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A lower bound on the maximum permanent in Λ_n^k

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Abstract

Let P_n^k be the maximum value achieved by the permanent over Λ_n^k, the set of (0, 1)-matrices of order n with exactly k ones in each row and column. Brègman proved that $P_n^k \leqslant k!^{n/k}$. It is shown here that $P_n^k \geqslant k!^t r!$ where $n = tk + r$ and $0 \leqslant r < k$. From this simple bound we derive that $(P_n^k)^{1/n} \sim k!^{1/k}$ whenever $k = \mathrm{o}(n)$ and deduce a number of structural results about matrices which achieve P_n^k. These include restrictions for large n and k on the number of components which may be drawn from Λ_{k+c}^k for a constant $c \geqslant 1$.

Our results can be directly applied to maximisation problems dealing with the number of extensions to Latin rectangles or the number of perfect matchings in regular bipartite graphs.

Keywords: Permanent; (0,1) matrix; Bregman bound; Merriell's conjecture; Regular bipartite graph; Latin rectangle

1. Introduction

If $A = (a_{ij})$ is a square matrix of order n, the *permanent* of A is given by

$$\mathrm{per}(A) = \sum_{\tau} \prod_{i=1}^{n} a_{i\tau(i)},$$

where the sum is over all permutations of $\{1, 2, \ldots, n\}$. We use Λ_n^k for the set of square (0, 1)-matrices of order n, with exactly k ones in each row and in each column. This paper is primarily concerned with the maximum value achieved by the permanent in Λ_n^k and the matrices which achieve that maximum. We define

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$$P_n^k = \max_{A\in\Lambda_n^k} \operatorname{per}(A)$$

and we say that $A \in \Lambda_n^k$ is a *maximising matrix* if $\operatorname{per}(A) = P_n^k$. For $B \in \Lambda_n^k$ we define the *complement* of B (denoted $\overline{B}$) by $\overline{B} = J_n - B \in \Lambda_n^{n-k}$ where $J_n \in \Lambda_n^n$ is the order n matrix consisting entirely of ones.

A theorem of Brègman [3], when applied to matrices in Λ_n^k, shows that

$$P_n^k \leqslant (k!)^{n/k}, \tag{1}$$

with equality if and only if k divides n. In Section 3 we treat the issue of how close this bound comes to being achieved in cases when equality does not hold.

The minimum value p_n^k achieved by the permanent in Λ_n^k has also been studied. Schrijver [16] showed that

$$p_n^k = \min_{A\in\Lambda_n^k} \operatorname{per}(A) \geqslant \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n. \tag{2}$$

For any given k, the base constant $(k-1)^{k-1}k^{2-k}$ is best possible.

Let $\sigma_i(A)$ be the sum of the permanents of all the order i submatrices of a matrix A. By convention we choose that $\sigma_0(A) = 1$ for every A. Any (0, 1)-matrix can be interpreted as a 'chessboard' with certain allowed and other prohibited positions. In this scenario σ_i always counts the number of arrangements of i non-attacking rooks placed on permitted squares (the ones marked with 1's). We follow Godsil [5] in defining the *rook polynomial* of a matrix A by

$$\rho(A) = \sum_{i=0}^{n}(-1)^i\sigma_i(A)x^{n-i}. \tag{3}$$

Note that this is just one of a number of polynomials which are called rook polynomials in the literature. Our main reason for preferring this definition is that it gives the very useful formula (4).

Define the linear operator $\mathscr{T}(\cdot)$ by

$$\mathscr{T}(f(x)) = \int_0^\infty \mathrm{e}^{-x} f(x)\,\mathrm{d}x.$$

Joni and Rota [9] and Godsil [5] independently showed that for any $A \in \Lambda_n^k$,

$$\operatorname{per}(\overline{A}) = \mathscr{T}(\rho(A)). \tag{4}$$

This remarkable formula can be viewed as an inclusion–exclusion result. The permanent on the left hand side is a sum of $n!$ terms, each of which is a product of 0's and 1's. The number of these terms with at least c factors equal to 0 is $(n-c)!\,\sigma_c(A)$, except that this overcounts terms with at least $c+1$ factors equal to 0. By inclusion–exclusion,

$$\operatorname{per}(\overline{A}) = n! - (n-1)!\sigma_1(A) + (n-2)!\sigma_2(A) - \cdots + (-1)^n 0!\sigma_n(A). \tag{5}$$

Since $\mathscr{T}(x^i) = i!$, the right hand side of (5) turns out to be precisely $\mathscr{T}(\rho(A))$.

For each $k \geqslant 0$, define

$$\ell_k = \rho(J_k) = (-1)^k k! \sum_{i=0}^{k} \binom{k}{i} \frac{(-x)^i}{i!}. \tag{6}$$

That is, ℓ_k is a Laguerre polynomial, normalised to be monic. A variant of (6) appears in [15, p. 171]. Note that the Laguerre polynomials are often (see [1], for example) defined by their orthogonality with respect to $\mathscr{T}(\cdot)$. Specifically,

$$\mathscr{T}(\ell_i \ell_j) = \begin{cases} i!^2 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Laguerre polynomials are intimately connected with rook theory. We have,

$$\rho(A) = \sum_{i=0}^{n} \sigma_i(\overline{A}) \ell_{n-i}. \tag{8}$$

Eq. (8) appears explicitly in [5]. The idea that a rook polynomial can be determined from the complementary board goes back at least as far as [15].

