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Impulsive stabilization of delay differential systems via the Lyapunov–Razumikhin method☆

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Abstract

This work studies global exponential stability of impulsive delay differential systems. By employing the Razumikhin technique and Lyapunov functions, several global exponential stability criteria are established for general impulsive delay differential equations. Our results show that delay differential equations may be exponentially stabilized by impulses. An example and its simulation are also given to illustrate our results.

Keywords: Razumikhin technique; Lyapunov function; Impulsive delay differential equations; Global exponential stability

1. Introduction

Impulsive delay differential equations arise in many applied fields such as control technology, communication networks, and biological population management and hence they have attracted considerable attention. See [1–4,6, 9–13] and the references therein. In recent years, stability of differential equations has been extensively studied. One of the most investigated problems in the stability analysis of such systems is exponential stability since it has played an important role in many areas such as designs and applications of neural networks and synchronization in secure communication [5,7,8].

On the other hand, various methods, such as LMI tools, Laplace transform, and Lyapunov functional or function methods (combined with the Razumikhin technique) and so on, have been successfully utilized in the investigation of exponential stability; see [1,4–6,12] for example. And the well-known Razumikhin technique has been successfully applied in the study of asymptotic and exponential stability of impulsive delay differential equations; see [9–13] and relevant references cited therein. However, to the best of our knowledge, there have been few results obtained for impulsive exponential stabilization of delay differential equations [8,13]. The aim of this work is to establish global exponential stability criteria for impulsive delay systems by employing the Razumikhin technique which illustrate that impulses do contribute to the stabilization of some delay differential systems.

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2. Preliminaries

Given a constant $\tau > 0$, we equip the linear space $PC([-\tau, 0], \mathbb{R}^n)$ with the norm $\|\cdot\|_\tau$ defined by $\|\psi\|_\tau = \sup_{-\tau \le s \le 0} \|\psi(s)\|$.

Consider the following impulsive system:

$$\begin{cases} x'(t) = F(t, x_t), & t \ne t_k, \\ \Delta x(t_k) = I_k(t_k, x_{t_k^-}), & k \in N, \\ x_{t_0} = \phi, \end{cases} \tag{2.1}$$

where $F, I_k : \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$; $\phi \in PC([-\tau, 0], \mathbb{R}^n)$; $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, with $t_k \to \infty$ as $k \to \infty$; $\Delta x(t) = x(t) - x(t^-)$; and $x_t, x_{t^-} \in PC([-\tau, 0], \mathbb{R}^n)$ are defined by $x_t(s) = x(t+s)$, $x_{t^-}(s) = x(t^- + s)$ for $-\tau \le s \le 0$, respectively.

In this work, we assume that functions $F, I_k, k \in N$, satisfy all necessary conditions for the global existence and uniqueness of solutions for all $t \ge t_0$ [2]. Denote by $x(t) = x(t, t_0, \phi)$ the solution of (2.1) such that $x_{t_0} = \phi$. We further assume that all the solutions $x(t)$ of (2.1) are continuous except at $t_k, k \in N$, at which $x(t)$ is right continuous, i.e., $x(t_k^+) = x(t_k), k \in N$.

Definition 2.1. Function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is said to belong to the class ν_0 if

(i) V is continuous in each of the sets $[t_{k-1}, t_k) \times \mathbb{R}^n$ and for each $x \in \mathbb{R}^n, t \in [t_{k-1}, t_k), k \in N$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists; and

(ii) $V(t, x)$ is locally Lipschitzian in all $x \in \mathbb{R}^n$, and for all $t \ge t_0$, $V(t, 0) \equiv 0$.

Definition 2.2. Given a function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, the upper right-hand derivative of V with respect to system (2.1) is defined by

$$D^+V(t, \psi(0)) = \limsup_{h\to 0^+} \frac{1}{h}[V(t + h, \psi(0) + hF(t, \psi)) - V(t, \psi(0))],$$

for $(t, \psi) \in \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n)$.

