Journal of Pure and Applied Algebra 16 (1980) 265–273

STABLE EQUIVALENCE AND RINGS WHOSE MODULES ARE A DIRECT SUM OF FINITELY GENERATED MODULES

Harlan L. HULLINGER*
Department of Mathematics, The University of Kansas, Lawrence, KS 66045, USA

Communicated by H. Bass
Received 6 December 1978

The representation theory of a ring Δ has been studied by examining the category of contravariant (additive) functors from the category of finitely generated left Δ-modules to the category of abelian groups. Closely connected with the representation theory of a ring is the study of stable equivalence, which is a relaxing of the notion of Morita equivalence. Here we relate two stably equivalent rings via their respective functor categories and examine left artinian rings with the property that every left Δ-module is a direct sum of finitely generated modules.

The representation theory of a ring Δ has been studied by examining the category of contravariant (additive) functors from the category of finitely generated left Δ-modules to the category of abelian groups (see [2, 3, 9]). Closely connected with the representation theory of rings is the study of stable equivalence, which is a relaxing of the notion of Morita equivalence. Here we relate two stably equivalent rings via their respective functor categories and examine left artinian rings with the property that every left Δ-module is a direct sum of finitely generated modules. But first, we recall the definition of stable equivalence and fix our notation.

For a ring Δ, ${}_\Delta\mathcal{M}$ ($\mathcal{M}_\Delta$) denotes the category of all left (right) Δ-modules, and mod Δ (mod Δ^{op}) denotes the category of finitely generated left (right) Δ-modules. For M, $N \in$ mod Δ, $P_\Delta(M, N)$ is the subgroup of $\mathrm{Hom}_\Delta(M, N)$ consisting of all the homomorphisms $f: M \to N$ that factor through projectives. The additive category of finitely generated left Δ-modules modulo projectives is **mod** Δ. Its objects (denoted $\boldsymbol{M}$) are the same as those of mod Δ, and its morphisms (denoted $\underline{f}$) are the members of the factor group

$$\mathbf{Hom}_\Delta(M, N) = \mathrm{Hom}_\Delta(M, N)/P_\Delta(M, N).$$

Two rings Δ and Δ' are said to be (*left*) *stably equivalent* in case **mod** $\Delta \approx$ **mod** Δ' (where $\approx$ denotes equivalence of categories).

If Δ is left artinian with (up to isomorphism) only a finite number of finitely generated indecomposable left Δ-modules, then Δ is said to be of *finite module type* (or *finite representation type*). It is well known that over a ring Δ of finite module type every left Δ-module is a direct sum of finitely generated modules. It is still not

* Present address: Department of Mathematics, Wright State University, Dayton, OH 45435, USA.

known if the converse is true. However, when Λ is an artin algebra much about the representation theory of Λ is known. M. Auslander [4] proved, by analysis of the functor category, that Λ is of finite module type if and only if every left Λ-module is a direct sum of finitely generated left Λ-modules. Further, he proved in [6] that if Λ and Λ' are stably equivalent artin algebras, then Λ is of finite module type if and only if Λ' is of finite module type.

In this paper we study the representation theory of left artinian rings. We first show that the global dimension and the dominant dimension of the functor categories of a ring determine when the ring has the property that every left module is a direct sum of finitely generated modules. This result helps determine whether a ring with the property that every left module is a direct sum of finitely generated modules is of finite module type. The main theorem employs the functor categories associated with two left artinian rings to prove that if Δ and Δ' are stably equivalent, then every left Δ-module is a direct sum of finitely generated modules if and only if every left Δ'-module is a direct sum of finitely generated modules. The essence of the proof (Lemma 5) is then used to show that if every left Δ-module is a direct sum of finitely generated modules, then the endomorphism ring of the minimal cogenerator in ${}_\Delta\mathcal{M}$ also has that property for its left modules. Finally, we prove the converse of a result of Fuller and Reiten [10] concerning noetherian and conoetherian conditions on a family of homomorphisms.