Throughout this paper we use $\oplus$ to denote direct sum and use the word *components* to describe the matrices which are summands in a direct sum. We also use $m \cdot A$ as shorthand for a direct sum of m copies of A. In formulae involving both '$\oplus$' and '$\cdot$' we give '$\cdot$' priority in order of evaluation over '$\oplus$'. Note that the rook polynomial is multiplicative on components, meaning that $\rho(A \oplus B) = \rho(A)\rho(B)$.

In Section 2 we prove some elementary bounds on the subpermanent sums σ_i. These are used in Section 3 to prove a simple lower bound on P_n^k and asymptotics for $(P_n^k)^{1/n}$. In Section 4 we recall conjectures due to Merriell as to which matrices achieve P_n^k. The two sections after that are devoted to proving various results about these maximising matrices. Some of our results support Merriell's ideas while others show their limitations. Finally, in Section 7 we provide a summary of the paper.

It is worth remarking that the results on permanents proved in this paper transfer to results on well known equivalent problems, such as those given in [12, Section 8.2]. In particular, maximising the permanent in Λ_n^k is equivalent to maximising the number of perfect matchings in a k-regular bipartite graph on $2n$ vertices and also to maximising the number of extensions of an $(n-k) \times n$ Latin rectangle to $(n-k+1) \times n$ Latin rectangles.

2. Elementary bounds on subpermanent sums

We derive, for subsequent use, some elementary bounds on the subpermanent sums $\sigma_i(A)$ for matrices $A \in \Lambda_n^k$. For $i \geqslant 1$ define

$$\xi_i = \left(\sigma_i(A) \big/ \tbinom{n}{i}\right)^{1/i}.$$

Since $\rho(A)$ has positive real roots [8] a classical result of Maclaurin [7, Theorem 52] tells us the ξ_i form a decreasing sequence. Since $\xi_1 = k$ and $\xi_n = (\mathrm{per}(A))^{1/n}$, our first lemma follows from (2):

Lemma 1. *For $A \in \Lambda_n^k$ and $1 \leqslant i \leqslant n$,*

$$\frac{(k-1)^{k-1}}{k^{k-2}} \leqslant \xi_i \leqslant k.$$

In the generality in which it is stated, both the upper and lower bounds in Lemma 1 are best possible. However, for small i the lower bound is improved in our next result.

Lemma 2. *For $A \in \Lambda_n^k$ and $i \leqslant k$,*

$$\sigma_i(A) \geqslant \binom{n}{i} \frac{k!}{(k-i)!}.$$

Proof. There are $\binom{n}{i}$ ways to choose i rows of A and each row has k positive entries. A simple inductive argument (or alternatively, Corollary 7.4.2 of [4]) shows that each selection of i rows contributes at least $k!/(k-i)!$ to $\sigma_i(A)$. $\square$

We also have this asymptotic result:

Lemma 3. *For $1 \leqslant k \leqslant n$ and fixed i*

$$\sigma_i(A) = \frac{1}{i!}(kn)^i - \mathrm{O}(k^i n^{i-1})$$

uniformly over $A \in \Lambda_n^k$ as $n \to \infty$. For arbitrary i, we always have $\sigma_i(A) \leqslant \frac{1}{i!}(kn)^i$.

Proof. Each $A \in \Lambda_n^k$ has kn ones, and σ_i counts sets of i of these in which no two ones are collinear (lie in the same row or the same column). The number of collinear pairs of ones is $\frac{1}{2}kn(2k-2)$, so

$$\binom{kn}{i} - \frac{1}{2}kn(2k-2)\binom{kn-2}{i-2} \leqslant \sigma_i(A) \leqslant \binom{kn}{i} \leqslant \frac{1}{i!}(kn)^i.$$

For any fixed i the lower bound is $\frac{1}{i!}(kn)^i - \mathrm{O}(k^i n^{i-1})$, proving the result. $\square$

3. Close to the Brègman bound

In this section we look at how close the Brègman bound (1) comes to being achieved. Note that Brègman himself showed that his bound for Λ_n^k is achieved exactly if and only if k divides n. Our first theorem gives a lower bound on the maximum permanent. This bound coincides with the Brègman upper bound when k divides n.

