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Journal of Computational and Applied Mathematics 179 (2005) 343–353

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

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Orthogonality of analytic polynomials: a little step further

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Received 13 January 2004

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Abstract

From the constellation mentioned in Jones and Njåstad (J. Comput. Appl. Math. 105 (1999) 51–91) we have chosen *orthogonality of polynomials* and *moment problems* enriching them with *operator theory* apparatus. Thus this essay resumes the theme of Szafraniec (J. Comput. Appl. Math. 49 (1993) 255) and culminates in updating it with the results of Stochel and Szafraniec (J. Funct. Anal. 159 (1998) 432).

Keywords: Analytic polynomials; Recurrence relation; Complex moment problem; Positive definiteness; Hessenberg matrix; Subnormal operator

1. Assorted examples

The general form, *not normalized* say, of the recurrence relation for polynomials orthogonal on the real line is

$$p_{n+1} = (A_n X + B_n) p_n - C_n p_{n-1}, \quad n = 0, 1, \dots \tag{1}$$

with A_n's and C_n's being positive; for shortening the expression we make the innocent assumption: $p_{-1} = 0$. Pretty often there is a need to normalize the polynomials in this or another way (sometimes to have them monic, sometimes of $\mathscr{L}^2$-norm 1, for instance). This always reflects the eventual form of the relation; for the orthonormal case it becomes symmetric, that is the associated matrix is Jacobi. A similar

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doi:10.1016/j.cam.2004.09.049

behaviour can be recognized in the other classical case, the unit circle one. In what follows we illustrate our intentions by special cases.

1.1. Hermite polynomials

The sequence of polynomials $\{H_n\}_n$ satisfy the *three*-term recurrence

$$2XH_n = H_{n+1} + 2nH_{n-1}, \quad n = 0, 1 \dots . \tag{2}$$

The normalization $h_n = (2^n n!)^{-1/2} H_n$ makes the relation *symmetric* (the associated tridiagonal matrix is symmetric, hence Jacobi):

$$Xh_n = \sqrt{\tfrac{1}{2}(n+1)}\, h_{n+1} + \sqrt{\tfrac{1}{2}n}\, h_{n-1}, \quad n = 0, 1, \dots . \tag{3}$$

Another normalization, $h_n^A = b_n(A)^{-1/2} H_n$, where, for $0 < A < 1$,

$$b_n(A) = \frac{\pi\sqrt{A}}{1-A}\left(2\frac{1+A}{1-A}\right)^n n!$$

brings relation (2) to the form

$$Zh_n^A = \sqrt{\frac{1+A}{1-A}}\sqrt{\tfrac{1}{2}(n+1)}\, h_{n+1}^A + \sqrt{\frac{1-A}{1+A}}\sqrt{\tfrac{1}{2}n}\, h_{n-1}^A. \tag{4}$$

This recurrence is *no longer* symmetric. However, they both lead to orthonormal sequences: the first, for $\{h_n\}_n$, benefits from the very classical orthonormality over *the real line* with respect to the Gaussian measure; the latter, for $\{h_n^A\}_n$, does it over *the complex plane* according to (cf. [5])

$$\int_{\mathbb{C}} h_m^A(z)\overline{h_n^A(z)} \exp\left[-(1-A)x^2 - \left(\frac{1}{A} - 1\right)y^2\right] \mathrm{d}x\,\mathrm{d}y = \delta_{m,n} \quad z = x + \mathrm{i}y.$$

1.2. The monomials

Though the monomials $p_n \overset{\mathrm{df}}{=} Z^n$, $n = 0, 1, \dots$, are never orthogonal on the real line, they enjoy a lot of orthogonality possibilities over the complex plane (including that over the unit circle). They are orthogonal with respect to any radially invariant measure on $\mathbb{C}$. Their recurrence is the simplest possible, the *one-term* relation

$$Zp_n = p_{n+1}, \quad n = 0, 1, \dots .$$

Normalization calls upon introducing some coefficients in the above like

$$Zp_n = \alpha_n p_{n+1}, \quad n = 0, 1, \dots$$

and it allows a unique orthonormal solution with a rotationally invariant orthogonality measure if and only if $\{|\alpha_0 \cdots \alpha_n|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence. The set of orthogonality measures may be pretty sizable and, among them, measures which are not rotationally invariant may appear as well.

