Half-Transitive Group Actions on Finite Graphs of Valency 4

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The action of a subgroup $G$ of automorphisms of a graph $X$ is said to be $\frac{1}{2}$-transitive if it is vertex- and edge- but not arc-transitive. In this case the graph $X$ is said to be $(G, \frac{1}{2})$-transitive. In particular, $X$ is $\frac{1}{2}$-transitive if it is $(\text{Aut} X, \frac{1}{2})$-transitive. The $\frac{1}{2}$-transitive action of $G$ on $X$ induces an orientation of the edges of $X$ which is preserved by $G$. Let $X$ have valency 4. An even length cycle $C$ in $X$ is a $G$-alternating cycle if every other vertex of $C$ is the head and every other vertex of $C$ is the tail of its two incident edges in the above orientation. It transpires that all $G$-alternating cycles in $X$ have the same length and form a decomposition of the edge set of $X$ (Proposition 2.4); half of this length is denoted by $r_G(X)$ and is called the $G$-radius of $X$. Moreover, it is shown that any two adjacent $G$-alternating cycles of $X$ intersect in the same number of vertices and that this number, called the $G$-attachment number $a_G(X)$ of $X$, divides $2r_G(X)$ (Proposition 2.6). If $X$ is $\frac{1}{2}$-transitive, we let the radius and the attachment number of $X$ be, respectively, the $\text{Aut} X$-radius and the $\text{Aut} X$-attachment number of $X$. The case $a_G(X)=2r_G(X)$ corresponds to the graph $X$ consisting of two $G$-alternating cycles with the same vertex sets and leads to an arc-transitive circulant graph (Proposition 2.4). If $a_G(X)=r_G(X)$ we say that the graph $X$ is tightly $G$-attached. In particular, a $\frac{1}{2}$-transitive graph $X$ of valency 4 is tightly attached if it is tightly $\text{Aut} X$-attached. A complete classification of tightly attached $\frac{1}{2}$-transitive graphs with odd radius and valency 4 is obtained (Theorem 3.4).

1. INTRODUCTORY REMARKS

Throughout this paper graphs are simple and, unless otherwise specified, undirected and connected. Furthermore, all graphs and groups are assumed to be finite. For group-theoretic terms not defined here we refer the reader to [12, 16].

If $X$ is a graph let $V(X)$ and $E(X)$ denote the respective sets of vertices and edges. For $v_1, \ldots, v_k \in V(X)$ and a positive integer $i$ we let $N^i(v_1, \ldots, v_k)$ denote the set of all vertices in $X$ at distance $i$ from the set $\{v_1, \ldots, v_k\}$. In

particular, we let $N(v_1, ..., v_k) = N^1(v_1, ..., v_k)$ be the set of neighbors of \{v_1, ..., v_k\}. A **circulant** is a Cayley graph of a cyclic group. For a positive integer $n$ and a symmetric subset $S = -S$ of $\mathbb{Z}_n$, we let $\text{Circ}(n, S)$ denote the circulant with vertex set $\{v_i; i \in \mathbb{Z}_n\}$ and edges of the form $v_i v_{i+s}$, $i, s \in \mathbb{Z}_n$. An automorphism of a graph $X$ is said to be $(m, n)$-semiregular if it has $m \geq 2$ orbits of length $n \geq 2$ and no other orbits. An $(m, n)$-**metacirculant** is a graph with an $(m, n)$-semiregular automorphism normalized by an automorphism which cyclically permutes its orbits. In short, a metacirculant is a graph with a transitive metacyclic subgroup of automorphisms.

A graph $X$ is said to be **vertex-transitive**, **edge-transitive**, and **arc-transitive**, respectively, if its automorphism group $\text{Aut} X$ acts vertex-transitively, edge-transitively, and arc-transitively. The first result linking vertex- and edge-transitivity to arc-transitivity is due to Tutte [14, 7.53, p. 59] who proved that a vertex-transitive and edge-transitive graph of odd valency is necessarily arc-transitive. In fact, it follows from Tutte's result that the automorphism group of a vertex- and edge-transitive but not arc-transitive graph must necessarily have two orbits on the arc set, with each orbit containing an arc corresponding to each edge; that is, in the terminology of [16, p. 24], it acts $\frac{1}{2}$-transitively on the arc set having two orbits (of equal length). For brevity reasons we shall thus say that a graph $X$ is **$\frac{1}{2}$-transitive** provided it is vertex- and edge- but not arc-transitive. More generally, by a $\frac{1}{2}$-transitive action of a subgroup $G$ of $\text{Aut} X$ on $X$ we shall throughout this paper always mean, albeit in a slight dissonance with its usual meaning in the literature [16, p. 24], a vertex- and edge- but not arc-transitive action on $X$. Furthermore, we shall say that the graph $X$ is $(G, \frac{1}{2})$-transitive if the group $G$ acts $\frac{1}{2}$-transitively on $X$. Tutte's question [14] as to whether his result extends to graphs of even valency or not was answered in 1970 by Bouwer [3] with a construction of a $\frac{1}{2}$-transitive graph of valency $2k$ for every $k \geq 2$. The smallest graph in his family has 54 vertices and valency 4. Some years later, Holt [6] found one with 27 vertices. More recently, Alspach, Nowitz and the author [1] proved that no smaller $\frac{1}{2}$-transitive graph exists and exhibited an infinite family of $\frac{1}{2}$-transitive graphs of valency 4 (containing the Holt's graph as well as the smallest Bouwer's graph). The problem of existence of $\frac{1}{2}$-transitive graphs with a primitive automorphism group was posed independently in [6, 7]. This question was answered by Praeger and Xu [10] in the affirmative. Moreover, an infinite family of such graphs was given by Taylor and Xu [13]. The problem of classifying $\frac{1}{2}$-transitive graphs of specific orders, notably $p^3$ and $pq$ where $p$ and $q$ are distinct primes, has also been considered (see [2, 15, 17]).

Despite all of these efforts, however, a classification of the entire class of $\frac{1}{2}$-transitive graphs is presently beyond our reach. Namely, in general
\(1/2\)-transitive graphs arise in the following context. Given a transitive permutation group \(G\) on a set \(V\), let \(A \neq \{(v, v): v \in V\}\) be a nontrivial orbital in the natural action of \(G\) on \(V \times V\) and let \(A' = \{(v, v): (v, v) \in A\}\) be the paired orbital of \(A\). The orbital graph \(X(G, A)\) of \(G\) relative to \(A\), is the graph with vertex set \(V\) and arc set \(A\). Of course, if \(A = A'\) is a self-paired orbital then \(X(G, A)\) can be viewed as an undirected graph which admits a vertex- and arc-transitive action of \(G\). On the other hand, if \(A \neq A'\) is a non-self-paired orbital then \(A \cap A' = \emptyset\) and the underlying undirected graph of the orbital graph \(X(G, A)\) is \((G, 1/2)\)-transitive. Thus, if \(G\) is the full automorphism group of the underlying undirected graph then this graph is \(1/2\)-transitive. Conversely, given an edge \(uv\) of a \(1/2\)-transitive graph \(X\) of valency \(2k\), the two arcs \((u, v)\) and \((v, u)\) give rise (via the action of \(\text{Aut} X\)) to two oriented graphs. These graphs are orbital graphs of \(\text{Aut} X\) relative to two paired orbitals of length \(k\).

The comments above suggest that classification results on \(1/2\)-transitive graphs would necessarily involve a deep understanding of the structure of transitive permutation groups with non-self-paired orbitals and the corresponding orbital graphs. Undoubtedly, an almost impossible task in general. Rather than imposing extra conditions on the possible order of such graphs we choose valency restriction and propose to study the structure of \(1/2\)-transitive graphs of smallest admissible valency 4, and more generally, graphs of valency 4 admitting \(1/2\)-transitive group actions.

There are two essentially different types of such graphs. Namely, given a graph \(X\) of valency 4 admitting a \(1/2\)-transitive action of a subgroup \(G\) of \(\text{Aut} X\), the restriction of the stabilizer \(G_v, v \in V(X)\), to the neighbor set \(N(v) = \{u, w, x, y\}\) may be isomorphic either to \(Z_2\) or to \(Z_2 \times Z_2\). Let \(\{x, y\}\) and \(\{u, w\}\) be the two orbits of \(G_v\) on \(N(v)\). In the first case, \(G_{N(v)}^{(x,y)} = \langle (xy)(uw) \rangle\) and it is easily seen that the restriction homomorphism is a monomorphism and so \(|G_v| = 2\). In the second case, \(G_{N(v)}^{(xy)} = \langle (xy), (uv) \rangle\) and the order \(|G_v|\) is not bounded as may be seen by the lexicographic products \(C_t [K_2], t \geq 3\), where \(C_t\) denotes the cycle of length \(t\). For each of the \(1/2\)-transitive graphs belonging to the infinite families constructed in [1, 11] the action of the automorphism group is of the first kind. Besides, all of these graphs are metacirculants. An example of a \(1/2\)-transitive graph of valency 4 which is not a metacirculant was found in [8]. Let us also mention that there is a one-to-one correspondence between \(1/2\)-transitive graphs of valency 4 having girth 3 and cubic one-regular graphs. Namely, it is not hard to see that \(Y\) is a cubic one-regular graph if and only if its line graph \(L(Y)\) is a \(1/2\)-transitive graph of valency 4 with vertex stabilizer \(Z_2\) (see [8]). In this sense the line graph of the first known example of a cubic one-regular graph with 432 vertices constructed by Frucht [5] is, at least implicitly, the first example of a \(1/2\)-transitive graph. (A graph is one-regular if its automorphism group acts regularly on the set of its arcs.) Also, the
line graphs of the one-regular graphs constructed in [4] provide further examples of \( \frac{1}{2} \)-transitive graphs of valency 4.

This paper has two objectives. First, in Section 2 graphs of valency 4 admitting a \( \frac{1}{2} \)-transitive group action are studied via a particular decomposition of their edge sets into cycles of equal even length (Proposition 2.4), called the alternating cycles relative to the group in question. Half of this length is called the radius of the graph relative to the group in question. It transpires that any two adjacent alternating cycles have the same number of common vertices (Proposition 2.6). A \( \frac{1}{2} \)-transitive graph of valency 4 is tightly attached if two adjacent alternating cycles, relative to the full automorphism group, have precisely half of their vertices in common. (See Section 2 for more detailed definitions of these concepts.) This brings us to the second objective of this paper. As our main result, we give a complete classification of tightly attached \( \frac{1}{2} \)-transitive graphs of valency 4 having odd radius (relative to the full automorphism group) (Theorem 3.4). In Section 3 a particular labeling of tightly attached graphs, suitable for the analysis of their structure, is given. In Sections 4 and 5 some preliminary results on the cycle structure of tightly attached graphs are proved, setting the stage for the proof of Theorem 3.4 in Section 6.

2. GENERAL RESULTS

We start with the following simple observation on \( \frac{1}{2} \)-transitive group actions.

**Proposition 2.1.** Let \( X \) be a \((G, \frac{1}{2})\)-transitive graph for some \( G \leq \text{Aut} \ X \). Then no element of \( G \) can interchange a pair of adjacent vertices in \( X \).

**Proposition 2.2.** Let \( X \) be a graph having an automorphism \( \rho \) with two orbits \( U, W \) of length \( n \geq 2 \) such that \( \{U, W\} \) is a bipartition of \( X \). Then \( X \) is not \( \frac{1}{2} \)-transitive.

**Proof.** Let \( u \in U \) and \( w \in W \) be adjacent and let \( u_i = u^i \cdot p, w_i = w^i \cdot p \) for each \( i \in \mathbb{Z}_n \). It is not difficult to see that the permutation \( \gamma \), which interchanges \( u \) with \( w_{n-i} \) for each \( i \in \mathbb{Z}_n \), is an automorphism of \( X \). But \( \gamma \) interchanges \( u \) and \( w \) and so \( X \) is not \( \frac{1}{2} \)-transitive by Proposition 2.1.

Let \( X \) be a graph of valency 4 admitting a \( \frac{1}{2} \)-transitive action of some subgroup \( G \) of \( \text{Aut} \ X \). Let \( D_{\bar{c}}(X) \) be one of the two oriented graphs associated with the action of \( G \) on \( X \), fixed from now on. Of course, as mentioned in the introductory section, \( D_{\bar{c}}(X) \) is an orbital graph of \( G \) relative to a non-self-paired orbital and \( X \) is its underlying undirected graph. For \( u, v \in V(X) \) such that \((u, v)\) is an arc in \( D_{\bar{c}}(X) \), we shall write...
$u \to v$ and say that $u$ is the tail of $(u,v)$ and a predecessor of $v$, and that $v$

is the head of $(u,v)$ and a successor of $u$. We shall say that a path $P$ in $X$

is an alternating path of $X$ relative to $G$, in short a $G$-alternating path of $X$,

if every other internal vertex of $P$ is the head and every other internal

vertex of $P$ is the tail (in $D_G(X)$) of its two incident edges. Similarly,

an even length cycle $C$ in $X$ is an alternating cycle of $X$ relative to $G$, in short

a $G$-alternating cycle if every other vertex of $C$ is the tail and every other

vertex of $C$ is the head (in $D_G(X)$) of its two incident edges. In particular,

a $G$-alternating cycle of $X$ is said to be an alternating cycle of $X$ in the case

$G = \text{Aut} X$.

**Lemma 2.3.** Let $X$ be a $(G, \frac{1}{2})$-transitive graph of valency 4 and let

$v_0, v_1, \ldots, v_{2k}$ be a $G$-alternating path of $X$ of even length $2k$. Then $v_0$ and $v_{2k}$

are not adjacent in $X$.

**Proof.** Since $v_{k+1}, v_{k-1}$ are either both predecessors or both successors

of $v_k$, the $\frac{1}{2}$-transitive action of $G$ implies the existence of an automorphism

$\tau \in G$ fixing $v_0$ and interchanging $v_{k-1}$ and $v_{k+1}$. It follows that $\tau$ inter-

changes $v_{k-1}$ and $v_{k+1}$ for $i = 1, 2, \ldots, k$ and so Proposition 2.1 implies that

$v_0$ and $v_{2k}$ are not adjacent.

**Proposition 2.4.** Let $X$ be a $(G, \frac{1}{2})$-transitive graph of valency 4 for

some subgroup $G$ of $\text{Aut} X$. Then there exists an integer $r \geq 2$ such that

(i) all $G$-alternating cycles of $X$ have length $2r$ and form a decomposition

of $E(X)$;

(ii) either $X$ has precisely two $G$-alternating cycles, both spanning

$V(X)$, which occurs if and only if $X \cong \text{Circ}(2r, \{1, -1, s, -s\})$ for some odd

$s \in Z; \{1, -1\}$ such that $s^2 \pm 1 = 0$; in this case $X$ is arc-transitive;

(iii) or $X$ has at least three $G$-alternating cycles, which are all induced

cycles, and if $C$ is a $G$-alternating cycle of $X$ then the constituent $G^{V(C)}$

has two orbits consisting of every other vertex of $C$, and is isomorphic to $Z^2$ if
$r = 2$, and to $D_{2r}$, the dihedral group of order $2r$, if $r \geq 3$;

**Proof.** Let us first deal with (i). Choose a vertex $v = v_0$ and let

$P = v_0v_1 \cdots v_{2m-1}$ be the largest $G$-alternating path with $v$ as its tail. Assume

first that $m$ is odd. Then $v_{m-1} \to v_{m-2}$. Let $w$ be the other successor of

$v_{m-1}$. By maximality of $P$ we have that $w = v_i$ for some $i = 0, 1, \ldots, m-3$. But $i$

cannot be odd for otherwise $v_i$ would have three predecessors and, moreover, $i$ cannot be even in view of Lemma 2.3. Therefore, $m = 2r$ must be even. Then $v_{m-2} \to v_{m-1}$. Let $w$ be the other predecessor of $v_{m-1}$. The maximality of $P$ implies $w = v_i$ for some $i = 0, 1, \ldots, m-3$. As above $i$

cannot be odd in view of Lemma 2.3 and so, since each vertex has only two successors, it follows that $w = v_0$. Hence, $C = v_0v_1 \cdots v_{2r-1}v_0$ is a
G-alternating cycle of \( X \). It is clear that \( C \) is the only \( G \)-alternating cycle containing any of its edges. Moreover, edge transitivity of \( X \) implies that each edge is contained on precisely one \( G \)-alternating cycle (of length \( 2r \)) and that all \( G \)-alternating cycles have length \( 2r \). Of course, they decompose \( E(X) \).