Theorem 4. *Let $n = tk + r$ with $t \geqslant 1$ and $0 \leqslant r < k$. Then $P_n^k \geqslant k!^t r!$*

Proof. It suffices to prove the case $t = 1$, as the remaining cases will then follow by appending $t-1$ copies of J_k. Let $Y \in \Lambda_k^{k-r}$ be arbitrary and consider $X = \overline{J_r \oplus \overline{Y}} \in \Lambda_{k+r}^k$. By combining (4), (6)–(8), we see that

$$\mathrm{per}(X) = \mathscr{T}\big(\rho(J_r)\rho(\overline{Y})\big) = \mathscr{T}\left(\ell_r \sum_{i=0}^{k} \sigma_{k-i}(Y)\ell_i\right) = r!^2\sigma_{k-r}(Y).$$

(Indeed, the reader is invited to draw a picture then derive this result by an elementary argument.) Now by employing Lemma 2, we have $\mathrm{per}(X) \geqslant r!^2\binom{k}{k-r}(k-r)! = k!r!$ □

This theorem tells us the Brègman bound is achieved to within a constant factor when n falls just short of being a multiple of k. In fact:

Corollary 5. *If $a = \mathrm{o}(k)$ and $n \equiv -a \bmod k$ then as $k \to \infty$,*

$$\frac{k!^{n/k}}{P_n^k} = \mathrm{O}(\mathrm{e}^a).$$

Proof. Using $r = k - a$ and Stirling's formula gives,

$$\frac{k!^{r/k}}{r!} = \mathrm{O}(1)\sqrt{\frac{k^{1-a/k}}{k-a}}\left(\frac{k}{k-a}\right)^{k-a} = \mathrm{O}(\mathrm{e}^a),$$

from which the Corollary follows. □

We can also use Theorem 4 to establish the nth root of the maximum permanent.

Theorem 6. *Suppose that $k = \mathrm{o}(n)$ as $n \to \infty$. Then*

$$\big(P_n^k\big)^{1/n} \sim k!^{1/k}.$$

Proof. Let $\mu = k!^{1/k}$ and note that Stirling's formula and $k = \mathrm{o}(n)$ imply that $\mu = \mathrm{o}(n)$. Suppose that $n = tk + r$ for $0 \leqslant r < k$, in which case Brègman's bound (1) and Theorem 4 tell us that

$$\mu \geqslant \big(P_n^k\big)^{1/n} \geqslant \big(k!^{(n-r)/k}r!\big)^{1/n} = \mu\left(\frac{r!}{\mu^r}\right)^{1/n}.$$

Let $f(r) = r!/\mu^r$ and note that $f(r+1)/f(r) = (r+1)/\mu$. It follows that for any given $\mu \geqslant 1$ the minimum of $f(r)$ over positive integers is achieved when $r = \lfloor\mu\rfloor$. Hence

$$\mu \geqslant \big(P_n^k\big)^{1/n} \geqslant \mu\left(\frac{\lfloor\mu\rfloor!}{\mu^{\lfloor\mu\rfloor}}\right)^{1/n} = \mu\mathrm{e}^{-\mathrm{O}(1)\lfloor\mu\rfloor/n} = \mu(1-\mathrm{o}(1)),$$

by using Stirling's formula again. This gives the desired asymptotic result. □

4. Merriell's conjectures

The question of finding the maximum permanent in Λ_n^k when k does not divide n is Problem 4 in [12]. An attempt to answer this question was made by Merriell [11], who solved the $k \leqslant 3$ case and conjectured a partial answer for larger values.

Let J_r and Z_r denote $r \times r$ blocks of ones and zeroes respectively. We use J without a subscript to denote a (not necessarily square) block of ones of unspecified, but implied dimensions. Also, let D_r denote the complement of the order r identity matrix. Building on Brègman's earlier result, Merriell showed that

$$B = (t - r) \cdot J_k \oplus r \cdot D_{k+1} \tag{9}$$

maximises the permanent in Λ_{kt+r}^k provided $0 \leqslant r < k \leqslant 3$ and $t \geqslant r$. In the case of Λ_5^3, the permanent is maximised by the circulant matrices. Bol'shakov [2] confirmed Merriell's results and extended them to a proof of (9) for all t when $r = 1$ and $k = 4$.

Merriell also made several conjectures which we discuss now. Suppose $k \leqslant n \leqslant 2k$ and that $k \geqslant 5$ or n is even. According to Conjectures 1 and 2 of [11], the maximum permanent in Λ_n^k should be achieved by a matrix with block structure

$$\begin{pmatrix} A & J \\ J & B \end{pmatrix}, \tag{10}$$

where A and B are square matrices with orders that differ by at most 1. Furthermore, A and B should be chosen to maximise their individual permanents.

Conjecture 3 of [11] can be stated as follows. Let $n = tk + r$ for integers $k \geqslant 5$, $t \geqslant 1$ and $r \geqslant 0$. Then the maximum permanent in Λ_n^k is achieved by

$$\begin{cases} (t - r) \cdot J_k \oplus r \cdot D_{k+1} & \text{when } r \leqslant \min\{t, k - 3\}, \\ (t - 1) \cdot J_k \oplus X_{k,r} & \text{when } r = k - 2 \text{ or } r = k - 1, \end{cases}$$

$$\text{where} \quad X_{k,k-2} = \begin{pmatrix} J & I_{k-1} \\ I_{k-1} & J \end{pmatrix} \quad \text{and} \quad X_{k,k-1} = \begin{pmatrix} J & Z_{k-1} \\ I_k & J \end{pmatrix}. \tag{11}$$

Unfortunately, Merriell's conjectures are known to be fatally flawed. Many known counterexamples are discussed in [10,17], and some more will result from Theorem 10. However, some of the results we prove, particularly Corollaries 9 and 13, are in the spirit of these conjectures, as is (13).