1.3. Newton polynomials

This is the case when the recurrence relation is of *two terms*. The polynomials are defined, with $h > 0$, as

$$N_n^h(z) \stackrel{\text{df}}{=} \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{\Gamma(h)\, n!}{\Gamma(h+n)} \, \dfrac{z(z-1)\cdots(z-n+1)}{n!} & \text{if } n \geqslant 1, \end{cases}$$

and their recurrence is

$$ZN_n^h = \sqrt{(n+1)(h+n)}N_{n+1}^h + nN_n^h, \quad n = 0, 1, \ldots .$$

They are orthonormal with respect to some measure (of unbounded support) on the complex plane, details in [3,6].

1.4. Conclusions

1° The same polynomials, the Hermite ones, may have orthogonality measures of rather contrasting support, though their recurrence relations (3) and (4) are both of the standard form (1). 2° The recurrence relation for polynomials orthonormal over the complex plane is no longer symmetric and may be of arbitrary length. 3° Orthonormality is usually in some reproducing kernel Hilbert space of analytic functions, in which the sequence constitutes a basis, and which is a proper subset of an $\mathscr{L}^2$ space provided such a space exists (this conclusion is behind the exhibited examples and is discussed in [12]).

2. The core

2.1. From the recurrence to moments. The Hessenberg operator

Suppose we are given a sequence $\boldsymbol{p} \stackrel{\text{df}}{=} \{p_n\}_n = 0 \subset \mathbb{C}[Z]$ such that $\deg\, p_n = n$ for $n = 0, 1, \ldots$. Therefore, $\{p_n\}_n$ is a Hamel basis of $\mathbb{C}[Z]$ and consequently

$$Zp_n = a_{0,n}p_0 + a_{1,n}p_1 + \cdots + a_{n+1,n}p_{n+1}, \quad n = 0, 1, \ldots \tag{5}$$

with

$$a_{n+1,n} \neq 0. \tag{6}$$

This is the *recurrence relation* for $\boldsymbol{p}$, which in case of the real line orthogonality reduces to the last three terms.

Declaring the polynomials p_n to be orthonormal means we are going to define an inner product $\langle \cdot\,, -\rangle_{\boldsymbol{p}}$ by extending sesquilinearly

$$\langle p_m, p_n\rangle_{\boldsymbol{p}} = \delta_{mn}, \quad m, n = 0, 1, \ldots; \tag{7}$$

this is in fact an inner product because $\boldsymbol{p}$ is a Hamel basis. Completing $\mathbb{C}[Z]$ with respect to this inner product, we come to the Hilbert space $\mathscr{H}_{\boldsymbol{p}}$ in which $\boldsymbol{p}$ becomes an orthonormal basis. Moreover, the recurrence coefficients can be expressed as

$$a_{k,n} = \langle Zp_n, p_k \rangle_{\boldsymbol{p}}. \tag{8}$$

The inner product $\langle \cdot, - \rangle_{\boldsymbol{p}}$ determines its moments [2] $\{c^{\boldsymbol{p}}_{m,n}\}_{m,n=0}^{\infty}$ by

$$c^{\boldsymbol{p}}_{m,n} \overset{\text{df}}{=} \langle Z^m, Z^n \rangle_{\boldsymbol{p}}, \quad m, n = 0, 1, \ldots \, . \tag{9}$$

The explicit expression for the moments can be calculated by means of the vectors $t_n \overset{\text{df}}{=} (t_{0,n}, t_{1,n}, \ldots)$, $n = 0, 1, \ldots$, which, in turn, can be obtained iterating (5) so as to come to the *recurrence* relation

$$t_{i,m} = \sum_{j=0}^{\infty} t_{j,m-1} a_{i,j}, \quad t_{0,0} = 1. \tag{10}$$

As $a_{i,k} = 0$ for $i < k + 1$, the above sum terminates with $j = i - 1$ and $t_{i,m} = 0$ for $i \geqslant m$.

