Binary operations on Jordan algebras and orthogonal normal models

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Abstract

Jordan algebras are used to present normal orthogonal models in a canonical form. Binary operations are defined on these algebras, which enable us to build complex models based on simpler ones. Properties of these operations and their relation with balanced models are studied.

The canonical model formulation is interesting because it leads to complete sufficient statistics. These statistics are then used to obtain estimators in order to test hypothesis on the model parameters.

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1. Introduction

Jordan algebras were introduced (see [1]) to provide an algebraic foundation for Quantum Mechanics. Later these structures were applied (see [2–5]) to study estimation problems, namely to obtain uniform minimum variance unbiased estimators (UMVUE). They are now called quadratic vector spaces because they are vector spaces, constituted by symmetric matrices that contain the square of every matrix in the space. For priority’s sake we will use the first name. We are interested in commutative Jordan algebras, where matrices commute. These algebras have (see [4]) unique principal basis constituted by orthogonal projection matrices, all of them mutually orthogonal.

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Furthermore, two binary operations are defined on Jordan algebras: a product (⊗) and a restricted product (∗). Since each Jordan algebra is associated with an orthogonal model, we can say that these operations act on models, that originate new models. In fact, the product ⊗ crosses models on which it acts whereas the product ∗ nests the second model in the first. We thus get two tools to build orthogonal models. Properties of these operations are analyzed, as well as some other interesting properties of Jordan algebras and their connections to orthogonal models. A stimulating overview of orthogonal models may be seen in the initial chapters of [6], whereas there are several monographs (for instance, see [7]) devoted to the use of Jordan algebras in statistics.

Jordan algebras enable a unified presentation of wide classes of normal models. A normal orthogonal model belongs to the class associated with a Jordan algebra if:

1. the mean vector \( \mu^n \) of the observation vector \( y^n \) belongs to the range space \( R(B) \) of a symmetric matrix \( B \) belonging to the algebra and having rank \( (B) < n \);
2. the covariance matrix \( V \) of \( y^n \) also belongs to that commutative Jordan algebra.

We will give a canonical formulation for such classes of models. The parameters in such canonical formulations will be the canonical parameters.

In the next section, operations on Jordan algebras—more exactly on their principal basis—will be defined allowing us to build complex Jordan algebras from simple ones, thus permitting the construction of complex orthogonal models from simple models. In the third section the canonical formulation (see [8]) of orthogonal models is presented, as well as its connection with the associated Jordan algebra, and UMVUE are obtained for the parameters of the model. Section 4 presents three kinds of models, built-up using the operations defined on Jordan algebras.

In our treatment of Jordan commutative algebras, the Kronecker matrix product will play a central role. This product was already considered by Khuri in [9]. The introduction of the binary operations lightens considerably the derivation of the matrices required for the statistical analysis. In this way complex models, such as those with two tiers of factors, become tractable.

2. Jordan algebras

Superscripts will be used to indicate the number of components of vectors. \( I_s \) will denote the \( s \times s \) identity matrix, \( J_s = 1^s 1^s \) and \( J_s = I_s - \frac{1}{s} J_s \), whereas \( T_s \) will be obtained by deleting the first line equal to \( \frac{1}{\sqrt{s}} 1^s \) of an orthogonal \( s \times s \) matrix.

Let \( g_1, \ldots, g_w \) be the ranks of the \( n \times n \) matrices \( Q_1, \ldots, Q_w \) in the principal basis of a commutative Jordan algebra. The identity matrix in that algebra will be \( U = \sum_{j=1}^w Q_j \). We will have \( U = I_n \) if and only if \( \sum_{j=1}^w g_j = n \). If \( \sum_{j=1}^w g_j < n \) we can join to the principal basis the matrix \( Q_{w+1} = I_n - \sum_{j=1}^w Q_j \), thus obtaining an expanded commutative Jordan algebra.

**Definition 1.** Given the families of matrices \( T_1 \) and \( T_2 \), \( T_1 \otimes T_2 \) will be the family of the Kronecker matrix products \( M_1 \otimes M_2, M_d \in T_d, d = 1, 2. \)

If \( \mathcal{N}(\mathcal{A}_d) = \{Q_{d,1}, \ldots, Q_{d,w_d} \} \) is the principal basis of the commutative Jordan algebra \( \mathcal{A}_d \), \( d = 1, 2 \),

\[
\mathcal{N}(\mathcal{A}_1 \otimes \mathcal{A}_2) = \mathcal{N}(\mathcal{A}_1) \otimes \mathcal{N}(\mathcal{A}_2)
\]
will be the principal basis of the commutative Jordan algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$. Moreover, it is easy to show that, because the Kronecker product of matrices is associative,

$$(\mathcal{T}_1 \otimes \mathcal{T}_2) \otimes \mathcal{T}_3 = \mathcal{T}_1 \otimes (\mathcal{T}_2 \otimes \mathcal{T}_3),$$

whenever the matrix families $\mathcal{T}_d$, $d = 1, 2, 3$.

