Abstract

In this paper we define multiresolution analysis (MRA) in function space \( L_p(\Omega, \mu) \) for \( p > 1 \), where \( \Omega \) is a compact subset in \( \mathbb{R}^n \) and \( \mu \) is a probability measure defined on \( \Omega \). We establish a general framework for construction of MRA in \( L_p(\Omega, \mu) \) and its conjugate space \( L_q(\Omega, \mu) \) with \( p^{-1} + q^{-1} = 1 \), \( p, q > 1 \).

Keywords: Multiresolution analysis; Wavelets; \( L_p \) space; Refinement equation

1. Introduction

In [5,6,8] C.A. Micchelli and Y. Xu constructed discontinuous wavelets on invariant set \( \Omega \) of \( \mathbb{R}^n \) and it is known that such wavelets are very useful for solutions of integral equations because multiscale methods based on these wavelets lead to linear systems with sparse coefficient matrices which have bounded condition numbers [1,7]. We know that establishing multiscale structure in \( L_p(\Omega, \mu) \) is important for studying the functions in \( L_p(\Omega, \mu) \) [2,3]. In this paper we construct multiresolution analysis (MRA) in space \( L_p(\Omega, \mu) \), where \( \Omega \) is a compact subset of \( \mathbb{R}^n \) and \( \mu \) is a probability measure defined on \( \Omega \) which has no atom and is absolutely continuous with respect to the Lebesgue measure and simultaneously obtain MRA structure in the conjugate space \( L_q(\Omega, \mu) \) of \( L_p(\Omega, \mu) \), with \( p^{-1} + q^{-1} = 1 \), \( p, q > 1 \). This paper is organized as follows: In Section 2, we give the definition of MRA in \( L_p(\Omega, \mu) \) and give some notations. In Section 3, we present a general framework for the construction of MRA in \( L_p(\Omega, \mu) \) and its conjugate space \( L_q(\Omega, \mu) \).
2. Some notations and properties

We denote by \( L^p(\Omega, \mu) \) the space of \( p \) integrable functions with respect to \( \mu \), i.e.,

\[
\|f\|_p = \|f\|_{L^p(\Omega, \mu)} := \left( \int_{\Omega} |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} < \infty, \quad f \in L^p(\Omega, \mu).
\] (2.1)

The dual space \( L^{*p}(\Omega, \mu) \) of \( L^p(\Omega, \mu) \) is isometrically isomorphic to \( L^q(\Omega, \mu) \), where \( g \in L^q(\Omega, \mu) \) is identified with a functional \( f^* \) on \( L^p(\Omega, \mu) \) by

\[
f^*(f) = \langle f, g \rangle = \int_{\Omega} fg \, d\mu.
\] (2.2)

We say that \( f \in L^p(\Omega, \mu) \) is orthogonal to \( g \in L^q(\Omega, E) \) if \( \langle f, g \rangle = 0 \). Let \( A = \{g_1, \ldots, g_N\} \subseteq L^p(\Omega, E), B = \{h_1, \ldots, h_N\} \subseteq L^q(\Omega, E) \), we say \( A \) and \( B \) are biorthogonal pair if

\[
\langle g_i, h_j \rangle = \delta_{ij}, \quad \text{for } i, j \in \{1, 2, \ldots, N\}.
\]

Now we give the definition of MRA in \( L^p(\Omega, \mu) \) similarly to [3]. Throughout this paper, we suppose \( p^{-1} + q^{-1} = 1, \ p, q > 1 \).

**Definition 2.1.** Let \( \Omega \subseteq \mathbb{R}^n \) is a compact subset, \( p > 1 \). We say subspaces \( \{F_k\}_{k=0}^\infty \) are Multiresolution Analysis (MRA) in \( L^p(\Omega, \mu) \), if they satisfy the following conditions:

(a) \( \bigcup_{k=0}^\infty F_k = L^p(\Omega, \mu) \),
(b) \( F_k \subseteq F_{k+1}, \ k \geq 0 \),
(c) \( \bigcap_{k=0}^\infty F_k = \{0\} \),

and we call \( F_0 \) an initial scale space for MRA.

We decompose \( \Omega \) into a finite number of measurable \( \Omega_i, i \in E_m := \{0, 1, \ldots, m-1\} \), such that \( \Omega = \bigcup_{i=0}^{m-1} \Omega_i \) and \( \text{meas}(\Omega_i \cap \Omega_j) = 0, \ i \neq j, \ i, j \in E_m \), where \( \text{meas}(A) \) denotes the Lebesgue measure of a set \( A \subseteq \mathbb{R}^d \). Just like in [6], we need \( m \) bijective maps \( \phi_i: \Omega \to \Omega_i, \ i \in E_m \), defined a.e. (which depend on \( \mu \)) for invariant set the existence of \( \phi_i \), see [4,5], such that the bounded linear operators defined by

\[
(P_i f)(x) = \begin{cases} \frac{1}{\sqrt{v_i}} f\left(\phi_i^{-1}(x)\right), & x \in \Omega_i, \\ 0, & x \notin \Omega_i, \end{cases}
\]

with

\[
v_i := \int_{\Omega_i} d\mu(x), \quad i \in E_m,
\]

satisfy the equation

\[
\frac{1}{v_i} \int_{\Omega_i} f\left(\phi_i^{-1}(x)\right) d\mu(x) = \int_E f(x) d\mu(x), \quad f \in L^p(\Omega, \mu).
\] (2.3)
The existence of these mappings follows from general measure theoretic considerations [9]. We can verify that each $P_i$ is an isometry from $L_p(\Omega, \mu)$ to $L_p(\Omega, \mu)$). Similarly, we define isometry operators from $L_q(\Omega, \mu)$ to $L_q(\Omega, \mu)$:

$$(\tilde{P}_i g)(x) = \begin{cases} \frac{1}{\sqrt{v_i}} g(\phi_i^{-1}(x)), & x \in \Omega_i, \\ 0, & x \notin \Omega_i, \end{cases}$$

with $v_i$ satisfying (2.3). Define a linear operator from $L_p(\Omega, \mu)$ to $L_p(\Omega, \mu)$:

$$G_i f := \sqrt{v_i} f \circ \phi_i, \quad i \in E_m, \quad f \in L_p(\Omega, \mu),$$

and operators from $L_q(\Omega, \mu)$ to $L_q(\Omega, \mu)$:

$$\tilde{G}_i g := \sqrt{v_i} g \circ \phi_i, \quad i \in E_m, \quad g \in L_q(\Omega, \mu).$$

Here we are using the notation $(f \circ g)(x) = f(g(x)), \ x \in \Omega$, for functional composition. For $f \in L_p(\Omega, \mu), g \in L_q(\Omega, \mu)$, since

$$\langle P_i f, g \rangle = \int_{\Omega_i} \frac{1}{\sqrt{v_i}} f(\phi_i^{-1}(x)) g(x) \, d\mu(x) = 0,$$

we know $\tilde{G}$ is the adjoint of the operator $P_i$. Similarly $G$ is the adjoint of the operator $\tilde{P}_i$.

**Lemma 2.1.** For operators $P_i$, $\tilde{P}_i$ and $G_i$, $\tilde{G}_i$, we have the following properties:

$$P_i G_i = \chi_{E_i} I_{L_p(\Omega, \mu)}, \quad \tilde{P}_i \tilde{G}_i = \chi_{E_i} I_{L_q(\Omega, \mu)}, \quad i \in E_m,$$

$$G_i P_j = \delta_{ij} I_{L_p(\Omega, \mu)}, \quad \tilde{G}_i \tilde{P}_j = \delta_{ij} I_{L_q(\Omega, \mu)}, \quad i, j \in E_m.$$  

**Proof.** We only prove the first statement for $P_i$ and $G_i$ of (2.6) and (2.7).

For any $f \in L_p(\Omega, \mu)$, if $x \in \Omega_i$, from the definition of $P_i$ ($i \in E_m$), we have

$$(P_i G_i f)(x) = P_i((G_i f)(x)) = \frac{1}{\sqrt{v_i}} (G_i f)(\phi_i^{-1}(x)) = \frac{1}{\sqrt{v_i}} \sqrt{v_i} f(\phi_i^{-1}(x)) = f(x),$$

if $x \notin \Omega_i$, obviously $(P_i G_i f)(x) = 0$. So the first statement of (2.6) is proved. For $x \in \Omega$, since

$$(G_i P_j f)(x) = G_i((P_j f)(x)) = \sqrt{v_i} (P_j f)(\phi_i(x)) = \sqrt{v_i} \frac{1}{\sqrt{v_i}} f(\phi_i^{-1}(\phi_i(x))) = f(x),$$

on the other hand, if $i \neq j$,

$$(G_i P_j f)(x) = \sqrt{v_i} (P_j g)(\phi_i(x)) = 0.$$ 

So the statement of (2.7) is proved. \(\square\)

We choose $m \times m$ orthogonal matrix

$$Q = (q_{ij})_{i,j \in E_m},$$

that is

$$QQ^T = Q^T Q = I,$$

and define a set of linear operators $T_i$ ($i \in E_m$) from $L_p(\Omega, \mu)$ to $L_p(\Omega, \mu)$ and linear operators $\tilde{T}_i$ ($i \in E_m$) from $L_q(\Omega, \mu)$ to $L_q(\Omega, \mu)$:
\[ T_i = \sum_{j=0}^{m-1} q_{ij} P_j, \quad i \in E_m, \quad (2.8) \]

\[ \tilde{T}_i = \sum_{j=0}^{m-1} q_{ij} \tilde{P}_j, \quad i \in E_m. \quad (2.9) \]

With orthogonality of matrix \( Q \) and \( \tilde{G}_i, G_i \) being the adjoint of \( P_i, \tilde{P}_i \) respectively, we know

\[ T_i^* = \sum_{j=0}^{m-1} q_{ij} \tilde{G}_j \quad (2.10) \]

and

\[ \tilde{T}_i^* = \sum_{j=0}^{m-1} q_{ij} G_j \quad (2.11) \]

are adjoint operators \( T_i \) and \( \tilde{T}_i \), respectively.

**Lemma 2.2.** For operators \( T_i, \tilde{T}_i, \) and \( T_i^*, \tilde{T}_i^* \), we have

1. \( \tilde{T}_i^* T_j = \delta_{ij} I_{L_p(\Omega, \mu)} \),
2. \( T_i^* \tilde{T}_j = \delta_{ij} I_{L_q(\Omega, \mu)} \),
3. \( \langle f, g \rangle = \sum_{j=0}^{m-1} \langle G_i f, \tilde{G}_i g \rangle \), for \( f \in L_p(\Omega, \mu), \ g \in L_q(\Omega, \mu) \).