The final expression for the moments looks like

$$c^{\boldsymbol{p}}_{m,n} = \sum_{i=0}^{\infty} t_{i,m} \bar{t}_{i,n} \tag{11}$$

and the right-hand-side sum terminates with $i = \min\{m, n\}$.

Notation: $\ell^2_0 \overset{\text{df}}{=} \operatorname{lin}\{e_n; n = 0, 1, \ldots\}$, where $\{e_n\}_{n=0}^{\infty}$ is the canonical zero–one basis in ℓ^2. Thus ℓ^2_0 is composed of all sequences with entries equal to 0 but a finite number.

The infinite matrix $(a_{ij})_{i,j=0}^{\infty}$ of the coefficients in (5) turns out to be of a *Hessenberg type* and it is apparently of the form

$$\begin{pmatrix} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ 0 & a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ 0 & 0 & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{12}$$

The matrix (12) represents[3] a densely defined operator $A_{\boldsymbol{p}}$ in ℓ^2 with $\mathscr{D}(A_{\boldsymbol{p}})$, its domain, to be safely chosen as ℓ^2_0. The unitary operator $U : \mathscr{H}_{\boldsymbol{p}} \mapsto \ell^2$ such that $Up_n = e_n$ establishes a unitary isomorphism

[2] The word ‘moments’ as it is used on many occasions in the literature on orthogonal polynomials stands routinely for a sequence, or less often a multisequence, of numbers, which can be easily attached to a linear or multilinear functional on polynomials, say; thus when exempted from a deeper context, it means practically very little. We are aware of this contentional trap; fortunately, our ‘moments’ will gain importance as soon as the integral representation problem enters the game.

[3] We do not make any distinction between Hessenberg matrices and the corresponding operators defined in this way, once we know the relationship.

between $\mathbb{C}[Z]$ and ℓ_0^2. Under these circumstances, it is clear that

$$UM_Z \subset A_{\boldsymbol{p}}U,$$

where M_Z stands for the operator of multiplication by the independent variable in $\mathscr{H}_{\boldsymbol{p}}$ with $\mathscr{D}(M_Z)=\mathbb{C}[Z]$, and, consequently,

$$Up(M_Z) \subset p(A_{\boldsymbol{p}})U \tag{13}$$

for any $p \in \mathbb{C}[Z]$.

Remark 1. Because (10) can be written[4] as $t_m = A_{\boldsymbol{p}}t_{m-1}$ performing the iteration with the initial vector $t_0 = e_0$, we can write (11) as

$$c^{\boldsymbol{p}}_{m,n} = \langle A^m_{\boldsymbol{p}}e_0, A^n_{\boldsymbol{p}}e_0\rangle.$$

2.2. From moments to the recurrence

Suppose we are given a matrix $\boldsymbol{c} \overset{\mathrm{df}}{=} (c_{m,n})^{\infty}_{m,n=0}$ of complex numbers, which is positive definite.[5] Occasionally we think of it as a bisequence to come closer to what appears in the moment theory; double live of some objects is sometimes acceptable.