Merriell's ideas developed in two natural directions from Brègman's proof that J_k's are advantageous for maximising permanents. One option is to try to maximise the number of copies of J_k, while the other approach is to make the components as close as possible in size to J_k. The latter approach dictates, where possible, taking r copies of D_{k+1}, where r is the remainder of n when divided by k.

Let $m \geqslant 5$ and $t \geqslant 2$ be integers and let $k = (m - 1)t$. The maximising matrices in Λ_{mt}^k were identified in [10], and are the complement of Brègman's maximising matrices in Λ_{mt}^t. It follows from (4) and (6) that $P_{mt}^k = \mathscr{T}((\ell_k)^m)$. Also, a standard result on derangements says that $\mathrm{per}(D_{k+1})$ is the nearest integer to

$$\frac{(k+1)!}{e}. \tag{12}$$

Hence in $\Lambda^k_{(m-1)t^2+t}$, for fairly small values of m and t, we can numerically compare the two approaches favoured by Merriell. Doing this shows that

$$\operatorname{per}(t \cdot D_{k+1}) > \operatorname{per}\big((t-1) \cdot J_k \oplus X\big) \tag{13}$$

for all $X \in \Lambda^k_{mt}$, at least whenever $mt \leqslant 500$. For each m the numerical trend suggests that the relative gap in (13) widens as t grows. This supports conjecture (11) by hinting that there may be no $o(k)$ bound on the number of copies of D_{k+1} which can occur in maximising matrices. However, in Theorem 10 we shall prove that Merriell erred slightly in the number of D_{k+1}'s he predicted.

5. Matrices with density below one half

In this section we look for maximising matrices in Λ^k_n where $k \leqslant \frac{1}{2}n$. In this case it is possible for the matrices to be partly decomposable and the evidence suggests that this is typically what happens. This observation is based on the small order examples computed in [10] and on [17], in which the follow results were proved. For any given k there is a finite set S_k of matrices such that, for arbitrary n, any maximising matrix in Λ^k_n is a direct sum of components chosen (possibly with repetition) from S_k. Fewer than k of the components of any maximising matrix in Λ^k_n differ from J_k.

In some inexact sense then, we know most of the structure of sparse maximising matrices and the focus now centres on characterising the "remnant" components which are not copies of J_k. Identifying these remnants in general is a difficult task, partly due to the tendency for small examples not to fit the general trend [10].

For our first theorem we need a result from [6], where it was proved in the context of extensions to Latin rectangles. For any (0, 1)-matrix A let $s(A)$ denote the number of order 2 submatrices of A which contain only 1's. For fixed c,

$$\operatorname{per}(\overline{A}) = n!\left(\frac{n-c}{n}\right)^n \exp\left(\frac{c}{2n} + \frac{c(3c-1)}{6n^2} + f + \frac{s(A)}{n^4} + \frac{(4c-2)s(A)}{n^5} + O(n^{-5})\right), \tag{14}$$

uniformly over $A \in \Lambda^c_n$, as $n \to \infty$. The function f is specified in [6], but we only need to know that it is independent of A and that $f = O(n^{-3})$. (Note that when (14) was quoted in [17] the '6' in the denominator of the $O(n^{-2})$ term was accidentally erased by the typesetters.)

Theorem 7. *For each pair (a, b) such that $0 \leqslant a < b - 1$ there exists $k_{a,b}$ such that $\operatorname{per}(U \oplus V) < \operatorname{per}(X \oplus Y)$ for every $k > k_{a,b}$ and choice of $U \in \Lambda^k_{k+a}$, $V \in \Lambda^k_{k+b}$, $X \in \Lambda^k_{k+a+1}$ and $Y \in \Lambda^k_{k+b-1}$.*

Proof. For a constant c and $k \to \infty$, (14) implies that

$$\begin{aligned}\mathrm{per}(A) &= (k+c)!\left(\frac{k}{k+c}\right)^{k+c} \exp\left(\frac{c}{2k+2c} + \frac{c(3c-1)}{6(k+c)^2} + \mathrm{O}(k^{-3})\right)\\ &= (k+c)!\left(\frac{k}{k+c}\right)^{k+c} \exp\left(\frac{c}{2k} + \frac{c}{6k^2} + \mathrm{O}(k^{-3})\right)\end{aligned}$$

uniformly over $A \in \Lambda_{k+c}^{k}$, given that $s(\overline{A}) = \mathrm{O}(k)$. It follows that

$$\begin{aligned}\frac{\mathrm{per}(U)\,\mathrm{per}(V)}{\mathrm{per}(X)\,\mathrm{per}(Y)} &= \left(\frac{k+a+1}{k+a}\right)^{k+a} \left(\frac{k+b-1}{k+b}\right)^{k+b-1} \exp\left(\mathrm{O}(k^{-3})\right)\\ &= \exp\left(\frac{a-b+1}{2k^2} + \mathrm{O}(k^{-3})\right).\end{aligned}$$

Hence $\mathrm{per}(U \oplus V) < \mathrm{per}(X \oplus Y)$ provided k is large enough and $a - b + 1 < 0$. □

Corollary 8. *For fixed t and all large enough k the maximising matrices in Λ_{2k+t}^{k} are either fully indecomposable or have two components whose orders differ by at most* 1. *In the latter case the components should be chosen to maximise their $s(\cdot)$ values.*

The comment about $s(\cdot)$ follows from (14) and the fact, proved in [17], that for general n and k any matrix maximising $s(\cdot)$ in Λ_n^k is the complement of a matrix maximising $s(\cdot)$ in Λ_n^{n-k}.