Besides the $\otimes$ product of algebras, we will consider a restricted product. Let $\{Q_{1,1}, \ldots, Q_{1,w_1}\}$ and $\{Q_{2,1}, \ldots, Q_{2,w_2}\}$ be the principal basis of the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. We put $U_1 = \sum_{j=1}^{w_1} Q_{1,j}$ and assume that $Q_{2,1} = \frac{1}{n_2} J_{n_2}$, then $\mathcal{A}_1 \ast \mathcal{A}_2$ will be the Jordan commutative algebra with the principal basis

$$\{Q_{1,1} \otimes Q_{2,1}, \ldots, Q_{1,w_1} \otimes Q_{2,1}\} \cup \{U_1 \otimes Q_{2,2}, \ldots, U_1 \otimes Q_{2,w_2}\}.$$  

(3)

Given the principal basis $\{Q_{3,1}, \ldots, Q_{3,w_3}\}$, with $Q_{3,1} = \frac{1}{n_3} J_{n_3}$, of another commutative Jordan algebra, it is easy to show that

$$(\mathcal{A}_1 \ast \mathcal{A}_2) \ast \mathcal{A}_3 = \mathcal{A}_1 \ast (\mathcal{A}_2 \ast \mathcal{A}_3).$$  

(4)

In what follows some very simple commutative Jordan algebras will play an important part. Let the components $a_1, \ldots, a_u$ of $a^u$ be positive integers. These integers will be the sizes of the matrices. Then $\mathcal{A}(a^u)$ will be the commutative Jordan algebra whose principal basis is of the form

$$Q_{h+1}(a^u) = \bigotimes_{k=1}^{u} Q_{h,k}(a_k); \quad h = 0, \ldots, u,$$

with, putting $a_0 = 1$,

$$Q_{h,k}(a_k) = \begin{cases} I_{a_k}, & k < h, \\ J_{a_k}, & k = h, \\ \frac{1}{\sqrt{a_k}} J_{a_k}, & k > h. \end{cases}$$

(6)

This principal basis will be represented by $\mathcal{N}(a^u)$. Putting

$$A_{h+1}'(a^u) = \bigotimes_{k=1}^{u} A'_{h,k}(a_k); \quad h = 0, \ldots, u,$$

with

$$A'_{h,k}(a_k) = \begin{cases} I_{a_k}, & k < h, \\ T_{a_k}, & k = h, \\ \frac{1}{\sqrt{a_k}} J_{a_k}, & k > h. \end{cases}$$

(8)

we will have

$$Q_h(a^u) = A_h(a^u) A'_h(a^u); \quad h = 1, \ldots, u + 1.$$  

(9)

It is easily seen that

$$Q_h(a^u) = Q_h(a^{u-1}) \otimes \frac{1}{a_u} J_{a_u}; \quad h = 0, \ldots, u - 1$$

(10)

and, with $c(a^{u-1}) = \prod_{h=1}^{u-1} a_h$, that

$$Q_u(a^u) = I_{c(a^{u-1})} \otimes J_{a_u}.$$  

(11)

Now $\mathcal{N}(a_u)$, the basis of $\mathcal{A}(a_u)$, is $\{\frac{1}{a_u} J_{a_u}, J_{a_u}\}$ so that according to the definition of operation $\ast$ we have

$$\mathcal{A}(a^u) = \mathcal{A}(a^{u-1}) \ast \mathcal{A}(a_u).$$  

(12)
Moreover, the sum of the matrices in $\mathfrak{N}(a_1)$ will be $I_{a_1}$. This leads, as it is easily seen, to the sum of the matrices in $\mathfrak{N}(a^n)$ being $I_{c(a^n)}$.

The family of matrices $A_h(a^n), h = 1, u + 1$ will be represented by $\mathfrak{A}_h(a^n)$. In the models with nesting and cross-nesting the $a_1, \ldots, a_u$ will be the number of factor levels.

A very simple example of these algebras is $\mathfrak{A}(r)$, with the principal basis

$$\mathfrak{N}(r) = \left\{ \frac{1}{r} J_r, J_r \right\}. \quad (13)$$

As we shall see, algebras $\mathfrak{A}_h(a^n)$ will be used to describe groups of nested factors and algebras $\mathfrak{A}_h(r)$ to describe the existence of replicates. To describe the crossing between groups of factors we will use the $\otimes$ product algebras $\bigotimes_{i=1}^{L_u} \mathfrak{A}(a^n_i)$. The components of $a^n_i$ will be $a_{i_1}, \ldots, a_{i_{u_i}}, l = 1, \ldots, L$. With $\mathfrak{A} = \mathfrak{A}(a^n_i)$, the matrices in $\mathfrak{N}(\bigotimes_{i=1}^{L_u} \mathfrak{A}_i)$ will correspond to the vectors in the set

$$\Gamma = \left\{ h^L : 0 \leq h \leq u_l; l = 1, \ldots, L \right\}. \quad (14)$$