Set $G_n \overset{\mathrm{df}}{=} \det(c_{i,j})^n_{i,j=0}$ and assume all the G_n's are positive. Set also

$$p^{\boldsymbol{c}}_n \overset{\mathrm{df}}{=} \frac{1}{\sqrt{G_nG_{n-1}}}\det\begin{pmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,n-1} & 1 \\ c_{1,0} & c_{1,1} & \cdots & c_{1,n-1} & Z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{n,0} & c_{n,1} & \cdots & c_{n,n-1} & Z^n \end{pmatrix}, \quad n = 0, 1, \ldots$$

with $G_0 \overset{\mathrm{df}}{=} 1$. Then the sequence $\boldsymbol{p}^{\boldsymbol{c}} \overset{\mathrm{df}}{=} \{p^{\boldsymbol{c}}_n\}_n$ is orthonormal with respect to the inner product $\langle\cdot, -\rangle_{\boldsymbol{c}}$ which extends sesquilinearly

$$\langle Z^m, Z^n\rangle_{\boldsymbol{c}} \overset{\mathrm{df}}{=} c_{mn}, \quad m, n = 0, 1, \ldots \tag{14}$$

to $\mathbb{C}[Z]$.

Remark 2. Here we have arrived upon the point when the two approaches, the present and that of the preceding section, meet (9) accords with (14). This gives rise to sometimes drop super- or subscripts indicating the logical path creating the symbols in question; no confusion guaranteed.

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[4] We abandon the usual notation of linear algebra in favour of shapeless operator theory encoding, also because our matrices are infinite dimensional, therefore operators; cf. footnote 3.

[5] This sometimes is called positive semidefinite.

Remark 3. Due to (8), the recurrence coefficients $a_{k,n}$ can be written down explicitly in a determinantal way as well. The recurrence, however, may break down (it can happen $a_{n+1,n} = 0$ for some n). This corresponds to the situation when some G_n is equal to 0 or, still another way, when the Gram–Schmidt orthonormalization loops in. While for orthogonality of analytic polynomials this is not very dramatic (the case of measures of finite mass points has to be excluded), in the several variable cases it creates a severe problem, cf. [4].

3. What is necessary

3.1. Some operator theory

A densely defined operator S in a Hilbert space $\mathscr{H}$ is said to be *subnormal* if there is another Hilbert space $\mathscr{K}$ which contains $\mathscr{H}$ isometrically, and a normal operator N in $\mathscr{K}$ such that $S \subset N$ (this means $\mathscr{D}(S) \subset \mathscr{D}(N)$ as for their respective domains and $Sf = Nf$ for $f \in \mathscr{D}(S)$). If E stands for the spectral measure of N, then

$$\langle p(S)f, q(S)g\rangle_{\mathscr{H}} = \int_{\mathbb{C}} p(z)\overline{q(z)}\langle E(\mathrm{d}z)f, g\rangle_{\mathscr{K}}, \quad p, q \in \mathbb{C}[Z], \quad f, g \in \mathscr{D}(S) \tag{15}$$

provided $S\mathscr{D}(S) \subset \mathscr{D}(S)$. This is the part of the spectral theorem subnormal operators inherit from their normal extensions and it fits in with our need of finding orthogonality measures if $S = A_{\boldsymbol{p}}$ as defined via the Hessenberg matrix (12) coming from the recurrence (5). Moreover, due to (11), the operator $A_{\boldsymbol{p}}$ is cyclic[6] with the cyclic vector e_0. This is a fortunate circumstance and it will help us later on.

The following is a necessary condition for subnormality.

Fact A. *Suppose $S\mathscr{D}(S) \subset \mathscr{D}(S)$. If S is subnormal then*

$$\sum_{m,n=0}^{k} \langle S^m f_n, S^n f_m\rangle \geqslant 0, \textit{ for any finite sequence } f_0, \ldots, f_k \textit{ in } \mathscr{D}(S). \tag{16}$$

The converse is not true!

3.2. The complex moment problem

A bisequence (which we sometimes prefer to see here as an infinite matrix) $\{c_{m,n}\}_{m,n=0}^{\infty}$ is said to be a *complex moment bisequence* if there exists a measure μ such that

$$c_{m,n} = \int_{\mathbb{C}} z^m \bar{z}^n \mu(\mathrm{d}z), \quad m, n = 0, 1, \ldots .$$

A result which is parallel to Fact A is as follows.