Corollary 9. *For each fixed $r \geqslant 2$ there is a k_r such that the following statements hold for any given $k > k_r$:*

(i) *There are finitely many n for which maximising matrices in Λ_n^k can contain a component from Λ_{k+c}^{k} for any c satisfying $2 \leqslant c \leqslant r$.*
(ii) *Maximising matrices in Λ_{tk+r}^{k}, where $t \geqslant r$, are either isomorphic to $(t-r) \cdot J_k \oplus r \cdot D_{k+1}$ or contain fewer than t components.*

Proof. By Theorem 7, if we choose k large enough then maximising matrices in Λ_n^k can never contain both J_k and some $X \in \Lambda_{k+c}^{k}$ (where $2 \leqslant c \leqslant r$) as components. Part (i) then follows because for large enough n these matrices must contain J_k, as was proved in [17]. For part (ii), let maximising $A \in \Lambda_{tk+r}^{k}$ have largest component C. Observe that components of A are of order at least k, so there cannot be more than t of them. Supposing there are t of them, C must have order $k + c$ for some $c \leqslant r$. If $c = 1$ then every component of A must be either J_k or D_{k+1}, so $(t-r) \cdot J_k \oplus r \cdot D_{k+1}$ is the only option. If $c > 1$ then there must be a J_k, otherwise the total orders of the components would exceed $t(k+1) \geqslant tk + r$. But the presence of J_k and C then contradicts the choice of k. □

When studying the preceding results it is worth noting that there is no known example of a maximising matrix in any Λ_n^k which has fewer than the maximum possible number, $\lfloor n/k \rfloor$, of components. [1] Corollary 9(ii) therefore suggests a weakened form of Merriell's conjecture (11). A limitation in that direction, though, is given by the following:

Theorem 10. *If* $1 \leqslant a = \mathrm{o}(\log k)$ *then for sufficiently large k no maximising matrix may contain* $k - a$ *components which are copies of* D_{k+1}.

Proof. Let $B = (k-a) \cdot D_{k+1} \in \Lambda_n^k$ where $n = (k-a)(k+1)$. Hence by (12),

$$\mathrm{per}(B) = \left(\frac{(k+1)!}{e} + \mathrm{o}(1)\right)^{k-a} = \left(\frac{(k+1)!}{e}\right)^{k-a} (1 + \mathrm{o}(1/k!)).$$

Hence

$$\frac{k!^{n/k}}{\mathrm{per}(B)} = \frac{1}{(1+\mathrm{o}(1/k!))}\left(\frac{k!^{1+1/k}e}{(k+1)!}\right)^{k-a} \sim \frac{\sqrt{2\pi k}}{e}.$$

The result now follows by comparison with Corollary 5. □

This last result means that if $3 \leqslant a = \mathrm{o}(\log k)$ and $n \equiv -a \bmod k$ then Merriell's conjecture (11) fails for all sufficiently large k, because it asserts that $\mathrm{per}(B) = P_n^k$.

6. Density greater than one half

If $k > \frac{1}{2}n$ then matrices in Λ_n^k are necessarily fully indecomposable but if we consider their complements there is some hope that they will be partly decomposable. From [17] we know that statements similar to those in the previous section hold. Namely, for any given k there is a finite set $\overline{S}_k$ of matrices such that, for arbitrary n, any maximising matrix in Λ_n^{n-k} is the complement of a direct sum of elements of $\overline{S}_k$. We call the matrices in the direct sum the *complementary components*. Fewer than k of the complementary components of any maximising matrix in Λ_n^{n-k} differ from J_k.

One difference from the previous section is that there are a number of examples known from [10] in which maximising matrices in Λ_n^k have fewer than the maximum possible number of complementary components. Maximising matrices in Λ_7^4 and Λ_9^5 are the complements of fully indecomposable matrices, while maximising matrices in Λ_6^4 and Λ_9^6 are the complements of matrices with just two components. Despite these examples, we can prove results similar to some of those in the previous section.

