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On the Structure of the Set of Solutions to Some Nonlinear Boundary-Value Problems

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1. INTRODUCTION

In recent years, much attention has been paid to the range set of nonlinear operators of the type $L + N$, where L is usually a linear differential operator and N is a nonlinear operator which is in some sense small or bounded compared with L. In many cases [23, 38, 42], the operator L has been uniformly elliptic with kernel and N has been a bounded Nemytsky operator on L^2.

In these papers, various necessary and sufficient conditions have been given for a function to be in the range of $L + N$. As in papers studying the multiplicity of solutions [3, 4, 27], a key factor has always been some condition involving the projection of the target function onto a finite-dimensional subspace.

In this paper, we will study the equation

$$Lu + Nu = h, \tag{1}$$

where L is a linear operator on L^2 and N is usually (but not necessarily) a uniformly Lipschitzian Nemytsky operator. Our object is to gain generic information on the set of functions u satisfying (1) for any given h. The main results assert the existence of a finite-dimensional subspace depending on L and N such that if the orthogonal projection of h does not belong to a set of measure zero in this space, then the solution set is actually a manifold of the same dimension as the Fredholm index (dim kernel L–dim(range $L)^{\perp}$).

A variation of these theorems may be obtained using the techniques of Smale, by verifying that the hypotheses of [39] are satisfied. In our situation, we avoid this approach since the methods of Section 2 (related to the alternative method)

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give more information on the structure of the manifold (see the first counterexample at the end of Section 3) and suggest the different results of Section 4, where we deduce the existence of a manifold for *all* values h, not merely almost all. Furthermore, our results apply to L^2, as opposed to the more restrictive Hölder spaces. We illustrate the generality of these methods in Section 3, in which we consider a variety of examples.

2. The Main Theorem

Let Ω be a connected open set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$; in fact, the results of this section apply more broadly to Ω, a smooth Riemannian manifold with or without boundary, or to systems of equations. We are concerned with the structure of the set of solutions u (a real-valued function on Ω) to the nonlinear equation

$$Lu + Nu = h, \tag{1}$$

where h is a given function in $L^2(\Omega)$, and L and N are, respectively, linear and nonlinear operators defined on L^2, or on dense subspaces of L^2.

The operator L is a partial differential operator defined on a dense subspace of smooth functions on Ω which satisfy certain linear homogeneous boundary conditions on $\partial\Omega$. We will seek solutions u satisfying these same boundary conditions. L is assumed to be a closed Fredholm operator. That is, L has a finite-dimensional kernel and a closed range of finite codimension; we define the index of L to be $i = \dim(\text{kernel}\, L) - \text{codim}(\text{range}\, L)$. Note that either of these dimensions may be zero or strictly positive. N is a Nemytsky operator of the form $Nu = f(x, u)$, where f satisfies Caratheodory conditions (measurable in x for all u, and continuous in u for almost every x) on $\Omega \times \mathbb{R}$. We also assume that the partial derivative $f_u(x, u)$ satisfies these conditions, and particularly that the essential supremum of $|f_u(x, u)|$ is bounded by a finite constant M_0. We require too that $f(x, 0)$ be in L^2; this implies that N is defined on all of L^2 and maps it (continuously and boundedly) into itself (see [41] for related details).

We make an additional assumption concerning the spectrum $\sigma(LL^*)$ of LL^*; being a closed positive self-adjoint operator [43], LL^* has a real nonnegative spectrum. We assume that for some $M_1 > M_0$, the part of $\sigma(LL^*)$ that lies within the interval $[0 \leqslant x \leqslant M_1^2]$ consists of isolated eigenvalues, each of finite multiplicity. This is certainly the case for the many situations in which L has compact resolvent; this would imply that all of $\sigma(LL^*)$ is isolated, as desired. For example, if Ω is a bounded set or a compact Riemannian manifold, and L is suitably elliptic (see [21, Chap. 10]), this would follow from elliptic regularity and Rellich's theorem [43, p. 281]. Ω need not be compact, however; on $\mathbb{R}^n$, if L is the time-independent Schrödinger operator $Lu = -\Delta u + V(x)u$, the potential V is assumed to be defined and continuous on $\mathbb{R}^n$, and $V(x)$ tends to ∞ as $|x|$ tends to ∞, then here too, L has an isolated spectrum, as needed. Our spectral assumption is a strict weakening of the requirement of compact resolvent; in the Schrödinger example, if we only assume that $V(x) \geqslant M_1$ for $|x|$ sufficiently large, then our spectral assumption holds [35] even though the entire spectrum is not necessarily isolated. Our assumption can even apply in the hyperbolic case, as we see in the next section.

The conclusion radically depends upon the index i. One might hope that the set of solutions to (1) would be a differentiable manifold of dimension i, and this generically turns out to be the situation. We first consider the important case $i = 0$, which applies in particular to self-adjoint L.

THEOREM 1 ($i = 0$). *Given the above hypotheses on L and N, there exists a certain finite-dimensional subspace S_1 of L^2 and its orthogonal complement S_2 with the following property. For any h_2 in S_2, there is a sparse (nongeneric) subset A of S_1 such that if h_1 is in S_1 but not in A, then for $h = h_1 + h_2$, the boundary-value problem* (1) *has a zero-dimensional manifold, i.e., a discrete set, as its solution set. The set A is sparse in the sense that it is of Lebesgue measure zero.*

Remark. Since the solution set is closed, it must therefore be isolated. This may be interpreted as a local uniqueness result. The solution set will appear embedded in a finite-dimensional subspace of L^2. If from other considerations, we know that the solutions obey an L^2 a priori bound, this will imply a similar bound in the subspace. The solution set is thus compact and hence finite.