With $\mathfrak{N}(a^L) = \{ Q_{l_1}, \ldots, Q_{l_{u_l}}, l = 1, \ldots, L \}$, we put

$$Q(h^L) = \bigotimes_{l=1}^{L} Q_{l,h}, \quad h^L \in \Gamma,$$  \quad (15)

being easy to see that

$$Q(h^L) = A(h^L)A'(h^L), \quad h^L \in \Gamma,$$  \quad (16)

with

$$A(h^L) = \bigotimes_{l=1}^{L} A_{l,h}, \quad h^L \in \Gamma.$$ \quad (17)

The family of these matrices will be represented by $\mathfrak{A}(\Gamma)$. These matrices may be used in the study of cross-nested designs (see [8]).

We point out that this algebra may be used to describe, as we shall see, the cross-nesting of $L$ groups with $u_1, \ldots, u_L$ nested factors. The principal matrices will be of the type $m \times m$, with $m$ being the number of level combinations for all factors. Moreover, it is easily seen that $Q(0^L) = \frac{1}{m^L} J_m$. The principal basis and the commutative Jordan algebra will be represented by $\mathfrak{N}(\Gamma)$ and $\mathfrak{A}(\Gamma)$, respectively.

Besides one tier cross-nesting we will consider two tiers cross-nesting. With $\Gamma_1$ and $\Gamma_2$ the sets of vectors corresponding to the two tiers, $\mathfrak{N}(\Gamma_d)$ and $\mathfrak{A}(\Gamma_d), d = 1, 2$, the respective principal basis and commutative Jordan algebra, we now have the commutative Jordan algebra $\mathfrak{A}(\Gamma_1 \ast \Gamma_2) = \mathfrak{A}(\Gamma_1) \ast \mathfrak{A}(\Gamma_2)$. With $\mathfrak{N}(\Gamma_d) = \left\{ Q(h^L_d) : h^L_d \in \Gamma_d \right\}$, the principal basis of the new algebra will be of the form

$$\mathfrak{N}(\Gamma_1 \ast \Gamma_2) = \left\{ Q(h^{L_1}_1) \otimes \left( \frac{1}{m_2} J_{m_2} \right), h^{L_1}_1 \in \Gamma_1 \right\} \cup \left\{ I_{m_1} \otimes Q(h^{L_2}_2), h^{L_2}_2 \in \Gamma_2 \setminus 0^{L_2} \right\}.$$  \quad (18)

The remaining family of matrices associated with this algebra will be

$$\mathfrak{A}(\Gamma_1 \ast \Gamma_2) = \left\{ A(h^{L_1}_1) \otimes \left( \frac{1}{m_2} J_{m_2} \right), h^{L_1}_1 \in \Gamma_1 \right\} \cup \left\{ I_{m_1} \otimes A(h^{L_2}_2), h^{L_2}_2 \in \Gamma_2 \setminus 0^{L_2} \right\}.$$  \quad (19)
Let $\mathcal{A}^0$ and $\mathcal{N}_0$ be the commutative Jordan algebra and the principal basis for a model before replicates are considered. If we take $r$ observations for all the treatments in the model, $\mathcal{N}_0$ will be replaced by

$$\mathcal{N}_0 \ast r = \left\{ Q^0 \otimes \left( \frac{1}{r} J_r \right), \; Q^0 \in \mathcal{N}_0 \right\} \cup \{ I_m \otimes J_r \}. \quad (20)$$

We also put $Q^\perp = I_m \otimes J_r$, thus

$$Q^\perp = (A^\perp)' A^\perp, \quad (21)$$

with $A^\perp = I_m \otimes T_r$. Moreover, we represent the new commutative Jordan algebra by $\mathcal{A}^0 \ast r$.

Let the principal matrices of the commutative Jordan algebra $\mathcal{A}$ be $Q_1, \ldots, Q_w$. Then

$$I_m = \sum_{j=1}^w Q_j = \sum_{j=1}^w A_j' A_j = P' P, \quad (22)$$

with

$$P' = [A_1' \; \cdots \; A_w'], \quad (23)$$

so that $P$ will be orthogonal. Given $M \in \mathcal{A}$, we will have

$$M = \sum_{j=1}^w v_j Q_j = \sum_{j=1}^w v_j A_j' A_j, \quad (24)$$

thus $P$ will diagonalize $M$ and the coefficients $v_j$, $j = 1, \ldots, w$ will become the distinct eigenvalues of the matrix $M$, their multiplicities being