**Proof.** For \( i, j \in E_m \), notice that

\[ \tilde{T}_i^* T_j = \tilde{T}_i^* \left( \sum_{s=0}^{m-1} q_{js} P_s \right) = \sum_{i=0}^{m-1} q_{ii} G_i \left( \sum_{s=0}^{m-1} q_{js} P_s \right) = \sum_{i=0}^{m-1} \sum_{s=0}^{m-1} q_{ii} q_{js} G_i P_s = \sum_{s=0}^{m-1} q_{ii} q_{js} I_{L_p(\Omega, \mu)} = \delta_{ij} I_{L_p(\Omega, \mu)}. \]

So (1) is proved. With similar argument (2) can be proved. For any \( f \in L_p(\Omega, \mu), \ g \in L_q(\Omega, \mu) \), from the first statement of (2.6), we have

\[ \langle f, g \rangle = \sum_{i=0}^{m-1} \langle P_i G_i f, g \rangle = \sum_{i=0}^{m-1} \langle G_i f, G_i g \rangle = \sum_{i=0}^{m-1} \langle G_i f, \tilde{G}_i g \rangle. \]

(3) is proved. \( \Box \)

Now we choose linear independent vector fields \( f = (f_1, \ldots, f_n)^T \) and \( \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n)^T \) with \( f_i \in L_p(\Omega, \mu) \) and \( \tilde{f}_i \in L_q(\Omega, \mu) \), \( i \in E_m \), which satisfy the matrix refinement equations

\[ G_i f = \sqrt[4]{\nu_i} A_i^T f, \quad i \in E_m, \quad (2.12) \]

and

\[ G_i \tilde{f} = \sqrt[4]{\nu_i} \tilde{A}_i^T \tilde{f}, \quad i \in E_m. \quad (2.13) \]
Here $A_i, \tilde{A}_i, i \in E_m$, are some prescribed $n \times n$ matrices and $v^T f = a, \tilde{v}^T \tilde{f} = \tilde{a}$, where $v, \tilde{v}$ are some constant vectors in $\mathbb{R}^n$ and $a, \tilde{a}$ are some nonzero constants. For the existence of $A_i$ and $\tilde{A}_i$ we refer to [5,6]. We assume that the components of $f$ and $\tilde{f}$ are chosen so that they satisfy the additional condition

$$\langle f_i, \tilde{f}_j \rangle = \delta_{ij}, \quad i, j \in E_m.$$  

The pair of sets $\mathcal{F}_0 = \{f_1, \ldots, f_n\}$ and $\tilde{\mathcal{F}}_0 = \{\tilde{f}_1, \ldots, \tilde{f}_n\}$ form our initial biorthogonal pair in $L_p(\Omega, \mu)$ and $L_q(\Omega, \mu)$. Set finite dimensional space in $L_p(\Omega, \mu)$: $F_0 = \text{span} \mathcal{F}_0$, and finite dimensional space in $L_q(\Omega, \mu)$: $\tilde{F}_0 = \text{span} \tilde{\mathcal{F}}_0$. We define:

$$F_{k+1} := \bigoplus_{i=0}^{m-1} T_i F_k, \quad k = 0, 1, \ldots,$$

and

$$\tilde{F}_{k+1} := \bigoplus_{i=0}^{m-1} \tilde{T}_i \tilde{F}_k, \quad k = 0, 1, \ldots.$$  

We want to use these pair of scale of subspaces $F_k$ in $L_p(\Omega, \mu)$ and $\tilde{F}_k$ in $L_q(\Omega, \mu)$ to generate, recursively in $k$, a scale of biorthogonal subspaces in $L_p(\Omega, \mu)$ and $L_q(\Omega, \mu)$ which was defined as Multiresolution Analysis of $L_p(\Omega, \mu)$ in Section 1. Now we present some properties of $F_k$ and $\tilde{F}_k$. The next lemma insures that these subspaces are nested.

**Lemma 2.3.** Let $F_k$ and $\tilde{F}_k$ be defined as above. Then

$$F_k \subseteq F_{k+1}, \quad \tilde{F}_k \subseteq \tilde{F}_{k+1}, \quad k = 0, 1, \ldots, \quad (2.14)$$

and

$$T_i^* F_{k+1} \subseteq F_k, \quad T_i^* \tilde{F}_{k+1} \subseteq \tilde{F}_k, \quad k = 0, 1, \ldots. \quad (2.15)$$

**Proof.** We only prove the statements for $F_k$. The proof for $\tilde{F}_k$ is similar. Define operators from $L_p(\Omega, \mu)$ to $L_p(\Omega, \mu)$:

$$H_i = \sum_{i=0}^{m-1} q_{ji} T_j, \quad i \in E_m. \quad (2.16)$$

For any $f \in L_p(\Omega, \mu)$, notice that

$$\sum_{i=0}^{m-1} H_i G_i f = \sum_{i=0}^{m-1} \sum_{l=0}^{m-1} \left( \sum_{j=0}^{m-1} q_{ji} q_{jl} \right) P_l G_i f = \sum_{i=0}^{m-1} \sum_{l=0}^{m-1} \delta_{li} P_l G_i f = \sum_{i=0}^{m-1} P_l G_i f,$$

from Lemma 2.1, we have

$$\sum_{i=0}^{m-1} H_i G_i f = \sum_{i=0}^{m-1} \chi_{E_i} f = f. \quad (2.17)$$

So $\sum_{i=0}^{m-1} H_i G_i = I_{L_p(\Omega, \mu)}$. From (2.12) and (2.17) we have

$$f = \sum_{j=0}^{m-1} \left( \sum_{i=0}^{m-1} \sqrt{q_{ji}} A_i^T \right) T_j f,$$
which means that $F_0 \subseteq F_1$. By induction, suppose $F_k \subseteq F_{k+1}$, $k \in N_0$, then we will see that $T_i$ $(i \in E_m)$ is one-to-one bounded linear operator from $L_p(\Omega, \mu)$ to $L_p(\Omega, \mu)$ which will be proved in Lemma 2.5 and has left inverse from Lemma 2.2. So we have

$$F_{k+1} = \bigoplus_{i=0}^{m-1} T_i F_k \subseteq \bigoplus_{i=0}^{m-1} T_i F_{k+1} = F_{k+2}.$$  