Remark. The crucial space S_1 is constructively realized as the linear span of the first few eigenvectors of LL^* (ordered by size of eigenvalues). As a finite-dimensional linear space, S_1 inherits a well-defined notion of "measure zero" from Lebesgue measure on Euclidean space; hence the solution set is "almost surely" an i-manifold, in the sense of probability. Alternatively, the sparse set A may be taken to be of first Baire category, or even a (single) closed set with empty interior.

THEOREM 2 ($i < 0$). *For the situation analogous to that of Theorem* 1, *the generic conclusion is that* (1) *has no solution whatever.*

TERMINOLOGY. In order to unify the theory, we define the empty set to be the only i-manifold for $i < 0$. We will hereafter refer to the above generic conclusion of unsolvability by describing the solution set to be an i-manifold (for $i < 0$).

Remark. Of course, if $i \geqslant 0$, the solution set may also be empty. In order to rule out this trivial situation, one must assume some additional hypotheses (for existence of solutions) to the very general hypotheses used here. Such hypotheses are widespread in the literature for $i \geqslant 0$; Theorem 2 explains why this is *not* the case for $i < 0$.

We now turn to the most interesting case, $i > 0$. This situation has been extensively studied in conjunction with nontrivial stable homotopy (see [32, 34, 37]). In this setting, the question arises of coordinatizing the i-manifold of solutions; we return to this question in Section 4. The analysis is more delicate here; while C^1 differential topology suffices for $i \leqslant 0$, C^r seems to be necessary here ($r > i$). We must make some additional assumptions in order to achieve this extra degree of smoothness.

We will assume (for Theorem 3) that the linear operator L is uniformly elliptic with continuous coefficients and that the underlying region Ω is compact. As usual, we assume that the (linear homogeneous) boundary conditions are complementing [2], of order (m_j) less than the order of $L(2m_0)$, and smooth $(C^{2m_0-m_j})$, $j = 1, 2,..., m_0$. These hypotheses will enable us to apply elliptic regularity. We also assume additional smoothness on $f(x, u)$, specifically that all partial derivatives $\partial^j f/\partial u^j$ satisfy Caratheordory conditions and are uniformly bounded on $\Omega \times \mathbb{R}$, for $j = 1, 2,..., i+2$. While Theorem 3 applies to the case of $i = 0$, it should be noted that the required smoothness of f is one order greater than is needed for Theorem 1.

THEOREM 3 ($i > 0$). *For operators L (of index $i > 0$) and N as specified above, the generic conclusion is that the set of solutions to* (1) *form a differentiable orientable i-manifold.*

We first prove Theorems 1 and 2 simultaneously, and then describe the modifications necessary to prove Theorem 3.

Proof of Theorems 1 *and* 2

We first split (1) into two simpler (but coupled) equations according to the procedure of Lyapunov and Schmidt. We view these equations from the standpoint of functional analysis as developed by Cesari and others [7–10, 15, 16] (the alternative method). Specifically, a problem such as (1) may be split into a finite-dimensional problem coupled with a (contraction) fixed-point equation; this idea is due to Cesari [10].

We first define two projection operators P and Q on L^2 by orthogonal projection onto the kernel of L and the orthogonal complement of the range of L, respectively. Writing $u = v + c\theta$, where $Pu = c\theta$ (we use $c\theta$ to denote $\sum c_i\theta_i$, for θ_i an orthonormal basis for the kernel of L), (1) becomes

$$Lv = h - N(v + c\theta). \tag{2}$$

We give the domain of L a new inner product, the graph inner product, defined by $\langle\langle f, g\rangle\rangle \equiv \langle f, g\rangle + \langle Lf, Lg\rangle$, where $\langle\ ,\ \rangle$ denotes the usual L^2 inner product. With respect to this inner product, the domain of L is a complete Hilbert space, and L is a continuous linear operator from its domain into L^2. Neither injective nor surjective as yet, L becomes injective if we restrict its domain to the orthogonal complement of its kernel and surjective if we consider its range to be the kernel of Q. By the open mapping theorem, L has a continuous inverse H from the kernel of Q into the domain of L and hence into L^2.

Equation (2) is equivalent to the two coupled equations

$$v = H(I - Q)\{h - N(v + c\theta)\} \tag{3}$$

and

$$Q\{h - N(v + c\theta)\} = 0, \tag{4}$$

which are called the auxiliary and the bifurcation equations, respectively. We will further decompose (3) as follows.