Definition 2.3. The trivial solution of system (2.1) is said to be globally exponentially stable if there exist some constants $\alpha > 0$ and $M \ge 1$ such that for any initial data $x_{t_0} = \phi$

$$\|x(t, t_0, \phi)\| \le M\|\phi\|_\tau e^{-\alpha(t-t_0)}, \quad t \ge t_0,$$

where $(t_0, \phi) \in \mathbb{R}_+ \times PC([-\tau, 0], \mathbb{R}^n)$.

3. The Lyapunov–Razumikhin method

In this section, we shall present some Razumikhin-type theorems on global exponential stability for system (2.1) based on the Lyapunov–Razumikhin method. Our results show that impulses play an important role in stabilizing delay differential systems.

Theorem 3.1. *Assume that there exist a function $V \in \nu_0$ and constants $p, c, c_1, c_2 > 0$ and $\alpha > \tau$, $\lambda > c$ such that*

(i) $c_1\|x\|^p \le V(t, x) \le c_2\|x\|^p$, *for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$;*

(ii) $D^+V(t, \varphi(0)) \le cV(t, \varphi(0))$, *for all $t \in [t_{k-1}, t_k), k \in N$, whenever $qV(t, \varphi(0)) \ge V(t + s, \varphi(s))$ for $s \in [-\tau, 0]$, where $q \ge e^{2\lambda\alpha}$ is a constant;*

(iii) $V(t_k, \varphi(0) + I_k(t_k, \varphi)) \le d_k V(t_k^-, \varphi(0))$, *where $d_k > 0$, $\forall k \in N$, are constants;*

(iv) $\tau \le t_k - t_{k-1} \le \alpha$ *and* $\ln(d_k) + \lambda\alpha < -\lambda(t_{k+1} - t_k)$.

Then the trivial solution of the impulsive system (2.1) is globally exponentially stable and the convergence rate is $\frac{\lambda}{p}$.

Proof. Choose $M \geq 1$ such that

$$c_2\|\phi\|_\tau^p < M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}e^{-\alpha c} < M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \leq qc_2\|\phi\|_\tau^p. \tag{3.1}$$

Let $x(t) = x(t, t_0, \phi)$ be any solution of system (2.1) with $x_{t_0} = \phi$, and $v(t) = V(t, x)$. We shall show

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_k-t_0)}, \quad t \in [t_{k-1}, t_k), k \in N. \tag{3.2}$$

We first show that

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}, \quad t \in [t_0, t_1). \tag{3.3}$$

From condition (i) and (3.1), we have, for $t \in [t_0 - \tau, t_0]$

$$v(t) \leq c_2\|x\|^p \leq c_2\|\phi\|_\tau^p < M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}e^{-\alpha c}.$$

If (3.3) is not true, then there must exist some $\bar{t} \in (t_0, t_1)$ such that

$$\begin{aligned} v(\bar{t}) &> M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} > M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}e^{-\alpha c} \\ &> c_2\|\phi\|_\tau^p \geq v(t_0 + s), \quad s \in [-\tau, 0], \end{aligned} \tag{3.4}$$

which implies that there exists some $t^* \in (t_0, \bar{t})$ such that

$$v(t^*) = M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}, \quad \text{and} \quad v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}, \quad t_0 - \tau \leq t \leq t^*; \tag{3.5}$$

and there exists $t^{**} \in [t_0, t^*)$ such that

$$v(t^{**}) = c_2\|\phi\|_\tau^p, \quad \text{and} \quad v(t) \geq c_2\|\phi\|_\tau^p, \quad t^{**} \leq t \leq t^*. \tag{3.6}$$

Then we obtain, for any $t \in [t^{**}, t^*]$

$$v(t + s) \leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \leq qc_2\|\phi\|_\tau^p \leq qv(t), \quad s \in [-\tau, 0], \tag{3.7}$$

and thus by condition (ii), we get $D^+v(t) \leq cv(t)$ for $t \in [t^{**}, t^*]$, and then we have $v(t^{**}) \geq v(t^*)e^{-\alpha c}$, i.e., $c_2\|\phi\|_\tau^p \geq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}e^{-\alpha c}$, which contradicts (3.1). Hence (3.3) holds and then (3.2) is true for $k = 1$.