To prove (ii) and (iii), consider first the setwise stabilizer \( G_{V(C)} \) of the above \( G \)-alternating cycle \( C \). Since \( G \) acts \( \frac{1}{2} \)-transitively there exists an automorphism \( \tau \in G \) satisfying the rule \( v_i \tau = v_{i+r} \) for each \( i \in \mathbb{Z}_2 \). Moreover, there must exist an automorphism, say \( \rho \in G \), taking the arc \( (v_0, v_1) \) into the arc \( (v_2, v_3) \). Consequently \( \rho \in G_{V(C)} \). Its restriction to \( V(C) \) maps according to the rule \( v_i \rho = v_{i+2} \) for each \( i \in \mathbb{Z}_2 \). Thus \( \rho^{-1} \tau \rho = \rho^{-1} \) on \( V(C) \). Assume first that \( C \) is an induced cycle. Then by transitivity all \( G \)-alternating cycles are induced cycles. Clearly, there must be at least three such cycles. Moreover, \( \langle \rho, \tau \rangle V(C) = G_{V(C)} \) and so the latter is isomorphic to \( Z_2^3 \) if \( r = 2 \) and to \( D_2 \) otherwise, with two orbits consisting of vertices \( v_i \) with even and odd indices, respectively. Assume now that \( C \) is not an induced cycle. In other words, there are \( i, j \in \mathbb{Z}_2 \) such that \( i \neq j + 1, j - 1 \) with \( v_i \) and \( v_j \) adjacent in \( X \). If \( i \) and \( j \) were both even, then either \( v_i \) or \( v_j \) would have three predecessors. Similarly, if \( i \) and \( j \) were both odd, then either \( v_i \) or \( v_j \) would have three predecessors. Therefore \( j - i = s \) must be odd.

Applying the group \( \langle \rho, \tau \rangle \) we can see that \( V(X) = V(C) \) and that \( v_k v_{k+s} \) is an edge of \( X \) for all \( k \in \mathbb{Z}_2 \). In other words, \( X \cong \text{Circ}(2r, \{1, -1, s, -s\}) \).

Moreover, for \( X \) to admit a \( \frac{1}{2} \)-transitive group action, two things must happen. First, the edges of the form \( v_{k+s}, k \in \mathbb{Z}_2 \) must generate a \( G \)-alternating cycle \( C' \) of \( X \) and, second, there must be an automorphism, say \( \sigma \in G \), mapping \( v_0 \) to \( v_1 \) and interchanging \( C \) and \( C' \). It is easily seen that \( \sigma \) obeys one of the following two rules: \( v_{k} \sigma = v_{k+s} \) for each \( k \in \mathbb{Z}_2 \) or \( v_{k} \sigma = v_{k-s} \) for each \( k \in \mathbb{Z}_2 \). In both cases the additional condition \( s^2 = \pm 1 \) is deduced. Also, it is clear that \( X \) satisfies the assumptions of Proposition 2.2 and so it must be arc-transitive. This completes the proof of Proposition 2.4.

The following result is an immediate consequence of Proposition 2.4(i).

**Corollary 2.5.** Let \( X \) be a \((G, \frac{1}{2})\)-transitive graph of valency 4 for some subgroup \( G \) of \( \text{Aut} X \). Then every vertex of \( X \) lies on two \( G \)-alternating cycles.

Proposition 2.4 above suggests the following definition. One half of the length of \( G \)-alternating cycles of a given graph \( X \) of valency 4 admitting a \( \frac{1}{2} \)-transitive action of a group \( G \) will be called the radius \( r(G,X) \) of \( X \) relative to \( G \), in short the \( G \)-radius of \( X \). In particular, if \( G = \text{Aut} X \) we shall call it the radius \( r(X) \) of \( X \). The next result captures the nature and the internal structure of the intersection of two adjacent \( G \)-alternating cycles.
Proposition 2.6. Let $X$ be a $(G, \frac{1}{2})$-transitive graph of valency 4 and $G$-radius $r$ for some subgroup $G$ of $\text{Aut} X$. Then the following statements hold:

(i) any two adjacent $G$-alternating cycles of $X$ intersect in the same number of vertices and these intersections form a block system of $G$;

(ii) if $C = v_0 v_1 \cdots v_{2r-1}$ is a $G$-alternating cycle of $X$ and $C'$ is the other $G$-alternating cycle of $X$ containing $v_0$, then there exists a divisor $k$ of $2r$ such that $V(C) \cap V(C') = \{v_k: j = 0, 1, \ldots, (2r/k) - 1\}$.

Proof. Consider two pairs of adjacent $G$-alternating cycles, say $C_1, C_2$ and $C_3, C_4$. Choose vertices $u \in V(C_1) \cap V(C_2)$ and $v \in V(C_3) \cap V(C_4)$. Let $\gamma \in G$ be such that $u \gamma = v$. By Corollary 2.5, we have that $\gamma$ takes $\{C_1, C_2\}$ to $\{C_3, C_4\}$ and so it maps $C_1 \cap C_2$ to $C_3 \cap C_4$. Hence $|V(C_1) \cap V(C_2)| \leq |V(C_3) \cap V(C_4)|$ and by reversing the roles of $u$ and $v$ we have that equality holds. To show that $B = V(C_1) \cap V(C_2)$ is a block, let $x \in G$ be such that $B x \cap B \neq \emptyset$. There are $w, y \in B$ such that $w x = y$ and since $C_1$ and $C_2$ are the two $G$-alternating cycles associated with both $w$ and $y$ it follows that $x$ either fixes or interchanges $V(C_1)$ and $V(C_2)$. So $B x = B$, proving (i).

To prove (ii), let $k$ be the smallest positive integer such that $v_k \in B = V(C) \cap V(C')$. Let $\rho, \tau \in G_{V(C)}$, mapping according to the rules $v_i \rho = v_{i+2}$ and $v_i \tau = v_{i-j}$ for each $i \in \mathbb{Z}_{2r}$, be two elements in $G$ whose restrictions to $V(C)$ generate $G_{V(C)}$. Clearly, $v_i \tau \rho^k = v_k$ and so $\tau \rho^k$ must fix both $C$ and $C'$ and therefore it has to fix $B$. Since $v_0 \tau \rho^k = v_{2k}$ we must therefore have $v_{2k} \in B$. Next, we have that $v_{2k} \tau \rho^{2k} = v_{3k}$ and so $\tau \rho^{2k}$ fixes both $C$ and $C'$ and so it fixes $B$. Since $v_{2k} \tau \rho^{2k} = v_{3k}$, it follows that $v_{3k} \in B$. Continuing this way we have that $\{v_{jk}: j = 0, 1, \ldots, (2r/k) - 1\} \subseteq B$. The equality then follows in view of the minimality of $k$, completing the proof of Proposition 2.6.

The above result is a basis for the following definition. Let $X$ be a graph of valency 4 admitting a $\frac{1}{2}$-transitive action of a group $G$. The intersection of two adjacent $G$-alternating cycles of $X$ will be called a $G$-attachment set of $X$ and its cardinality the $G$-attachment number $a_G(X)$ of $X$. In particular, a $G$-attachment set of $X$ is said to be an attachment set of $X$ in the case $G = \text{Aut} X$. Similarly, the $G$-attachment number of $X$ is said to be the attachment number $a(X)$ of $X$ in the case $G = \text{Aut} X$. It follows by Proposition 2.4(ii) that the maximum attachment number $a_G(X) = 2r_G(X)$ is attained when $X$ has precisely two $G$-alternating cycles, that is when $X$ is a particular arc-transitive circulant on $2r_G(X)$ vertices. On the other hand, if $X$ has at least three $G$-alternating cycles then, in view, of Proposition 2.6, we have that $a_G(X)$ is a proper divisor of $2r_G(X)$ satisfying $1 \leq a_G(X) \leq r_G(X)$. The two extremal cases are of particular interest. If $a_G(X) = 1$, we say that $X$ is loosely $G$-attached and if $a_G(X) = r_G(X)$ we say...
that $X$ is tightly $G$-attached. Furthermore, we say that $X$ is loosely attached and tightly attached if it is $G$-loosely attached and $G$-tightly attached, respectively, and $G = \text{Aut} X$. For example, the $\frac{1}{2}$-transitive graph constructed in [8] and the line graphs of the cubic one-regular graphs constructed in [4] are all loosely attached $\frac{1}{2}$-transitive graphs of valency 4. Moreover, the $\frac{1}{2}$-transitive graphs constructed in [1, 11] are all tightly attached.

The following is a straightforward consequence of Propositions 2.4(iii) and 2.6(ii).

**Corollary 2.7.** Let $X$ be a $(G, \frac{1}{2})$-transitive graph of valency 4 for some subgroup $G$ of $\text{Aut} X$, let $X$ have at least three $G$-alternating cycles and let $A$ be a $G$-attachment set of $X$ of cardinality $a = a_G(X)$. Then the constituent $G_A^4$ is isomorphic to $Z_2^2$ if $a = 2$ and to $D_{2a}$ if $a \geq 3$.

The block system arising from Proposition 2.6 fails to be an imprimitivity system in the case of loose attachment. Nevertheless, there exists an easy combinatorial argument showing that a $\frac{1}{2}$-transitive group action on a graph of valency 4 is necessarily imprimitive. We wrap up this section by giving a particular imprimitivity block system in such graphs. (Let us also remark that imprimitivity of these groups follows from [9, Theorem 5], where transitive groups with a suborbit of length 2 are characterized.)

Given a graph of valency 4 admitting a $\frac{1}{2}$-transitive action of a group $G$, we define a relation $R$ on $V(X)$ by letting $uRv$ if $u = v$ or if $u$ and $v$ are antipodal vertices on some $G$-alternating cycle of $X$. The equivalence classes of the transitive hull of $R$ will be called the transversals of $X$ relative to $G$, in short the $G$-transversals of $X$. They are called transversals of $X$ in the case $G = \text{Aut} X$.

**Proposition 2.8.** Let $X$ be a $(G, \frac{1}{2})$-transitive graph of valency 4 for some subgroup $G$ of $\text{Aut} X$ and let $r$ be the $G$-radius of $X$. Let $\mathcal{T}$ be the set of $G$-transversals of $X$ and let $\mathcal{T}^#$ $T \in \mathcal{T}$. Then the following statements hold:

(i) the constituent $G_T^r$ is a regular group and, moreover, if $r$ is odd then $G_T^r$ is cyclic, and if $r$ is even then $|T|$ is even too and $G_T^r$ is isomorphic to $Z_2^2$ if $|T| = 2$, to $Z_2^3$ if $|T| = 4$ and to $D_2^r$ otherwise;

(ii) the set $\mathcal{T}$ is an imprimitivity system for $G$.

**Proof.** Let us first prove (i). The fact that $G_T^r$ is a semiregular group is straightforward to check as every element of $G$ fixing a given vertex must also fix both of its antipodal vertices on the corresponding $G$-alternating cycles of $X$. Consider two antipodal vertices $u, v \in V(X)$ on a $G$-alternating cycle $C$ of $X$. Let $C'$ be the other $G$-alternating cycle containing $v$ and let $w$ be the antipodal vertex of $v$ on $C'$. Suppose that $u$ has two successors on
C. Assume first that $r$ is odd. Then the predecessors of $v$ are both on $C$ and the successors of $v$ are both on $C'$. Choose $\sigma \in G$ so as to map $u$ into $v$. It follows that $\sigma v = w$. Continuing this way we see that $G_T^\sigma = \langle \sigma \rangle^T$ is cyclic and transitive, and thus regular. On the other hand, if $r$ is even then the successors of $v$ are both on $C$ and the predecessors of $v$ are both on $C'$. In particular, it follows that $|T|$ is even. Let $\pi \in G$ interchange $u$ and $v$. If $|T| = 2$ then $G_T^\pi = \langle \pi \rangle^T$ is cyclic. Let $|T| \geq 4$ and let $\sigma \in G$ take $u$ into $w$. It follows that $G_T^\sigma = \langle \sigma, \pi \rangle^T$ is a transitive and thus regular group, which is isomorphic to $Z_2^2$ if $|T| = 4$ and is dihedral otherwise.

Let us now deal with (ii). First of all, we must have that $|\mathcal{F}| \geq 2$ for otherwise $V(X)$ would be a $G$-transversal and, hence, by (i), $G$ would be a regular group. Moreover, each $T \in \mathcal{F}$ contains at least two elements. Thus to prove (ii) it suffices to show that each $T \in \mathcal{F}$ is a block of $G$. Observe that every element of $G$ preserves antipodality and therefore if $T \in \mathcal{F}$ and $\pi \in G$ then $Tx \in \mathcal{F}$. If $T \cap Tx \neq \emptyset$ and $Tx \neq T$ then there are antipodal vertices $u, v \in T$ such that $ux \in T$ and $vx \notin T$, contradicting the fact that $Tx$ is a $G$-transversal. Hence, $T$ is a block of $G$, completing the proof of Proposition 2.8.

### 3. A LABELING OF TIGHTLY ATTACHED GRAPHS

Let $X$ be a tightly $G$-attached $(G, \frac{1}{2})$-transitive graph of valency 4 for some subgroup $G \leq Aut X$, let $r \geq 2$ and $t \geq 3$, respectively, be the $G$-radius and the number of $G$-alternating cycles of $X$, let $\mathcal{A}$ be the imprimitivity system of $G$-attachment sets of length $r$ of $X$ and let $K$ be the kernel of the action of $G$ on $\mathcal{A}$. We have the following result.

**Proposition 3.1.** Let $X, G, \mathcal{A}, K, r, t$ have the meaning described above. If $r = 2$ then $X \cong C_2[2K_1]$. If $r \geq 3$ then there are automorphisms $\rho, \tau, \sigma \in G$ such that the following statements hold:

(i) $K = \langle \rho, \tau \rangle \cong D_2$, and moreover, $\rho$ is $(t, r)$-semiregular, that is its restriction to each $A \in \mathcal{A}$ is an $r$-cycle, and $\tau$ is an involution inverting each $G$-alternating cycle and such that $\rho^s = \rho^{-1}$;

(ii) $\sigma$ cyclically permutes the $G$-attachment sets (and the $G$-alternating cycles) of $X$ so that $\langle \sigma \rangle K \cap \langle \sigma \rangle \cong Z_2 \cong G/K$ and $\rho^s = \rho^t$ for some $s \in Z_2^*$, and moreover, $X$ is a $(t, r)$-metacirculant;

(iii) $G = \langle \rho, \tau, \sigma \rangle$ and its order is $2tr$.

**Proof.** Let us first observe that the claim is clearly true for $r = 2$. So we shall now assume that $r \geq 3$. Let $A \in \mathcal{A}$ be a $G$-attachment set of $X$ and let
C and C′ be the two G-alternating cycles of X intersecting in A. We claim that

\[ \text{if } \gamma \in G \text{ fixes } A \text{ setwise then } \gamma \in K \quad (1) \]

and

\[ \text{if } \gamma \in G \text{ fixes } A \text{ pointwise then } \gamma = 1. \quad (2) \]

Observe that for each vertex in A the orientations (in DG(X)) of its two incident edges on C are opposite from those on C′. Hence if \( \gamma \) fixes A setwise then it must fix both C and C′ setwise and so it fixes the G-attachment sets \( V(C) \setminus A \) and \( V(C') \setminus A \) setwise. Continuing this way we have \( \gamma \in K \), proving (1). As for (2), if \( \gamma \) fixes A pointwise, then clearly (1) implies \( \gamma \in K \). So we may apply Proposition 2.4(iii) to deduce that \( \gamma \) fixes \( V(C) \setminus A \) and \( V(C') \setminus A \), pointwise. Continuing this way we have \( \gamma = 1 \).