Let $H = UR$ be the polar decomposition of H; U is a unitary operator, and R is a positive self-adjoint operator. (Information about polar decompositions may be found in [36; 17, pp. 68–69; 29]). R is the unique positive square root of H^*H. The spectrum of R is real and nonnegative, and the part of it contained in $[M_2 \leqslant x < \infty)$ consists solely of a finite number of isolated eigenvalues of finite multiplicity, where $M_2^{-1} = M_1$. This follows from the spectral hypothesis on LL^* via the spectral mapping theorem. Defining a new (orthogonal) projection operator P_1 onto the span of the eigenvectors of R corresponding to the eigenvalues in $[M_2, \infty)$ we may write $R = R_1 + R_2$, where $R_1 = P_1 R$, $R_2 = (I - P_1)R$. It should be noted that $I - P_1$ is an orthogonal projection onto the "other eigenspaces" of R; R is self-adjoint so its eigenspaces are perpendicular. Let $H = H_1 + H_2$, where $H_1 = UR_1$, $H_2 = UR_2$, and similarly write $v = v_1 + v_2 = UP_1U^{-1}v + U(I - P_1)U^{-1}v$, respectively; (3) decomposes into the equivalent system

$$v_1 = H_1(I - Q)\{h - N(v_1 + v_2 + c\theta)\}, \tag{5}$$

$$v_2 = H_2(I - Q)\{h - N(v_1 + v_2 + c\theta)\}. \tag{6}$$

Equations (4) and (5) are finite dimensional and can therefore be analyzed by transversality and "counting dimensions"; while (6) is infinite dimensional, it is a contraction mapping fixed-point equation. Equation (1) is equivalent to the simultaneous solution of (4), (5), and (6).

We first solve (6). For each h, v_1, $c\theta$, we regard (6) as a fixed-point equation for v_2 in the subspace closure (range H_2) of the Hilbert space L^2. The nonlinear operator $v_2 \to H_2(I - Q)\{h - N(v_1 + v_2 + c\theta)\}$ is a smooth C^1 uniform contraction, with Lipschitz constant $\leqslant M_0/M_1 < 1$. The contraction mapping fixed-point theory [16] implies that (for each h, v_1, $c\theta$) there is a unique solution $v_2 = v_2(h, v_1, c\theta)$ to (6), and that this solution is a C^1 function of h, v_1, $c\theta$. We may therefore restrict our attention to the simultaneous solution of (4) and (5), regarding (6) as solved, although the v_2 appearing in (4) and (5) must now be considered as defined uniquely and smoothly (if somewhat obliquely) by (6). In principle, v_2 can be obtained as the limit of iterations of the contraction map defined above.

Concerning the C^1 smoothness of the contraction mapping (and hence the smoothness of the solution v_2 of (6)), a word of explanation is necessary. The only nonlinear operator is the Nemytsky operator N, which has a (linear) Gateaux differential DN mapping L^2 into $L(L^2, L^2) = \{$bounded linear transformation from L^2 into $L^2\}$; see [41, Chaps. 3, 18–20] for the results mentioned in this paragraph. $DN(u)[v]$ is an element of L^2 which depends linearly on v, nonlinearly on u, and continuously on (u, v) in $L^2 \times L^2$. DN is not a Frechet differential (unless N is linear); N is not Frechet differentiable on L^2, and the closest thing one has to a second Gateaux derivative is not even everywhere defined on L^2, even for $f(x, u)$ of C^∞ smoothness. This is the primary complication in the $i > 0$ case, for which C^{i+1} smoothness is necessary. For the time being, all first derivatives are (linear) Gateaux derivatives, continuous in the above sense. We remark that $DN(u)$ is an element of $L(L^2, L^2)$ which does *not* depend continuously on u, if $L(L^2, L^2)$ is given the uniform operator (norm) topology. However, $DN(u)$ does depend continuously on u if $L(L^2, L^2)$ is given the strong operator topology.

We wish now to solve the finite-dimensional equations (4) and (5), having disposed of the remaining infinite-dimensional problem in (6). Let $\{\alpha_i\}$ be orthonormal eigenvectors of R spanning the range of P_1, $i = 1,..., m$; these are the first few nonzero eigenvectors of LL^*. Let $\{\psi_j\}$ be an orthonormal basis for the orthogonal complement of the range of L, $j = 1,..., i_2 =$ codim (range L). Recall that $\{\theta_j\}$ is an orthonormal basis for the kernel of L, $j = 1,..., i_1 =$ dim(kernel L). We will identify the four spaces range P_1, range UP_1, (range $L)^\perp$, and kernel L with $\mathbb{R}^m$, $\mathbb{R}^{m'}$, $\mathbb{R}^{i_2}$, $\mathbb{R}^{i_1}$, respectively. We use these spaces to build larger Euclidean spaces, as follows. Let $S_1 = \mathbb{R}^{i_2} \times \mathbb{R}^m$, let $X = \mathbb{R}^{i_1} \times \mathbb{R}^{m'}$, and let $Y = \mathbb{R}^{i_2} \times \mathbb{R}^{m'} \times \mathbb{R}^{i_2} \times \mathbb{R}^{m'}$.

Fix h_2 in the orthocomplement of S_1 as in the statement of the theorems. We define a map ρ: $S_1 \to C^1(X, Y)$ by

$$\begin{aligned}\rho(d\psi, e\alpha)(c\theta, v_1) = (0, v_1, Q\{h_2 + d\psi + e\alpha - N(v_1 + v_2 + c\theta)\}, \\ H_1(I - Q)\{h_2 + d\psi + e\alpha - N(v_1 + v_2 + c\theta)\}).\end{aligned} \tag{7}$$

In this expression, $d\psi$, $e\alpha$, $c\theta$ are used to denote finite sums $\sum d_i\psi_i$, etc; $(d\psi, e\alpha)$ is in S_1, $(c\theta, v_1)$ is in X. It should be noted that Eqs. (4) and (5) are solved (for $h = h_2 + d\psi + e\alpha = h_2 + h_1$, $v = v_1 + v_2$) if and only if $\rho(d\psi, e\alpha)(c\theta, v_1)$ is in the "diagonal" set W in Y, where $W = \{(a, b, a, b)$; a in $\mathbb{R}^{i_2}$, b in $\mathbb{R}^{m'}\}$.