$$g_j = \text{rank}(Q_j); \quad j = 1, \ldots, w. \quad (25)$$

It is also easy to see that the Moore–Penrose inverse of $M$ will be

$$M^+ = \sum_{j=1}^w v_j^+ Q_j, \quad (26)$$

with $v_j^+ = v_j^{-1}$ whenever $v_j \neq 0$, and $v_j^+ = 0$, whenever $v_j = 0$, so that

$$\det(M) = \prod_{j=1}^n v_j^{g_j}. \quad (27)$$

From (26) we get

$$M^{-1} = \sum_{j=1}^{w+1} v_j^{-1} Q_j, \quad (28)$$

whenever $M$ is regular. Even when the sum $\sum_{j=1}^w Q_j$ of the matrices in the principal basis of a commutative Jordan algebra is not equal to $I_n$, expression (24) and, consequently, expression (26) will hold. Thus, any commutative Jordan algebra will contain the Moore–Penrose inverses of its matrices. This is an alternative proof of the same result in [7, pp. 11–13].
3. Models

Model orthogonality is based on orthogonal partitions of the sample subspace such that the orthogonal projections of the observation vectors on the subspaces in those partitions are uncorrelated. Let us write $\bigoplus$ for orthogonal direct sum of subspaces. Then, we have the orthogonal partitions

$$\mathbb{R}^{n} = \bigoplus_{i=1}^{w+1} \nabla_i.$$  \hfill (29)

Let the orthogonal projection matrices on the subspaces $\nabla_i, \ldots, \nabla_{w+1}$ be $Q_1, \ldots, Q_{w+1}$. Then $g_i = \text{rank}(Q_i) = \dim(\nabla_i)$, $i = 1, \ldots, w+1$, as well as $\sum_{i=1}^{w+1} g_i = n$ and as $\sum_{i=1}^{w+1} Q_i = I_n$. If the columns of $A_i$ constitute an orthonormal basis for $\nabla_i$ we will have $Q_i = A_iA_i^\prime$ as well as $A_i^\prime A_i = I_{g_i}$, $i = 1, \ldots, w+1$. Moreover, with $0_{r,s}$ as the null $r \times s$ matrix, we will have $A_i^\prime Q_j = 0_{g_i,n}$ whenever $i \neq j$.

Let us now establish

**Proposition 1.** A normal orthogonal model associated with the orthogonal partition in (29) (as with the corresponding Jordan commutative algebra) has the canonical form

$$Y^n = \sum_{i=1}^{w+1} A_i \eta_i^{g_i},$$

where vectors $\eta_i^{g_i}, i = 1, \ldots, w+1$, are normal, independent, with mean vectors $\lambda_i^{g_i}, i = 1, \ldots, w+1$, and covariance matrices $\gamma_i I_{g_i}, i = 1, \ldots, w+1$.

We write $\eta_i^{g_i} \sim \mathcal{N}(\lambda_i^{g_i}, \gamma_i I_{g_i}), i = 1, \ldots, w+1$.

**Proof.** Let $Q^*$ be the orthogonal projection matrix in the sub-space that contains the observation mean vector $\mu^n$. Since $Q^*$ belongs to the algebra we will have $Q^* = \sum_{i=1}^{w+1} c_i Q_i$, with $c_i = 0$ or $c_i = 1$, $i = 1, \ldots, w+1$. We can assume without loss of generality that $Q^* = \sum_{i=1}^{m} Q_i$. Thus,

$$\mu^n = Q^* \mu^n = \sum_{i=1}^{m} Q_i \mu^n = \sum_{i=1}^{m} A_i A_i^\prime \mu^n = \sum_{i=1}^{m} A_i \lambda_i^{g_i},$$

where $\lambda_i^{g_i}$ is the mean vector of $\eta_i^{g_i} = A_i Y^n, i = 1, \ldots, m$. Moreover, $\lambda_i^{g_i} = 0^{g_i}, i = m+1, \ldots, w+1$, will be the mean vector of $\eta_i^{g_i} = A_i^\prime Y^n, i = 1, \ldots, m$. Then

$$Y^n = I_n Y^n = \sum_{i=1}^{w+1} Q_i Y^n = \sum_{i=1}^{w+1} A_i A_i^\prime Y^n = \sum_{i=1}^{w+1} A_i \eta_i^{g_i}.$$  \hfill (30)

To complete the proof we have only to point out that $\eta_i^{g_i}, i = 1, \ldots, w+1$, will be normal and independent because, as it is easily seen, their cross-covariance matrices are null and their covariance matrices are $\gamma_i I_{g_i}, i = 1, \ldots, w+1$. $\square$

As one could see during the proof, we have

$$\mu^n = \sum_{i=1}^{m} A_i \lambda_i^{g_i}.$$  \hfill (31)
Moreover, because the \( \eta^g_{ij} \) are independent, the covariance matrix of \( Y^n \) will be
\[
V = \sum_{i=1}^{w+1} \gamma_i Q_i. \tag{32}
\]
Thus (see [8]),
\[
\begin{cases}
\det(V) = \prod_{i=1}^{w+1} \gamma_i^{g_{ii}}, \\
V^{-1} = \sum_{i=1}^{w+1} \gamma_i^{-1} Q_i.
\end{cases} \tag{33}
\]

The following result will be useful, because it shows that, for every \( \mu^n \) and \( V \), the canonical parameters \( \lambda^g_{ij}, i = 1, \ldots, m \), and \( \gamma_i, i = 1, \ldots, w + 1 \), are unique.