Thus $F_k$ $(k \geq 0)$ are nested in $k$. As for (2.15), we only verify that

$$G_l F_{k+1} \subseteq F_k, \quad l \in E_m, \quad k \in N_0,$$

for any $g \in F_{k+1}$. By the definition of $F_{k+1}$, there are $m$ functions $g_i \in F_k$, $i \in E_m$, such that

$$g = \sum_{i=0}^{m-1} T_i g_i.$$  

Applying the linear operators $G_l$, $l \in E_m$, we obtain

$$G_l g = \sum_{i=0}^{m-1} G_l T_i g_i.$$  

From (2.7) in Lemma 2.1, we have

$$G_l g = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} q_{ij} G_l P_j g_i = \sum_{i=0}^{m-1} q_{ii} g_i \in F_k.$$  

From the arbitrary of $g$ in $F_{k+1}$, we have proved $G_l F_{k+1} \subseteq F_k$, $l \in E_m$, $k \in N_0$. So the lemma is proved. $\Box$

The following lemma will be used later.

**Lemma 2.4.** Suppose the subsets $G := \{g_1, \ldots, g_N\}$ in $L_p(\Omega, \mu)$ and $\tilde{G} := \{\tilde{g}_1, \ldots, \tilde{g}_N\}$ in $L_q(\Omega, \mu)$ are biorthogonal pair. Then the sets

$$G = \{T_i g_j : i = 0, 1, \ldots, m-1, \; j = 1, 2, \ldots, N\}$$

and

$$\tilde{G} = \{\tilde{T}_j \tilde{g}_i : i = 0, 1, \ldots, m-1, \; j = 1, 2, \ldots, N\}$$

are likewise a biorthogonal pair.

**Proof.** Since

$$\langle T_i g_s, \tilde{T}_j \tilde{g}_t \rangle = \int_{\Omega} (T_i g_s)(x)(\tilde{T}_j \tilde{g}_t)(x) \, d\mu(x)$$

$$= \sum_{l=0}^{m-1} \sum_{l'=0}^{m-1} q_{il} q_{jl'} \int_{\Omega} (P_l g_s)(x)(\tilde{P}_{l'} \tilde{g}_t)(x) \, d\mu(x)$$

$$= \sum_{l=0}^{m-1} \sum_{l'=0}^{m-1} q_{il} q_{jl'} \langle P_l g_s, \tilde{P}_{l'} \tilde{g}_t \rangle.$$
From the fact that \( \tilde{G}_i \) is the adjoint of operator \( P_i \), we have

\[
\langle T_i g_s, \tilde{T}_j \tilde{g}_t \rangle = \sum_{l=0}^{m-1} \sum_{l'=0}^{m-1} q_{il} q_{jl'} \langle g_s, \tilde{G}_i \tilde{P}_{l'} \tilde{g}_t \rangle.
\]

By using (2.7) and the biorthogonality of \( G \) and \( \tilde{G} \), we have

\[
\langle T_i g_s, \tilde{T}_j \tilde{g}_t \rangle = \sum_{l=0}^{m-1} \sum_{l'=0}^{m-1} q_{il} q_{jl'} \langle g_s, \delta_{ll'} \tilde{g}_t \rangle = \sum_{l=0}^{m-1} q_{il} \delta_{ll} = \delta_{ij} \delta_{st}.
\]

In the last step, we have used the orthogonality of matrix \( Q \).

\[\Box\]

**Lemma 2.5.** For operators \( T_i \) and \( \tilde{T}_i \), the following statements hold:

(i) \( \| T_i \|_p = \left( \sum_{j=0}^{m-1} |q_{ij}|^p \right)^{\frac{1}{p}} \), \( \| \tilde{T}_i \|_q = \left( \sum_{j=0}^{m-1} |q_{ij}|^q \right)^{\frac{1}{q}} \), and \( T_i, \tilde{T}_i \) (\( i \in E_m \)) are one-to-one linear bounded operators.

(ii) For any subset \( X \subseteq L_p(\Omega, \mu) \) and subset \( \tilde{X} \subseteq L_q(\Omega, \mu) \), we have

\[
T_i(X) \cap T_j(X) = \{0\} \quad (i \neq j),
\]

\[
\tilde{T}_i(\tilde{X}) \cap \tilde{T}_j(\tilde{X}) = \{0\} \quad (i \neq j).
\]

**Proof.** For any \( f \in L_p(\Omega, \mu) \),

\[
\| T_i f \|_p^p = \int_{\Omega} \left| \sum_{j=0}^{m-1} \frac{1}{\sqrt{\nu_j}} q_{ij} f(\phi_j^{-1}(x)) \chi_{\Omega_j}(x) \right|^p \, d\mu(x)
\]

\[= \sum_{j=0}^{m-1} \int_{\Omega_j} |q_{ij}|^{\frac{1}{p}} \left| f(\phi_j^{-1}(x)) \right|^p \, d\mu(x)\]

\[= \sum_{j=0}^{m-1} |q_{ij}|^{\frac{1}{p}} \int_{\Omega_j} \left| f(\phi_j^{-1}(x)) \right|^p \, d\mu(x)\]

\[= \sum_{j=0}^{m-1} |q_{ij}|^{\frac{1}{p}} \int_{\Omega} \left| f(x) \right|^p \, d\mu(x)\]

\[= \sum_{j=0}^{m-1} |q_{ij}|^{\frac{1}{p}} \cdot \| f \|^p.
\]

From the above equation, we know \( T_i \) is one-to-one operator. So (i) is proved.