Theorem 11. *Let* $b \geqslant a \geqslant 1$ *be fixed integers and* $\varepsilon > 0$ *a fixed real number. Then for all sufficiently large* k,

[1] However, see the beginning of the next section in this regard.

$$\mathrm{per}(\overline{X \oplus Y}) = k!^2 k^{a+b} \left(\frac{b}{a}\right)^{(b-a)/2} I_{b-a}\bigl(2\sqrt{ab}\bigr)\chi \tag{15}$$

for every $X \in \Lambda_{k+a}^k$ *and* $Y \in \Lambda_{k+b}^k$. *Here* χ *depends on* X *and* Y *but satisfies* $|\chi - 1| < \varepsilon$, *and* I_n *denotes the modified Bessel function of the first kind of order* n.

Proof. Using (15) as the definition of χ, we are required to show that as $k \to \infty$ we can confine χ within the interval $(1 - \varepsilon, 1 + \varepsilon)$. By using (4), (8) and (7) in turn we have

$$\begin{aligned}\mathrm{per}(\overline{X \oplus Y}) &= \mathscr{T}(\rho(X)\rho(Y))\\ &= \mathscr{T}\left(\sum_{i=0}^{k+a} \sigma_{k+a-i}(\overline{X})\ell_i \sum_{i=0}^{k+b} \sigma_{k+b-i}(\overline{Y})\ell_i\right)\\ &= \sum_{i=0}^{k+a} i!^2 \sigma_{k+a-i}(\overline{X})\sigma_{k+b-i}(\overline{Y}).\end{aligned} \tag{16}$$

Noting that $\overline{X} \in \Lambda_{k+a}^a$ and $\overline{Y} \in \Lambda_{k+b}^b$, we use Lemma 3 to estimate the subpermanent sums. For a fixed j,

$$\begin{aligned}&(k+a-j)!^2 \sigma_j(\overline{X})\sigma_{b-a+j}(\overline{Y})\\ &\quad= (k+a-j)!^2 \frac{(a(k+a))^j (b(k+b))^{b-a+j}}{j!(b-a+j)!}(1 + \mathrm{O}(k^{-1}))\\ &\quad= \frac{k!^2 k^{a+b} (ab)^j b^{b-a}}{j!(b-a+j)!}(1 + \mathrm{O}(k^{-1})).\end{aligned} \tag{17}$$

Note that by definition of I_n we have

$$\sum_{j=0}^{\infty} \frac{(ab)^j}{j!(b-a+j)!} = (ab)^{(a-b)/2} I_{b-a}\bigl(2\sqrt{ab}\bigr). \tag{18}$$

The range of χ can now be found by reversing the summation in (16), evaluating it using (17), and comparing the result to (18). Since Lemma 3 guarantees that (17) is an upper bound for all j, we find that $\chi \leqslant 1 + \mathrm{O}(k^{-1})$. Moreover, the same results allow us the accuracy to take any finite number of terms from the infinite sum in (18), and hence bound χ within an arbitrary constant of the upper bound. □

We can use Theorem 11 to prove a complementary result to Theorem 7.

Theorem 12. *For each pair* (a, b) *such that* $0 \leqslant a < b - 1$ *there exists* $k_{a,b}$ *such that* $\mathrm{per}(\overline{U \oplus V}) < \mathrm{per}(\overline{X \oplus Y})$ *for every* $k > k_{a,b}$ *and choice of* $U \in \Lambda_{k+a}^k$, $V \in \Lambda_{k+b}^k$, $X \in \Lambda_{k+a+1}^k$ *and* $Y \in \Lambda_{k+b-1}^k$.

Proof. We first prove the $a = 0$ case, where $U = J_k$. From (16) and Lemma 3 we find

$$\operatorname{per}(\overline{U \oplus V}) = k!^2\sigma_b(\overline{V}) = k!^2\frac{b^b(k+b)^b}{b!}(1 - \mathrm{O}(k^{-1}))$$
$$= k!^2k^b\frac{b^b}{b!}(1 - \mathrm{O}(k^{-1})).$$

By comparison, Theorem 11 says that

$$\operatorname{per}(\overline{X \oplus Y}) = k!^2k^b(b-1)^{(b-2)/2}I_{b-2}\big(2\sqrt{b-1}\big)\chi.$$

Note that $(b-1-j)!/(b-2+j)! \geqslant (b-1)^{1-2j}$ by the AM–GM inequality and hence

$$(b-1)^{(b-2)/2}I_{b-2}\big(2\sqrt{b-1}\big)$$
$$= \sum_{j=0}^{\infty}\frac{(b-1)^{b-2+j}}{j!(b-2+j)!}$$
$$> \frac{(b-1)^{b-1}}{(b-1)!}\sum_{j=0}^{b-1}\frac{(b-1)!}{j!(b-1-j)!}(b-1)^{j-1}\frac{(b-1-j)!}{(b-2+j)!}$$
$$\geqslant \frac{(b-1)^{b-1}}{(b-1)!}\sum_{j=0}^{b-1}\binom{b-1}{j}(b-1)^{-j} = \frac{b^b}{b!}.$$

The result for $a = 0$ then follows from the strict inequality above.