**Proposition 2.** We have \( \sum_{i=1}^{m} A_i a^g_{ij} = \sum_{i=1}^{m} A_i b^g_{ij} \) if and only if \( a^g_{ij} = b^g_{ij}, i = 1, \ldots, m \), and \( \sum_{i=m+1}^{w+1} u_i Q_i = \sum_{i=m+1}^{w+1} v_i Q_i \) when and only when \( u_i = v_i, i = m + 1, \ldots, w + 1 \).

**Proof.** For either part of the thesis it is sufficient to establish the necessary condition, because the corresponding sufficient condition is self-evident. Starting with the first part, since \( \nabla_i \cap \nabla_j \cap \nabla_j = \{0^n\} \), if \( \sum_{i=1}^{m} A_i a^g_{ij} = \sum_{i=1}^{m} A_i b^g_{ij} \), i.e., if \( A_i (b^g_{ij} - a^g_{ij}) = -\sum_{j \neq i} A_j (b^g_{ij} - a^g_{ij}) \in \nabla_i \cap \nabla_j \cap \nabla_j \), we have \( A_i (b^g_{ij} - a^g_{ij}) = 0^n \) as well as \( b^g_{ij} - a^g_{ij} = A_i A_i (b^g_{ij} - a^g_{ij}) = 0^n \), \( i = 1, \ldots, m \), so the first part is established. Moreover, if \( \sum_{i=m+1}^{w+1} u_i Q_i = \sum_{i=m+1}^{w+1} v_i Q_i \), we have \( u_i Q_i = Q_i (\sum_{i=m+1}^{w+1} u_j Q_j) = Q_i (\sum_{i=m+1}^{w+1} v_j Q_j) = v_i Q_i \), thus \( u_i = v_i, i = m + 1, \ldots, w + 1 \), and the proof is complete. \( \square \)

We now establish a result which will play a central part in the inference. This result is an alternative to the well known result by Seely (see [5]) and Zmysłony (see [10]).

**Theorem 1.** For the observations vector
\[
Y^n = \sum_{j=1}^{w+1} A_j \eta^g_{ij}, \tag{34}
\]
where the random vectors \( \eta^g_{ij} \) are independent, normal, with the mean vectors \( \lambda^g_{ij}, j = 1, \ldots, m \), \( 0^g \), \( j = m + 1, \ldots, w \), and the covariance matrices \( \gamma_j Q_j, j = 1, \ldots, w \), we have the density
\[
n(y^n | \mu^n, V) = \frac{\exp \left\{ -\frac{1}{2} \left( \sum_{i=1}^{m} \frac{\| \eta^g_{ij} - \lambda^g_{ij} \|^2}{\gamma_i} + \sum_{i=m+1}^{w+1} \frac{\eta^g_{ij}}{\gamma_i} \right) \right\}}{(2\pi)^n \prod_{i=1}^{w+1} \gamma_i^{g_{ii}}}, \tag{35}
\]
with complete sufficient statistics \( \tilde{\eta}^g_{ij} = A_j y^n, j = 1, \ldots, m \), and \( s_j = \| A_j y^n \|^2, j = m + 1, \ldots, w \).

**Proof.** As we saw, \( A_j' Y^n = \eta^g_{ij}, i = 1, \ldots, w + 1 \), \( A_j' \mu^n = \lambda^g_{ij}, i = 1, \ldots, m \), and \( A_j' \mu^n = 0^g \), \( i = m + 1, \ldots, w + 1 \). We also get
\[
A_j' V^{-1} A_i = A_j' \left( \sum_{j=1}^{w+1} \gamma_j^{-1} Q_j \right) A_i = \gamma_i^{-1} A_j' A_i' = \gamma_i^{-1} I_{g_i}, \quad i = 1, \ldots, w + 1 \tag{36}
\]
so that
\[
(y^n - \mu^n)'V^{-1}(y^n - \mu^n) = \sum_{i=1}^{w+1} \frac{1}{\gamma_i} (y^n - \mu^n)'A_iA_i'(y^n - \mu^n)
\]
\[
= \sum_{i=1}^{m} \frac{\|\gamma_i^{gi} - \lambda_i^{gi}\|^2}{\gamma_i} + \sum_{i=m+1}^{w+1} \frac{s_i}{\gamma_i},\tag{37}
\]
where \(\gamma_i^{gi} = A_i'Y^n\) and \(s_i = \|A_i'Y^n\|^2\). Thus, the model’s density will be (35) and (see [11, pp. 31–32]) we have the set of complete sufficient statistics \(\gamma_i^{gi}, i = 1, \ldots, m,\) and \(s_i, i = m + 1, \ldots, w + 1\). □