P. Gabriel [11] proved that the functor category is equivalent to the category of modules over a ring R. Naturally, the ring R need not have an identity, but it is always a ring with enough idempotents (in the terminology of [8]). We work in this setting. In particular, let Δ be a ring with identity and $\{U_\alpha \mid \alpha \in A\}$ a set containing one isomorphic copy of each finitely generated left Δ-module. Let ${}_\Delta U = \bigoplus_A U_\alpha$ and, writing homomorphisms on the right, define

$$R = \{r: {}_\Delta U \to {}_\Delta U \mid U_\alpha r = 0 \text{ a.e.}\}$$

(where "a.e." means for all but a finite number of $\alpha \in A$). We see that $\{e_\alpha \mid \alpha \in A\}$, the idempotents of decomposition for $\bigoplus_A U_\alpha$, is a complete set of idempotents for R. We write $R = R(\text{mod}\,\Delta)$ and call R the *functor ring of the finitely generated left Δ-modules*, or just the (left) functor ring of Δ.

We will freely use notation and facts from [9]. In particular, we shall make use of the left exact covariant (additive) functor

$$\hat{\text{H}}\text{om}_\Delta({}_\Delta U_R, -): {}_\Delta\mathcal{M} \to {}_R\mathcal{M}$$

defined by

$$\hat{\text{H}}\text{om}_\Delta({}_\Delta U_R, {}_\Delta M) = \{\varphi: {}_\Delta U \to {}_\Delta M \mid U_\alpha\varphi = 0 \text{ a.e.}\}$$

and, for each $g: {}_\Delta M \to {}_\Delta M'$ in ${}_\Delta\mathcal{M}$,

$$g_* = \hat{\text{H}}\text{om}_\Delta({}_\Delta U_R, g): \varphi \to \varphi g \quad (\varphi \in \hat{\text{H}}\text{om}_\Delta(U, M)).$$

The functor $\hat{\mathrm{Hom}}_\Delta(U, -)$ has left adjoint

$$({}_\Delta U \otimes_R -) : {}_R\mathcal{M} \to {}_\Delta\mathcal{M}.$$

Also, recall that a ring R with enough idempotents is called *semiperfect* if R has a complete set of local idempotents, i.e. $R = \bigoplus_C Rg_\gamma$ where $g_\gamma R g_\gamma$ is a local ring. For example, if Δ is left artinian then $R = R(\mathrm{mod}\ \Delta)$ is semiperfect. Similarly, the ring R is called *left perfect* if R is semiperfect and the Jacobson radical of R, $J = J(R)$, is left T-nilpotent. For example, Fuller [8] has proved

Theorem 1. *Let $R = R(\mathrm{mod}\ \Delta)$. The functor ring R is left perfect if and only if every left Δ-module is a direct sum of finitely generated modules.*

We begin our work by showing the effect of the left global dimension and the left dominant dimension of the functor ring on Δ.

Theorem 2. *Let Δ be a left artinian ring and $R = R(\mathrm{mod}\ \Delta)$. Then the following are equivalent*:

(a) *Every left Δ-module is a direct sum of finitely generated modules.*

(b) *The left global dimension of R is at most* 2 *and the left dominant dimension of R is at least* 2.

Proof. (a) $\Rightarrow$ (b). Note that for any R-module K, ${}_\Delta U \otimes_R K$ must be isomorphic to a direct sum of finitely generated left Δ-modules, say $\bigoplus_B Ue_\beta$. Hence,

$$\hat{\mathrm{Hom}}_\Delta(U, U \otimes_R K) \cong \bigoplus_B Re_\beta$$

is projective. Thus, given an exact sequence $0 \to {}_RK \to {}_RP_1 \to {}_RP_2$ with ${}_RP_i$ projective, we may apply [9, 1.1.2] and the five lemma to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & P_1 & \longrightarrow & P_2 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
0 & \longrightarrow & 0 & \longrightarrow & \hat{\mathrm{Hom}}_\Delta(U, U \otimes_R K) & \longrightarrow & \hat{\mathrm{Hom}}_\Delta(U, U \otimes_R P_1) & \longrightarrow & \hat{\mathrm{Hom}}_\Delta(U, U \otimes_R P_2)
\end{array}$$