For any \( g \in T_i(X) \cap T_j(X) \) (\( i \neq j \)), there exist two functions \( g_i, g_j \in X \subseteq L_p(\Omega, \mu) \) such that

\[
g = T_i g_i = T_j g_j.
\]

\[ q_{ik}g_i(\phi_k^{-1}(x)) - q_{jk}g_j(\phi_k^{-1}(x)) \chi_{\Omega_k}(x) = 0. \]

So for any \( k \in E_m, x \in \Omega_k \),

\[ q_{ik}g_i(\phi_k^{-1}(x)) = q_{jk}g_j(\phi_k^{-1}(x)). \]

Thus from bijection of \( \phi_k \), for any \( u \in \Omega, k \in E_m \), we have

\[ q_{ik}g_i(u) = q_{jk}g_j(u). \]

But from the orthogonality of matrix \( Q \), we have

\[ 0 \equiv \sum_{k=0}^{m-1} q_{ik}q_{jk}g_i(u) = \sum_{k=0}^{m-1} q_{ik}^2g_j(u) = g_j(u). \]

With similar method \( g_i(u) \equiv 0 \), thus \( g \equiv 0 \). The lemma is proved. \( \square \)

**Lemma 2.6.** Suppose \( \mathcal{A} \) and \( \mathcal{B} \) are finite dimensional subspaces of space \( L_p(\Omega, \mu) \), \( \mathcal{A} \subseteq \mathcal{B} \). \( \tilde{\mathcal{A}} \) and \( \tilde{\mathcal{B}} \) are finite dimensional subspaces of space \( L_q(\Omega, \mu) \), \( \tilde{\mathcal{A}} \subseteq \tilde{\mathcal{B}} \). \( \mathcal{A}, \tilde{\mathcal{A}} \) have bases \( \mathcal{A} = \{a_1(x), \ldots, a_L(x)\} \), \( \tilde{\mathcal{A}} = \{\tilde{a}_1(x), \ldots, \tilde{a}_L(x)\} \), respectively. \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \) are a biorthogonal pair. \( \mathcal{B} \) and \( \tilde{\mathcal{B}} \) have bases \( \mathcal{B} = \{b_1(x), \ldots, b_N(x)\} \), \( \tilde{\mathcal{B}} = \{\tilde{b}_1(x), \ldots, \tilde{b}_N(x)\} \) (\( L < N \)) such that \( \mathcal{B} \) and \( \tilde{\mathcal{B}} \) are also a biorthogonal pair. Then there are \( N - L \) functions \( a_{L+1}(x), \ldots, a_N(x) \) in \( L_p(\Omega, \mu) \) and \( N - L \) functions \( \tilde{a}_{L+1}(x), \ldots, \tilde{a}_N(x) \) in \( L_q(\Omega, \mu) \), such that \( (a_1(x), \ldots, a_N(x)), (\tilde{a}_1(x), \ldots, \tilde{a}_N(x)) \) are bases of \( \mathcal{B} \) and \( \tilde{\mathcal{B}} \), respectively. \( (a_1(x), \ldots, a_N(x)) \) and \( (\tilde{a}_1(x), \ldots, \tilde{a}_N(x)) \) are biorthogonal pairs.

**Proof.** Since \( \mathcal{A} \subseteq \mathcal{B} \), \( \tilde{\mathcal{A}} \subseteq \tilde{\mathcal{B}} \), \( \{a_1(x), \ldots, a_L(x)\} \) and \( \{\tilde{a}_1(x), \ldots, \tilde{a}_L(x)\} \) are the bases of \( \mathcal{A} \) and \( \tilde{\mathcal{A}} \), respectively, and \( \{b_1(x), \ldots, b_N(x)\}, \{\tilde{b}_1(x), \ldots, \tilde{b}_N(x)\} \) are the bases of \( \mathcal{B} \) and \( \tilde{\mathcal{B}} \), then there exist two \( L \times N \) matrix \( M_1 = (m_{ij})_{1 \leq i \leq L, 1 \leq j \leq N}, \tilde{M}_1 = (\tilde{m}_{ij})_{1 \leq i \leq L, 1 \leq j \leq N} \) with \( \text{Rank}(M_1) = \text{Rank}(\tilde{M}_1) = L \) such that

\[ (a_1(x), \ldots, a_L(x))^T = M_1(b_1(x), \ldots, b_N(x))^T, \]
\[ (\tilde{a}_1(x), \ldots, \tilde{a}_L(x))^T = \tilde{M}_1(b_1(x), \ldots, b_N(x))^T. \]

Set

\[ M_1 = (m_1, \ldots, m_L)^T, \quad \tilde{M}_1 = (\tilde{m}_1, \ldots, \tilde{m}_L)^T, \]
\[ m_i = (m_{i1}, \ldots, m_{iN})^T, \quad \tilde{m}_i = (\tilde{m}_{i1}, \ldots, \tilde{m}_{iN})^T, \quad i = 1, 2, \ldots, L. \]