According to the Blackwell–Lehmann–Scheffé theorem, the \(\gamma_i^{gi}, i = 1, \ldots, m,\) and the \(\gamma_i = s_i/g_i, i = m + 1, \ldots, w + 1\) are UMVUE for the mean vectors \(\lambda_i^{gi}, i = 1, \ldots, m,\) and the variance components \(\gamma_i, i = m + 1, \ldots, w + 1\). To avoid over-parametrization, we assume that
\[
\gamma_i = \sum_{j=m+1}^{w+1} b_{i,j} \gamma_j, \quad i = 1, \ldots, m,\tag{38}
\]
so that we will also have the UMVUEs
\[
\gamma_i = \sum_{j=m+1}^{w+1} b_{i,j} \gamma_j, \quad i = 1, \ldots, m.\tag{39}
\]
The estimable vectors will be of the form \(\psi_i^{gi} = B_i \lambda_i^{gi}, i = 1, \ldots, m,\) for which we have the UMVUEs \(\tilde{\psi}_i^{gi} = B_i \gamma_i^{gi}, i = 1, \ldots, m.\)

The joint distribution of the \(A_i'Y^n, i = 1, \ldots, w + 1,\) is normal and, because their cross-covariance matrices are null, they will be independent. Thus the \(\gamma_i^{gi} = A_i'Y^n, i = 1, \ldots, m,\) and the \(\gamma_i = s_i/g_i, i = m + 1, \ldots, w + 1,\) will be independent. Moreover, the \(\gamma_i^{gi} = A_i'Y^n\) and the \(\gamma_i = s_i/g_i, i = 1, \ldots, m,\) will also be independent, as well as the \(\psi_i^{gi}\) and the \(\gamma_i,\) \(i = 1, \ldots, m,\) where \(\psi_i^{gi} = B_i \lambda_i^{gi}\) is an estimable vector, \(i = 1, \ldots, m.\) It may be interesting to point out that we may take \(B_i = 1_{g_i},\) so that \(\lambda_i^{gi}\) is itself an estimable vector. If \(\text{rank}(B_i) = r_i,\) \(B_iB_i'\) will be positive definite and \(\psi_i^{gi} = B_i \lambda_i^{gi}\) will be a regular estimable vector. In what follows we restrict ourselves to such estimable vectors only.

4. Model build-up

4.1. Nested designs

If there are \(u\) factors we will have \(a_1\) levels for the first factor, each of which nests \(a_2\) levels of the second factor, and so on. There will be \(c(h) = \prod_{k=1}^{h} a_k\) level combinations for the first \(h\) factors so that the total number of level combinations will be \(m = c(u).\) Moreover, each combination of the levels of the first \(h\) factors nests \(b(h) = \frac{m}{c(h)}\) level combinations of the remaining factors. Assuming there are \(r\) replications, our model may be written as
\[
Y^n = \sum_{h=0}^{u} X_h \beta^{c(h)} + e^n,\tag{40}
\]
with \( n = mr \) and
\[
X_h = \left( \bigotimes_{k=1}^{u} X_{h,k} \right) \otimes I^r, \quad (41)
\]
where
\[
X_{h,k} = \begin{cases} I_{a_k}, & k \leq h, \\ I_{b_k}, & k > h \end{cases} \quad (42)
\]
and \( X_0 = I^n \).

Vector \( \beta_0 \) will have as a sole component the general mean value \( \mu \), whereas the \( c(h) \) components of \( \beta_h^{(h)} \) will be the effects of the \( h \)th factor and \( e^n \) the error vector. If the first \( v \) factors have fixed effects, vectors \( \beta_1^{(v+1)}, \ldots, \beta_v^{(u)} \) and \( e^n \) will be independent normal, with null mean vectors and the covariance matrices \( \sigma_k^2 I_{c(k)}, \quad k = v + 1, \ldots, u, \) and \( \sigma_1^2 I_n \), respectively. The vector \( Y^n \) will be normal with the mean vector
\[
\mu^n = 1^n \beta_0^n + \sum_{h=1}^{v} X_h \beta_h^{(h)} \quad (43)
\]
and the covariance matrix
\[
V = \sum_{h=v+1}^{u} \sigma_h^2 M_h + \sigma_1^2 I_n, \quad (44)
\]
where, taking \( X_0 = I^n \),
\[
M_h = X'_h X_h = b_h \sum_{k=0}^{h} Q_k; \quad h = 0, \ldots, u, \quad (45)
\]
with \( Q_0, \ldots, Q_u \) as the matrices constituting the principal bases of \( \mathcal{A}(a^n) * \mathcal{A}(r) \). We thus see that this model is associated with this basis. The application of the general theory to this case is straightforward.