Now Corollary 2.7 ensures the existence of automorphisms \( \rho \) and \( \tau \) of X whose respective restrictions to A are an \( r \)-cycle and an involution generating the constituent \( G'_t, \) a transitive dihedral group of order 2r. More precisely, a conjugation by \( \tau^t \) inverts \( \rho^t \). By (1) we have \( \langle \rho, \tau \rangle \subseteq K \). To prove the converse let \( \gamma \in K \). Then clearly \( \gamma^t \in \langle \rho, \tau \rangle^t \) and so \( \gamma^t = \delta^t \) for some \( \delta \in \langle \rho, \tau \rangle \). But then \( \gamma^t \delta^{-1} \) fixes A pointwise and so (2) implies that \( \gamma = \delta \in \langle \rho, \tau \rangle \). Moreover, since \( \rho^t \) is an \( r \)-cycle, (2) implies that the restriction of \( \rho \) to any G-attachment set must be an \( r \)-cycle. Similarly, since the restriction of \( \tau \) to A is an involution inverting \( \rho \), the same holds for any other G-attachment set. This proves (i).

To prove (ii), choose \( \sigma \in G \) such that \( A \sigma \cap (V(C) \setminus A) \neq \emptyset \). Because of the particular orientation of the edges (in \( D_G(X) \)), used above in the proof of (1), we have that \( \sigma \) maps A and \( V(C') \setminus A \), respectively, to \( V(C) \setminus A \) and A. Repeating this argument we have that \( \sigma \) cyclically permutes the G-attachment sets and the G-alternating cycles of X. This together with (1) implies that \( G/K \) is regular on \( \mathcal{A} \). More precisely, \( G/K \cong \mathbb{Z}_t \times \mathbb{Z}_r \cong \langle \sigma \rangle/K \cap \langle \sigma \rangle \). Note that \( \rho^t \sigma \in K \) and because of the structure of K we must have \( \rho^t \sigma \in \langle \rho \rangle \) and so there exists \( s \in \mathbb{Z}_r \) such that \( \rho^s = \rho^t \). Of course, X is a \( (t, r) \)-metacirculant.

Finally, (iii) follows from (i) and (ii).

The above result enables us to give a labeling of graphs of valency 4 admitting a \( \frac{1}{2} \)-transitive group action with respect to which these graphs have odd radius and are tightly attached. To that end we first define a class of graphs. Let \( t \geq 3 \) be an integer, \( s \geq 3 \) be an odd integer and let \( s \in \mathbb{Z}_r^* \) satisfy \( s^t = \pm 1 \). The graph \( X(s, t, r) \) is defined to have the vertex set \( \{ v'_i : i \in \mathbb{Z}_s, j \in \mathbb{Z}_r \} \) and edges of the form \( v'_i, v'_{i+1}, v'_{i\pm j}, v'_{i+1} (i \in \mathbb{Z}_s, j \in \mathbb{Z}_r) \). It
is easily checked that the permutations \( \rho, \sigma, \) and \( \tau \) mapping according to the rules

\[
    v'_i \rho = v'_{i+1}, \quad i \in \mathbb{Z}_t, \quad j \in \mathbb{Z}_r. \quad (3)
\]
\[
    v'_i \sigma = v'_{i+1}, \quad i \in \mathbb{Z}_t, \quad j \in \mathbb{Z}_r. \quad (4)
\]
\[
    v'_i \tau = v'_{i'}, \quad i \in \mathbb{Z}_t, \quad j \in \mathbb{Z}_r. \quad (5)
\]

are automorphisms of \( X(s; t, r) \). We use the symbols \( H(s; t, r) \) and \( \text{Att} X(s; t, r) \), respectively, to denote the group \( \langle \rho, \sigma, \tau \rangle \) and the largest subgroup of \( \text{Aut} X(s; t, r) \) having the set of orbits \( V_i = \{ v'_j : j \in \mathbb{Z}_r \}, i \in \mathbb{Z}_t \), of \( \rho \) as an imprimitivity block system. Of course, \( H(s; t, r) \leq \text{Att} X(s; t, r) \).

We remark that for brevity reasons the parameters \( s, t, r \) will occasionally be omitted in the above notation \( H(s; t, r) \), causing no ambiguity.

The next two propositions establish a 1–1 correspondence between the graphs \( X(s; t, r) \) and the graphs of valency 4 admitting a \( \frac{1}{2} \)-transitive group action relative to which they are tightly attached with odd radius. The proof of the first of these propositions is an immediate consequence of the above definitions.

**Proposition 3.2.** With the above notation, the graph \( X(s; t, r) \) is a tightly \( H(s; t, r) \)-attached \( \frac{1}{2} \)-transitive graph. Moreover, the corresponding radius is \( r \) and the corresponding attachment sets and transversals are, respectively, the orbits \( V_i = \{ v'_j : j \in \mathbb{Z}_r \}, i \in \mathbb{Z}_t \), of \( \rho \) and the sets \( \{ v'_i : i \in \mathbb{Z}_t \}, j \in \mathbb{Z}_r \).

**Proposition 3.3.** Let \( t \geq 3 \) be an integer and \( r \geq 3 \) be an odd integer. Let \( X \) be a graph of valency 4 and \( G \) be a subgroup of \( \text{Aut} X \) such that \( X \) is tightly \( G \)-attached \( (G, \frac{1}{2}) \)-transitive with \( r \) and \( t \) as its respective \( G \)-radius and the number of \( G \)-alternating cycles. Then there exists \( s \in \mathbb{Z}_r^* \) such that \( X \cong X(s; t, r) \).

**Proof.** Let \( \mathcal{A} \) be the set of \( G \)-attachment sets of \( X \) and let \( K \) be the kernel of the action of \( G \) on \( \mathcal{A} \). We are going to choose the automorphisms \( \rho, \sigma, \) and \( \tau \) (existing by Proposition 3.1) in a particular way, so as to deduce the isomorphism of \( X(s; t, r) \) with \( X(s; t, r) \) for some \( s \in \mathbb{Z}_r^* \). Choose a vertex \( v \in V(X) \) and let \( \tau \in K \) be the involution fixing \( v \). Letting \( T \) be the \( G \)-transversal containing \( v \), we have that \( T \) is precisely the set of fixed points of \( \tau \). Let \( A \) be the \( G \)-attachment set containing \( v \) and let \( C \) be one of the two \( G \)-alternating cycles generated by \( A \) (containing \( v \)). Choose \( \sigma \) in such a way that it maps \( v \) to its antipodal vertex on \( C \). (In other words, \( \sigma \) is chosen in such a way that \( \sigma^T \) generates \( G_T^* \).) Clearly, \( \sigma^T \in \langle \tau \rangle \). This enables us to label the vertices of \( X \) as follows. First, let \( v_0 = v\sigma^i \) for all
Next, we choose $\rho \in K$ so as to map $v$ to one of the two neighbors on $C$ of its antipodal vertex $\psi$. Of course $K = \langle \rho, \tau \rangle$. Let $s \in \mathbb{Z}_r^*$ be such that $\rho^s = \rho \tau$. Set $v'_i = v_i^0 \rho^s$ for all $i \in \mathbb{Z}_r$ and $j \in \mathbb{Z}_s$. It is easily checked that $\rho$ and $\sigma$ satisfy the respective rules (3) and (4). As for $\tau$, note that $\rho^s = \rho^{-1}$ and so one can easily see that $v'_i \tau = v_i^{-1}$ for all $i \in \mathbb{Z}_r$ and $j \in \mathbb{Z}_s$. Besides, since $s' \in \langle \tau \rangle$ it follows that $s' = \pm 1$. Let us now check the adjacencies in $X$. By assumption we have that $v_0^s = v_0^0 \sigma$ is adjacent to $v_0^1 = v_0^0 \rho$. Applying $\rho^{-1}$ we get that $v_0^{-1}$ is a neighbor of $v_0^0$ and then applying $\tau$ we obtain that $v_0^1$ is a neighbor of $v_0^0$, too. Finally, using the automorphisms $\sigma^\rho, i \in \mathbb{Z}_r$, $j \in \mathbb{Z}_s$, on these adjacencies we have that $v_j^i$ is a neighbor of $v_j^{i+1}$ and $v_j^{-i}$ for all $i \in \mathbb{Z}_r$ and $j \in \mathbb{Z}_s$, completing the proof of Proposition 3.3.

In the rest of this paper we shall be gradually working our way towards the classification of tightly attached $\frac{1}{2}$-transitive graphs of valency 4 with odd radius. By Propositions 3.2 and 3.3, such a classification amounts to determining all triples $(s; t, r)$ giving rise to a $\frac{1}{2}$-transitive graph $X(s; t, r)$. We shall prove the following result.

**Theorem 3.4.** A graph $X$ is a tightly attached $\frac{1}{2}$-transitive graph of valency 4 and odd radius $r$ if and only if $X \cong X(s; t, r)$, where $t \geq 3$ and $s \in \mathbb{Z}_r^*$ satisfies $s' = \pm 1$, and moreover none of the following conditions is fulfilled:

(i) $s^2 = \pm 1$;

(ii) $(s, t, r) = (2; 3, 7)$;

(iii) $(s, t, r) = (s; 6, 7k)$, where $k \geq 1$ is odd, $(7, k) = 1$, $s^6 = 1$, and there exists a unique solution $q \in \{s, -s, 1/s, -1/s\}$ of the equation $x^2 + x - 2 = 0$ such that $7(q - 1) = 0$ and $q \equiv 5 \pmod{7}$.

We remark that the cases $t = 3$ and $t = 4$ of Theorem 3.4 may be deduced from the results proven, respectively, in [1, 11]. Let us also mention that for each odd integer $k$ coprime with 7 there exists a unique $q$ such that the condition (iii) of Theorem 3.4 is satisfied. An infinite family of arc-transitive graphs $X(q; 6, 7k)$ is thus generated, with the smallest member being the graph $X(5; 6, 7) \cong X(2; 6, 7)$. Furthermore, it is interesting to note that $X(2; 3, 9)$ is in fact isomorphic to the smallest $\frac{1}{2}$-transitive graph constructed by Holt [6], whereas $X(2; 6, 9)$, the double cover of $X(2; 3, 9)$, is isomorphic to the smallest Bouwer’s graph [3]. (The double cover of a graph $Y$ has vertex set $V' \cup V''$, where $|V'| = |V(Y)| = |V''|$, and edges of the form $u'v'$ and $w'v''$ with $uv \in E(Y)$.)

We end this section by making our first assault on Theorem 3.4. As an immediate consequence of the result proven below we have that the condition $s^2 = \pm 1$ forces the graphs $X(s; t, r)$ to be arc-transitive.
Proposition 3.5. Let \( t \geq 3 \) be an integer, \( r \geq 3 \) be an odd integer and let \( s \in \mathbb{Z}_r^* \) satisfy \( s^t = \pm 1 \). Let \( X = X(s; t, r) \), \( H = H(s; t, r) \) and let \( \omega \) be the permutation of \( V(X) \) mapping according to the rule:

\[
v^i_1 \omega = v^i_{t-i}, \quad i \in \mathbb{Z}_r, \quad f \in \mathbb{Z}_r.
\]

Then the following is true:

(i) if \( s^t = \pm 1 \) then \( \text{Att } X(s; t, r) = \langle H, \omega \rangle \) acts arc-transitively;

(ii) if \( s^t \neq \pm 1 \) then \( \text{Att } X(s; t, r) = H \) acts \( \frac{1}{2} \)-transitively.

Proof. Let \( G = \text{Aut } X, \mathcal{A} = \{ V_i : i \in \mathbb{Z}_r \} \) and \( L = \text{Att } X(s; t, r) \). Let \( \bar{L} = \text{Ker}(L \rightarrow L) = \langle \rho, \tau \rangle \). The structure of \( X \) implies that either \( \bar{L} \cong \mathbb{Z}_r \) or \( \bar{L} \cong D_2 \). In the first case we must have that \( \bar{L} = \langle \sigma \rangle \) and \( L = H \) acts \( \frac{1}{2} \)-transitively. In the second case we must have that \( L = \langle H, \omega \rangle \). In fact, we shall prove that \( \bar{L} = \langle H, \omega \rangle \) in this case, implying that \( L \) acts arc-transitively.

It follows from the above comments that it suffices to characterize those values of \( s \) for which the group \( L \) is dihedral. So assume this is the case and choose \( \alpha \in L \) of order a power of 2 fixing \( v_0 \) and interchanging the sets \( V_i \) and \( V_{i-1} \), for all \( i \neq 0 \). (Such an element always exists.) We have that \( \alpha \) interchanges the sets \( N(v_0^i) \cap V_i = \{ v_i^1, v_i^{-1} \} \) and \( N(v_0^i) \cap V_{i-1} = \{ v_i^{t-1}, v_i^{-t-1} \} \). Moreover, \( \alpha \) interchanges the sets \( N(v_0^i, v_0^{-1}) \cap V_0 = \{ v_0^1, v_0^{-2} \} \) and \( N(v_0^{i-1}, v_0^{t-1}) \cap V_0 = \{ v_0^{2t-1}, v_0^{2-1} \} \). Continuing this way we can see that \( \alpha \) interchanges the sets \( \{ v_i^j, v_i^{-j} \} \) and \( \{ v_i^{t-j}, v_i^{-t-j} \} \) for \( i = 0, 1 \) and all \( j \in \mathbb{Z}_r \). But \( N(v_0^i, v_0^{-1}) \cap V_{i-1} = \{ v_i^{t-j} : e, f \in \{ 1, -1 \} \} \) and \( N(v_0^{i-1}, v_0^{t-1}) \cap V_{i-1} = \{ v_i^{t-j} : e, f \in \{ 1, -1 \} \} \). Thus \( s^{-1}(\pm 1 \pm s^{-1}) = \pm j \pm s^{-1} \) and so \( s^t = \pm 1 \).

It remains to check that \( s^t = \pm 1 \) forces \( \omega \in G \). In fact, we have that \( N(v_0^i) = \{ v_0^{i-1}, v_0^{-i+1}, v_0^{i-1}, v_0^{-i+1} \} \) and \( N(v_i^j) = \{ v_i^{j-1}, v_i^{-j+1}, v_i^{j-1}, v_i^{-j+1} \} \). Since \( s^t = \pm 1 \) these two sets are equal and so \( \omega \in G \). In particular, \( L = \langle H, \omega \rangle \) and so \( L \) acts arc-transitively, completing the proof of Proposition 3.5. \( \square \)

Corollary 3.6. Let \( t \geq 3 \) be an integer, \( r \geq 3 \) be an odd integer and let \( s \in \mathbb{Z}_r^* \) satisfy \( s^t = \pm 1 \). If \( s^t = \pm 1 \) then \( X(s; t, r) \) is arc-transitive.

4. GRAPHS \( X(s; t, r) \) WITH NONCOILED GIRTH AT MOST 6

Throughout this section we let \( t \geq 3 \) be an integer, \( r \geq 3 \) be an odd integer and we let \( s \in \mathbb{Z}_r^* \) satisfy \( s^t = \pm 1 \).

Proposition 4.1. \( X(-s; t, r) \cong X(s; t, r) \cong X(s^{-1}; t, r) \).
The first isomorphism is obvious. As for the second one, letting \( \{ v'_i : i \in \mathbb{Z}_t \} \) and \( \{ u'_i : i \in \mathbb{Z}_t \} \) be the respective vertex sets of \( X(s, t, r) \) and \( X(s', t, r) \), it may be easily checked that the mapping \( \phi : v'_i \to u'_{i+1} \) is a graph isomorphism.