Here \( M^T \) means the transform of matrix \( M \). First we extend linear independent vectors sets \( \{m_1, \ldots, m_L\} \) and \( \{\tilde{m}_1, \ldots, \tilde{m}_L\} \) into the bases \( \{m_1, \ldots, m_L, m_{L+1}', \ldots, m_N'\} \) and \( \{\tilde{m}_1, \ldots, \tilde{m}_L, \tilde{m}_{L+1}', \ldots, \tilde{m}_N'\} \) of \( \mathbb{R}^N \). From the biorthogonality of \( A \) and \( \tilde{A} \) and \( B \) and \( \tilde{B} \), we have

\[ \text{Gram}(m_1, \ldots, m_L; \tilde{m}_1, \ldots, \tilde{m}_L) := \{m_i, \tilde{m}_j\}_{1 \leq i, j \leq L} = I_{L \times L}. \]

Here \( \langle m_i, \tilde{m}_j \rangle = \sum_{l=1}^N m_{il}\tilde{m}_{jl} \), \( I_{L \times L} \) is identity matrix. Now we construct a vector in \( \mathbb{R}^N \)

\[ m_{L+1} := \sum_{l=0}^L \beta_{l+1}m_l + m_{L+1}' \] which has the property

\[ \langle m_{L+1}, \tilde{m}_i \rangle = 0, \quad \text{for} \quad i = 1, \ldots, L. \]
In fact, if \( \langle m'_{L+1}, \hat{m} \rangle = 0 \), for \( i = 1, \ldots, L \), then we can choose \( \beta_{(L+1)l} = 0 \), for \( l = 1, \ldots, L \), and \( \hat{m}_{L+1} = m'_{L+1} \); otherwise, if there is some \( i = 1, \ldots, L \) such that \( \langle m'_{L+1}, \hat{m} \rangle \neq 0 \), then the linear system

\[
\sum_{l=0}^{L} \langle m_l, \hat{m} \rangle \beta_{(L+1)l} + \langle m'_{L+1}, \hat{m} \rangle = 0, \quad j = 1, \ldots, L,
\]

has a unique nonzero solution \( \{\beta_{(L+1)l} : l = 1, \ldots, L\} \) due to the nonsingularity of the matrix \( \langle (m_l, \hat{m}) \rangle_{1 \leq l, j \leq L} = I_{L \times L} \). It is obvious that \( \{m_1, \ldots, m_L, m_{L+1}\} \) is linear independent. We want to construct \( \hat{m}_{L+1} \), such that \( \langle \hat{m}_{L+1}, m_j \rangle = 0 \), for \( j = 1, \ldots, L \), and \( \langle \hat{m}_{L+1}, m_{L+1} \rangle = 1 \).

In fact there is a vector \( \hat{m}'_s \) with \( s \in \{L+1, \ldots, N\} \) such that \( \langle \hat{m}'_s, m_{L+1} \rangle \neq 0 \); otherwise \( m_{L+1} \) will be zero vector because \( \{\hat{m}_1, \ldots, \hat{m}_L, \hat{m}'_{L+1}, \ldots, \hat{m}'_N\} \) are bases vectors set in \( \mathbb{R}^N \). By permutation, we choose \( m^*_{L+1} = \langle \hat{m}, m_{L+1} \rangle \), then \( \langle m^*_{L+1}, m_{L+1} \rangle = 1 \). If \( \langle m^*_{L+1}, m_j \rangle = 0 \), for \( j = 1, 2, \ldots, L \), then we choose \( \hat{m}_{L+1} = m^*_{L+1} \); otherwise, if there exists \( t \in \{1, \ldots, L\} \) such that \( \langle m^*_{L+1}, m_t \rangle \neq 0 \), notice that the linear system

\[
\sum_{l=1}^{L} \tilde{\beta}_{L+1,l} \langle \hat{m}_l, m_j \rangle + \langle m^*_{L+1}, m_j \rangle = 0, \quad j = 1, \ldots, L,
\]

has a unique nonzero solution \( \{\tilde{\beta}_{L+1,l} : 1 \leq l \leq L\} \) because of the nonsingularity of matrix \( \langle (\hat{m}_l, m_j) \rangle_{1 \leq l, j \leq L} \), so we can choose

\[
\hat{m}_{L+1} = \sum_{l=1}^{L} \tilde{\beta}_{L+1,l} \hat{m}_l + m^*_{L+1},
\]

which satisfies: \( \langle \hat{m}_{L+1}, m_j \rangle = 0 \), but \( \langle \hat{m}_{L+1}, m_{L+1} \rangle = 1 \) and \( \{\hat{m}_1, \ldots, \hat{m}_L, \hat{m}_{L+1}\} \) is linear independent. Consequently by induction, we can construct \( \{m_{L+2}, \ldots, m_N\} \) and \( \{\hat{m}_{L+2}, \ldots, \hat{m}_N\} \) such that

\[
\langle m_l, \hat{m}_j \rangle = \langle \hat{m}_l, m_j \rangle = 0, \quad \text{for } i = L + 1, \ldots, N, \ j = 1, \ldots, L,
\]

and

\[
\langle m_l, \hat{m}_l \rangle = \delta_{ij}, \quad \text{for } i, l = L + 1, \ldots, N.
\]

