\mathbf{R}} G_z^* v = \lim_{m \nearrow +\infty} \int_0^T \int_{\mathbf{R}} G_z^*(x,t)\, v(x,t) \exp(-x^2/m)\, dx\, dt$$

$$\leqslant \sup_{\varepsilon > 0} \int_0^T \int_{\mathbf{R}} G_{z_\varepsilon}^* v_\varepsilon \leqslant C_{25}. \tag{6.19}$$

For we also have $F_z^* \in L^1(\mathbf{R} \times]0, T[)$ (cf. (3.5), (2.8), and Remark 2.5), by (6.18) we deduce $v \in W^{1,1}(0, T; L^1(\mathbf{R}))$ and (6.1). Then, by (6.1) and (6.17), it is easy to infer that v fulfils (3.4). Moreover, as v can be represented in the form (6.3), by (2.4), (2.6), and (2.9) it follows that $w_z^* v \in C^0([0, T]; L^1(\mathbf{R}))$.

It remains to prove that v, z satisfy (3.2) ((3.1) will be a consequence thanks to (6.11), (6.14), and (2.2)). Observe first that

$$\lambda(z_\varepsilon) \to \lambda(z) \qquad \text{strongly in } C^0([0, T]) \text{ as } \varepsilon \searrow 0, \tag{6.20}$$

since λ is uniformly continuous in any compact subset of $\mathbf{R}$ (cf. (2.2)) and (6.14) holds. Then it suffices to show, for instance, that

$$\int_{\mathbf{R}} w(x + z_\varepsilon(\cdot))\, v_\varepsilon(x, \cdot)\, dx$$

$$\to \int_{\mathbf{R}} w(x + z(\cdot))\, v(x, \cdot)\, dx \qquad \text{weakly star in} \quad L^\infty(0, T) \tag{6.21}$$

as $\varepsilon \searrow 0$. To this aim, note that for $\phi \in L^1(0, T)$ and $m \geqslant 1$ we have

$$\left| \int_0^T \phi \int_{\mathbf{R}} \{ w(x + z_\varepsilon(\cdot))\, v_\varepsilon(x, \cdot) - w(x + z(\cdot))\, v(x, \cdot) \}\, dx \right|$$

$$\leqslant \left| \int_0^T \phi(t) \int_{\mathbf{R}} w(x + z_\varepsilon(t))\, v_\varepsilon(x, t) \{ 1 - \exp(-x^2/m) \}\, dx\, dt \right|$$

$$+ \left| \int_0^T \int_{\mathbf{R}} \phi(t) \exp(-x^2/m) (w(x + z_\varepsilon(t)) - w(x + z(t)))\, v_\varepsilon(x, t)\, dx\, dt \right|$$

$$+ \left| \int_0^T \int_{\mathbf{R}} \phi(t) \exp(-x^2/m)\, w(x + z(t)) (v_\varepsilon(x, t) - v(x, t))\, dx\, dt \right|$$

$$+ \left| \int_0^T \phi(t) \int_{\mathbf{R}} w(x + z(t))\, v(x, t) \{ \exp(-x^2/m) - 1 \}\, dx\, dt \right|. \tag{6.22}$$

Thanks to (6.14), (2.4), (3.30), (6.3), (2.8), (2.9), we have that

$$\left| \int_0^T \phi(t) \int_{\mathbf{R}} w(x+z_\varepsilon(t))\, v_\varepsilon(x,t)\{1-\exp(-x^2/m)\}\, dx\, dt \right|$$

$$+ \left| \int_0^T \phi(t) \int_{\mathbf{R}} w(x+z(t))\, v(x,t)\{\exp(-x^2/m)-1\}\, dx\, dt \right|$$

$$\leqslant C_{26} m^{-1/2} \int_0^T \int_{\mathbf{R}} |\phi(t)|\, (1+x^2)(v_\varepsilon(x,t)+v(x,t))\, dx\, dt$$

$$\leqslant C_{27} m^{-1/2},$$

for some positive constants C_{26} and C_{27} independent of m, since $1-\exp(-\xi^2) \leqslant |\xi|$ for any $\xi \in \mathbf{R}$. As, for m fixed, the remaining two members on the right hand side of (6.22) go to 0 as $\varepsilon \searrow 0$ (cf. (2.4), (6.14), (6.7)), then for an arbitrary $\phi \in L^1(0, T)$

$$\lim_{\varepsilon \searrow 0} \int_0^T \phi \int_{\mathbf{R}} \{ w(x+z_\varepsilon(\cdot))\, v_\varepsilon(x,\cdot) - w(x+z(\cdot))\, v(x,\cdot)\}\, dx = 0$$

and (6.21) is proved. By (6.20) and (6.21), taking the limit in

$$\lambda(z_\varepsilon(\cdot)) = \int_{\mathbf{R}} w(x+z_\varepsilon(\cdot))\, v_\varepsilon(x,\cdot)\, dx, \tag{6.23}$$

we get (3.2) and this completes the proof.