It should be noted that the derivatives involved here are continuous (linear) Gateaux derivatives; it follows from the finite dimensionality of the domain spaces S_1 and X that these are actually (continuous) Frechet derivatives. If an operator F between two Banach spaces E_1 and E_2 has a linear Gateaux derivative

DF: $E_1 \to L(E_1, E_2)$ which is continuous, not only from $E_1 \times E_1 \to E_2$ as we have, but from $E_1 \to L(E_1, E_2)$ with respect to the uniform operator (norm) topology on $L(E_1, E_2)$, then DF is a continuous Frechet derivative [41, Theorem 3.3]. The two types of continuity are equivalent if E_1 is finite dimensional.

For any h_1 in S_1, $\rho(h_1)$ maps X into Y. We wish to conclude that for "most" h_1 in S_1, $[\rho(h_1)]^{-1}(W)$ is a C^1 submanifold of X of dimension i. If $\rho(h_1)$ is transverse to W, standard "submanifold" theorems (see [1, p. 45], [14, p. 28], or [20, p. 22]) imply this desired result. Also, we can conclude that $\rho(h_1)$ is transverse to W for "most" h_1 using standard "density" and "stability" theorems ([1, pp. 47–48; 14, pp. 68, 35; 20, pp. 74–75]); the hypotheses are easily verified except one, that ρ: $S_1 \times X \to Y$ is transverse to W. The distinction here is to be made between ρ defined on $S_1 \times X$ and $\rho(h_1)$ defined on X. The space S_1 was chosen in such a way to facilitate this.

Let $D_1\rho$ be the partial derivative of ρ with respect to the S_1 argument; this is a map from $S_1 \times X$ into $L(S_1, Y)$. In order to conclude that ρ is transverse to W, it suffices to show for any (s_1, x) in $S_1 \times X$ with $\rho(s_1, x)$ in W, that the vector space Y is spanned by the totality of the vectors in W and the vectors in the range of $D_1\rho(s_1, x)$. The vectors in W are of "diagonal" form (a, b, a, b); we claim that the range of $D_1\rho(s_1, x)$ contains all vectors of the form $(0, 0, a, b)$, which will finish the proof of the theorem.

We first rewrite (6) in the form

$$v_2 = H_2(I - Q)\{h_2 + d\psi + e\alpha - N(v_1 + v_2 + c\theta)\}, \tag{8}$$

where v_2 depends on h_2, $d\psi$, $e\alpha$, v_1, and $c\theta$. We compute the derivative of the fixed point v_2 with respect to the variables $d\psi$ and $e\alpha$, and it is zero. For example,

$$\frac{\partial v_2}{\partial d_i} = H_2(I - Q)\left\{\psi_i - \frac{\partial f}{\partial u}\frac{\partial v_2}{\partial d_i}\right\},$$

where the function $\partial f/\partial u$ is evaluated at $(x, v_1 + v_2 + c\theta)$. Therefore

$$\frac{\partial v_2}{\partial d_i} = \left[I + H_2(I - Q)\frac{\partial f}{\partial u}\right]^{-1}(H_2(I - Q)\,\psi_i) = 0$$

since ψ_i is in the kernel of $I - Q$. Similarly $H_2(I - Q)\alpha_i = 0$. This fact makes it much easier to determine $D_1\rho$, since we may now ignore any contributions to $D_1\rho$ from the dependence of v_2 on $d\psi + e\alpha$. We have

$$\begin{aligned} D_1\rho(s_1, x)(d\psi + e\alpha) &= (0, 0, Q\{d\psi + e\alpha\}, H_1(I - Q)\{d\psi + e\alpha\}) \\ &= (0, 0, d\psi, H_1(e\alpha)\} = (0, 0, d\psi, UR(e\alpha)). \end{aligned} \tag{9}$$

This establishes Theorems 1 and 2.

Proof of Theorem 3

This proof proceeds along the same lines as the previous ones. The additional requirement is that the map ρ must be $(i+1)$-differentiable in the sense of Frechet in order to apply the "submanifold" theorems of differential topology. This will follow from standard theorems of advanced calculus [31, p. 575] if we can establish that the finite-dimensional range map $\phi(d, e, v_1, c) = Q\{h_2 + d\psi + e\alpha - f(v_1 + v_2 + c\theta)\}$ admits continuous $(i+2)$nd partial derivatives with respect to its (finitely many) domain variables. We need the same result for the map given by the fourth component of ρ. We sketch the proof that ϕ admits continuous second-order partial derivatives; the extension to higher order is similar.

The main problem here is that v_2 depends on v_1 and c (as well as d and e) and we must verify that v_2 is partially differentiable in these variables. The function $v_2(v_1, c)$ is defined implicitly by (8); we differentiate (8) with respect to c_i and obtain

$$\frac{\partial v_2}{\partial c_i} = -H_2(I-Q)\left\{\frac{\partial f}{\partial u}\left(\frac{\partial v_2}{\partial c_i} + \theta_i\right)\right\}, \tag{10}$$

where the function $\partial f/\partial u$ is evaluated at the argument $(x, v_1 + v_2 + c\theta)$ and $\partial v_2/\partial c_i$ is evaluated at (v_1, c). By a standard bootstrap argument using the elliptic regularity of L and the boundedness of $\partial f/\partial u$, we conclude from (10) that $\partial v_2/\partial c_i$ is uniformly bounded (L^∞). This argument also requires that the functions θ_i are bounded, which follows from elliptic regularity too. We establish that the first partials of v_2 are continuous in c by solving (10):

$$\frac{\partial v_2}{\partial c_i} = \left[I + H_2(I-Q)\frac{\partial f}{\partial u}\right]^{-1}\left(-H_2(I-Q)\frac{\partial f}{\partial u}\theta_i\right). \tag{11}$$

The derivatives of v_2 with respect to v_1 are handled similarly.