### 4.2. Single tier cross-nested designs

We assume that there are \( L \) groups of \( u_1, \ldots, u_L \) factors with nesting in the groups with more than one factor. The model may be written as
\[
Y^n = \sum_{h^L \in \Gamma} X(h^L) \beta^{c(h^L)}(h^L) + e^n, \quad (46)
\]
where \( c(h^L) = \prod_{l=1}^{L} c(l, h_l), \quad c(0^L) = 1, \quad m = c(u^L) = 1 \) and \( X(0^L) = I^m \) and
\[
X(h^L) = \left( \bigotimes_{l=1}^{L} \bigotimes_{k=1}^{h_l} X_{l,k} \right) \otimes I^r, \quad h^L \in \Gamma. \quad (47)
\]
The sole component of \( \beta^1(0^L) \) will be the general mean value \( \mu \). If \( h^L \) has a unique non null component indexing a factor, \( \beta^{c(h^L)}(h^L) \) will correspond to the effects of that factor, otherwise it will correspond to the interactions between the components of the factors indexed by the non null components of \( h^L \). If the non null components of \( h^L \) index fixed effects, we write \( h^L \in \Gamma_f \).
and $\beta^{(h^L)}(h^L)$ will be fixed. Otherwise, we put $h^L \in \Gamma_r$ and $\beta^{(h^L)}(h^L)$ will be assumed normal, with null mean value and covariance matrix $\sigma^2(h^L)I_{c(h^L)}$. These vectors and $e^n$ are assumed to be independent and $e^n$ to be normal with null mean vector and covariance matrix $\sigma^2I_n$. Thus, the mean vector and the covariance matrix of $Y^n$ will be

$$
\begin{align*}
    \mu^n &= \sum_{h^L \in \Gamma_f} X(h^L)\beta^{(h^L)}(h^L), \\
    V &= \sum_{h^L \in \Gamma_r} \sigma^2 M(h^L),
\end{align*}
$$

with

$$
M(h^L) = X'(h^L)X(h^L) = b(h^L) \sum_{k^L \leq h^L} Q(h^L),
$$

where $b(h^L) = r \prod_{l=1}^L b(l, h_l)$. Since

$$
I_n = \sum_{k^L \in \Gamma_r} Q(k^L) + Q^\perp,
$$

we see that the model is associated with the algebra $\mathcal{A}(\Gamma) * \mathcal{A}(u^n)$ and that $V$ can be rewritten as

$$
V = \sum_{k^L \in \Gamma_r} \gamma(k^L)Q(k^L) + \sigma^2 Q^\perp,
$$

with

$$
\gamma(k^L) = \sigma^2 + \sum_{h^L : k^L \leq h^L} b(h^L)\sigma^2(h^L).
$$

With $\oplus(h^L)$ the set of vectors $k^L \in \Gamma$ with components such that $h_l \leq k_l \leq \min\{h_l, u_l\}$, $l = 1, \ldots, L$, we have (see [8])

$$
\begin{align*}
    \sigma^2(h^L) &= \frac{1}{b(h^L)} \sum_{k^L \in \oplus(h^L)} (-1)^{m(k^L, h^L)} \gamma(h^L), \quad k^L \neq u^L, \\
    \sigma^2(u^L) &= \frac{1}{b(u^L)} (\gamma(u^L) - \sigma^2),
\end{align*}
$$

where $m(k^L, h^L)$ is the number of components of $k^L$ that exceed the corresponding components of $h^L$. It is now straightforward to apply the general theory to these models.

In particular, this kind of models include all factorial models, with any number of factors and levels. In order to define such models, we have only to consider $u_l = 1$, $l = 1, \ldots, L$, so that all families of nested factors have only one factor (and no nesting occurs).

### 4.3. Double tier cross-nested designs

In the first tier we have $L_1$ groups of nested factors that cross. Each of these $m_1$ level combinations of the first tier nests $L_2$ groups of nested factors. Let us assume that all factors have random effects. The model can be written as

$$
Y^n = \sum_{h_1^{L_1} \in \Gamma_1} X(h_1^{L_1})\beta^{(h_1^{L_1})}(h_1^{L_1}) + \sum_{h_2^{L_2} \in \Gamma_2 \setminus \{0^{L_2}\}} X(h_2^{L_2})\beta^{m_1c_2}(h_2^{L_2}) + e^n,
$$

where the indexes 1 and 2 refer to the tiers and their models, and

$$
\begin{align*}
    X(h_1^{L_1}) &= (\bigotimes_{l=1}^{L_1} X_{1,l,h_l}) \otimes 1^{m_2} \otimes 1^r, \quad h_1^{L_1} \in \Gamma_1, \\
    X(h_2^{L_2}) &= I_{m_1} \otimes (\bigotimes_{l=1}^{L_2} X_{2,l,h_l}) \otimes 1^r, \quad h_2^{L_2} \in \Gamma_2.
\end{align*}
$$
Namely, $X(0^{L-1}) = 1^n$ with $n = m_1m_2r$ and the sole component of $\beta^i(0^{L-1})$ will be the general mean $\mu$.