Let now (v, z) be the solution of Problem (P_4). As Problems (P_1) and (P_2) are equivalent (see Proposition 3.1), setting

$$u(x,t) := v(x-z(t), t) \qquad \text{for a.e.} \quad (x,t) \in \mathbf{R} \times]0, T[, \tag{6.24}$$

one could ask if (u, z) solves the hyperbolic problem (1.1)–(1.3), (2.11). The answer is negative in general. Indeed (cf. the formulation of (P_4) and (6.3)) v is not sufficiently smooth in order to define time and space partial derivatives of u. Nevertheless, assuming data to be more regular, the answer is positive, as stated below.

THEOREM 6.3. *Assume* (2.1)–(2.10) *hold. Moreover, let* u_0, F, G *satisfy*

$$u_0 \in W^{1,\infty}(\mathbf{R}), \tag{6.25}$$

$$\forall R>0\ \exists M_5(R)>0\colon\ \forall (x_1,t,\xi), (x_2,t,\xi) \in \mathbf{R} \times [0,T] \times [-R,R]$$
$$|F(x_1,t,\xi) - F(x_2,t,\xi)| + |G(x_1,t,\xi) - G(x_2,t,\xi)| \leqslant M_5(R)\, |x_1 - x_2|. \tag{6.26}$$

Then there exists a unique pair (u, z), *with* $u \in C^{0,1}(\mathbf{R} \times [0, T])$ *and* $z \in C^{0,1}([0, T])$, *satisfying* (1.1)–(1.3), (2.11), (2.17).

Proof. Let u be defined by (6.24). Then, by (6.3), (6.25)–(6.26), it is not difficult to check that (1.1) is satisfied. The other conditions (1.2)–(1.3) and (2.11) follow directly from (3.1)–(3.2) and (3.4). Conversely if u solves (1.1)–(1.3) and (2.11), the function v, related to u by (6.24), gives a solution of (P_4) and Theorem 6.1 allows us to conclude the proof.

Remark 6.1. By Theorem 6.3 we get an existence and uniqueness result for the hyperbolic problem which differs from those of [8, 13, 4]. Indeed here u_0 and F may not have a compact support (in $\mathbf{R}$) like, e.g., in [4]. On the other hand some growth conditions on G (not needed in the quoted cases) have to be required. Finally, we recall that assumptions (6.25)–(6.26) are related to the fact that hyperbolic equations do not have smoothing effects on data.

7. Approximation of the Muscle Contraction Problem

In this section we consider the hyperbolic problem describing muscle contraction already mentioned in the Introduction (see (1.1)–(1.3)). Assuming the same hypotheses of [4], we approximate this problem via a parabolic regularization and prove convergence of the approximating solutions.

It will be useful to recall the assumptions on data stated in [4] and the detailed formulation of the problem as well.

We assume that (2.1)–(2.5), (2.8)–(2.10), (6.25), and

$$F, G \in C^0(\mathbf{R} \times [0, T] \times \mathbf{R}), \tag{7.1}$$

$$\begin{gathered} \forall R > 0 \quad \exists M_6(R) > 0\colon \quad \forall (x_1, t, \xi_1), (x_2, t, \xi_2) \\ \in [-R, R] \times [0, T] \times [-R, R] \\ |F(x_1, t, \xi_1) - F(x_2, t, \xi_2)| + |G(x_1, t, \xi_1) - G(x_2, t, \xi_2)| \\ \leqslant M_6(R)\{ |x_1 - x_2| + |\xi_1 - \xi_2| \}, \end{gathered} \tag{7.2}$$

$$\exists N > 0\colon u_0(x) = \mathscr{F}(x, t) = 0 \qquad \forall |x| \geqslant N, \quad \forall t \in [0, T]. \tag{7.3}$$

hold. Let us consider the hyperbolic muscle problem formulated as follows (cf., e.g., [4]).

Problem (P_5). *Find* $u \in C^{0,1}(\mathbf{R} \times [0, T])$ *and* $z \in C^{0,1}([0, T])$ *such that*

$$u(\cdot, t) \text{ has a compact support for any } t \in [0, T], \tag{7.4}$$

$$a < \int_{\mathbf{R}} w(x)\, u(x, t)\, dx < b \qquad \forall t \in [0, T], \tag{7.5}$$

$$z(t) = L\left(\int_{\mathbf{R}} w(x)\, u(x, t)\, dx\right) \qquad \forall t \in [0, T], \tag{7.6}$$

$$u_t(x, t) + z'(t)\, u_x(x, t) + G_z(x, t)\, u(x, t)$$
$$= F_z(x, t) \qquad \text{for a.e.} \quad (x, t) \in \mathbf{R} \times]0, T[, \tag{7.7}$$

$$u(x, 0) = u_0(x) \qquad \text{for a.e.} \quad x \in \mathbf{R}, \tag{7.8}$$

where F_z, G_z are defined by (2.15).