We now consider the second partial derivatives $\partial^2 v_2/\partial c_i\,\partial c_j$. We emphasize that these are directional (Gateaux) derivatives at this stage. A routine calculation yields, from (10),

$$\frac{\partial^2 v_2}{\partial c_i\,\partial c_j} = -H_2(I-Q)\left\{\frac{\partial^2 f}{\partial u^2}\left(\frac{\partial v_2}{\partial c_i} + \theta_i\right)\left(\frac{\partial v_2}{\partial c_j} + \theta_j\right) + \frac{\partial f}{\partial u}\left(\frac{\partial^2 v_2}{\partial c_i\,\partial c_j}\right)\right\}. \tag{12}$$

As in (10), bootstrap arguments imply that these second partial derivatives are bounded (L^∞), and we solve explicitly for the derivative:

$$\frac{\partial^2 v_2}{\partial c_i\,\partial c_j} = \left[I + H_2(I-Q)\frac{\partial f}{\partial u}\right]^{-1}\left(-H_2(I-Q)\left\{\frac{\partial^2 f}{\partial u^2}\left(\frac{\partial v_2}{\partial c_i} + \theta_i\right)\left(\frac{\partial v_2}{\partial c_j} + \theta_j\right)\right\}\right). \tag{13}$$

This demonstrates the continuous dependence of the second partials on the (finite number of) variables c, and similarly, v_1. This method extends to derivatives of order $\leqslant i + 2$.

We return to the problem of showing that $\phi(d, e, v_1, c)$ has continuous $(i + 2)$nd partial (Gateaux) derivatives. This follows from the chain rule and the results of the last paragraphs. Since ϕ has finite-dimensional domain and range, ϕ (and therefore ρ) has an $(i + 1)$st Fréchet derivative.

All of the manifolds S_1, X, Y, W, involved in the definition of ρ are orientable, since they are Euclidean spaces. By the argument of [14, pp. 100–101], the manifold of solutions is orientable. This concludes the proof of Theorem 3.

3. Examples

First we consider the problem studied in [23]. Let L be a strongly elliptic self-adjoint differential operator on a bounded region Ω with smooth (C^∞) boundary and coercive boundary conditions. We assume that the function $f(u)$ is smooth (C^1), has finite asymptotic limits $f(-\infty), f(\infty)$, and in addition satisfies $f(-\infty) < f(s) < f(+\infty)$, for all s, $-\infty < s < +\infty$. Then a necessary and sufficient condition that the equation

$$Lu + f(u) = h(x) \tag{1}$$

in $L^2(\Omega)$ has solutions is that

$$f(+\infty) \int_{\theta>0} \theta + f(-\infty) \int_{\theta<0} \theta > \int h\theta > f(-\infty) \int_{\theta>0} \theta + f(+\infty) \int_{\theta<0} \theta \tag{14}$$

for all θ in the kernel of L, $\| \theta \| = 1$.

The proof of [23] implies the existence of an a priori bound on the solutions u. Since the index of L is zero, our theorem shows that there exists a finite-dimensional subspace S_1 of L^2 spanned by a finite number of eigenvectors of L such that for any given $h_2 \perp S_1$ the equation $Lu + f(u) = h_1 + h_2$ admits only a finite number of solutions for almost all $h_1 \in S_1$.

The function f need not necessarily be uniformly Lipschitzian, if it is eventually monotone in the right way. Consider the Dirichlet problem $-\Delta u + u^3 - u = h(x)$ on a bounded region Ω, where h is assumed to be (pointwise) bounded and we require $u = 0$ on $\partial\Omega$. While the function $f(u) = u^3 - u$ is not uniformly Lipschitzian, one can use the weak maximum principle to conclude that any solution to this problem obeys an a priori bound $\| u \|_{L^\infty} \leqslant M$. We may therefore replace $f(u) = u^3 - u$ by a smooth function $g(u)$ identical to $f(u)$ for $| u | \leqslant M$, but which tends nicely to asymptotic limits $g(-\infty), g(\infty)$. The existence of solutions to the modified Dirichlet problem for $-\Delta u + g(u) = h(x)$ follows easily from the Schauder fixed-point theorem; these solutions are precisely the same as the solutions to the original problem, and our analysis applies to the multiplicity of solutions.

The Nemytsky operator N may depend on derivatives of u as well, of order less than the order of L. For example, consider the Dirichlet problem for $-\Delta u + f(x, u, \nabla u) = h(x)$ on a bounded region Ω, $u = 0$ on $\partial\Omega$. If we assume that f and its first partial derivatives f_u, $f_{\nabla u}$ satisfy the Caratheodory conditions on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, that $|f_u|$ and $|f_{\nabla u}|$ are uniformly bounded, and that $f(x, 0, 0)$ is in L^2, a "Theorem 1" could then be proved. We could even allow f to depend on second derivatives of u, provided the dependence was sufficiently small.