[6] An operator S with invariant domain is called *cyclic* with a cyclic vector f_0 if $\mathscr{D}(S) = \mathrm{lin}\{\boldsymbol{p}(S)f_0;\ p \in \mathbb{C}[Z]\}$.

Fact B. *If* $\{c_{m,n}\}_{m,n=0}^{\infty}$ *is a complex moment bisequence, then*

$$\sum_{m,n,p,q=0}^{k} c_{m+q,n+p}\xi_{m,n}\bar{\xi}_{p,q} \geqslant 0 \text{ for any finite bisequence } \{\xi_{m,n}\}_{m,n=0}^{k} \text{ in } \mathbb{C}. \tag{17}$$

The converse is not true either!

3.3. *The link*

The interrelation between subnormality and the complex moment problem is simple and very useful.

Fact C (*Stochel and Szafraniec* [7]). *A cyclic operator S with a cyclic vector* f_0 *is subnormal if and only if* $\{\langle S^m f_0, S^n f_0\rangle\}_{m,n=0}^{\infty}$ *is a complex moment bisequence.*

Any solution to either of these two affects the other. We are going to exploit this kind of brotherhood in the sequel: either the Hessenberg operator (*ergo* the recurrence relation) or the would-be moments will be examined so as to squeeze out of it representing measures to exist. One has to point out that they are both very resisting objects. Anyway, we can establish the following.

Link. *Given a polynomial sequence* $\boldsymbol{p} = \{p_n\}_n$ *with* deg $p_n = n$, *the following conditions are equivalent*:

(i) *the operator* $A_{\boldsymbol{p}}$ *is a subnormal operator with a cyclic vector* e_0;
(ii) *there is a measure* μ on $\mathbb{C}$ *such that* $\boldsymbol{p}$ *is a sequence of polynomials orthonormal with respect to* μ;
(iii) $c_{\boldsymbol{p}}$ *is a complex moment sequence.*

Proof. Suppose $A_{\boldsymbol{p}}$ and A_Q is subnormal and E is the spectral measure of some of its normal extensions in $\mathscr{K}$, say. Then, due to (15), (13) and (7), we have

$$\begin{aligned}\int_{\mathbb{C}} p_m(z)\overline{p_n(z)}\mu(\mathrm{d}z) &= \int_{\mathbb{C}} p_m(z)\overline{p_n(z)}\langle E(\mathrm{d}z)e_0, e_0\rangle_{\mathscr{K}} = \langle p_m(A_{\boldsymbol{p}})e_0, p_n(A_{\boldsymbol{p}})e_0\rangle_{\ell^2}\\ &= \langle Up_m(M_Z)U^{-1}e_0, Up_n(M_Z)U^{-1}e_0\rangle_{\ell^2}\\ &= \langle p_m(M_Z)1, p_n(M_Z)1\rangle_{\boldsymbol{p}} = \langle p_n, p_n\rangle_{\boldsymbol{p}}\\ &= \delta_{m,n}\end{aligned}$$

where $\mu \overset{\mathrm{df}}{=} \langle E(\cdot)e_0, e_0\rangle$. This establishes (ii).

Suppose $\boldsymbol{p}$ is orthonormal in some $\mathscr{L}^2(\mu)$. Denote by M_Z^μ the operator of multiplication by Z in $\mathscr{L}^2(\mu)$; M_Z^μ is densely defined as functions of compact support are dense in $\mathscr{L}^2(\mu)$, it is apparently normal. Let $V : \mathscr{H}_{\boldsymbol{p}} \mapsto \mathscr{L}^2(\mu)$ be the inclusion mapping which is an isometry with respect to the corresponding norms (this is so due to orthonormality of $\boldsymbol{p}$). Then $VM_Z \subset M_Z^\mu V$. By (13), we have $VU^{-1}A_{\boldsymbol{p}} \subset M_Z^\mu VU^{-1}$. Because VU^{-1} is an isometry of ℓ^2 into $\mathscr{L}^2(\mu)$, after proper identification we get subnormality of $A_{\boldsymbol{p}}$.