This problem has a unique solution (u, z). Moreover, setting

$$v(x, t) := u(x + z(t), t) \qquad \forall (x, t) \in \mathbf{R} \times]0, T[, \tag{7.9}$$

the pair (v, z) solves Problem (P_4) (cf., e.g., [4]). In order to approximate Problem (P_5), we consider the following sequence. Let $\varepsilon \in]0, 1/N]$ and let

$$G_\varepsilon(x, t, \xi)$$
$$:= \begin{cases} \min\{G(-N, t, \xi) + (N - x)/\varepsilon, G(-1/\varepsilon, t, \xi)\} & \text{if } -x > 1/\varepsilon \\ \min\{G(-N, t, \xi) + (N - x)/\varepsilon, G(x, t, \xi)\} & \text{if } N \leqslant -x \leqslant 1/\varepsilon \\ G(x, t, \xi) & \text{if } |x| < N \\ \min\{G(N, t, \xi) + (x - N)/\varepsilon, G(x, t, \xi)\} & \text{if } N \leqslant x \leqslant 1/\varepsilon \\ \min\{G(N, t, \xi) + (x - N)/\varepsilon, G(1/\varepsilon, t, \xi)\} & \text{if } x > 1/\varepsilon \end{cases} \tag{7.10}$$

for any $(x, t, \xi) \in \mathbf{R} \times [0, T] \times \mathbf{R}$, where N is defined in (7.3). Note hat G_ε still satisfies (7.1)–(7.2) and besides F, G_ε fulfil (2.5)–(2.7), where M_2 and h possibly depend on ε. Let us consider the following approximation problem.

Problem (P_ε). *Find* $u_\varepsilon \in C^0([0, T]; H) \cap L^2(0, T; V)$, $z_\varepsilon \in C^{0,1}([0, T])$ such that $(u_\varepsilon)_t \in L^2(0, T; V') + L^1(0, T; H)$, $wu_\varepsilon \in C^0([0, T]; \; L^1(\mathbf{R}))$, $(G_\varepsilon)_{z_\varepsilon} u_\varepsilon \in L^2(0, T; H)$, and satisfying (2.11)–(2.14) with u, z, F_z, G_z replaced by u_ε, z_ε, F_{z_ε}, $(G_\varepsilon)_{z_\varepsilon}$, respectively, where

$$F_{z_\varepsilon}(x, t) := F(x, t, z_\varepsilon(t)), \qquad (G_\varepsilon)_{z_\varepsilon}(x, t) := G_\varepsilon(x, t, z_\varepsilon(t))$$
$$\forall (x, t) \in \mathbf{R} \times]0, T[. \tag{7.11}$$

By Theorem 2.1, Problem (P_ε) admits a unique solution satisfying (2.17). Recalling (7.9), we set

$$v_\varepsilon(x, t) := u_\varepsilon(x + z_\varepsilon(t), t) \qquad \forall (x, t) \in \mathbf{R} \times]0, T[, \tag{7.12}$$

and $(v_\varepsilon, z_\varepsilon)$ solves Problem $(\mathbf{P}_2)$, where G is replaced by G_ε (cf. Proposition 3.1).

Remark 7.1. The approximating sequence G_ε has been chosen in order to be both globally bounded by a linear function having slope $1/\varepsilon$ and globally Lipschitz continuous (see (2.6) (2.7)). The chosen slope will be useful in the passage to the limit, as we show later.

THEOREM 7.1. *Let $(v_\varepsilon, z_\varepsilon)$ and (v, z) be defined by (7.12) and Problem (P_ε) and by (7.9) and Problem (P_5), respectively. Then we have*

$$v_\varepsilon \to v \qquad \textit{weakly star in } L^\infty(0, T; L^2(\mathbf{R})) \cap L^\infty(\mathbf{R} \times]0, T[), \tag{7.13}$$

$$z_\varepsilon \to z \qquad \textit{weakly star in } W^{1,\infty}(0, T) \tag{7.14}$$

as ε goes to zero.