Furthermore \( \{m_1, \ldots, m_L, \hat{m}_{L+1}, \ldots, \hat{m}_N\} \) and \( \{\hat{m}_1, \ldots, \hat{m}_L, \hat{m}_{L+1}, \ldots, \hat{m}_N\} \) are two basis of \( \mathbb{R}^N \). Set

\[
M = (m_1, \ldots, m_L, m_{L+1}, \ldots, m_N)^T, \quad \tilde{M} = (\hat{m}_1, \ldots, \hat{m}_L, \hat{m}_{L+1}, \ldots, \hat{m}_N)^T.
\]

Then \( M^T \tilde{M} = I_{N \times N} \) and \( \tilde{M}^T M = I_{N \times N} \). Let

\[
(a_{L+1}(x), \ldots, a_N(x)) = (m_{L+1}, \ldots, m_N)^T (b_1(x), \ldots, b_N(x))
\]

and

\[
(\tilde{a}_{L+1}(x), \ldots, \tilde{a}_N(x)) = (\hat{m}_{L+1}, \ldots, \hat{m}_N)^T (b_1(x), \ldots, b_N(x))
\]

then \( \{a_1(x), \ldots, a_N(x)\} \) and \( \{\tilde{a}_1(x), \ldots, \tilde{a}_N(x)\} \) are biorthogonal pair and are two bases of \( B, \tilde{B} \) because \( M \) and \( \tilde{M} \) are nonsingular matrices. The lemma is proved. \( \square \)
3. Construction of multiresolution analysis

We continue discussion in previous content. Now we prove the main result.

**Theorem 3.1.** Assume that $F_k$ and $\tilde{F}_k$ $(k \geq 0)$ are defined in Section 2, then

(i) there exists $n(m-1)$ linear independent functions $\mathcal{W}_0 = \{\omega_1, \ldots, \omega_{n(m-1)}\} \subseteq F_1$ and $n(m-1)$ linear independent functions $\tilde{\mathcal{W}}_0 = \{\tilde{\omega}_1, \ldots, \tilde{\omega}_{n(m-1)}\} \subseteq \tilde{F}_1$ such that $\mathcal{W}_0$ and $\tilde{\mathcal{W}}_0$ are biorthogonal pair.

(ii) Denote $W_0 = \text{span} \mathcal{W}_0$, $\tilde{W}_0 = \text{span} \tilde{\mathcal{W}}_0$, then $W_0 \perp \tilde{F}_0$, $\tilde{W}_0 \perp F_0$.

(iii) Let

$$W_{k+1} = \bigoplus_{i=0}^{m-1} T_i W_k, \quad k = 0, 1, \ldots,$$

$$\tilde{W}_{k+1} = \bigoplus_{i=0}^{m-1} \tilde{T}_i \tilde{W}_k, \quad k = 0, 1, \ldots,$$

then

$$F_{k+1} = F_k \oplus W_k, \quad k = 0, 1, \ldots,$$

$$\tilde{F}_{k+1} = \tilde{F}_k \oplus \tilde{W}_k, \quad k = 0, 1, \ldots$$

and $W_k \perp F_k$, $\tilde{W}_k \perp F_k$, and $\mathcal{W}_k$ and $\tilde{\mathcal{W}}_k$ are biorthogonal pair, here

$$\mathcal{W}_k := \{T_e \omega_j : e = (\epsilon_{k-1}, \ldots, \epsilon_0) \in E_m, T_e \omega_j = T_{\epsilon_{k-1}} \cdots T_{\epsilon_0} \omega_j, \quad j = 1, 2, \ldots, n(m-1)\},$$

$$\tilde{\mathcal{W}}_k := \{\tilde{T}_e \tilde{\omega}_j : e = (\epsilon_{k-1}, \ldots, \epsilon_0) \in E_m, \tilde{T}_e \tilde{\omega}_j = \tilde{T}_{\epsilon_{k-1}} \cdots \tilde{T}_{\epsilon_0} \tilde{\omega}_j, \quad j = 1, 2, \ldots, n(m-1)\}.$$

**Proof.** We only prove the statement for $F_k$. We know

$$F_0 = \text{span}\{f_1, \ldots, f_n\},$$

$$\tilde{F}_0 = \text{span}\{\tilde{f}_1, \ldots, \tilde{f}_n\},$$

$$F_1 = \text{span}\{T_i f_j : i = 0, 1, \ldots, m-1, j = 1, \ldots, n\},$$

$$\tilde{F}_1 = \text{span}\{\tilde{T}_i \tilde{f}_j : i = 0, 1, \ldots, m-1, j = 1, \ldots, n\}.$$

From Lemmas 2.2 and 2.5, we know sets $\mathcal{F}_1 = \{T_i f_j : i \in E_m, j = 1, \ldots, n\}$ and $\tilde{\mathcal{F}}_1 = \{\tilde{T}_i \tilde{f}_j : i \in E_m, j = 1, \ldots, n\}$ are the basis of $F_1$ and $\tilde{F}_1$, respectively. Lemma 2.4 means that $\mathcal{F}_1$ and $\tilde{\mathcal{F}}_1$ are biorthogonal pair. So from Lemma 2.6, we know there exists biorthogonal pair $\{\omega_1, \ldots, \omega_{n(m-1)}\}$ and $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_{n(m-1)}\}$. Set

$$W_0 = \text{span}\{\omega_1, \ldots, \omega_{n(m-1)}\},$$

$$\tilde{W}_0 = \text{span}\{\tilde{\omega}_1, \ldots, \tilde{\omega}_{n(m-1)}\}$$

then $W_0 \perp \tilde{F}_0$, $\tilde{W}_0 \perp F_0$, and
\[ F_1 = F_0 \oplus W_0, \]
\[ \tilde{F}_1 = \tilde{F}_0 \oplus \tilde{W}_0. \]

So (i) and (ii) are proved. To prove (iii), set \( W_1 = \bigoplus_{i=1}^{m-1} T_i W_0 \), we want to prove \( F_2 = F_1 \oplus W_1 \).