to see that K is projective and the left global dimension of R is at most 2. Now, since the injective envelope of each simple left Δ-module is finitely generated, there exists an idempotent $e^2 = e \in R$ such that ${}_\Delta Ue$ is isomorphic to the minimal cogenerator in ${}_\Delta\mathcal{M}$. Thus, there exists a monomorphism

$$0 \to U \to Ue^B.$$

But Δ is left artinian, so Ue^B is isomorphic to a direct sum of indecomposable left Δ-modules, and Ue^B embeds in $Ue^{(A_1)}$ for some set A_1. Applying the same argument to the module $Ue^{(A_1)}/U$, we obtain an exact sequence

$$0 \to U \to Ue^{(A_1)} \to Ue^{(A_2)}.$$

By applying $\hat{\mathrm{Hom}}_\Delta(U, -)$ and [9, 1.3.1] to the sequence, it follows that the left dominant dimension of R is at least 2.

(b) ⇒ (a). First we show that the injective envelope of each simple left Δ-module is finitely generated. Since the left dominant dimension is at least 2, there exists an embedding

$$0 \to {}_RR \xrightarrow{\varphi} {}_RX$$

where ${}_RX$ is both projective and injective. Let ${}_\Delta Ue_\alpha$ be a simple left Δ-module. Since $e_\alpha\varphi : Re_\alpha \to X$ is non-zero,

$$U \otimes_R (e_\alpha\varphi) : Ue_\alpha \to U \otimes_R X$$

is a non-zero embedding. But ${}_RX \cong \bigoplus_C Re_\gamma$ and $U \otimes_R X \cong \bigoplus_C Ue_\gamma$; hence the minimal cogenerator embeds in a finite number of the Ue_γ (see [9, 1.3.1]). Therefore, the minimal cogenerator is finitely generated; say it is isomorphic to ${}_\Delta Ue$. Finally, let $M \in {}_\Delta\mathcal{M}$. As before we obtain an exact sequence

$$0 \to \hat{\mathrm{Hom}}_\Delta(U, M) \to Re^{(A_1)} \to Re^{(A_2)}$$

for some sets A_1 and A_2. Then $\hat{\mathrm{Hom}}_\Delta(U, M)$ is projective since the left global dimension of R is at most 2. Thus

$$\hat{\mathrm{Hom}}_\Delta(U, M) \cong \bigoplus_B Re_\beta \quad \text{and} \quad M \cong U \otimes_R \hat{\mathrm{Hom}}_\Delta(U, M) \cong \bigoplus_B Ue_\beta.$$

It is not difficult to see that if we raise the dominant dimension of $R = R(\mathrm{mod}\,\Delta)$ while keeping the left global dimension of R at most 2, then Δ is semisimple. Similarly, if we lower the global dimension while maintaining the left dominant dimension of R at 2 or more, then Δ is semisimple. Hence

Corollary 3. *Let Δ be a ring with identity and $R = R(\mathrm{mod}\,\Delta)$. If every left Δ-module is a direct sum of finitely generated modules and Δ is not of finite module type, the left global dimension and the left dominant dimension of R must both equal* 2.

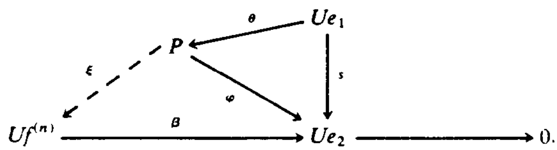
Next we examine the connection between two stably equivalent rings, Δ and Δ', and their respective functor categories. Unless otherwise stated, Δ will be a ring with identity, $R = R(\mathrm{mod}\,\Delta)$, and $f^2 = f \in R$ such that ${}_\Delta Uf \cong {}_\Delta\Delta$. (We will call f the *idempotent in R that generates Δ*). Respectively, Δ' is also a ring with identity, $R' = R'(\mathrm{mod}\,\Delta')$, and $f' \in R'$ is the idempotent that generates Δ'. Let

$$I = \{r \in R \mid r \text{ factors through a projective } \Delta\text{-module}\}.$$

It is easy to check directly that I is a two-sided ideal in R. In fact, $I = RfR$.