Let us consider the action of the group \( H = H(s, t, r) \) on the set of 2-paths of \( X(s, t, r) \). First, any 2-path whose endvertices belong to the same attachment set \( V_i, i \in \mathbb{Z}_t \), will be called an \textit{anchor}. The group \( H \) has two orbits on the set of all anchors of \( X \): one containing the anchor \( v_0^0v_1v_2^0 \) and the other containing the anchor \( v_0^2v_1v_2^1 \). They will be denoted by \( \text{Anc}^+(s, t, r) \) and \( \text{Anc}^-(s, t, r) \), respectively, and their elements will be called positive anchors and negative anchors, respectively (see Figs. 1a,b).

Let \( \text{Anc}(s, t, r) = \text{Anc}^+(s, t, r) \cup \text{Anc}^-(s, t, r) \). Any other 2-path has each of the three vertices in different \( H \)-attachment sets. Any 2-path belonging to the same \( H \)-orbit as \( v_0^0v_1v_2^0 \) is said to be a \textit{glide} (see Fig. 1c) and any 2-path belonging to the same \( H \)-orbit as \( v_0^0v_1v_2^1 \) a \textit{zigzag} (see Fig. 1d).

To summarize, \( \text{Anc}^+(s, t, r), \text{Anc}^-(s, t, r), \text{Gli}(s, t, r) \) and \( \text{Zig}(s, t, r) \) are the four \( H \)-orbits on the set of 2-paths of \( X \). For brevity reasons, we shall be omitting the parameters \( s, t, r \) in the above notations for 2-paths of \( X \). Such a use of symbols \( \text{Anc}, \text{Anc}^+, \text{Anc}^-, \text{Gli}, \text{and Zig} \) should cause no ambiguity.

**Proposition 4.2.** Let \( X = X(s; t, r) \), where \( t \geq 3, r \geq 3 \) is odd and \( s \in \mathbb{Z}^* \) satisfies \( s't' = \pm 1 \). Then \( \text{Att} X \) is the largest subgroup of automorphisms of \( X \) fixing the set \( \text{Anc} \).

**Proof.** Let \( M \) be the largest subgroup of \( \text{Aut} X \) fixing the set \( \text{Anc} \). Clearly, \( \text{Att} X \leq M \). To prove that \( \text{Att} X = M \) it therefore suffices to show that \( \mathcal{A} = \{ V_i : i \in \mathbb{Z}_t \} \) is an imprimitivity block system of \( M \). Let \( \pi \in M \), let \( i \in \mathbb{Z}_t \) and suppose that \( V_i \cap V_j \pi \neq \emptyset \). Let \( u \in V_i \cap V_j \pi \). Since \( \text{Anc} = \text{Anc} \),

\[
\begin{align*}
\text{(a)} & \quad \text{Fig. 1. Two-paths in } X(s; t, r). \\
\text{(b)} & \\
\text{(c)} & \\
\text{(d)} &
\end{align*}
\]
we must have that \((V_i \cap N^2(u)) x \subseteq V_i\). Continuing this way we see that 
\((V_i \cap N^{2k}(u)) x \subseteq V_i\) for each \(k\) and thus \(V_i x = V_i\). Hence, \(V_i\) is a block of \(M\).

Propositions 3.5 and 4.2 together imply the following result.

**Corollary 4.3.** Let \(X = X(s; t, r)\), where \(t \geq 3, r \geq 3\) is odd and \(s \in Z^*_r\) satisfies \(s' = \pm 1\). The following statements are equivalent:

(i) \(\text{Aut } X = H(s; t, r)\);

(ii) \(X\) is \(\frac{1}{2}\)-transitive;

(iii) \(s^2 \neq \pm 1\) and \(\text{Aut } X\) fixes the set of anchors \(\text{Anc}\) of \(X\).

**Lemma 4.4.** Let \(X = X(s; t, r)\), where \(t \geq 3, r \geq 3\) is odd and \(s \in Z^*_r\) satisfies \(s' = \pm 1\) and \(s^2 \neq \pm 1\). Then no automorphism of \(X\) maps \(\text{Anc}\) to either \(\text{Gli}\) or \(\text{Zig}\).

**Proof.** Suppose that there exists an automorphism \(\varphi\) of \(X\) such that \(\text{Anc} \varphi = \text{Gli}\). Then let \(P_1\) and \(P_2\) be two consecutive anchors of \(X\) (one positive and one negative) and let \(Q_1\) and \(Q_2\) be their respective images under \(\varphi\). Since \(\text{Gli}\) is an orbit of the action of \(H = H(s; t, r)\) on 2-paths of \(X\), there exists \(\beta \in H\) such that \(Q_1 \varphi = Q_2\). It follows that \(s^2 \beta \varphi^{-1}\) takes \(P_1\) to \(P_2\) and, moreover, it fixes the set \(\text{Anc}\). But \(s^2 \neq \pm 1\) and so Propositions 3.5 and 4.2 together imply that \(s^2 \beta \varphi^{-1} \in H\), a contradiction as \(P_1\) and \(P_2\) belong to different \(H\)-orbits. A similar contradiction is obtained if we assume that \(\text{Anc} \varphi = \text{Zig}\).
other words a cycle of length $d$ is coiled if its trace is $n^d$. The proof of the next result is straightforward.

**Proposition 4.5.** Let $X = X(s; t, r)$, where $t \geq 3$, $r \geq 3$ is odd and $s \in Z^*_+$ satisfies $s' = \pm 1$. Then positive and negative anchors alternate on each noncoiled cycle of $X$.

Let $\mathcal{C}$ be a set of cycles in $X$, closed under the action of the group $H$ (that is, $\mathcal{C}$ is a union of $H$-orbits of cycles in $X$) and let $P$ be a path of $X$. The $\mathcal{C}$-frequency $\nu(\mathcal{C}, P)$ of $P$ is the number of cycles from $\mathcal{C}$ containing $P$. In particular, if $d \geq 3$ is an integer and $\mathcal{C}$ is the set of all $d$-cycles in $X$, we let the $d$-frequency $\nu(d, P)$ of $P$ equal to $\nu(\mathcal{C}, P)$. The proof of the next result is omitted.

**Proposition 4.6.** Let $X = X(s; t, r)$, where $t \geq 3$, $r \geq 3$ is odd and $s \in Z^*_+$ satisfies $s' = \pm 1$, let $\mathcal{C}$ be a union of $\text{Aut}_X$-orbits of cycles of $X$ and let $P$ and $Q$ be two paths of $X$ permutable by an automorphism of $X$. Then $\nu(\mathcal{C}, P) = \nu(\mathcal{C}, Q)$.

Assume now that $P$ is one of the 2-paths of $X$. Since $\text{Anc}^+$, $\text{Anc}^-$, Gli, and Zig are $H$-orbits and since, in view of Proposition 4.5, every cycle in $X$ contains the same number of positive and negative anchors, we may define $v_a(\mathcal{C})$, $v_g(\mathcal{C})$, and $v_z(\mathcal{C})$ to be the respective $\mathcal{C}$-frequencies of anchors, glides, and zigzags in $X$, and in particular, we may define $v_a(d)$, $v_g(d)$, and $v_z(d)$ to be the respective $d$-frequencies of anchors, glides, and zigzags in $X$. For example, let $\mathcal{C}$ be an $H$-orbit and let $C \in \mathcal{C}$ contain $a$ anchors. Then every cycle in $\mathcal{C}$ contains $a$ anchors and since there are $2tr$ anchors in $X$ we have the following formula for the $\mathcal{C}$-frequencies of anchors, as well as glides and zigzags, in $X$:

$$v_x(\mathcal{C}) = \frac{|C|a_{x,c}}{2tr}, \quad x \in \{a, g, z\}, \quad C \in \mathcal{C}. \quad (7)$$

**Lemma 4.7.** Let $X = X(s; t, r)$, where $t \geq 3$, $r \geq 3$ is odd and $s \in Z^*_+$ satisfies $s' = \pm 1$ and $s^2 \neq \pm 1$, and let $\mathcal{C}$ be an $\text{Aut}_X$-orbit of $d$-cycles, $d \geq 3$, in $X$. If $v_a(\mathcal{C}) \neq v_g(\mathcal{C})$, $v_z(\mathcal{C})$ then $X$ is $\frac{1}{2}$-transitive. In particular, $v_a(d) \neq v_g(d)$, $v_z(d)$ implies that $X$ is $\frac{1}{2}$-transitive.

**Proof.** The assumptions of this lemma force that Anc is fixed by $\text{Aut}_X$ and the result follows by Corollary 4.3.

The noncoiled girth of $X = X(s; t, r)$ is the length of its shortest noncoiled cycle. Since, for example, $v_0^0 v_1^0 v_2^0 v_3^0 v_0^1 v_1^0 v_2^0 v_3^0 v_0^2 v_1^0 v_2^0 v_3^0$ is a noncoiled 8-cycle
of $X$ and since noncoiled cycles are necessarily of even length, it follows that the noncoiled girth of $X$ is either 4, 6, or 8. Clearly, the first case happens if and only if $s = \pm 1$, and so it follows by Corollary 3.6, that, if $X$ is $\frac{1}{2}$-transitive, its noncoiled girth is either 6 or 8. The analysis of $\frac{1}{2}$-transitivity of $X$ will be based on a careful consideration of these two cases. To end this section the case of noncoiled girth 6 is taken care of.

**Proposition 4.8.** Let $X = X(s; t, r)$, where $t \geq 5$, $r \geq 3$ is odd and $s \in \mathbb{Z}_*^+$ satisfies $s' = \pm 1$ and $s^2 \neq \pm 1$, have noncoiled girth 6. Then $X$ is $\frac{1}{2}$-transitive unless $X \cong X(2; 6, 7k), k \in \{1, 3\}$, in which case it is arc-transitive.

*Proof.* Let $G = \text{Aut} X$ and recall that $H = H(s; t, r)$. Since $s^2 \neq \pm 1$ we have

$$r \geq 7.$$  

(8)

It is easy to check that a noncoiled 6-cycle in $X$ must have one of the following two traces. First, if it intersects precisely three consecutive $H$-attachment sets $V_1, V_{s+1}, V_{s+2}$ its trace is $a'nun$ and, second, if it intersects precisely four consecutive $H$-attachment sets $V_1, V_{s+1}, V_{s+2}, V_{s+3}$ then its trace is $an'an'$. Also, a coiled 6-cycle may only exist for $t = 6$.

In Table I below all $H$-orbits of noncoiled 6-cycles in $X$ are given. For each orbit we list the corresponding trace, a representative $C$ belonging to that orbit, a necessary and sufficient condition on $s$ for that orbit to exist, the length of the orbit, and finally its code.

We observe that no two conditions on $s$ from rows 1 to 4 as well as no two conditions on $s$ from rows 5 to 8 of Table I may hold simultaneously. So suppose that two rows in Table I give rise to 6-cycles in $X$ simultaneously. By Proposition 4.1 we may then assume that $s = 2$. Thus (8)

<table>
<thead>
<tr>
<th>Row</th>
<th>Trace</th>
<th>Respective $C$</th>
<th>Condition</th>
<th>Orbit</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a'nun$</td>
<td>$v_1^2 v_0^2 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3$</td>
<td>$s - 2 = 0$</td>
<td>$tr$</td>
<td>$a'zaz$</td>
</tr>
<tr>
<td>2</td>
<td>$a'nun$</td>
<td>$v_1^2 v_0^2 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3$</td>
<td>$s + 2 = 0$</td>
<td>$tr$</td>
<td>$a'xog$</td>
</tr>
<tr>
<td>3</td>
<td>$a'nun$</td>
<td>$v_1^2 v_0^2 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3$</td>
<td>$2s - 1 = 0$</td>
<td>$tr$</td>
<td>$a'zaz$</td>
</tr>
<tr>
<td>4</td>
<td>$a'nun$</td>
<td>$v_1^2 v_0^2 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3$</td>
<td>$2s + 1 = 0$</td>
<td>$tr$</td>
<td>$a'xog$</td>
</tr>
<tr>
<td>5</td>
<td>$an'am'$</td>
<td>$v_1^2 v_0^2 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3$</td>
<td>$s^2 + s + 1 = 0$</td>
<td>$tr$</td>
<td>$ag'og^2$</td>
</tr>
<tr>
<td>6</td>
<td>$an'am'$</td>
<td>$v_1^2 v_0^2 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3$</td>
<td>$s^2 - s - 1 = 0$</td>
<td>$tr$</td>
<td>$ag'xog$</td>
</tr>
<tr>
<td>7</td>
<td>$an'am'$</td>
<td>$v_1^2 v_0^2 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3$</td>
<td>$s^2 - s + 1 = 0$</td>
<td>$tr$</td>
<td>$ax'zaz$</td>
</tr>
<tr>
<td>8</td>
<td>$an'am'$</td>
<td>$v_1^2 v_0^2 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3 v_1^3$</td>
<td>$s^2 + s - 1 = 0$</td>
<td>$tr$</td>
<td>$az'zaz$</td>
</tr>
</tbody>
</table>
implies that the condition in row 5 must be satisfied and so \( r = 7 \). Also, \( t = 3k \) for some \( k \geq 2 \). Therefore we have that

\[
either X \cong X(2, 3k, 7) \text{ for some } k \geq 2
\]
or a single condition from Table I holds. \hfill (9)

We are going to distinguish two different cases.

**Case 1.** \( X \) contains no coiled 6-cycle. Suppose first that two conditions from Table I hold simultaneously. Then (9) implies \( X \cong X(2, 3k, 7) \), \( k \geq 3 \). Computing the 6-frequencies of 2-paths in \( X \) we have \( v_6(6) = 3 \), \( v_4(6) = 2 \), and \( v_2(6) = 1 \) and so, by Lemma 4.7, \( X \) is \( \frac{1}{2} \)-transitive. Assume now that a single condition from Table I holds. In view of Lemma 4.7 we may assume that a condition in row 6 or row 8 is satisfied. In both cases each 2-path in \( X \) is contained on a unique 6-cycle, that is \( v_6(6) = v_4(6) = v_2(6) = 1 \). Assume that the condition in row 6 is satisfied. Then the refinement of the corresponding 6-cycles is \( a^-gza^+g \). We observe that a 3-path of \( X \) is contained on a 6-cycle if and only if its refinement is \( gz, a^-g, or a^-z \). Let \( \alpha \in G \) and let \( C \) be an arbitrary alternating cycle of \( X \). Clearly, no 3-subpath of \( C \) is contained on a 6-cycle and so \( Cx \) contains no 3-subpaths with refinements \( gz, a^-g, or a^-z \). If \( Cx \) contains a 3-subpath with refinement \( a^-g \), then it must contain a 3-subpath with refinement \( ga^- = a^-g \), which is impossible, or \( za^- = a^-z \), the latter forcing the existence of a 3-subpath with refinement \( gz \), again impossible. The same argument is used to see that \( Cx \) contains no 3-subpath with refinement \( a^-z \). But then, in view of Lemma 4.4 we must have \( \text{re}(Cx) = (a^-a^-) = \text{re}(C) \), that is \( Cx \) is an alternating cycle of \( X \). We argue in the same way if the condition in row 8 is satisfied. We conclude that \( G \) fixes \( \text{Anc} \) and so \( X \) is \( \frac{1}{2} \)-transitive by Corollary 4.3.