The equivalence of (i) and (iii) is the matter of Fact C. □

Remark 4. If one wants to go the other way around, starting from the (prospective) moments getting to orthonormality, the alike link is easy to state and to prove.

4. What is sufficient

4.1. Non-iterative methods

Here are some results which do not need higher powers of the Hessenberg operator to get involved or are based on a very truncation of the complex moment problem.

Our first approach is based on a deep-rooted theorem of Bishop ([1]; for a more contemporary proof and a much extended version of it, see [13]). It can be viewed as a sort of approximation result.

Theorem 5. *Given an infinite Hessenberg matrix $A=(a_{i,j})_{i,j}$. Then the operator A is subnormal if (and only if) for every $\varepsilon>0$ and for every finite subset I of $\{0,1,\ldots\}$ there is an infinite Hessenberg matrix $A'=(a'_{i,j})_{i,j}$ which as an operator is subnormal and such that*

$$\sum_{k}^{\infty}|a'_{i,k}-a_{i,k}|^2<\varepsilon, \quad i\in I. \tag{18}$$

Proof. Condition (18) implies that A' is in a strong operator topology neighbourhood of A. The rest follows from the aforesaid result of Bishop. □

Denote by INT the collection of all polynomial sequences which are orthonormal in some $\mathscr{L}^2$-space. Then a more explicit version of Theorem 5 follows.

Corollary 6. *Given a sequence of polynomials* $\boldsymbol{p}$ *satisfying* (5) *and* (11), $\boldsymbol{p}\in$ INT *if (and only if) for every* ε *and for every finite subset I of* $\{0,1,\ldots\}$ *there is* $\boldsymbol{p}'\in$ INT *with the recurrence relation*

$$Zp'_n=a'_{0,n}p'_0+a'_{1,n}p'_1+\cdots+a'_{n+1,n}p'_{n+1}, \quad n=0,1,\ldots, \quad a'_{n+1,n}\neq 0$$

such that (18) *holds.*

The next result is of different nature, though still no higher power of the Hessenberg is required. It, in turn, relies on the Markoff–Kakutani fix point.

Theorem 7 (*Szafraniec [11]*). *Given a Hessenberg matrix A, the operator A is subnormal if (and only if) there is a family* $\{\mu_f\}_{f\in\ell_0^2}$ *of positive measures satisfying*

$$\langle A^m f, A^n f\rangle=\int_{\mathbb{C}} z^m\bar{z}^n\mu_f(\mathrm{d}z), \quad m,n=0,1$$

and such that

$$\mu_{f+g}+\mu_{f-g}-2\mu_f \text{ is a positive measure for every } f,g\in\ell_0^2.$$

In terms of ‘moments’, it takes the following form. First notation: for $c=\{c_{m,n}\}_{m,n=0}^{\infty}$ and a polynomial $p=\sum_{i,j=0}^{n} p_{i,j} Z^i \bar{Z}^j$, set

$$c_{m,n}(p) \stackrel{\mathrm{df}}{=} \sum_{i,j=0}^{n} p_{i,j} c_{m+i,n+j}, \quad m,n=0,1,\ldots\ .$$

Corollary 8. *c is a complex moment bisequence if and only if there is a family* $\{\mu_p\}_{p\in\mathbb{C}[Z,\bar{Z}]}$ *of measures satisfying*

$$c_{m,n}(p)=\int_{\mathbb{C}} z^m \bar{z}^n \mu_p(\mathrm{d}z), \quad m,n=0,1 \tag{19}$$

and such that

$$\mu_{p+q}+\mu_{p-q}-2\mu_p \text{ is a positive measure for every } p,q\in\mathbb{C}[Z,\bar{Z}]. \tag{20}$$

The point here is that one has to solve a family of *very truncated* complex moment problems of the form (19) which can be solved numerically leading to measures μ_p with finite mass points. The only constraint is for them to fulfil the consistency condition (20). The same refers to Theorem 7.