Another situation which received some attention [11, 15, 24] is the case for which f does not (at least in the limit) pass through the eigenvalues of L. For example, in [22] the Dirichlet problem $\Delta u + f(u) = h(x)$ in a bounded region Ω was studied under the assumptions that f was C^∞ and that $\lambda_n < f'(-\infty) < f'(+\infty) < \lambda_{n+1}$; λ_n and λ_{n+1} are two consecutive eigenvalues of $-\Delta$ (the eigenvalues are arranged in increasing order). Again, our theorem demonstrates the existence of a finite-dimensional subspace S_1 spanned by eigenvalues of $-\Delta$ such that if $h = h_1 + h_2$, $h_2 \perp S_1$, then for a given h_2, the Dirichlet problem has at most finitely many solutions (for almost all h_1 in S_1).

Theorem 1 also applies to hyperbolic equations which have been studied by these methods. For example, the equation (for which $0 \leqslant x \leqslant \pi$, $0 \leqslant t$, h periodic in t of period π, c a positive constant)

$$u_{tt} - u_{xx} - cu + f(x, t, u) = h(x, t),$$

with boundary conditions (15)

$$u(x, t) = u(x, t + \pi), \qquad u(0, t) = u(\pi, t) = 0$$

has been studied in [28] under the hypothesis $f_u < c$. (We are really working on the compact manifold $[0, \pi] \times S^1$ with boundary $(\{0\} \times S^1) \cup (\{\pi\} \times S^1)$.) If f is sufficiently smooth to satisfy the hypotheses of Theorem 1, and also $\sup |f_u(x, t, u)| < c$, then we may conclude that there exists a subspace S_1 of $L^2([0, \pi] \times S^1)$ of finite dimension, spanned by the eigenvectors of $u_{tt} - u_{xx}$, $\phi_{ij}^1 = \sin ix \cos jt$, $\phi_{ij}^2 = \sin ix \sin jt$, where i and j are integers, $i > 0$, $j \geqslant 0$, and $|i^2 - j^2| < c$. For any $h_2 \perp S_1$, and $h = h_1 + h_2$, Eq. (15) has finitely many solutions for almost all $h_1 \in S_1$.

There are many other problems of index zero which may be treated by this method including parabolic linear operators [26].

We now pass on to some genuinely non-self-adjoint problems of nonzero index. We consider the equation

$$\begin{aligned} \Delta^2 u + f(u) &= h(x, y) \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial x} = \frac{\partial}{\partial n}\Delta u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{16}$$

where Ω is a smooth bounded region in $\mathbb{R}^2$, withoutl oss of generality of unit measure, and n indicates the outward unit normal on $\partial\Omega$. This operator has kernel spanned by $\{1, y, y^2\}$ and cokernel spanned by $\{1\}$. We assume that f is bounded and for simplicity depends only on u, that $f(-\infty) < f(s) < f(+\infty)$, and that f is C^4. We observe first that by projection onto the vector 1, we have that

$$f(-\infty) < \int h < f(+\infty) \tag{17}$$

is necessary for the existence of solutions. Theorem 3 guarantees that if the equation admits a solution then the solution set is generically a manifold of dimension 2. Here we examine in more detail the structure of this manifold.

THEOREM 4. *Condition* (17) *is sufficient for the existence of solutions to* (16). *Furthermore, given any* c_2, c_3 *in* $\mathbb{R}$, *there exists at least one function* $u_{c_2c_3} + c_2y + c_3y^2$, *where* $u_{c_2c_3}$ *is orthogonal to the space spanned by* $\{y, y^2\}$.

Proof. Let P be a projection onto the space spanned by $\{1, y, y^2\}$ and Q be a projection onto the space spanned by $\{1\}$. As usual in the alternative method since L: $(I - P)L^2 \to (I - Q)L^2$ is invertible, we have a partial inverse H: $(I - Q)L^2 \to (I - P)L^2$, and Eq. (16) is equivalent to the pair of equations

$$v = H(I - Q)\{h - N(v + c\theta)\} \tag{3}$$

and

$$Q\{h - N(v + c\theta)\} = 0. \tag{4}$$

Since both sides of (3) belong to $(I - Q)L^2$ we may combine (3) and (4) into the equation

$$T(u) \equiv v - H(I - Q)\{h - N(v + c\theta)\} - Q\{h - N(v + c\theta)\} = 0, \tag{18}$$

where $v = (I - P)u$ and $u = c_1 + c_2y + c_3y^2 + v$. We fix c_2 and c_3 and write $w = c_1 + v$, so that $u = w + c_2y + c_3y^2$. Writing $T_{c_2c_3}(w) = Tu$, we may consider (18) as an equation $T_{c_2c_3}(w) = 0$ in w. Both w and $T_{c_2c_3}(w)$ are orthogonal to the span of y, y^2.