Vectors $\beta^{c_i}(h_1^{L_1}) (h_1^{L_1})$, with $h_1^{L_1} \neq 0^{L_1}$, $\beta^{m_1c_2}(h_2^{L_2}) (h_2^{L_2})$, with $h_2^{L_2} \neq 0^{L_2}$, and $e^n$ are assumed to be normal, with null mean vectors and the covariance matrix $\sigma^2(h_1^{L_1}) I_{c_1}(h_1^{L_1})$, $\sigma^2(h_2^{L_2}) I_{m_1c_2}(h_2^{L_2})$ and $\sigma^2 I_n$, respectively.

Thus, the application of the general theory to this case is also straightforward.

Reasoning as for the single tier models, it can be shown that the relevant algebra is now $A(\Gamma_1 \ast \Gamma_2) \ast A(a^n)$ and that the mean vector and the covariance matrix of the observation vector are

$$\begin{align*}
\mu^n &= 1^n \mu, \\
V &= \sum_{h_1^{L_1} \in \Gamma_1} \gamma(h_1^{L_1}) Q^*(h_1^{L_1}) + \sum_{h_2^{L_2} \in \Gamma_2 \setminus \{0^{L_2}\}} \gamma(h_2^{L_2}) Q^*(h_2^{L_2}) + \sigma^2 Q, \\
\sigma^2(h_1^{L_1}) &= \frac{1}{b^*(h_1^{L_1})} \sum_{k_1^{L_1} \in \Gamma_1(h_1^{L_1})} (-1)^{m(h_1^{L_1}, k_1^{L_1})} \gamma(k_1^{L_1}), \\
\sigma^2(h_2^{L_2}) &= \frac{1}{b^*(h_2^{L_2})} \sum_{k_2^{L_2} \in \Gamma_2(h_2^{L_2})} (-1)^{m(h_2^{L_2}, k_2^{L_2})} \gamma(k_2^{L_2}),
\end{align*}$$

where $Q^*(h_1^{L_1}) = Q(h_1^{L_1}) \frac{1}{m_2} J_{m_2} \otimes \frac{1}{r} J_r$, $Q^*(h_2^{L_2}) = I_{m_1} \otimes Q(h_2^{L_2}) \otimes \frac{1}{r} J_r$.

and

$$\begin{align*}
\gamma(h_1^{L_1}) &= v + \sum_{k_1^{L_1} \in \Gamma_1} b^*(k_1^{L_1}) \sigma^2(k_1^{L_1}), \\
\gamma(h_2^{L_2}) &= \sigma^2 + \sum_{k_2^{L_2} \in \Gamma_2} b^*(k_2^{L_2}) \sigma^2(k_2^{L_2}).
\end{align*}$$

with $v = \sigma^2 + \sum_{h_2^{L_2} \in \Gamma_2} b^*(h_2^{L_2}) \sigma^2(h_2^{L_2})$, $b^*(k_1^{L_1}) = m_2 r b(k_1^{L_1})$ and $b^*(k_2^{L_2}) = r b(k_2^{L_2})$. Finally, we have

$$\begin{align*}
\sigma^2(h_1^{L_1}) &= \frac{1}{b^*(h_1^{L_1})} \sum_{k_1^{L_1} \in \Gamma_1(h_1^{L_1})} (-1)^{m(h_1^{L_1}, k_1^{L_1})} \gamma(k_1^{L_1}), \\
\sigma^2 &= \frac{1}{m_2 r} (\gamma(u_1^{L_1}) - v),
\end{align*}$$

as well as

$$\begin{align*}
\sigma^2(h_2^{L_2}) &= \frac{1}{b^*(h_2^{L_2})} \sum_{k_2^{L_2} \in \Gamma_2(h_2^{L_2})} (-1)^{m(h_2^{L_2}, k_2^{L_2})} \gamma(k_2^{L_2}), \\
\sigma^2 &= \frac{1}{r} (\gamma(u_2^{L_2}) - \sigma^2).
\end{align*}$$

Thus, the application of the general theory to this case is also straightforward.

The extension to more than two tiers presents no difficulty. We have confined the case to two tiers to avoid overloading the presentation, and because we expect it to be important in applications.

References