**Case 2.** \( X \) contains a coiled 6-cycle. Of course \( t = 6 \). Suppose first that precisely one of the conditions in rows 5 to 8 of Table I holds. In particular, \( v_6(6) = 1 \). By Proposition 4.1, we may assume that either \( s^2 + s + 1 = 0 \) or \( s^2 - s - 1 = 0 \). In the first case we have \( v_6(6) \geq 2 \). Moreover, \( s^2 = 1 \) and so \( s^3 + s^4 + s^5 + s^6 = 1 = 0 \), as well as \( s^3 - s^4 + s^5 - s^6 + 1 = 0 \), giving us coiled 6-cycles with respective codes \( g^2gzg \) and \( z^5 \). Thus \( v_6(6) \geq 2 \) and so Lemma 4.7 implies that \( X \) is \( \frac{1}{2} \)-transitive. In the second case we have \( s^3 - s^4 + s^5 - s - 1 = 0 \), as well as \( s^3 - s^4 + 4 + 1 = 0 \), giving us coiled 6-cycles with respective codes \( g^2gz^2 \) and \( g^2gzg^2z \). It follows that \( v_6(6) < v_6(6) \) and again Lemma 4.7 implies that \( X \) is \( \frac{1}{2} \)-transitive.

In view of (9) we may now assume that either \( X \cong X(2, 6, 7) \) or precisely one of the conditions in rows 1 to 4 of Table I holds. Hence, by Proposition 4.1 we may assume \( s = 2 \) and so \( X \) contains 6-cycles with code \( a^2zaz \).
Since $2^6 = \pm 1$, it follows that $r$ divides 63 or 65 and so, by (8), we have $r \in \{7, 9, 13, 21, 63, 65\}$. We now prove that in all of these cases, except for $r \in \{7, 21\}$, the graph $X$ is $\frac{1}{2}$-transitive. Clearly, $X$ has no coiled 6-cycles for $r = 65$ and so $v_s(6) = 2$, $v_r(6) = 1$, and $v_g(6) = 0$. Thus $X$ is $\frac{1}{2}$-transitive by Lemma 4.7. If $r = 63$ there is only one $H$-orbit of coiled 6-cycles, and its code is $g^6$. Thus $v_r(6) = 2$ and $v_g(6) = 1 = v_s(6)$ and again $X$ is $\frac{1}{2}$-transitive by Lemma 4.7. Next, if $r = 9$, it may be seen that $X$ contains two $H$-orbits of coiled 6-cycles with respective codes $g^6$ and $g^2g^7$; forcing $v_r(6) = 2 = v_s(6)$ and $v_g(6) = 3$. In particular, $G$ fixes the set Zig. But then each of the three $H$-orbits of 6-cycles is fixed by $G$ and so $G$ must fix the sets Anc and Gli, too. Hence $X$ is $\frac{1}{2}$-transitive by Corollary 4.3. If $r = 13$ then it may be inferred that there is precisely one $H$-orbit of coiled 6-cycles in $X$; its code is $g^6gzgz$. It follows that $v_r(6) = 1$ and $v_g(6) = 2 = v_s(6)$. Hence, Gli is fixed by $G$. But then $G$ must also fix each of the two $H$-orbits of 6-cycles in $X$, implying that both Anc and Zig are preserved by $G$. Hence $X$ is $\frac{1}{2}$-transitive by Corollary 4.3. The remaining two cases $r \in \{7, 21\}$ give rise to arc-transitive graphs belonging to the infinite family $X(s; 6, 7, k)$, $k \geq 1$, in Theorem 3.4(iii). The details will be provided in Section 6. Besides, let us also remark that $X(2, 6, 7)$ is isomorphic to the double cover of $X(2, 3, 7)$, the latter being the line graph of the Heawood graph.

Combining the comments preceding the statement of Proposition 4.8 with the information given by Table I, we obtain the following result.

**Corollary 4.9.** The graph $X(s; t, r)$, where $t \geq 3$, $r \geq 3$ is odd and $s \in \mathbb{Z}^*$ satisfies $s' = \pm 1$, has noncoiled girth 8 if and only if none of the conditions below are satisfied:

(i) $s \in \{1, -1, 2, -2, 1/2, -1/2\}$;

(ii) $s^2 \pm s \pm 1 = 0$.

Consequently, if $X(s; t, r)$ has noncoiled girth 8 then $r \geq 17$.

5. CYCLES OF LENGTH 8 IN $X(s; t, r)$

Throughout this section we let $t \geq 5$ be an integer, $r \geq 3$ be an odd integer, and $s \in \mathbb{Z}^*$ satisfy $s' = \pm 1$. We analyze 8-cycles in the graphs $X(s; t, r)$. The understanding of the structure of such cycles is crucial in the analysis of $\frac{1}{2}$-transitivity of these graphs. A careful inspection gives us that an 8-cycle in $X(s; t, r)$ must intersect three, four, five, six, or eight consecutive attachment sets $V_i$, $i \in Z_t$. In the first case its trace is either $anananan$ (with the corresponding cycle being a canonical 8-cycle existing for all triples $(s; t, r)$—see row 1 of Table II) or $a'nan$. In the second case
its trace is either $a'n^2$ or $ana'n^2$ and in the third case its trace is $anan^3$. In the fourth case its trace is $a'n^6$ and the cycle may only exist for $t = 6$. Finally, in the fifth case its trace is $n^3$ and the cycle is coiled and may only exist for $t = 8$. In Tables II and III below we list all $H(s, t, r)$-orbits of 8-cycles in $X(s, t, r)$, other than those having traces $a'n^6$ or $n^3$. For each orbit we list the corresponding trace, a representative $C$, a necessary and sufficient condition on $s$ for that orbit to exist, the length of the orbit, and finally its code.

The next lemma gives us some details on the simultaneous existence of different $H(s, t, r)$-orbits of 8-cycles in $X(s, t, r)$.

**Lemma 5.1.** Let $X = X(s, t, r)$, where $t \geq 5, r \geq 3$ is odd, and $s \in Z^*$ satisfies $s' = \pm 1$ and $s^2 \neq \pm 1$, have noncoiled girth 8 and let $H = H(s, t, r)$. Then $X$ has at most

<table>
<thead>
<tr>
<th>Table II</th>
</tr>
</thead>
<tbody>
<tr>
<td>8-Cycles of Traces $ananaman$, $a'nam$, $a'n'an^2$, and $ana'n^3$ in $X(s, t, r)$</td>
</tr>
<tr>
<td>Row</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>1</td>
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<tr>
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<tr>
<td>3</td>
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<tr>
<td>19</td>
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<tr>
<td>20</td>
</tr>
<tr>
<td>21</td>
</tr>
</tbody>
</table>
### Table III

8-Cycles of Trace $an^3m^1$ in $X(s, t, r)$

<table>
<thead>
<tr>
<th>Row</th>
<th>Respective C</th>
<th>Condition</th>
<th>Orbit length</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a^2_1 b_1 c_1 d_1 e_1 f_1 g_1 h_1$</td>
<td>$s^3 + s^2 + s + 1 = 0$</td>
<td>$tr$</td>
<td>$ag agz^3$</td>
</tr>
<tr>
<td>2</td>
<td>$a^2_1 b_1 c_1 d_1 e_1 f_1 g_2 h_1$</td>
<td>$s^3 - s^2 - s - 1 = 0$</td>
<td>$tr$</td>
<td>$ag z h ag z^2$</td>
</tr>
<tr>
<td>3</td>
<td>$a^2_1 b_1 c_1 d_1 e_1 f_2 g_1 h_1$</td>
<td>$s^3 - s^2 + s + 1 = 0$</td>
<td>$tr$</td>
<td>$age az^3$</td>
</tr>
<tr>
<td>4</td>
<td>$a^2_1 b_1 c_1 d_1 e_1 f_2 g_2 h_1$</td>
<td>$s^3 + s^2 - s - 1 = 0$</td>
<td>$tr$</td>
<td>$agez agz^3$</td>
</tr>
<tr>
<td>5</td>
<td>$a^2_1 b_1 c_1 d_1 e_2 f_1 g_1 h_1$</td>
<td>$s^3 + s^2 + s - 1 = 0$</td>
<td>$tr$</td>
<td>$agez agz^3$</td>
</tr>
<tr>
<td>6</td>
<td>$a^2_1 b_1 c_1 d_2 e_1 f_1 g_1 h_1$</td>
<td>$s^3 - s^2 + s - 1 = 0$</td>
<td>$tr$</td>
<td>$agez agz^3$</td>
</tr>
<tr>
<td>7</td>
<td>$a^2_1 b_1 c_1 d_2 e_1 f_2 g_1 h_1$</td>
<td>$s^3 - s^2 + s - 1 = 0$</td>
<td>$tr$</td>
<td>$agez agz^3$</td>
</tr>
<tr>
<td>8</td>
<td>$a^2_1 b_1 c_1 d_2 e_2 f_1 g_1 h_1$</td>
<td>$s^3 + s^2 + s - 1 = 0$</td>
<td>$tr$</td>
<td>$agez agz^3$</td>
</tr>
</tbody>
</table>

(i) one $H$-orbit of 8-cycles of trace $a^5$nan;
(ii) one $H$-orbit of 8-cycles of trace $a^n z^3 m^2$ in rows 6 to 11 of Table II;
(iii) one $H$-orbit of 8-cycles of trace $a^n z^3 m^2$ in rows 12 to 17 of Table II;
(iv) one $H$-orbit of 8-cycles of trace $an z^3 m^2$;
(v) one $H$-orbit of 8-cycles of trace $an^3 z^2$ in rows 1 to 8 of Table III;
(vi) and at most one $H$-orbit of 8-cycles of trace $an^3 z^2$ in rows 9 to 20 of Table III, unless $s^3 = \pm 1$ when $X$ has two such $H$-orbits.

**Proof.** Using Corollary 4.9, the statements (i)-(iv) are straightforward to check. Suppose that two conditions in rows 1 to 8 of Table III hold simultaneously. If there is a match of the corresponding coefficients in $s^3 \pm s^2 \pm s \pm 1 = 0$ in two places, then it follows that $s^3 = \pm 1$. On the other
hand, if one or three of the above coefficients coincide, we obtain, by adding them or subtracting one from another, a contradiction with the fact that \( s \neq 0 \), proving (v). To prove (vi), let us first note that two conditions in rows 9 to 19 of Table III may only hold simultaneously if the corresponding constant terms are of opposite sign. Next, assuming that two conditions in rows 9 to 12 of Table III or two conditions in rows 13 to 16 of Table III hold simultaneously, we obtain a contradiction with the fact that \( s^3 \neq \pm 1 \). Assume now that a condition from rows 9 to 13 and a condition from rows 13 to 16 of Table III hold simultaneously. Hence, we have either \( s^3 + s^2 + 1 = 0 \) and \( s^3 - s - 1 = 0 \), or \( s^3 + s^2 - 1 = 0 \) and \( s^3 + s + 1 = 0 \). This gives us a total of eight combinations to check, all leading to a contradiction. For example, suppose that \( s^3 + s^2 + 1 = 0 \) and \( s^3 - s - 1 = 0 \). Adding these two equations we have \( s^2 + s + 2 = 0 \), and subtracting the second from the first and multiplying the result by \( s - 1 \) we obtain \( 2s^2 + s - 1 = 0 \). It follows that \( s = -\frac{1}{2} \) and \( s^2 = 3 \), implying \( r = 11 \), contradicting Corollary 4.9. We leave out the details of the remaining seven possibilities. Finally, suppose that a condition in rows 9 to 16 of Table III together with \( s^3 = \pm 1 \) holds. If \( s^3 + s^2 + 1 = 0 \) then \( s^3 - 1 = 0 \) gives us either \( s = 0 \) or \( s = \pm 2 \). On the other hand, if \( s^3 + s^2 - 1 = 0 \) then \( s^3 + 1 = 0 \) implies either \( s^2 = 0 \) or \( 2s = \pm 1 \). These contradictions complete the proof of Lemma 5.1.

The next corollary is an immediate consequence of (7) and Lemma 5.1.

**Corollary 5.2.** Let \( X = X(s, t, r) \), where \( t \geq 5, r \geq 3 \) is odd and \( s \in \mathbb{Z}_r^* \) satisfies \( s^t = \pm 1 \) and \( s^2 \neq \pm 1 \), have noncoiled girth 8:

(i) If \( \mathcal{C} \) is the set of all 8-cycles in \( X \) whose trace is either \( a^3n^2a^3n^2 \) or \( ana^2ana^2 \), then \( v_0(\mathcal{C}) = v_4(\mathcal{C}) + v_2(\mathcal{C}) \) is an even number;

(ii) If \( \mathcal{C} \) is the set of all 8-cycles in \( X \) whose trace is \( ana^2a^3n^2 \), then \( v_0(\mathcal{C}) = \frac{1}{2}v_2(\mathcal{C}) + v_4(\mathcal{C}) \) is a number.

The rest of this section is devoted to the analysis of coiled 8-cycles in \( X(s, 8, r) \). Each such cycle \( C \) arises from a relation of the form \( \sum_{i=0}^{3} w_i s^i = 0 \), with \( w_i \in \{1, -1\} \). A binary sequence of length 8 is associated with such a relation, where 1 and 0 correspond, respectively, to coefficients 1 and -1. Let \( \mathcal{W}_C \) denote the set of all such sequences generating a cycle belonging to the same \( H(s, 8, r) \)-orbit as \( C \) and let \( w \in \mathcal{W}_C \). We observe that, for \( s^3 = 1 \), the set \( \mathcal{W}_C \) consists precisely of all those sequences obtained from \( w \) by any number of transformations of the following two types: first, a cyclic rotation and, second, a replacement of each 1 by a 0 and vice versa. It follows that \( |\mathcal{W}_C| \in \{2, 4, 8, 16\} \) in this case. On the other hand, if \( s^3 = -1 \) then \( \mathcal{W}_C \) consists of all those sequences obtained from \( w \) by any number of transformations of the following type: shifting each term
one place to the right and replacing the first term by the symbol differing
from the last term. It follows that $|\mathcal{W}_C| = 16$ in this case.

**Proposition 5.3.** Let $X = X(s; 8, r)$ and $H = H(s; 8, r)$, where $r \geq 3$ is
odd and $s \in \mathbb{Z}_r^*$ satisfies $s^8 = -1$. Then $X$ has a coiled 8-cycle if and only if
$r = 17$ and $s$ generates $\mathbb{Z}_r^*$, that is $s \in \{3, -3, 5, -5, 6, -6, 7, -7\}$. Moreover,
if that is the case and if $\mathcal{C}$ is the corresponding $H$-orbit then
$|\mathcal{C}| = 2tr = 16r$ and $v_{\mathcal{C}}(\mathcal{C}) + v_{\mathcal{C}}(\mathcal{C}) = 8$.

**Proof.** In view of the comments preceding the statement of Proposition
5.3, the set of all binary sequences of length 8 decomposes into 16
equivalence classes, each consisting of 16 sequences and coinciding with the
set $\mathcal{W}_C$ for some coiled 8-cycle $C$ in $X$. The representatives of these 16
equivalence classes are

1. $1^8$, 2. $101^8$, 3. $1^801^4$, 4. $1^401^4$, 5. $1^401^3$, 6. $1^801^3$, 7. $1^802^4$,
8. $10101^4$, 9. $101^201^3$, 10. $101^301^3$, 11. $12010^21^2$, 12. $1201201^2$,

Let $C$ be a coiled 8-cycle in $X$. Then $\mathcal{W}_C$ is one of the above equivalence
classes. Analyzing each of these 16 possibilities, we end up getting either a
contradiction or one of the possibilities given in the statement of the
proposition. For example, if $\mathcal{W}_C$ contains sequence 1, then it must also
contain the sequence 01$^7$. But subtracting the latter from sequence 1 we get
2 = 0. Similar contradictions may also be obtained from the sequences 2, 7,
8, 12, 14, 15, and 16. On the other hand, the sequences 3, 4, 5, 6, 9, 10, 11,
and 13 imply $r = 17$ with $s$, respectively, being equal to $-7, -6, -3, -5$,
6, 3, 5, and 7. For example, let us check the sequence 3. Assuming that it
belongs to $\mathcal{W}_C$, we have that the latter also contains the sequences $01^8$
and $01^7$. Now subtracting the relation corresponding to the sequence
$01^8$ from the relation corresponding to the sequence 3 we obtain $s^8 - s^8 + 1 = 0$.
Similarly, combining together the sequence $01^8$ and the
sequence 3 we obtain $s^4 + s^4 - 1 = 0$. Now, using these two relations we
have, by computation, that $s^4 = -2$ and so $-1 = s^8 = 16$. Thus $r = 17$ and
$s = -7$. The details for the remaining seven possibilities are omitted. It is
easy to check that the lengths of the relative $H$-orbits are as claimed.