Now observe that since N is bounded, we have for some constant M that $\| H(I - Q)\{h - N(w + c_2y + c_3y^2)\| \leq M$, where $\| \|$ denotes L^2 norm. Also, if $\| v \| = R_2$, we have that $\langle T_{c_2c_3}(w), v\rangle \geq R_2{}^2 - MR_2 > 0$ for R_2 sufficiently large; $\langle\ ,\ \rangle$ denotes L^2 inner product. By estimates similar to those of [23, 42], it follows from (17) that the nonlinearity $h - N$ satisfies the following inequality: Given $R_2 > 0$, there exists $R_1 > 0$ such that if $\| v + c_2y + c_3y^2 \| \leq R_2$ and $| c_1 | > R_1$, then $\langle h - N(v + c_1 + c_2y + c_3y^2), c_1\rangle < 0$. From this we may conclude that the map $T_{c_2c_3}$ has Leray–Schauder degree 1 on the region $C_{R_1R_2} = \{v + c_1 + c_2y + c_3y^2$, where $\| v + c_2y + c_3y^2 \| \leq R_2$, $| c_1 | \leq R_1\}$ in $(\text{span}(y, y^2))^\perp$. This proves that for fixed c_2, c_3 Eq. (18) has at least one solution in $(\mathrm{span}(y, y^2))^\perp$. Thus the original equation (16) possesses a two parameter family of solutions $u_{c_2c_3} = c_1 + c_2 y + c_3 y^2 + v$.

Using the techniques of [6, 38], it can be shown that the set of solutions to (16) meets every sufficiently large sphere in L^2 (in fact, in a continuum of points). This precludes the use of stable homotopy theory [32] in this type of problem.

More generally this type of theorem is true for any L for which $(\mathrm{range}\, L)^\perp \subseteq (\mathrm{kernel}\, L)$. The necessary and sufficient condition would then be that

$$f(+\infty)\int_{\theta>0}\theta + f(-\infty)\int_{\theta<0}\theta > \int h\theta > f(-\infty)\int_{\theta>0}\theta + f(+\infty)\int_{\theta<0}\theta \quad (14)$$

be satisfied for all θ in $(\mathrm{range}\, L)^\perp$, $\|\theta\| = 1$. One can prove this type of theorem for situations where $(\mathrm{range}\, L)^\perp \nsubseteq (\mathrm{kernel}\, L)$ but instead some basis functions of kernel L possess regions of positivity and negativity in common with the basis functions of $(\mathrm{range}\, L)^\perp$ (see [38] for details).

An analysis similar to the preceding shows that if we consider the equation $Lu + Nu = h$, where L is a partial differential operator on Ω satisfying $\mathrm{range}\, L = L^2(\Omega)$, then there exists a finite-dimensional subspace S_1 with an associated projection operator P_1 such that if $P_1 h$ does not belong to a set of measure 0 in S_1, then the solution set is an i-manifold of the form $\{u_1 + u_2$ such that u_1 ranges over all of kernel L, and $u_2 \perp \mathrm{kernel}\, L\}$. An example of this is given by the oblique derivative problem ($i = 2$)

$$\Delta u + \arctan u = h \quad (19)$$

on a bounded smooth region Ω in $\mathbb{R}^2$, with boundary condition $\partial u/\partial x = 0$ on $\partial\Omega$. See [21, pp. 265–267] for details on this and also for examples of negative index.

These methods apply in similar problems of structure, as in the following. Consider the one-parameter family of Neumann problems $\Delta u + \lambda u + \arctan u = h$ in a bounded region Ω, where we require $\partial u/\partial n$ to vanish on $\partial\Omega$, and $\sup |h| < \pi/2$. For each fixed real λ, the Landesman–Lazer theorem [23] guarantees the existence of solutions $u_\lambda(x)$. One can prove by our methods that for generic h, the solution set $\{(\lambda, u_\lambda(x)\}$ forms a smooth 1-manifold in $\mathbb{R} \times L^2$.

We conclude with two counterexamples showing how the "generic" qualification is necessary. Consider the Dirichlet problem

$$\begin{aligned} -x'' + f(x) &= 0, \\ x(0) = x(\pi) &= 0, \end{aligned} \quad (20)$$

where f is smooth, satisfies $f(s) = -s$ for $-1 \leqslant s \leqslant 1$, $|f'| \leqslant 2$, and the limits $f'(\pm\infty)$ exist and are both positive. In this case, it is clear that instead of a discrete set, Eq. (20) has a solution set containing $\{\alpha \sin t,\ |\alpha| \leqslant 1\}$. However, since the relevant space S_1 occurring in the proof of the theorem will in this case be the one-dimensional subspace spanned by $\sin t$, we may conclude that the problem for $-x'' + f(x) = \beta \sin t$ admits only a finite number of solutions for almost all β. This is a much sharper conclusion than can be obtained by the Baire-category techniques of [39].

Our second example shows that the solution set need not be a manifold at all. Consider

$$\begin{aligned} -x'' + f(x) &= 0, \\ x'(0) = x'(\pi) &= 0, \end{aligned} \tag{21}$$

where f is smooth, bounded, $\sup |f'| < 1$, and $\{f^{-1}(0)\}$ is not a zero-manifold but a collection of points with an accumulation point. It is easy to verify (see [27]) that the only solutions of (21) are constant solutions, and thus the solution set of (21) is as undesirable as $\{f^{-1}(0)\}$.

4. Small Parameter Arguments ($i \geqslant 0$)

We consider in this section examples for which the Nemytsky operator defined by f is suitably small. These hypotheses may be achieved, for example, by assuming that f involves a small parameter, and that $f(x, u) = \epsilon g(x, u)$, where ϵ may be chosen small.