Now we assume that (3.2) holds for $k = 1, 2, \ldots, m$ ($m \in N,\ m \geq 1$), i.e.

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_k-t_0)}, \quad t \in [t_{k-1}, t_k), k = 1, 2, \ldots, m. \tag{3.8}$$

From condition (iii) and (3.8), we have

$$\begin{aligned} v(t_m) &\leq d_m v(t_m^-) \\ &< e^{-\lambda\alpha}e^{-\lambda(t_{m+1}-t_m)}M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}. \end{aligned} \tag{3.9}$$

Next, we shall show that (3.2) holds for $k = m + 1$, i.e.

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}, \quad t \in [t_m, t_{m+1}). \tag{3.10}$$

For the sake of contradiction, suppose (3.10) is not true. Then we define

$$\bar{t} = \inf\{t \in [t_m, t_{m+1}) | v(t) > M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}\}.$$

From (3.9), we know $\bar{t} \neq t_m$. By the continuity of $v(t)$ in the interval $[t_m, t_{m+1})$, we have

$$v(\bar{t}) = M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \quad \text{and} \quad v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)}, \quad t \in [t_m, \bar{t}]. \tag{3.11}$$

From (3.9), we have

$$v(t_m) < e^{-\lambda\alpha}M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} < v(\bar{t}),$$

which implies that there exists some $t^* \in (t_m, \bar{t})$ such that

$$v(t^*) = e^{-\lambda\alpha}M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \quad \text{and} \quad v(t^*) \leq v(t) \leq v(\bar{t}), \quad t \in [t^*, \bar{t}]. \tag{3.12}$$

Then we know $t+s \in [t_{m-1}, \bar{t}]$ for $t \in [t^*, \bar{t}]$ and $s \in [-\tau, 0]$ since $\tau \le t_k - t_{k-1} \le \alpha$. By (3.8) and (3.11), we get, for $t \in [t^*, \bar{t}]$,

$$\begin{aligned} v(t+s) &\le M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)} \\ &= M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} e^{\lambda(t_{m+1}-t_m)} \\ &\le e^{\lambda\alpha} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} \\ &= e^{2\lambda\alpha} v(t^*) \le q v(t), \quad s \in [-\tau, 0]. \end{aligned}$$

Then from condition (ii), we get $D^+v(t) \le cv(t)$; since $\lambda > c$, we have, from (3.12)

$$v(\bar{t}) \le v(t^*)e^{\alpha c} = e^{-\lambda\alpha} M\|\phi\|_\tau^p e^{-\lambda(t_{m+1}-t_0)} e^{\alpha c} < v(\bar{t}),$$

which is a contradiction. This implies the assumption is not true, and hence (3.2) holds for $k = m+1$. Thus by mathematical induction, we obtain that (3.2) holds, and hence we have

$$v(t) \le M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k).$$

Then by condition (i), we get

$$\|x\| \le M^*\|\phi\|_\tau e^{-\frac{\lambda}{p}(t-t_0)}, \quad t \in [t_{k-1}, t_k), k \in N,$$

where $M^* \ge \max\{1, [\frac{M}{c_1}]^{\frac{1}{p}}\}$; this implies that the trivial solution of system (2.1) is globally exponentially stable with convergence rate $\frac{\lambda}{p}$. □

Remark 3.1. If the condition $\lambda > c$ is removed in Theorem 3.1, then we have to require $q \ge \max\{e^{\alpha c}, e^{2\lambda\alpha}\}$ in condition (ii) and condition (iv) to be strengthened. The details are stated in the following result whose proof is similar and thus omitted.

Theorem 3.2. *Assume that there exist a function $V \in \nu_0$ and constants $p, c, c_1, c_2, \lambda > 0$ and $\alpha > \tau$ such that*

(i) $c_1\|x\|^p \le V(t,x) \le c_2\|x\|^p$, *for any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$;*
(ii) $D^+V(t,\varphi(0)) \le cV(t,\varphi(0))$, *for all $t \in [t_{k-1}, t_k), k \in N$, whenever $qV(t,\varphi(0)) \ge V(t+s,\varphi(s))$ for $s \in [-\tau, 0]$, where $q \ge \max\{e^{\alpha c}, e^{2\lambda\alpha}\}$ is a constant;*
(iii) $V(t_k, \varphi(0) + I_k(t_k,\varphi)) \le d_k V(t_k^-, \varphi(0))$, *where $d_k > 0$, $\forall k \in N$, are constants;*
(iv) $\tau \le t_k - t_{k-1} \le \alpha$ *and* $\ln(d_k) + (\lambda + c)\alpha < -\lambda(t_{k+1} - t_k)$.