To see this, we need only show that $I \subseteq RfR$ since the other containment is obvious. Let $s \in I$, and let e_1 and e_2 be idempotents in R such that $e_1 s = s = s e_2$. If s

factors through a projective module P, then s factors through a finitely generated free module, say $Uf^{(n)}$, i.e.



Then letting π_i and ι_i be the projections and injection for $Uf^{(n)}$, we have

$$s = e_1\theta\xi 1_{Uf^{(n)}}\beta e_2 = e_1\theta\xi \sum_{i=1}^{n} \pi_i f \iota_i \beta e_2 = \sum_{i=1}^{n} e_1\theta\xi\pi_i f\iota_i\beta e_2$$

so that $s \in RfR$.

Now let $\mathscr{S} = \mathscr{S}(\textbf{mod}\,\Delta)$ and $\mathscr{S}' = \mathscr{S}'(\textbf{mod}\,\Delta')$ be the respective functor rings created from $\textbf{mod}\,\Delta$ and $\textbf{mod}\,\Delta'$. Clearly if Δ is stably equivalent to Δ', then $\mathscr{S}$ must be isomorphic to $\mathscr{S}'$ due to the category equivalence of $\textbf{mod}\,\Delta$ and $\textbf{mod}\,\Delta'$. But the objects of mod Δ and $\textbf{mod}\,\Delta$ are the same, and $s \in \mathscr{S}$ is zero if and only if the map s factors through a projective Δ-module. Thus we see that $\mathscr{S}$ is isomorphic to R/RfR, and that

$$R/RfR \cong \mathscr{S} \cong \mathscr{S}' \cong R'/R'f'R'.$$

Conversely, if R/RfR is merely Morita equivalent to $R'/R'f'R'$ (written $R/RfR \approx R'/R'f'R'$) then $\textbf{mod}\,\Delta \approx \textbf{mod}\,\Delta'$ (see [1, Exercise 20.6]).

Proposition 4. *The ring Δ is stably equivalent to Δ' if and only if $R/RfR \approx R'/R'f'R'$.*

The connection between two stably equivalent rings and their associated functor rings stated in Proposition 4 is the motivation for this paper; however, the following lemma is the essence of all our work.

Lemma 5. *Let Δ be left artinian, $R = R(\text{mod}\,\Delta)$, and f the idempotent in R that generates Δ. The functor ring R is left (right) perfect if and only if R/RfR is left (right) perfect.*

Proof. If R is left perfect, then so is the factor ring R/RfR. For the converse, assume that R/RfR is left perfect. Since Δ is left artinian, we need only prove that the Jacobson radical of R, $J(R)$, is left T-nilpotent.

Let $a_1, a_2, a_3, \ldots$ be a sequence in $J(R)$ which, without loss of generality, may be viewed as

$$Ue_1 \xrightarrow{a_1} Ue_2 \xrightarrow{a_2} Ue_3 \longrightarrow \cdots$$

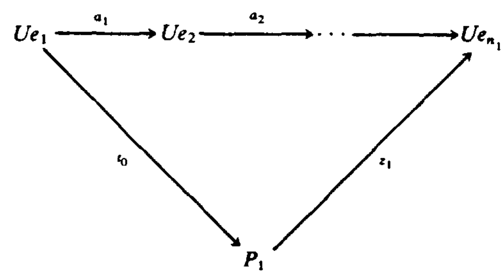
for some finitely generated left Δ-modules Ue_i $(i = 1, 2, 3 \ldots)$. Then

$$Ue_1 \xrightarrow{a_1} Ue_2 \xrightarrow{a_2} Ue_3 \longrightarrow \cdots$$

is a sequence in $J(R/RfR)$. But $J(R/RfR)$ is left T-nilpotent, so there exists an n_1 with $a_1 \cdots a_{n_1} = 0$. This means that there exists a projective Δ-module P_1 and maps

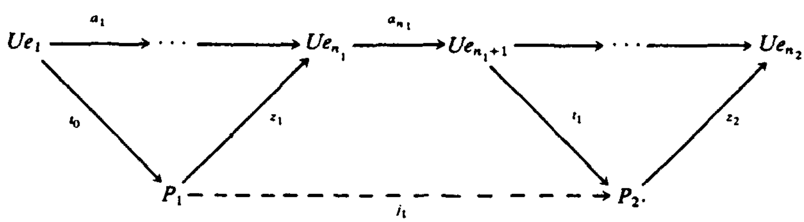
$$t_0 : Ue_1 \longrightarrow P_1, \qquad z_1 : P_1 \longrightarrow Ue_{n_1}$$

such that



commutes.