**Proposition 5.4.** Let $X = X(s; 8, r)$ and $H = H(s; 8, r)$, where $r \geq 3$ is
odd and $s \in \mathbb{Z}_r^*$ satisfies $s^8 = 1$ and $s^2 \neq \pm 1$. An 8-cycle $C$ in $X$ is coiled if and
only if $\mathcal{W}_C$ coincides with one of the following six equivalence classes: $1^8$,
$1010101$, $1^20^21^2$, $1^20^21^4$, and $1^20102^210$. Moreover,

(i) the codes of the $H$-orbits of 8-cycles corresponding to these six
classes are, respectively $g^s$, $z^s$, $gzgzg$, $g^2z^2\bar{g}^2z^2$, $g^2z^2\bar{g}^2z$, and $z^s\bar{g}^2z\bar{g}$.
(ii) the relations generating \( \mathcal{W}_C \) are, respectively,
\[
(1 - s) \sum_{i=0}^{7} s^i x^i = 0, \quad (1 + s^3)(1 - x^2) = 0, \quad s^4 = -1, \quad s^8 = 1, \quad \text{and} \quad s^9 = 1.
\]

(iii) and the lengths of the corresponding \( H \)-orbits are, respectively, 1, 1, 2, 4, 4, and 4.

Proof. Since \( s^2 \neq \pm 1 \), we must have \( r \geq 7 \). Let \( C \) be a coiled 8-cycle in \( X \) and let \( \sum_{i=0}^{7} w_i s^i = 0 \), with \( w_i \in \{1, -1\} \), be one of the relations generating \( C \). We claim that
\[
\begin{align*}
    w_{i+4} &= w_i, \quad i = 0, 1, 2, 3, \quad \text{or} \quad w_{i+4} &= -w_i, \quad i = 1, 2, 3. \quad (10)
\end{align*}
\]

Of course, \( \sum_{i=0}^{7} w_i s^i = 0 \), for each \( j \in \{0, 1, \ldots, 7\} \). For \( j = 0 \) we write
\[
\sum_{i=0}^{3} w_i s^i + s^3 \sum_{i=0}^{3} w_i s^i = 0,
\]
put the second term to the right-hand side, square the expression, and use the fact that \( s^8 = 1 \) to obtain
\[
\left( \sum_{i=0}^{3} w_i s^i \right)^2 = \left( \sum_{i=0}^{3} w_i s^i + s^3 \right)^2. \quad (11)
\]
Since \( w_i \in \{1, -1\} \), the coefficients in (11) belong to the set \( \{0, 2, -2\} \). Moreover, for each \( i \in \{0, 1, 2, 3\} \) precisely one of the coefficients \( w_i + w_{i+4} \) and \( w_i - w_{i+4} \) in (11) must be 0. To prove (10) we have to show that one of the above factors in (11) consists of four terms and the other is identical with 0. If that is not the case then the above two factors either both consist of two terms or one consists of a single term and the other of three terms.

The first possibility gives us one of these three cases: \( (\pm s^3 \pm s^2) \) \( (\pm 1 \pm s) = 0 \), or \( (\pm s^3 \pm s)(\pm s^2 \pm 1) = 0 \), or \( (\pm s^3 \pm 1)(\pm s^2 \pm s) = 0 \). Note that if the latter is true then taking \( j = 1 \) we obtain the first case, reducing the analysis to two cases only. In the first case we must have \( (\pm s \pm 1) = 0 \) and so, since \( s^2 \neq \pm 1 \), we must have \( (s \pm 1)^2 = 0 \). Thus \( s^2 = \pm 2s - 1 \) and by computation, \( s^4 = \pm 4s - 3 \) and \( 1 = s^8 = \pm 8s - 7 \), implying \( s = \pm 1 \), a contradiction. In the second case we have either \( s^2 = 1 \), implying that the first of the two factors in (11) equals 0, or \( (\pm s^2 \pm 1)^2 = 0 \). By computation, \( s^2 = \pm 2s - 1 \) and so \( 1 = s^8 = \pm 4s^2 \pm 4s^4 + 1 = \pm 4s^2 - 3 \), implying \( s^2 = \pm 1 \), a contradiction. We may now assume that one of the two factors in (11) consists of a single term and the other of three terms, giving us one of these three equations: \( \pm s^2 \pm s \pm 1 = 0 \), \( \pm s^2 \pm s \pm 1 = 0 \), or \( \pm s^2 \pm s^2 \pm 1 = 0 \). In fact, letting \( j \) run through all the elements of \( \{0, 1, 2, 3\} \) we end up getting a system of equations \( Z \cdot x = 0 \), where \( Z \) is a circulant matrix with the first row \( \{0, z_2, z_1, z_0\} \), for some \( z_0, z_1, z_2 \in \{-1, 1\} \), and \( s = [s^3, s^2, s, 1]^t \). It my be easily inferred that this system has no solution. These contradictions prove (10).

As the next step, we identify among the 16 equivalence classes of binary sequences of length 8 the ones which are associated with relations satisfying
There are precisely six of them with the corresponding representatives: 1, 1000, 10101010, 3, 130130, and 1404. The corresponding codes are, respectively, \( g^8, z^8, (gz)^4, (g^2z^2)^2, (g^3z)^2 \), and \((z^3g)^2\). Moreover, a more detailed analysis of the above six equivalence classes gives us the following necessary and sufficient conditions for the existence of a coiled 8-cycle \( C \) associated with a given binary sequence. For sequence 1 we have \( i = 7, i = 0, s_i = 0 \), for sequence 2 we have \((1 \& s) \sum_{i=0}^{3} s_i = 3, i = 0, s_2 = 0\), for sequence 3 we have \((1 + s^4)(1 - s^2) = 0\), for sequence 4 we have \( s^4 = -1\), and for sequences 5 and 6 we have \( s^4 = 1\). It is then easy to check that the lengths of the relative \( H \)-orbits are as claimed.

**Corollary 5.5.** Let \( X = X(s; 8, r) \), where \( t \geq 5, r \geq 3 \) is odd, and \( s \in \mathbb{Z}^* \) satisfies \( s^6 = 1 \) and \( s^2 \neq \pm 1 \), and let \( \mathcal{C} \) be the set of all coiled 8-cycles in \( X \). Then

(i) \( s^4 = 1 \) implies \( v_g(\mathcal{C}) + v_z(\mathcal{C}) \in \{8, 9\} \);

(ii) \( s^4 = -1 \) implies \( v_g(\mathcal{C}) + v_z(\mathcal{C}) = 8 \);

(iii) \( s^4 \neq \pm 1 \) implies \( v_g(\mathcal{C}) + v_z(\mathcal{C}) \in \{1, 2, 3\} \).

**Proof.** We use (7) and Proposition 5.4. Suppose first that \( s^4 = 1 \). Then there are no coiled 8-cycles with sequence 3 or sequence 4, but there are coiled 8-cycles with sequences 5 and 6. Besides, there may also be either a coiled cycle with sequence 1 or a coiled cycle with sequence 2, but not both. Namely, if \( s^4 = 1 \) then the relations \( \sum_{i=0}^{3} s_i = 0 \) and \((1 - s) \sum_{i=0}^{3} s_i^2 = 0\), combined together imply \( s^2 = \pm 1 \), a contradiction. This gives us (i). Suppose now that \( s^4 = -1 \). Then there are no coiled 8-cycles with sequences 5 or 6, but there are coiled 8-cycles with sequences 1, 2, 3, and 4, implying (ii). Finally, assume that \( s^4 \neq \pm 1 \). In this case we have no coiled 8-cycle with sequences 4, 5, and 6. Moreover, we can have coiled 8-cycles with at most two of the remaining sequences. Namely, combining together the relations \( \sum_{i=0}^{3} s_i = 0, (1 + s^4)(1 - s^2) = 0, \) and \((1 - s) \sum_{i=0}^{3} s_i^2 = 0\), we obtain \( s^4 = -1 \). It follows that \( v_g(\mathcal{C}) + v_z(\mathcal{C}) \in \{1, 2, 3\} \), completing the proof of Corollary 5.5.

We end this section with a result on the frequencies relative to 8-cycles of trace \( a^n b^6 \).

**Lemma 5.6.** Let \( X = X(s; 6, r) \) and \( H = H(s; 6, r) \), where \( t \geq 5, r \geq 3 \) is odd, and \( s \in \mathbb{Z}^* \) satisfies \( s^6 = \pm 1 \) and \( s^2 \neq \pm 1 \). Let \( \mathcal{C} \) be an \( H \)-orbit of 8-cycles in \( X \) of trace \( a^n b^6 \). Then \( v_g(\mathcal{C}) = 2 \) and \( v_g(\mathcal{C}) + v_z(\mathcal{C}) = 6 \).

**Proof.** Since no nonidentity element of \( H \) may fix a cycle in \( \mathcal{C} \), we have that \( |\mathcal{C}| = 2tr = 12r \). The result follows as cycles in \( \mathcal{C} \) have two anchors and six nonanchors.
6. PROVING THEOREM 3.4

**Lemma 6.1.** Let \( X = X(s; t, r) \), where \( t \geq 5, r \geq 3 \) is odd, and \( s \in \mathbb{Z}_r^* \) satisfies \( s^8 = \pm 1 \) and \( s^2 \neq \pm 1 \), have noncoiled girth 8. Then either \( v_\epsilon(8) \neq v_\delta(8) \) or \( v_\epsilon(8) \neq v_\gamma(8) \) or \( X \) is \( \frac{1}{2} \)-transitive.

**Proof.** First, by Corollary 4.9,

\[ r \geq 17. \quad (12) \]

Let \( H = H(s; t, r) \) and let \( \mathcal{C} \) denote the set of all 8-cycles in \( X \). We are going to distinguish three different cases depending on whether \( X \) contains coiled 8-cycles and cycles of trace \( a^n \) or not. Let \( \epsilon = v_\epsilon(\mathcal{C}) \), where \( \mathcal{C} \) is the set of all 8-cycles in \( X \) having trace \( a^n \) or \( aana^n \), and let \( \delta = v_\delta(\mathcal{C}) \), where \( \mathcal{C} \) is the set of all 8-cycles in \( X \) having trace \( anan^3 \). Note that \( \epsilon \) is even and \( \delta \leq 5 \), in view of Corollary 5.2.

**Case 1.** \( X \) has no coiled 8-cycles and no 8-cycles of trace \( a^n \). Then each 8-cycle of \( X \) belongs to an \( H \)-orbit in Tables II and III. Let us first assume that one of the conditions in rows 2 to 5 of Table II is satisfied. Using Corollary 5.2 and taking into account the canonical 8-cycles and the 8-cycles of trace \( a^n \), we have \( v_\gamma(8) + v_\delta(8) = 2 + \epsilon + \delta > 0 \), since \( \delta \leq 5 \), and the result follows.

Assume now that none of the conditions in rows 2 to 5 of Table II holds. A similar argument as above gives us \( v_\delta(8) = 2 + \epsilon + \delta \) and \( v_\epsilon(8) + v_\delta(8) = 2 + \epsilon + 3\delta \). Supposing that \( v_\lambda(8) = v_\epsilon(8) = v_\delta(8) \), we have \( \delta = \epsilon + 2 \). In particular, \( \epsilon \) and \( \delta \) are of the same parity. But \( \epsilon \) is even and so we only have two possibilities: either \( \epsilon = 0 \) and \( \delta = 2 \) or \( \epsilon = 2 \) and \( \delta = 4 \). In the first case it follows that \( \mathcal{C} \) consists of the canonical 8-cycles and the 8-cycles belonging to an \( H \)-orbit, say \( \mathcal{D} \), arising from one of the rows 9 to 16 of Table III. But then \( v_\lambda(\mathcal{D}) = 1 \) and \( v_\epsilon(\mathcal{D}) = 1 \) and \( v_\delta(\mathcal{D}) = v_\epsilon(\mathcal{D}) \). Hence \( v_\lambda(\mathcal{D}) \) differs from both \( v_\epsilon(\mathcal{D}) \) and \( v_\delta(\mathcal{D}) \). In the second case it follows that \( \mathcal{C} \) consists of the canonical 8-cycles, an \( H \)-orbit of 8-cycles arising from one of the rows 8 to 11 or 14 to 17 of Table II, and an \( H \)-orbit of 8-cycles arising from one of the rows 17 to 20 of Table III. In particular, \( s^3 = \pm 1 \). Moreover, in view of Proposition 4.1, we may assume that \( \mathcal{C} \) contains either the \( H \)-orbit from row 8 or the one from row 9 of Table II. In other words, \( s^3 + s^2 + s = 0 \). It follows that \( s^3 + s^2 + s = 0 \) and so \( s^3 + 2s + 1 = 0 \), implying that \( \mathcal{C} \) must also contain an \( H \)-orbit from rows 18 to 21 of Table II, contradicting \( \epsilon = 2 \).

**Case 2.** \( X \) contains a coiled 8-cycle. Of course, \( t = 8 \) and so \( s^8 = \pm 1 \). Hence \( s^3 \neq \pm 1 \), forcing

\[ \delta \leq 3. \quad (13) \]
Let us first assume that $s^8 = -1$. Then by Proposition 5.3 we have $r = 17$ and $s \in \{3, -3, 5, -5, 6, -6, 7, -7\}$. By Proposition 4.1, we may assume that $s = 3$ or $s = 5$. Suppose $s = 3$. Checking Tables II and III we have that $\mathcal{C}$ consists of the canonical 8-cycles, the $H$-orbits of 8-cycles arising from rows 3 and 13 of Table II and rows 5 and 14 of Table III, and the $H$-orbit of coiled 8-cycle with code $z^3 g^2 z^2 g$. Thus, in particular $\varepsilon = 4$ and $\delta = 3$. By computation, $v_a(8) = 12$, $v_g(8) = 9$, and $v_z(8) = 15$. Suppose now that $s = 5$. It may be seen that $\mathcal{C}$ consists precisely of the canonical 8-cycles, the 8-cycles belonging to the $H$-orbits arising from rows 12 and 21 of Table II and row 10 of Table III, and, finally, the 8-cycles with code $z^4 g^2 z g$. It follows that $\varepsilon = \delta = 3$ and $\varepsilon = \delta = 4$.