Then the trivial solution of the impulsive system (2.1) is globally exponentially stable and the convergence rate is $\frac{\lambda}{p}$.

Remark 3.2. It is well known that, in the stability theory of delay differential equations, the condition $D^+V(t,x) \le cV(t,x)$ allows the derivative of the Lyapunov function to be positive which may not even guarantee the stability of a delay differential system (see [9,13] and Example 4.1). However, as we can see from Theorems 3.1 and 3.2, impulses have played an important role in exponentially stabilizing a delay differential system.

Next, we shall apply the previous theorems to the following linear impulsive delay system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau(t)), & t \in [t_{k-1}, t_k), \\ \Delta x(t) = C_k x(t^-), & t = t_k, k \in N, \\ x_{t_0} = \phi, \end{cases} \tag{3.13}$$

where $t - \tau(t)$ is strictly increasing on $\mathbb{R}_+$ and $0 \le \tau(t) \le \tau$.

Corollary 3.1. *Suppose there exist some constants $\alpha, \lambda > 0$ such that*

(i) *for some constant $q \ge e^{2\lambda\alpha}$, $\lambda_{\max}(A) + q^{\frac{1}{2}}\|B\| < \frac{\lambda}{2}$;*

(ii) $\tau \le t_k - t_{k-1} \le \alpha$ *and*

$$\ln\|I + C_k\| + \frac{\lambda\alpha}{2} < -\frac{\lambda}{2}(t_{k+1} - t_k). \tag{3.14}$$

Then system (3.13) is globally exponentially stable and its convergence rate is $\frac{\lambda}{2}$.

Proof. It follows from Theorem 3.1 on choosing $V(x) = \|x\|^2$.

Corollary 3.2. *Suppose there exist some constants* $\alpha, \lambda > 0$ *such that*

(i) *there exists some constant* $q > 0$ *such that* $q \ge \max\{e^{c\alpha}, e^{2\lambda\alpha}\}$, *where* $c = 2(\lambda_{\max}(A) + q^{\frac{1}{2}}\|B\|)$;
(ii) $\tau \le t_k - t_{k-1} \le \alpha$ *and*

$$\ln\|I + C_k\| + \frac{\alpha}{2}(\lambda + c) < -\frac{\lambda}{2}(t_{k+1} - t_k). \tag{3.15}$$

Then system (3.13) is globally exponentially stable and its convergence rate is $\frac{\lambda}{2}$.

Proof. It follows from Theorem 3.2 on choosing $V(x) = \|x\|^2$. □

4. An example

In this section, we give an example and its simulation to illustrate our results.

Example 4.1. Consider the following linear impulsive delay system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx\left(t - \frac{1}{40}(1 + e^{-t})\right), & t \ne k \\ \Delta x(t) = C_k x(t^-), \quad t = k, k \in N, \\ x_{t_0} = \phi, \end{cases} \tag{4.1}$$

where

$$A = \begin{bmatrix} 0.1 & 0.2 & -0.1 \\ 0.2 & 0.15 & 0.3 \\ 0 & 0.24 & 0.1 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.12 & 0.03 & 0 \\ 0.12 & -0.2 & 0.05 \\ 0 & 0.14 & -0.1 \end{bmatrix}$$

and

$$C_k = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.8 & 0 \\ 0 & 0 & -0.4 \end{bmatrix}.$$

Then $\lambda_{\max}(A) = 0.4388$, $\|B\| = [\lambda_{\max}(BB^T)]^{\frac{1}{2}} = 0.2905$ and $\|I + C_k\| = 0.6$. Choosing $q = 2, \lambda = 1.7, \tau = 0.05, \alpha = 0.2$, we find that the conditions of Corollary 3.1 hold:

(i) $q = 2 \ge e^{2\lambda\alpha} = 1.9739$, $\lambda_{\max}(A) + q^{\frac{1}{2}}\|B\| = 0.8496 < \frac{\lambda}{2} = 0.85$;
(ii) $0.05 = \tau \le t_k - t_{k-1} \le \alpha = 0.2$, $\ln\|I + C_k\| + \frac{\lambda\alpha}{2} = -0.6808 < -\frac{\lambda}{2}(t_{k+1} - t_k) = -0.17$.