For $1 \leqslant a < b-1$ our theorem will follow if we can show that

$$\left(\frac{b}{a}\right)^{(b-a)/2}I_{b-a}\big(2\sqrt{ab}\big) < \left(\frac{b-1}{a+1}\right)^{(b-a-2)/2}I_{b-a-2}\big(2\sqrt{ab+b-a-1}\big). \tag{19}$$

The inequality (19) can be computed with enough precision to confirm it whenever $a + b < 100$. So we assume that $a + b \geqslant 100$ and divide into two cases according to whether or not

$$b - a > \tfrac{42}{23}(b+a)^{1/5} \tag{20}$$

holds. In the first case we suppose that it does. In particular this means that

$$b - a \geqslant \left\lceil \tfrac{42}{23}100^{1/5}\right\rceil = 5. \tag{21}$$

Recursively define polynomials $u_i(t)$ in the variable $t = (b-a)/(b+a)$ by

$$u_{i+1}(t) = \frac{1}{2}t^2(1-t^2)\frac{\mathrm{d}}{\mathrm{d}t}u_i(t) + \frac{1}{8}\int_0^t(1-5t^2)u_i(t)\,\mathrm{d}t,$$

for $i \geqslant 1$ and $u_0(t) = 1$. Then by (21) and a result of Olver [13, Eq. (7.18)] there is some $\varepsilon_{a,b}$ satisfying

$$|\varepsilon_{a,b}| \leqslant \frac{12\,566}{(b-a)^{15}} \tag{22}$$

for which

$$\begin{aligned}
&\left(\frac{b}{a}\right)^{(b-a)/2} I_{b-a}\big(2\sqrt{ab}\big)\\
&\quad= \frac{\mathrm{e}^{b+a}}{\sqrt{2\pi(b+a)}}\left[\sum_{i=0}^{14}\frac{u_i(t)}{(b-a)^i} + \varepsilon_{a,b}\right]\\
&\quad= \frac{\mathrm{e}^{b+a}}{\sqrt{2\pi(b+a)}}\left[X - \frac{(b-a)^2}{(b+a)^3}Y + \frac{(b-a)^4}{(b+a)^6}Z + R_{a,b} + \varepsilon_{a,b}\right].
\end{aligned} \tag{23}$$

Here X, Y and Z are polynomials in $(b+a)^{-1}$ with non-negative coefficients found by extracting the three lowest order terms from each of the polynomials $u_i(t)$. All higher order terms contribute to $R_{a,b}$. Using monotonicity, we substitute $b + a = 100$ into X, Y and Z to prove $0 < X - 1 < 1/795$, $0 < Y - 5/24 < 1/243$ and $0 < Z - 385/1152 < 1/51$.

To bound $R_{a,b}$ we observe that it is a sum of 78 terms of the form $c(b-a)^i(b+a)^{-j}$ for integers i and j satisfying $j - 2 > i \geqslant 6$ and a coefficient c depending only on the term. We bound the magnitude of such a term by $|c|100^{i-j+2}(b-a)(b+a)^{-3}$ on the basis that $b - a < b + a$ and $b + a \geqslant 100$. Separating the terms according to their sign shows that

$$|R_{a,b}| < \frac{(b-a)}{86(b+a)^3}. \tag{24}$$

Also (20), (22) and $b + a \geqslant 100$ together show that

$$|\varepsilon_{a,b}| < \left(\frac{23}{42}\right)^{16}\frac{12\,566}{(b+a)^{1/5}}\frac{(b-a)}{(b+a)^3} < \frac{(b-a)}{3(b+a)^3}. \tag{25}$$

We are at last in a position to judge what happens when a and b are replaced by $a + 1$ and $b - 1$, respectively. Making this change to the quantity in brackets in (23) will increase it by some quantity Δ. Using (24) and (25) to bound $R_{a,b}$, $R_{a+1,b-1}$, $\varepsilon_{a,b}$ and $\varepsilon_{a+1,b-1}$, we find that

$$\begin{aligned}
\Delta > {}& \frac{(b-a)^2 - (b-a-2)^2}{(b+a)^3}Y - \frac{(b-a)^4 - (b-a-2)^4}{(b+a)^6}Z\\
&- \frac{b-a+b-a-2}{(b+a)^3}\left(\frac{1}{3} + \frac{1}{86}\right)
\end{aligned}$$

$$> \frac{b-a-1}{(b+a)^3}\left[4Y - \frac{8(b-a)^2}{(b+a)^3}Z - \frac{2}{3} - \frac{2}{86}\right]$$
$$> \frac{b-a-1}{(b+a)^3}\left[\frac{5}{6} - \frac{8}{100}\left(\frac{385}{1152} + \frac{1}{51}\right) - \frac{2}{3} - \frac{2}{86}\right]$$
$$> \frac{b-a-1}{9(b+a)^3}.$$

Since Δ is positive we conclude that (19) holds in the case under consideration.

This leaves the second case, in which (20) does not hold. Throughout this case we will implicitly use the same term by term technique for finding bounds as we used in the derivation of (24), except that now we use $\frac{42}{23}(b+a)^{1/5}$ rather than $b+a$ as the upper bound for $b-a$.