Proof. Using the same techniques as in Sections 3, 4, it is not difficult to check that there are two positive constants C_{28} and $\tilde{K}$, independent of ε, such that

$$\|v_\varepsilon\|_{L^\infty(0, T; L^2(\mathbf{R}))} + \varepsilon^{1/2} \left\| \frac{dv_\varepsilon}{dx} \right\|_{L^2(\mathbf{R} \times]0, T[)}$$

$$+ \int_0^T \int_{\mathbf{R}} G_\varepsilon(x + z_\varepsilon(t), t, z_\varepsilon(t))\, v_\varepsilon(x, t)\, dx\, dt \leqslant C_{28}, \tag{7.15}$$

$$\int_{\mathbf{R}} |x|^\alpha\, v_\varepsilon(x, t)\, dx \leqslant C_{28} \qquad \forall t \in [0, T], \quad \forall \alpha \in [0, 2], \tag{7.16}$$

$$\|z_\varepsilon\|_T \leqslant \tilde{K}. \tag{7.17}$$

To prove the boundedness of z'_ε, we can show, for instance, that the function Γ_{z_ε} (see (4.23)) is bounded independently of ε (cf. (4.24) and (5.6)–(5.10)). From (2.4)–(2.5), (7.2), (7.3), (7.17), and (4.23) it follows that there is a constant C_{29}, independent of ε, such that

$$|\Gamma_\varepsilon(t, \tau)| \leqslant C_{29} \left(1 + \int_{\mathbf{R}} (1 + |y|)\, G(y + z_\varepsilon(\tau), \tau, z_\varepsilon(\tau))\, v_\varepsilon(y, \tau)\, dy \right)$$

$$\forall \tau \in [0, t], \quad \forall t \in [0, T]. \tag{7.18}$$

By (7.10), (7.17) we have that

$$(1 + |y|)\, G_\varepsilon(y + z_\varepsilon(t), t, z_\varepsilon(t))$$

$$\leqslant C_{30} + C_{31} \varepsilon^{-1} \{ (|y| - N - \tilde{K})^+ \}^2 \qquad \forall t \in [0, T], \quad \forall y \in \mathbf{R}, \tag{7.19}$$

where $(\cdot)^+$ denotes the positive part and, for instance,

$$C_{30} = \max\{(1+|y|)\,G(y,t,\xi) \colon |y| \leqslant N + 2\tilde{K},\ t \in [0,T],\ |\xi| \leqslant \tilde{K}\},$$

$$C_{31} = 2(2\tilde{K} + N + 1) + (2\tilde{K})^{-1}(4\tilde{K} + N + 1).$$

Then, by (7.16), (7.18)–(7.19), we only have to find a constant C_{32}, independent of ε, satisfying

$$\int_{\mathbf{R}} \varepsilon^{-1}\{(|x| - N - \tilde{K})^+\}^2\, v_\varepsilon(x,t)\,dx \leqslant C_{32} \qquad \forall t \in [0,T]. \tag{7.20}$$

To this aim, in Eq. (3.3) (with v and G replaced by v_ε and G_ε, respectively) we take as a test function $\psi_m(x) = \{(|x| - N - \tilde{K})^+\}^2 \exp(-x^2/m)$ and integrate by parts in space and time. Taking (3.4), (7.3), and (7.17) into account, we get

$$\begin{aligned}
&\varepsilon^{-1}\int_{\mathbf{R}} \{(|x| - N - \tilde{K})^+\}^2 \exp(-x^2/m)\, v_\varepsilon(x,t)\,dx \\
&\quad + \varepsilon^{-1}\int_0^t \int_{\mathbf{R}} \{(|x| - N - \tilde{K})^+\}^2 \exp(-x^2/m) \\
&\quad \times G_\varepsilon(x + z_\varepsilon(\tau), \tau, z_\varepsilon(\tau))\, v_\varepsilon(x,\tau)\,dx\,d\tau \\
&= \int_0^t \int_{\{|x| > N + \tilde{K}\}} v_\varepsilon(x,\tau)\{2 - 8m^{-1}|x|\,(|x| - N - \tilde{K}) \\
&\quad + (4x^2 m^{-2} - 2m^{-1})(|x| - N - \tilde{K})^2\} \\
&\quad \times \exp(-x^2/m)\,dx\,d\tau \qquad \forall t \in [0,T].
\end{aligned} \tag{7.21}$$

It is now a standard matter to prove that the left hand side of (7.21) is bounded independently of m (cf. (7.16)). Taking the limit as $m \nearrow +\infty$, the Beppo Levi Monotone Convergence Theorem allows us to obtain (7.20). Hence we have

$$\|z'_\varepsilon\|_T \leqslant C_{23} \tag{7.22}$$

for some constant C_{33} independent of ε, and moreover (see (7.20))

$$v_\varepsilon(x,t) \to 0 \qquad \text{for a.e.} \quad |x| \geqslant N + \tilde{K} \quad \text{and} \quad t \in [0,T].$$

Observing that

$$G_\varepsilon \to G \qquad \text{uniformly in any compact subset of } \mathbf{R} \times [0,T] \times \mathbf{R},$$

we can pass to the limit as in the previous section and the theorem is proved.

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