Assume now that $s^8 = 1$. Suppose first that a condition in rows 2 to 5 of Table II is satisfied. By Proposition 4.1 we can assume $s = 3$. Checking out the conditions from rows 6 to 21 of Table II, we see that only the one in row 12 does not contradict (12). But if $2s^2 + 1 = 0$ for $s = 3$ then $r = 19$ and clearly $s^3 \neq 1$ in $Z_{s^3}^*$. Hence $\varepsilon = 0$ and then, by computation, $v_a(8) = 5 + \delta$ and $v_g(8) = 3 + \delta + c$ for some $c \in \{1, 2, 3, 8, 9\}$, in view of Corollary 5.5. But then $2v_a - v_g(8) = 7 - \delta - c \neq 0$, in view of (13). Thus either $v_a(8) \neq v_g(8)$ or $v_a(8) \neq v_z(8)$. Suppose now that none of the conditions in rows 2 to 5 of Table II is satisfied. If $\varepsilon = 0$ then clearly no 3-path with code $a^2$ is contained on an 8-cycle. On the other hand, a 3-path with code $ag$ or $az$ is contained on an 8-cycle. This implies that, for each $\pi \in G$, either $\pi = \text{Aut} \mathcal{X}$ or $\pi \subseteq \text{Gli} \cup \text{Zig}$. But the latter would mean that $\pi$ maps each alternating cycle of $\mathcal{X}$ into a coiled cycle, contradicting the fact that coiled cycles have length divisible by 8. Hence $\pi$ is fixed by $G$ and so $\mathcal{X}$ is $\frac{1}{4}$-transitive by Corollary 4.3. We may therefore assume that $\varepsilon \geq 2$. We now show that $\varepsilon = 2$. This is done by checking all of the conditions in rows 6 to 21 of Table II against the fact that $s^8 = 1$. Clearly, each of the conditions in rows 6, 7, 12, and 13 of Table II contradicts (12). Suppose that one of the conditions in rows 18 to 21 of Table II holds. If $s^8 + 2s - 1 = 0$ then, by computation, $1 = s^8 + 8s - 7$, implying $s = \pm 1$. A similar contradiction is obtained if $s^8 + 2s - 1 = 0$. Now, since by subtracting any of the conditions in rows 8 to 11 of Table II from any of the conditions in
rows 14 to 17 of Table II, one of the conditions in rows 18 to 21 of Table II is obtained, it follows that precisely one of the conditions in rows 8 to 11 and 14 to 17 of Table II holds. So $\varepsilon = 2$ and, moreover, the equation $x^2 + x - 2 = 0$ has precisely one solution in the set \( \{ s, -s, s^{-1}, -s^{-1} \} \).

By Proposition 4.1 we may assume that $x^2 + s = 0$. Supposing that $x^2 + s = 0$, we have, by computation, $s^8 = 3x - 14$ and so $3x = 15$.

But then $0 = 3x^2 + 3s + 6 = 18s + 6$ and so $9s + 3 = 0$ which together with $3s = 15$ implies $48 = 0$, contradicting (12). Suppose now $x^2 + 2s = 0$. Since $\varepsilon = 2$, it follows that $v_{e}(8) = 4 + \delta$ and $v_{e}(8) = 4 + 3\delta + c$ for some $c \in \{ 1, 2, 3, 8, 9 \}$. Assuming that $v_{e}(8) = v_{e}(8)$, we have $4 = \delta + c$ and so $c \leq 3$ by Corollary 5.5. It may be inferred that no condition from rows 1 to 20 of Table III is compatible with conditions $x^2 + s = 0$ and $s^8 = 1$, the sole exception being the one in row 6. But the latter then forces $5x = 5$ and so $s^6 = (-s + 2)^2 = -5s + 6 = 1$, implying $c \in \{ 8, 9 \}$ by Corollary 5.5, contradicting $c \leq 3$ above.

Case 3. $X$ contains an 8-cycle of trace $\alpha^2 \eta^\delta$. Of course, $t = 6$. Let us assume that $v_{e}(8) = v_{e}(8)$. Supposing first that a condition in rows 2 to 5 of Table II is satisfied, we may let $s = 3$, by Proposition 4.1. It follows that $v_{e}(8) = 5 + \varepsilon + \delta + 2d$ and $v_{e}(8) = 4 + 3\delta + c$ for some $d \geq 1$, giving us $\varepsilon + 7 = \delta + 2d$. Since $\varepsilon$ is even, we must have that $\delta$ is odd. Therefore one of the conditions in rows 1 to 8 of Table III must be satisfied. But each of these conditions contradicts (12), as may be easily checked. We may therefore suppose that none of the conditions in rows 2 to 5 of Table II holds. Then $v_{e}(8) = 2 + \varepsilon + \delta + 2d$ and $v_{e}(8) + v_{e}(8) = 2 + 3\delta + 6d$, for some $d \geq 1$, giving us $\varepsilon + 2 = \delta + 2d$. Since $\varepsilon$ is even, the same holds for $\delta$. We are going to distinguish two subcases.

Subcase 3.1. $\varepsilon > 0$. As in Case 2 we first show that $\varepsilon = 2$. This is done by checking all of the conditions in rows 6 to 21 of Table II against the fact that $s^6 = \pm 1$. Clearly, each of the conditions in rows 6, 7, 12, and 13 of Table II contradicts (12). Suppose that one of the conditions in rows 18 to 21 of Table II holds. If $s^6 = 2s + 1 = 0$, that is if $(s + 1)^2 = 0$, then the condition in row 4 or the one in row 7 of Table III holds, contradicting $\delta$ even. If $s^6 = 2s - 1 = 0$ then, by Proposition 4.1, we may let $s^2 + 2s - 1 = 0$. Say $s^2 = 1$. By computation, $1 = s^6 = -70s + 29$, forcing $35s = 14$. But then $0 = 35s^2 + 35s - 70 = 14s - 7$ and so $28s = 14 = 35s$. A similar contradiction is obtained if $s^6 = -1$. Since by subtracting any of the conditions in rows 8 to 11 of Table II from any of the conditions in rows 14 to 17 of Table II, one of the conditions in rows 18 to 21 of Table II is obtained, it follows that precisely one of the conditions in rows 8 to 11 and 14 to 17 of Table II holds. Hence $\varepsilon = 2$ and thus the equation $x^2 + x - 2 = 0$ has a unique solution in \( \{ s, -s, s^{-1}, -s^{-1} \} \).
In view of Proposition 4.1 we may now assume that \( s^2 + s \pm 2 = 0 \). Supposing first that \( s^2 + s + 2 = 0 \), we have by computation, \( s^6 = -5s + 2 \). If \( s^6 = 1 \) then \( 5s = 1 \) which, combined again with \( s^2 + s + 2 = 0 \), gives us \( r = 7 \), contradicting (12). If \( s^6 = -1 \) then \( 5s = 3 \) and it may be deduced that \( r = 37 \) and \( s = 8 \). One can see that \( \% \) consists of precisely four \( H \)-orbits of 8-cycles: the canonical 8-cycles, the 8-cycles arising from row 9 of Table II, and two \( H \)-orbits of 8-cycles of trace \( a^7n^6 \). Thus \( v_a = 8 \).

But each 8-cycle of trace \( a^7n^6 \) has an odd number of glides as well as zigzags, forcing the frequencies \( v_a(8) \) and \( v_a(8) \) to be odd. We may therefore assume that \( s^2 + s - 2 = 0 \). Then \( s^6 = -21s + 22 \). If \( s^6 = -1 \) then \( 21s = 23 \), which combined with \( s^2 + s - 2 = 0 \) gives us \( s = -2 \), contradicting Corollary 4.9. We are left with \( s^2 + s - 2 = 0 \) and \( s^6 = 1 \). We may deduce that \( 21s = 21 \). Clearly, \( \delta \in \{ 0, 2 \} \). In fact, \( \delta = 0 \). Namely, since \( s^6 = 1 \) we have that none of the conditions in rows 9 to 16 of Table III may hold.

If, say \( s^6 = \pm 1 \), then \( s = \pm s + 1 \). Thus we must have \( \delta = 0 \) and so \( d = 2 \). This means that there are precisely two \( H \)-orbits of 8-cycles of trace \( a^7n^6 \) in \( X \). Since \( s^6 = 1 \), the 8-cycles belonging to these two \( H \)-orbits have an even number of both glides and zigzags. Supposing that the code of one of them is \( a^7g^6 \), in order to have \( v_a(8) = v_a(8) \), it follows that the code of the other one must be \( a^7g^6 \). Namely, the relative frequencies of glides and zigzags equal 1 for the canonical 8-cycles as well as for the \( H \)-orbit arising from row 11 of Table II. The existence of these two \( H \)-orbits of 8-cycles of trace \( a^7n^6 \) implies that \( s^2 + s + 3\) has two glides and four zigzags. There is a total of 15 possibilities for the refinements of such an orbit: 1. \( a^2a^+z^2g^2z \), 2. \( a^4a^+z^2g^2z^2 \), 3. \( a^4a^{-z^2g^2z} \), 4. \( a^4a^{-z^2g^2z} \), 5. \( a^4a^{-z^2g^2z} \), 6. \( a^4a^{-z^2g^2z} \), 7. \( a^4a^{-z^2g^2z} \), 8. \( a^4a^{-z^2g^2z} \), 9. \( a^4a^{-z^2g^2z} \), 10. \( a^4a^{-z^2g^2z} \), 11. \( a^4a^{-z^2g^2z} \), 12. \( a^4a^{-z^2g^2z} \), 13. \( a^4a^{-z^2g^2z} \), 14. \( a^4a^{-z^2g^2z} \), 15. \( a^4a^{-z^2g^2z} \). With each of these refinements a relation on \( s \) of order 5 is associated which, together with conditions \( s^2 + s - 2 = 0 \) and \( 21s = 21 \), brings us to a contradiction. To illustrate the argument we analyze the first refinement. The relation is \( s^2 + s + 3 = 0 \) implies \( s^4 = -s + 2 \) and \( s^3 = 3s - 2 \). Thus \( s = -5s + 6 \), and \( s = 11s - 10 \). Putting these expressions into the above relation we get \( s = -3 \), a contradiction. The remaining 14 refinements are done in a similar fashion. We omit the details.

Subcase 3.2. \( \varepsilon = 0 \). It follows that \( \delta = 0 \) and \( d = 1 \) and so \( \% \) consists of exactly two \( H \)-orbits of 8-cycles: the orbit of canonical 8-cycles and an orbit of 8-cycles with trace \( a^7n^6 \). If the set of canonical 8-cycles were fixed
by $G$, the same would be true for the set Anc, forcing $X$ to be $\frac{1}{2}$-transitive by Corollary 4.3. Hence we may assume that $G$ acts transitively on the set of 8-cycles $\mathcal{C}$ of $X$. Besides, in order to have $v_a(8) = v_z(8)$, the 8-cycles in the $H$-orbit with trace $a^2n^b$ must have three glides and three zigzags. In particular, $s^b = -1$. Using Proposition 4.1 we may assume that this orbit has one of the following seven refinements: 1. $a^a - g^2z^3$, 2. $a^a - g^2z^3g$, 3. $a^a - g^2z^3g^z$, 4. $a^a - g^2z^3g^2z$, 5. $a^a - g^2z^3g^2zg$, 6. $a^a - g^2z^3g^2zg^z$, 7. $a^a - g^2z^3g^2zg^zg$. Assign to each 3-path on a given 8-cycle of $X$ a positive integer, that is the number of 8-cycles containing it. This way each 8-cycle must be assigned the same sequence. Analyzing each of the seven possibilities above we see that this only happens for refinement number 5, where the associated 8-sequence is $(1, 2, 1, 2, 1, 2, 1, 2)$. But then the corresponding relation on $s$ is $s^2 - s^2 + s^2 + s + 3 = 0$. Multiplying it by $s^b$ and using the fact that $s^b = -1$ we get $s^2 + 3s^2 - s^3 + s^2 + s - 1 = 0$. Subtracting this relation from the above relation we have $-4s^2 + 4 = 0$ and so $s^2 = 1$, forcing $s^2 = -1$, a contradiction. This completes the proof of Lemma 6.1.

We are now ready to prove Theorem 3.4.

Proof of Theorem 3.4. In view of Proposition 3.3 we only need to show that $X(s, t, r)$ is arc-transitive if and only if one of conditions (i) to (iii) in the statement of Theorem 3.4 is satisfied. But in view of Corollary 3.6 and the results of [1, 11] (see the comments following the statement of Theorem 3.4) it suffices to show that $X = X(s, t, r)$, where $t \geq 5$ and $s^2 \neq \pm 1$, is arc-transitive if and only if the condition (iii) holds.

Let $G = \text{Aut } X$ and $H = H(s, t, r)$. We shall first assume that $X$ is arc-transitive and deduce that the parameters $s$, $t$, and $r$ satisfy the assumptions of (iii). (Following that we shall prove the converse too by showing that the graphs $X(s, t, r)$ satisfying these assumptions are in fact arc-transitive.) First, Proposition 4.8 allows us to restrict ourselves to graphs with noncoiled girth 8. Now, since $X$ is not $\frac{1}{2}$-transitive we have that $H$ is a proper subgroup of $G$ and, moreover, Lemma 6.1 implies that either $v_a(8) \neq v_z(8)$ or $v_z(8) \neq v_g(8)$. It follows that either Gli and Anc $\vee$ Zig or Zig and Anc $\vee$ Gli are orbits of the action of $G$ on 2-paths of $X$. By Proposition 4.1 we lose nothing in assuming the latter holds.

Denote by $\mathcal{E}$ the orbit of the action of $G$ on 8-cycles in $X$ containing canonical 8-cycles. Of course, $v_a(\mathcal{E}) = v_z(\mathcal{E})$. Since the two zigzags on a canonical 8-cycle are antipodal, a careful inspection extracts the following $H$-orbits of 8-cycles from Tables II and III, shown in Table IV, that might belong to $\mathcal{E}$. Our argument splits into two cases, depending on whether $\mathcal{E}$ contains 8-cycles with trace $a^2n^b$ or not.
### TABLE IV

<table>
<thead>
<tr>
<th>Row</th>
<th>Trace</th>
<th>Respective C</th>
<th>Condition</th>
<th>Orbit length</th>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>ananana</td>
<td>$\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8$</td>
<td>none</td>
<td>tr</td>
<td>agazagaz</td>
</tr>
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<td>2</td>
<td>$a^n\varepsilon_1^2\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8$</td>
<td>$\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8$</td>
<td>$s^4 + s - 2 = 0$</td>
<td>tr</td>
<td>$a^2\varepsilon_1^2g$</td>
</tr>
<tr>
<td>3</td>
<td>$a^n\varepsilon_1^2\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8$</td>
<td>$\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8$</td>
<td>$2s^2 - s - 1 = 0$</td>
<td>tr</td>
<td>$a^2\varepsilon_1^2g$</td>
</tr>
<tr>
<td>4</td>
<td>$a^n\varepsilon_1^2\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8$</td>
<td>$\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8$</td>
<td>$s^4 + s - s - 1 = 0$</td>
<td>tr</td>
<td>$a^3\varepsilon_1^2g$</td>
</tr>
<tr>
<td>5</td>
<td>$a^n\varepsilon_1^2\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8$</td>
<td>$\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\varepsilon_6\varepsilon_7\varepsilon_8$</td>
<td>$s^4 - 1 = 0$</td>
<td>$2tr$</td>
<td>$a^2\varepsilon_1^2g$</td>
</tr>
</tbody>
</table>

**Case 1.** $\%$ has no 8-cycles with trace $a^n\varepsilon$. Suppose first that $\%$ contains no coiled 8-cycles either. Then $\%$ contains no $H$-orbit of 8-cycle outside Table IV. Observe that any two of the conditions in rows 2, 3, and 5 (a row is a row of Table IV) imply the third one. Thus assuming that $\%$ contains the $H$-orbit from row 5 we either have that $\%$ contains the $H$-orbits from both rows 2 and 3 or from none of them. But the latter is impossible for then $v_4(\%) > v_4(\%)$. Moreover, if $\%$ contains the $H$-orbits from both rows 2 and 3, then the $H$-orbit from row 4 must also be contained in $\%$, for otherwise $v_4(\%) > v_4(\%)$. But the conditions from rows 4 and 5 together imply $s = 1$. Hence the only possibility is for $\%$ to consist of the canonical 8-cycles and the $H$-orbit from row 4. In particular, this implies that 3-paths with code $a^2$ are not contained on an 8-cycle in $\%$ whereas 3-paths with code $ag$ are. It follows that the set of alternating cycles of $X$ is fixed by $G$ and so $G$ fixes the set $Anc$. Thus $X$ is $\frac{1}{2}$-transitive by Corollary 4.3, a contradiction.