Applying this same argument to the sequence $a_{n_1+1}, a_{n_1+2}, \ldots$, we get the commutative diagram



By continuing this process we obtain a sequence of finitely generated projective left Δ-modules $P_1, P_2, \ldots$ and two sequences of maps $t_0, t_1, \ldots$ and $z_1, z_2, \ldots$ contained in R such that $j_i = z_i a_{n_i} t_i \in J(R)$. But then j_i is contained in the radical of $\mathrm{End}(P_i \oplus P_{i+1})$. Since Δ is artinian, j_i may be regarded as a matrix whose entries come from $\imath(\Delta)$, the radical of Δ. But $\imath(\Delta)$ is left T-nilpotent, so by the Köning Graph Theorem there exists a q such that $j_1 j_2 \cdots j_q = 0$. Thus

$$a_1 a_2 \cdots a_{n_{q+1}} = t_0 j_1 \cdots j_q z_q = 0,$$

and it follows that $J(R)$ is left T-nilpotent and R is left perfect. The parenthetical case follows similarly.

We now prove our main result. Namely,

Theorem 6. *Let Δ and Δ' be stably equivalent left artinian rings. Every left Δ-module is a direct sum of finitely generated Δ-modules if and only if every left Δ'-module is a direct sum of finitely generated Δ'-modules.*

Proof. We have just seen that every left Δ-module is a direct sum of finitely generated left Δ-modules if and only if $R = R(\text{mod}\, \Delta)$, $R/RfR \approx R'/R'f'R'$, and $R' = R'(\text{mod}\, \Delta')$ are all left perfect if and only if every left Δ'-module is a direct sum of finitely generated left Δ'-modules.

The two other immediate results we promised make use of the Auslander–Bridger transpose [5] and some of its properties that can be found in [4] and [5] or derived by standard diagram chasing techniques. We recall the definition of the transpose. Let Δ be semiperfect and M a finitely presented left Δ-module that contains no non-zero projective direct summands. Let

$$P_1 \xrightarrow{d} P_0 \longrightarrow M \longrightarrow 0$$

be a minimal projective resolution of M and $(-)^* = \text{Hom}_\Delta(-, \Delta)$. The *transpose of* M is a right Δ-module $\text{Tr}(M)$ such that

$$P_0^* \xrightarrow{d^*} P_1^* \longrightarrow \text{Tr}(M) \longrightarrow 0$$

is exact. $\text{Tr}(M)$ is also finitely presented with no non-zero projective direct summands, and M is indecomposable if and only if $\text{Tr}(M)$ is.

Let $g: M \to M'$ where neither M nor M' contains any projective direct summand. Then there exists a pair of maps (g_0, g_1) which makes the diagram of minimal presentations commute, i.e.,

$$\begin{array}{ccccccc}
P_0 & \longrightarrow & P_1 & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g} & & \\
P_0' & \longrightarrow & P_1' & \longrightarrow & M' & \longrightarrow & 0.
\end{array}$$

Although the pair (g_0, g_1) need not be unique, $\text{Tr}(g): \text{Tr}(M') \to \text{Tr}(M)$ making

$$\begin{array}{ccccccc}
P_1'^* & \longrightarrow & P_0'^* & \longrightarrow & \text{Tr}(M') & \longrightarrow & 0 \\
\downarrow{\scriptstyle g_1^*} & & \downarrow{\scriptstyle g_0^*} & & \downarrow{\scriptstyle \text{Tr}(g)} & & \\
P_1^* & \longrightarrow & P_0^* & \longrightarrow & \text{Tr}(M) & \longrightarrow & 0
\end{array}$$

commute is unique once the pair (g_0, g_1) is chosen. Thus, it is easy to check (see [4]) that when Δ is artinian, $\mathrm{Tr}:\mathbf{mod}\,\Delta \to \mathbf{mod}(\Delta^{\mathrm{op}})$ is a duality.