We again employ an asymptotic expansion based on the work of Olver [14, p. 269]. For any non-negative integer n and real number $z > \max\left\{\frac{9}{2}n^2, 50\right\}$,

$$I_n(z) = \frac{e^z}{\sqrt{2\pi z}}\left[1 + \sum_{i=1}^{5} \frac{(4n^2-1^2)(4n^2-3^2)\cdots(4n^2-(2i-1)^2)}{i!(-8z)^i} + \delta_1\right], \tag{26}$$

where $3|\delta_1|$ does not exceed the magnitude of the final term of the sum.

We next use $a+b \geqslant 100$ and the negation of (20) to argue that we are entitled to apply (26) in the examples of present interest. We begin by showing that $a > \frac{9}{10}b$ since otherwise

$$\tfrac{1}{10}b \leqslant b-a \leqslant \tfrac{42}{23}(b+a)^{1/5} \leqslant \tfrac{42}{23}(2b)^{1/5}$$

which is impossible since $b \geqslant 50$. Now $a > \frac{9}{10}b$ means that

$$2\sqrt{ab+b-a-1} > 2\sqrt{ab} > 2\sqrt{\tfrac{9}{10}}b > \sqrt{\tfrac{9}{10}}(a+b) > 90 \tag{27}$$

and hence

$$\frac{(b-a)^2}{2\sqrt{ab}} \leqslant \left(\frac{42}{23}\right)^2 \frac{\sqrt{10}(a+b)^{2/5}}{3(a+b)} < \frac{2}{9}. \tag{28}$$

Together, (27) and (28) are the justification we sought. They entitle us to use (26) to get

$$\frac{I_{b-a-2}\left(2\sqrt{ab+b-a-1}\right)}{I_{b-a}\left(2\sqrt{ab}\right)}$$
$$= 1 + \frac{4v}{w} + \frac{8v^2}{w^2} + \frac{80v^3+21v}{6w^3} + \frac{128v^4+84v^2+9v}{6w^4}$$
$$+ \frac{15\,488v^5+17\,280v^3+2880v^2+2745v}{480w^5} + \delta_2, \tag{29}$$

where $v = b - a - 1$, $w = a + b$ and $|\delta_2| < \frac{1}{5}vw^{-3}$. We also have,

$$\frac{a+1}{b-1}\left(\frac{ab-a}{ab+b}\right)^{(b-a)/2} = 1 - \frac{4v}{w} + \frac{8v^2}{w^2} - \frac{80v^3+16v}{6w^3} + \frac{64v^4+32v^2}{3w^4} - \frac{4v(11v^2+1)(11v^2+9)}{15w^5} + \delta_3, \tag{30}$$

where $|\delta_3| < \frac{1}{9}vw^{-3}$. The right hand sides of both (29) and (30) lie in the interval $\left[\frac{4}{5}, \frac{5}{4}\right]$. Multiplying these two equations, we get

$$\frac{a+1}{b-1}\left(\frac{ab-a}{ab+b}\right)^{(b-a)/2} \frac{I_{b-a-2}\left(2\sqrt{ab+b-a-1}\right)}{I_{b-a}\left(2\sqrt{ab}\right)} = 1 + \frac{(b-a-1)}{(a+b)^3}\left(\frac{5}{6} - \delta_4\right),$$

for some δ_4 satisfying $|\delta_4| < \frac{2}{3} < \frac{5}{6}$. This completes the proof of (19) and hence also the theorem. □

Corollary 13. *Let t be fixed as $k \to \infty$. Maximising matrices in Λ_{2k+t}^{k+t} either have a fully indecomposable complement or else have two complementary components whose orders differ by at most* 1.

Corollary 14. *Let t and $\varepsilon > 0$ be fixed as $k \to \infty$. If $t = 2a$ for an integer a then*

$$P_{2k+t}^{k+t} \geqslant k!^2 k^t I_0(t)\chi$$

while if $t = 2a + 1$ then

$$P_{2k+t}^{k+t} \geqslant k!^2 k^t \sqrt{\frac{a+1}{a}} I_1\left(2\sqrt{a(a+1)}\right)\chi,$$

where $|\chi - 1| < \varepsilon$, as before.

It seems likely that matrices with fully indecomposable complement will not exceed these values and hence that the $\geqslant$ signs in Corollary 14 can be replaced by equality signs.

7. Summary

We have shown in Theorem 4 that P_n^k, the maximum permanent in Λ_n^k, is at least $k!^t r!$ when $n = tk + r$ and $0 \leqslant r < k$. We used this in Theorem 6 to find the asymptotic value of the nth root of P_n^k and later to prove some structural results about matrices achieving P_n^k. We showed in Corollary 8 that two relatively small components should be made the same order, and proved a corresponding result

(Corollary 13) for matrices with density greater than 1/2. Furthermore, in Corollary 9 we deduced that the only small components which can occur in abundance are J_k, a block of ones, and D_{k+1}, complement of the identity matrix. We also slightly improved, in Theorem 10, the known upper bound on the number of D_{k+1}'s.

Finally note that, although J_k's are generally to be favoured when building maximising matrices, we have seen examples in the results just mentioned (Corollaries 8 and 13) and in Eq. (13) where the best arrangement does not use the maximum possible number of J_k's.

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