From the definition of \( F_2 = \bigoplus_{i=1}^{m-1} T_i F_1 \) and the fact that \( T_i \) is one-to-one and has left inverse operator, we know

\[ F_2 = \bigoplus_{i=0}^{m-1} T_i F_0 \oplus \bigoplus_{i=1}^{m-1} T_i W_0 = F_1 \oplus W_1. \]

Suppose that \( F_{k+1} = F_k \oplus W_k \), then

\[ F_{k+2} = \bigoplus_{i=0}^{m-1} T_i F_{k+1} = \bigoplus_{i=0}^{m-1} T_i (F_k \oplus W_k) = \left( \bigoplus_{i=0}^{m-1} T_i F_k \right) \oplus \left( \bigoplus_{i=0}^{m-1} T_i W_k \right) = F_{k+1} \oplus W_{k+1}. \]

Thus by induction, we prove the theorem. \( \square \)

To finish our final statement about MRA, we introduce the following notations. Let \( k \) be a positive integer, for every \( \varepsilon_1, \ldots, \varepsilon_k \in E_m \), we set

\[ \mathbf{e} := (\varepsilon_1, \ldots, \varepsilon_k)^T \in E_m. \]

We define the set

\[ \Omega_{\mathbf{e}} = \phi_{\mathbf{e}}(\Omega) := (\phi_{\varepsilon_k} \circ \cdots \circ \phi_{\varepsilon_1})(\Omega) \]

and the diameter

\[ \delta_k(\Omega) := \max \{ \text{diam } \Omega_{\mathbf{e}} : \mathbf{e} \in E_m^k \}, \]

where for any set \( S \subseteq \mathbb{R}^d \),

\[ \text{diam } S := \sup \{ \|x - y\|_2 : x, y \in S \}. \]

**Theorem 3.2.** Suppose that the mapping \( \phi_i, i \in E_m \), is Hölder continuous and

\[ \lim_{k \to \infty} \delta_k(\Omega) = 0, \]

then the following statements hold:

(i) \( \text{cl}_{L^p(\Omega, \mu)} \left( \bigcup_{k=0}^{\infty} F_k \right) = \text{cl}_{L^p(\Omega, \mu)} \left( F_0 \oplus \bigcup_{k=0}^{\infty} W_k \right) = L_p(\Omega, \mu), \)

(ii) \( \text{cl}_{L^q(\Omega, \mu)} \left( \bigcup_{k=0}^{\infty} \tilde{F}_k \right) = \text{cl}_{L^q(\Omega, \mu)} \left( \tilde{F}_0 \oplus \bigcup_{k=0}^{\infty} \tilde{W}_k \right) = L_q(\Omega, \mu). \)

**Proof.** From Theorem 3.1, we know that

\[ F_{k+1} = F_k \oplus W_k, \quad \tilde{F}_{k+1} = \tilde{F}_k \oplus \tilde{W}_k. \]
So we only prove the statement
\[ cL_p(\Omega, \mu) \left( \bigcup_{k=0}^{\infty} F_k \right) = L_p(\Omega, \mu). \]

It is obvious that
\[ \Omega = \bigcup \Omega_e, \quad e \in E_m^k, \]
and \( \text{meas}\{E_e \cap E_{e'}\} = 0, \ e \neq e', \ e, e' \in E_m^k \). We follow the reasoning method in the proof of Theorem 3.2 of [7].

Set
\[ \tilde{F}_k := \text{span}\{\chi_{\Omega_e} g : e \in E^m, \ g \in F_0\}. \quad (3.3) \]

We want to prove \( \tilde{F}_k = F_k \). We need only to prove that \( \tilde{F}_k \subseteq F_k \) since \( \dim \tilde{F}_k = m^k \dim F_0 = \dim F_k \), this would imply that \( \tilde{F}_k = F_k \). With the similar method and refinement equation in [7], we can prove
\[ \chi_{\Omega_e} f = H_{f_1} \cdots H_{f_k} A_{e_1}^T \cdots A_{e_k}^T f, \]
which means \( \chi_{\Omega_e} g \in F_k \) (\( g \in F_0 \)). So \( \tilde{F}_k \subseteq F_k \). We know from Section 2, that \( v^T f = a \) where \( a \) is a nonzero constant and
\[ \chi_{\Omega_e} \in F_k, \quad e \in E_m^k. \]

But we know a linear span of the set of characteristic functions
\[ \{ \chi_{\Omega_e} : e \in E_m^k, \ k = 0, 1, \ldots \} \]
is dense in \( L_p(\Omega, \mu) \). So we have proved the result. The theorem is proved. \( \square \)

From Lemma 2.5 and Theorems 3.1 and 3.2, now we can state the main result of this paper.

**Theorem 3.3.** Subspaces \( \{F_k\}_{k=0}^{\infty} \subseteq L_p(\Omega, \mu) \) and subspaces \( \{\tilde{F}_k\}_{k=0}^{\infty} \subseteq L_q(\Omega, \mu) \) are multiresolution analysis of \( L_p(\Omega, \mu) \) and \( L_q(\Omega, \mu) \), respectively.

**References**