Consider first the case for which L may have nontrivial kernel and for which the range of L is all of $L^2(\Omega)$; for example, the operator L appearing in Eq. (19). We consider the equation

$$Lu + Nu = h(x) \tag{22}$$

on the bounded region Ω, where $Nu = f(x, u)$ is a Nemytsky operator for which both f and f_u satisfy Caratheodory conditions on $\Omega \times \mathbb{R}$. We demand also that $\sup |f_u| < \alpha_1^{1/2}$, where α_1 is the first nonzero eigenvalue of LL^*, and that $f(x, 0)$ is in L^2. The operator L admits a (right) partial inverse H from L^2 onto $(I - P)L^2$ with $\| H \| = 1/\alpha_1^{1/2}$. Using the notation of Section 2, the bifurcation equation (4) is trivially satisfied because $Q = 0$. We need only solve the auxiliary equation

$$v = H\{h - N(v + c\theta)\}. \tag{3}$$

It is clear that for each $c\theta$ in the kernel of L, the map $T_c: v \to H\{h - N(v + c\theta)\}$ is a uniform contraction, and therefore there exists a unique v depending continuously on c. Therefore, the solution set to (22) is of the form $\{c\theta + v(c)\}$, where $c\theta$ ranges over the entire kernel of L. This is a (topological) manifold equal in dimension to L's kernel, and this proves that the manifold admits a Cartesian representation. That is, the manifold is the graph of a continuous function from the kernel of L into its orthogonal complement in L^2. With additional smoothness and regularity hypotheses as for Theorem 3, it can be shown that this is a smooth manifold. Also, the above remarks apply even if the kernel of L is infinite dimensional, as long as zero is an isolated point in the spectrum of LL^*.

We finally return to the problem of Section 3:

$$\begin{aligned} \Delta^2 u + f(u) &= h(x, y) \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial x} = \frac{\partial \Delta u}{\partial n} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{23}$$

where Ω is a smooth bounded region in $\mathbb{R}^2$ of unit measure, h is in L^2, and f is C^1. As before, P and Q are orthogonal projections onto the kernel of $L = \operatorname{span}\{1, y, y^2\}$ and the cokernel of $L = \operatorname{span}\{1\}$, respectively. Equation (23) is equivalent to the simultaneous solution of

$$v = H(I - Q)\{h - N(v + c\theta)\} \tag{3}$$

and

$$Q\{h - N(v + c\theta)\} = 0, \tag{4}$$

where H is the partial inverse of L, and we have an inequality $\| Hw \|_{L^\infty} \leqslant K \| w \|$. Here w is in $(I - Q)L^2$, Hw is in the Sobolev space H^4, which injects into L^∞ by the usual Sobolev estimates, and K is a positive constant independent of w.

As before, we fix c_2, c_3 in $\mathbb{R}$, and consider, for each c_1 in $\mathbb{R}$, the map T_{c_1} mapping $(I - P)L^2$ into itself by $T_{c_1}(v) = H(I - Q)\{h - N(v + c\theta)\}$. We assume that $\sup | f' | \leqslant M$; for M sufficiently small, the map T_{c_1} is a contraction and therefore has a unique fixed point $v = v(c_1)$ depending continuously on c_1 (as well as on c_2, c_3). Furthermore, a calculation yields

$$\| v(c_1) - v(c_1')\| \leqslant \frac{\| H \| M}{1 - \| H \| M} | c_1 - c_1' |. \tag{24}$$

Here $\| H \|$ denotes the operator norm of H on L^2. Because of Eq. (3), we get

$$\| v(c_1) - v(c_1')\|_{L^\infty} \leqslant \frac{KM}{1 - \| H \| M} | c_1 - c_1' |. \tag{25}$$

This "Lipschitz" constant $KM/(1 - \| H \| M)$ can be made arbitrarily small by assuming M to be sufficiently small.

We now study Eq. (4), which takes the form

$$I_{c_2, c_3}(c_1) \equiv \int h - f(c_1 + c_2 y + c_3 y^2 + v(c_1)) = 0. \tag{26}$$

We claim that I_{c_2,c_3} is a strictly decreasing function of c_1. Fixing $c_2, c_3, c_1 > c_1'$, $I_{c_2,c_3}(c_1') - I_{c_2,c_3}(c_1) = \int f(c_1 + c_2 y + c_3 y^2 + v(c_1)) - f(c_1' + c_2 y + c_3 y^2 + v(c_1'))$. But $c_1 + v(c_1) - c_1' - v(c_1') \geqslant c_1 - c_1' - \| v(c_1) - v(c_1')\|_{L^\infty} \geqslant (c_1 - c_1')(1 - (KM/1 - \| H \| M) > 0$ for M sufficiently small. We conclude from this that for fixed c_2, c_3, Eq. (26) has at most one solution, which, if it exists, depends continuously on c_2 and c_3.

Some additional hypotheses are necessary to ensure existence of solutions. If we assume also that $\inf f' > 0$, then $I_{c_2,c_3}(\pm\infty) = \mp\infty$, and so (23) has solutions for all h. Alternatively, we could assume only that $f' > 0$, and that Landesman–Lazer-type inequalities are satisfied by f and h; this would also imply existence for all such h. For example, we could assume that f is bounded, $f(-\infty) < f(s) < f(\infty)$, and such an inequality would take the form of Eq. (17). In either case, for each c_2, c_3, there is a function $c_1(c_2, c_3) + v(c_1(c_2, c_3))$ in $\{1\} \times (I - P)L^2$ depending continuously on c_2, c_3 so that $c_2 y + c_3 y^2 + c_1(c_2, c_3) + v(c_1(c_2, c_3))$ is a solution of (23). These are all of the solutions to (23); as in the previous example, for *all* h, we have a 2-manifold of solutions and this manifold has a Cartesian representation.

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