4.2. Iterative methods

By this we mean results which involve all the powers of the Hessenberg operator in question or need to solve the whole complex moment problem. To calculate powers of an infinite matrix is an iterative process in matrix multiplication. It may not be an easy task in general, but for those who are lucky enough it may become a delightful way to proceed.

Actually, what we want to do here is to try to reverse Fact A or Fact B, depending on circumstances. One case is relatively simple, that of bounded support.

Theorem 9. *Given a sequence of polynomials* $\boldsymbol{p}$ *satisfying* (5) *and* (11), $\boldsymbol{p}\in\mathsf{INT}$ *if* $A_{\boldsymbol{p}}$ *satisfies* (16) *on* ℓ_0^2 *and*

$$\sum_{n=0}^{\infty} |a_{k,n}|^2 < M \text{ with } M \text{ independent of } k=0,1,\ldots\ . \tag{21}$$

The orthonormality measure is unique and compactly supported.

Proof. Condition (21) guarantees the operator $A_{\boldsymbol{p}}$ to be bounded, which together with (16) ensures its subnormality, cf. [2]. Now Link makes the conclusion. □

For the moment approach we have a necessary and sufficient condition, cf. [9,10].

Theorem 10. *c is a complex moment sequence with a unique compactly supported measure if and only if it satisfies* (17) *and for some nonnegative a and* α

$$|c_{m,n}| \leqslant a\alpha^{m+n}, \quad m,n=0,1,\ldots\ .$$

A solution (in fact, one of the two) which is in [8] gives a complete characterization of complex moment bisequences and, in parallel, of unbounded subnormal. Let us state it for the moment problem, see [8, Theorem 1]; the operator version, which would be applicable to the Hessenberg matrix, is that of [8, Corollary 36].

Theorem 11. *A sequence* $\{c_{m,n}\}_{m,n=0}^{\infty} \subset \mathbb{C}$ *is a complex moment sequence if and only if there exists* $\{\tilde{c}_{m,n}\}_{m+n=0}^{\infty} \subset \mathbb{C}$ *such that*

$$c_{m,n} = \tilde{c}_{m,n} \quad \text{for } m, n = 0, 1, \ldots$$

and

$$\sum_{\substack{m+n \geqslant 0 \\ p+q \geqslant 0}} \tilde{c}_{m+q,n+p} \lambda_{m,n} \bar{\lambda}_{p,q} \geqslant 0 \quad \text{for any finite } \{\lambda_{m,n}\}_{m+n \geqslant 0} \subset \mathbb{C}. \tag{22}$$

The perspectives. Condition (17) allows one to extend the inner product $\langle \cdot, - \rangle_{\boldsymbol{p}}$ from $\mathbb{C}[Z]$ to $\mathbb{C}[Z, \bar{Z}]$, which is the background for further analysis. This brings up a question of completing the sequence $\boldsymbol{p}$, and the recurrence relation (5) and (11) at once, to a sequence (or rather a doubly indexed sequence) of polynomials from $\mathbb{C}[Z, \bar{Z}]$ to a Hamel basis therein; this would result in completing the aforementioned recurrence as well.

Condition (22) calls for further extension: complete the above to polynomials in $z \in \mathbb{C} \backslash \{0\}$ and $w \in \mathbb{T}$, which are of the form

$$\sum_{m+n \geqslant 0} \alpha_{m,n} z^m \bar{z}^n + \sum_{k+l \geqslant 0} \beta_{k,l} \delta_{k+l,0} w^k, \quad \alpha_{m,n}, \beta_{k,l} \in \mathbb{C},$$

where $\delta_{k+l,0}$ is the Kronecker symbol; this is a suggestion which comes out from [8, Remark 7]. The problem of how to implement this is challenging. Anyway, it comes close to the frontier of the *Iubilatus* research terrain.

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