Thus by Corollary 3.1, we know that the trivial solution of (4.1) is globally exponentially stable with convergence rate 0.85.

Furthermore, we can also find that the conditions of Corollary 3.2 hold:

(i) $c = 2(\lambda_{\max}(A) + q^{\frac{1}{2}}\|B\|) = 1.6992$, $q = 2 \ge \max\{e^{c\alpha}, e^{2\lambda\alpha}\} = 1.9739$;
(ii) $0.05 = \tau \le t_k - t_{k-1} \le \alpha = 0.2$, $\ln\|I + C_k\| + \frac{(\lambda+c)\alpha}{2} = -0.1709 < -\frac{\lambda}{2}(t_{k+1} - t_k) = -0.17$.

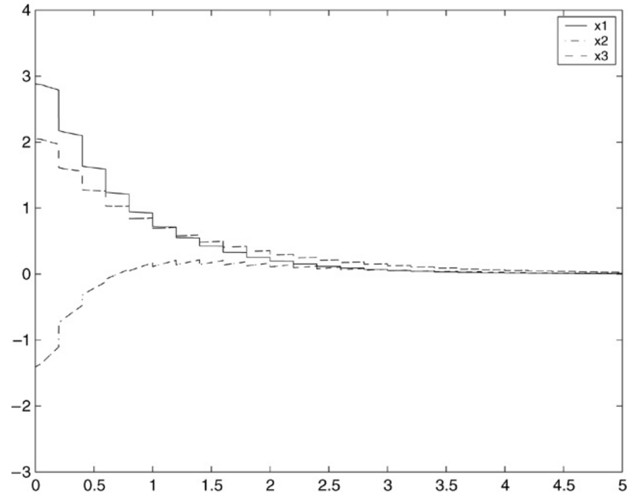

Fig. 1. Impulse-stabilized system.

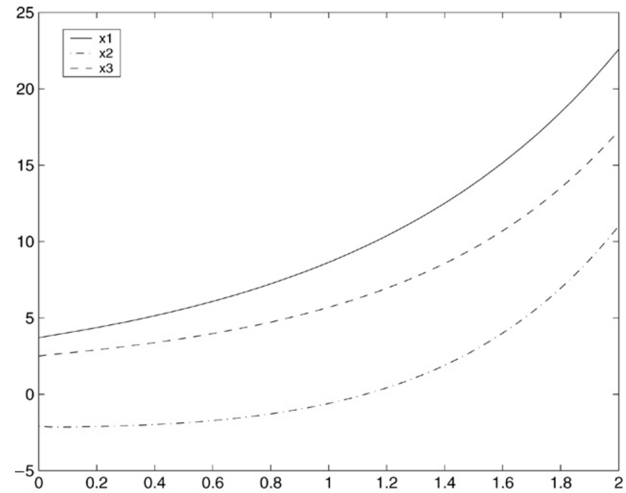

Fig. 2. System without impulses.

Then we know from Corollary 3.2 that the trivial solution of (4.1) is globally exponentially stable with convergence rate 0.85.

The numerical simulation of this impulsive delay differential equation with the initial function $(3.7H(t), -2.1H(t), 2.502H(t))^T$, where $H(t)$ is the Heaviside step function, is given in Fig. 1; the graph of the solution of the corresponding system without impulses is given in Fig. 2.

Remark 4.1. As we see from the above pictures, the trivial solution of system (4.1) without impulses is unstable; however, after impulsive control, the trivial solution becomes globally exponentially stable. This implies that impulses may be used to exponentially stabilize some delay differential systems.

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