Suppose now that $\%$ contains coiled 8-cycles. Of course, $t = 8$. Moreover, the code of coiled 8-cycles belonging to $\%$ must be $g'zg'$ and so their contribution to the number $v_4(\%)$ is precisely 3. It follows that $\%$ must contain $H$-orbits from rows 2 and 3, for otherwise $v_4(\%) > v_4(\%)$. But then $s^4 = 1$ and since $s^4 = \pm 1$ we have $s^4 = \pm 1$, a contradiction.

**Case 2.** $\%$ contains a cycle with trace $a^n\varepsilon$. Of course, $t = 6$. We now show that (iii) holds. Clearly, an 8-cycle with trace $a^n\varepsilon$ belonging to $\%$ must have the code $a^2g'zg'$. But this code is incompatible with the condition $s^4 = -1$ and so

$$s^4 = 1.$$  \hspace{1cm} (14)

Since the isomorphism $\phi: X(x; t, r) \rightarrow X(s^{-1}; t, r)$ in the proof of Proposition 4.1 preserves glides and zigzags, but interchanges the respective sets of
positive and negative anchors, we may assume that the refinement of the above 8-cycles is $a^+a^-gaz$. Hence the corresponding condition on $s$ is

\[ s^2 + s^4 + s^3 + s^2 - s - 3 = 0. \]  

(15)

Since $\nu_1(\mathcal{C}) = \nu_0(\mathcal{C})$, the existence of 8-cycles with refinement $a^+a^-gaz$ implies that $\mathcal{C}$ contains either the $H$-orbit from row 2 of Table IV or the one from row 3 of Table IV, and moreover, it does not contain the $H$-orbit from row 5 of Table IV. Hence, in view of the observation made on the conditions in rows 2, 3, and 5 of Table IV in Case 1, precisely one of rows 2 and 3 gives rise to 8-cycles in $\mathcal{C}$. It is easily seen that $2x^2 - s - 1 = 0$ and (15) together imply $s^2 = 1$. On the other hand, the condition

\[ s^2 + s - 2 = 0 \]  

(16)

is compatible with (15). Combined together they imply

\[ 7(s - 1) = 0. \]  

(17)

To summarize, Table V gives the three $H$-orbits of 8-cycles whose union is $\mathcal{C}$.

A 4-path in $X$ which contains no zigzags as 2-subpaths will be called admissible if it is contained on an 8-cycle from $\mathcal{C}$ and inadmissible otherwise. From Table V we have that a 4-path is admissible if and only if its refinement is one of the following: $g^3, ga^-a^+, ga^+g, a^-a^+a^-$, and $a^+ga^-$. The inadmissible 4-paths have refinements: $g^2a^+, g^2a^-, ga^-g, ga^+a^-$, and $a^+a^-a^+$. We now show that for each $v \in V(X)$

\[ K = \ker(G_v \to G_v^{N(x)}) \neq 1 \]  

has exponent 2. (18)

Suppose first that there exists $\gamma \in G \setminus H$ such that $\Anc^+ \cap \Anc^+\gamma = \emptyset$ and $\Anc^- \cap \Anc^-\gamma = \emptyset$. The (in)admissibility of 4-paths of $X$ then implies that either $\Anc^+\gamma \cap \Gli \neq \emptyset$ or $\Anc^-\gamma \cap \Gli \neq \emptyset$. In the first case let $P_1$ be

<table>
<thead>
<tr>
<th>Row</th>
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<th>Condition</th>
<th>Length</th>
<th>Refinement</th>
</tr>
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<td>1</td>
<td>$\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0, \varepsilon_4^0, \varepsilon_5^0, \varepsilon_6^0$</td>
<td>None</td>
<td>$tr$</td>
<td>$a^+ga^-za^+ga^-$</td>
</tr>
<tr>
<td>2</td>
<td>$\varepsilon_1^2, \varepsilon_2^2, \varepsilon_3^2, \varepsilon_4^2, \varepsilon_5^2, \varepsilon_6^2$</td>
<td>$s^2 + s - 2 = 0$</td>
<td>$tr$</td>
<td>$a^+a^-a^-gazg$</td>
</tr>
<tr>
<td>3</td>
<td>$\varepsilon_1^3, \varepsilon_2^3, \varepsilon_3^3, \varepsilon_4^3, \varepsilon_5^3, \varepsilon_6^3$</td>
<td>$s^2 + s^2 + s$</td>
<td>$2tr$</td>
<td>$a^+a^-gazg^2z$</td>
</tr>
</tbody>
</table>
a positive anchor of \(X\) whose image under \(\gamma\) is a glide and let \(C\) be the alternating cycle containing \(P_1\). Furthermore, for each \(i \in \{1, 2, 3, 4, 5\}\), let \(P_{3+i}\) denote the \(2(i+1)\)-subpath of \(C\) containing \(P_i\) as the central 2-subpath. Now, taking into account the (in)admissibility of 4-paths of \(X\) and the fact that \(\text{Anc}^- \cap \text{Anc}^- \gamma = \emptyset\), we have that \(P_2\), its refinement being \(a^r a^r a^- a^r\), must be mapped by \(\gamma\) to a 4-path with refinement \(g^3\). But then \(P_3\), having refinement \(a^r a^r a^- a^r a^r\), must be mapped by \(\gamma\) to a 6-path with refinement \(a^- g^3 a^- a^- g^3 a^- g^3 a^r\). Continuing this way we have that \(\text{ref}(P_{3+i}) = a^r a^r a^- a^r g^3 a^- a^- g^3 a^- a^- g\) and \(\text{ref}(P_{5+i}) = g a^- g^3 a^- a^- g a^- g\). But since the refinement of an admissible 4-path, whose central 2-subpath is a glide, is either \(g^3\) or \(a^- g a^-\) we may see that \(\text{ref}(P_{5+i})\) would have to be \(a^- g a^- g^3 a^- a^- g a^- g\), contradicting the fact that \(\text{Anc}^- \cap \text{Anc}^- \gamma = \emptyset\). A similar contradiction is obtained if \(\text{Anc}^- \gamma \cap \text{Gli} \neq \emptyset\). We may therefore assume that, for each \(\gamma \in G\), either \(\text{Anc}^+ \gamma \cap \text{Anc}^+ \neq \emptyset\) or \(\text{Anc}^- \gamma \cap \text{Anc}^- \neq \emptyset\). But then, since \(\text{Anc}^+\) and \(\text{Anc}^-\) are orbits of the action of \(H\) on 2-paths of \(X\), there must exist \(t \in G\) fixing an anchor of \(X\), say \(u_1u_2u_3\). Hence, there exists \(t' \in tH\) fixing each of the vertices \(u_1, u_2, u_3\), and so, since Zig is fixed by \(G\), also the other two neighbors of \(u_2\). Since \(t' \neq 1\) this proves the first part of (18), that is the unfaithfulness of \(G\). To show that in fact \(K\) has exponent 2, let \(v = v_0\) and let \(v \in K\) be arbitrary. Since Zig is fixed by \(G\), we have that \(v^2\) fixes pointwise the set \(N^2(v)\), too. In particular we have that \(v^2\) fixes each of the vertices \(v_1^2, v_1, v_1^0, v_1^1\), and \(v_1^2\), and thus the corresponding 4-path with refinement \(a^r a^- a^-\). But there is a unique 8-cycle in \(X\) containing this 4-path, forcing this 8-cycle, with refinement \(a^- a^r a^- z g a^- g z\), to be fixed pointwise by \(v^2\). In particular, \(v^2\) fixes \(v_1^2\) as well as all of its neighbors and so the 4-path \(v_1^2 + v_1^2 v_1^0 v_1^0 v_1^2\) is fixed pointwise by \(v^2\). Continuing this sort of argument, we see that \(V_0 \cup V_1\) is fixed pointwise. But then clearly, \(v^2 = 1\), thus proving (18).

Let \(C_0\) denote the alternating cycle spanned by \(V_0 \cup V_1\). The fact that Zig is fixed by \(G\) combined together with (18) implies the existence of an involution \(x \in G\) which fixes the vertices \(v_1^0, v_1^1, v_1, v_1^2\), and interchanges \(v_1^0\) and \(v_1^2\). In particular, \(x\) maps the admissible 4-path \(v_1 v_0^1 v_1^2 v_1^3\) with refinement \(a^- a^r a^- a^-\) onto the 4-path \(v_1 v_0^1 v_1^2 v_1^3\) with refinement \(a^- g a^- g a^-\). But then the succeeding inadmissible 4-path \(v_1^0 v_1^1 v_1^2 v_1^3\) with refinement \(a^- a^r a^- a^-\) must be mapped by \(x\) onto the 4-path \(v_1^0 v_1^1 v_1^2 v_1^3\) with refinement \(g a^- g^- a^- g\), and thus the succeeding admissible 4-path \(v_1^0 v_1^1 v_1^2 v_1^3\) with refinement \(a^- a^- a^- a^-\) must be mapped by \(x\) onto the 4-path \(v_1^0 v_1^1 v_1^2 v_1^3\) with refinement \(g a^- g^- g a^- g\). Continuing this way, using Proposition 4.5 and the (in)admissibility of 4-paths of \(X\), we can see that \(\text{ref}(C_0 x) = (a^- g a^- a^- g g g a^- a^- g a^- g)^{\infty}\). Among others this forces 7 to divide \(r\) and so

\[ r = 7k \quad \text{for some odd integer} \quad k \geq 1. \]
Combining together (16) and (17) we may express the powers of \( s \) as
\[
x^2 = -s + 2, x^3 = 3s - 2, x^4 = 2s - 1, \quad \text{and} \quad x^5 = 4s - 3.
\]
Now following the above period of length 14 in the refinement \((a^-ga'^+agga'^+agga'^+agg)^r\) of \( C_0 \), \( \alpha \) enables us to get the rule for the action of \( \alpha \) on \( V_0 \cup V_1 \). We argue along these lines. The image under \( \alpha \) of the subpath \( v_0v_1^{10}v_1^{11}v_1^{12}v_1^{13}v_0 = v_0v_1^{11}v_1^{12}v_1^{13}v_0 \) of length 14 in \( C_0 \) is the subpath \( v_0v_1^{14}v_2^{+1}v_2^{+2}v_2^{+3}v_2^{+4}v_2^{+5}v_2^{+6}v_2^{+7}v_2^{+8}v_2^{+9}v_2^{+10}v_2^{+11}v_2^{+12}v_2^{+13}v_0 \) (see Fig. 2) and the same pattern repeats itself every 14 vertices. It follows that the vertices \( v_0^{14m}, m \in \mathbb{Z}_+ \), are fixed by \( \alpha \), and since \( r \) is odd this means that the vertices \( v_0^{14m}, m \in \mathbb{Z}_+ \), are fixed by \( \alpha \). Similarly, for each \( m \in \mathbb{Z}_+ \), all the vertices of the form \( v_1^{7m+1} \) are fixed by \( \alpha \). Next, for each \( m \in \mathbb{Z}_+ \), we have that \( \alpha \) interchanges the vertices \( v_1^{7m+2} \) and \( v_1^{7m+2+\alpha} \). Continuing this way along the above period of length 14, we end up getting the first two columns of Table VI, where the rule for \( \alpha \) is given.

We now show that
\[
s \equiv 5 \pmod{7}. \quad (20)
\]

Consider the 4-path \( Q = v_0^{-2s}v_5^{-3}v_0^3v_5^{-4}v_0^{-1}v_5^{-1}v_5^{-4}v_0^{-3}v_5^{-s}v_0^{-1}v_5^{-r}v_0^{-3} \), with refinement \( a^-a'^+a'^- \). It follows that \( \text{re}(Qs) \) is either \( a^-a'^+a'^- \) or \( ga'^+g \).

In the first case we have that \( \alpha \) fixes the vertices \( v_0^{14} \) and \( v_0^{-2s} \). But then we may deduce from the first column of Table VI that \( s \equiv 0 \pmod{7} \) contradicting, in view of (19), the fact that \( s \in \mathbb{Z}_+^* \). Hence \( \text{re}(Qs) = ga'^+g \) and so we have \( v_0^{14} \alpha = v_0^{14}, v_5^{-1} \alpha = v_5^{-1}, v_5^{-s} \alpha = v_5^{-s}, v_0^{-1} \alpha = v_0^{-1} \) and so \( s \equiv 5 \pmod{7} \), by the first column of Table VI. (Note that, since by assumption the noncoiled girth of \( X \) is 8, we now have \( k \geq 5 \) in (19), by Proposition 4.8).

Having proved (20), it remains to show that the integer \( k \) in the expression for \( r = 7k \) in (19) is coprime with 7. But this is clearly the case for \( r \) divides

![Fig. 2. A subpath with refinement (a^-ga'^+agga'^+agga'^+agg) in C_0x.](image-url)
TABLE VI

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>j</td>
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<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>0</td>
<td>$v_0^j$</td>
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<td>$v_2^j$</td>
<td>$v_3^j$</td>
<td>$v_4^j$</td>
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</tr>
<tr>
<td>1</td>
<td>$v_0^{j+2(i-1)}$</td>
<td>$v_1^{j+2(i-1)}$</td>
<td>$v_2^{j+2(i-1)}$</td>
<td>$v_3^{j+2(i-1)}$</td>
<td>$v_4^{j+2(i-1)}$</td>
<td>$v_5^{j+2(i-1)}$</td>
</tr>
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<td>$v_0^{j+5(i-1)}$</td>
<td>$v_1^{j+4(i-1)}$</td>
<td>$v_2^{j+4(i-1)}$</td>
<td>$v_3^{j+4(i-1)}$</td>
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<td>$v_1^{j+2(i-1)}$</td>
<td>$v_2^{j+2(i-1)}$</td>
<td>$v_3^{j+2(i-1)}$</td>
<td>$v_4^{j+2(i-1)}$</td>
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<td>$v_2^{j+4(i-1)}$</td>
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<td>$v_5^{j+4(i-1)}$</td>
</tr>
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<td>$v_4^{j+4(i-1)}$</td>
<td>$v_5^{j+4(i-1)}$</td>
</tr>
</tbody>
</table>

This completes the proof of the fact that the condition (iii) is a necessary condition for arc-transitivity of those graphs $X(s, t, r)$ which do not satisfy conditions (i) or (ii).

To prove that the converse holds, we first complete the remaining columns of Table VI, using (20) and the fact that $\sigma$ is an involution. For example, letting $j \equiv 1 \pmod{7}$ and $i = 0$ we have that $v_4^{j+2(i-1)}$ must be mapped to $v_0^j$. Since $s \equiv 5 \pmod{7}$, this implies that $v_4^j$ must be mapped to $v_0^{j+5(i-1)}$ whenever $l \equiv 3 \pmod{7}$, giving us the entry in the third row and fifth column of Table VI. In the same way the entries in rows 2, 3, 6, and 7 of the third column, in rows 2 and 7 of the fourth column, in rows 3 and 5 of the fifth column and the entry in the first row of the last column of Table VI are obtained. To get the remaining entries we apply the following argument, shown in the case of the first entry of the third column. Let $j \equiv 0 \pmod{7}$. Then the two neighbors $v_4^{j+s}$ and $v_4^{j-s}$ of $v_4^j$ in $V_1$ are mapped by $\sigma$ to $v_0^{j+s+2(i-1)}$ and $v_0^{j-s+5(i-1)}$ whose only common neighbor is precisely $v_4^j$, the entry in the first row and third column of Table VI. Using this approach we can eventually complete all entries in Table VI, thus getting the formula for $\sigma$. The only thing one is now left to do is to check that Table VI really defines an automorphism of the graph $X(s, 6, 7k)$ when $k \geq 1$ is odd and coprime with 7 and $s \in Z^*_7$ satisfies $s^6 = 1$, $s^2 + s - 2 = 0$, $7(s-1) = 0$, and $s \equiv 5 \pmod{7}$. We leave this to the reader.

ACKNOWLEDGMENTS

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I am grateful to the Grateful Dead and the late Jerry Garcia for their music, a source of my inspirations.
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