If there is a bimodule ${}_\Delta W_\Lambda$ that defines a Morita duality (as discussed in [1, § 24], for example), then Δ is said to be a *left Morita ring*. We specify the second ring by saying that Δ is left Morita to Λ. Recall from [9, Lemma 3.1] that if Δ is left artinian and left Morita to Λ, then the functor ring of the finitely generated left Δ-modules is isomorphic to the functor ring of the finitely generated right Λ-modules. We now have

Theorem 7. *Let Δ be a ring with identity and assume every left Δ-module is a direct sum of finitely generated modules. If ${}_\Delta Ue$ is the minimal cogenerator in ${}_\Delta\mathcal{M}$, then $\Lambda = \mathrm{End}(Ue)$ has the property every left Λ-module is a direct sum of finitely generated modules.*

Proof. Clearly Δ is left Morita to Λ, and Λ is right artinian. We first show that Λ is also left artinian. Auslander has proved that when the left global dimension of $R = R(\mathrm{mod}\,\Delta)$ is at most 2, each Re_α is noetherian. Thus, by Theorem 2, ${}_R Re = {}_R\hat{\mathrm{H}}\mathrm{om}_\Delta(U, Ue)$ is left noetherian, hence

$$\Lambda = \mathrm{End}(Ue) \cong eRe$$

is itself left noetherian and so left artinian. Let $\mathcal{S} = \mathcal{S}(\mathrm{mod}\,\Lambda^{\mathrm{op}})$ with $h \in \mathcal{S}$ the idempotent that generated Δ_Λ, and let $T = T(\mathrm{mod}\,\Lambda)$ with $k \in T$ the idempotent that generates ${}_\Lambda\Lambda$. Then $R \cong \mathcal{S}$ since Δ is left Morita to Λ, and $\mathcal{S}/\mathcal{S}h\mathcal{S} \approx T/TkT$ via the transpose duality. The result follows from these isomorphisms, Theorem 1, and Lemma 5.

Thus when studying the representation of a ring Δ with the property that every left Δ-module is a direct sum of finitely generated modules, we may assume Δ is both right artinian and right Morita. For Δ is of finite module type if and only if the endomorphism ring of the minimal cogenerator, $\mathrm{End}({}_\Delta Ue)$, is of finite module type.

Finally, Auslander [3] says that a family of Δ-homomorphisms is *noetherian* if, given a sequence of maps

$$M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} M_2 \longrightarrow \cdots$$

in the family with $\varphi_0\varphi_1 \cdots \varphi_i \neq 0$ for all i, there is an integer n such that φ_k is an isomorphism for $k \geq n$. Analogously, the family is *conoetherian* in case given any sequence

$$\cdots \longrightarrow M_2 \xrightarrow{\varphi_1} M_1 \xrightarrow{\varphi_2} M_0$$

with $\varphi_i \cdots \varphi_1\varphi_0 \neq 0$ for all i, there is an integer n such that φ_k is an isomorphism for $k \geq n$. In [10], Fuller and Reiten proved that for an artinian ring Δ, if the family of homomorphisms between finitely generated indecomposable right Δ-modules is noetherian, then the family of epimorphisms between finitely generated indecomposable left Δ-modules is conoetherian. Not only have we proved both their result and its converse, but we have a slightly more general result. Namely,

Theorem 8. *Let Δ be artinian. The family of homomorphisms between finitely generated indecomposable left Δ-modules is noetherian if and only if the family of homomorphisms between finitely generated indecomposable right Δ-modules is conoetherian.*

Proof. Let $R = R(\text{mod}\,\Delta)$ with $f \in R$ the idempotent that generates ${}_{\Delta}\Delta$, and let $\mathcal{S} = \mathcal{S}(\text{mod}\,\Delta^{\text{op}})$ with $h \in \mathcal{S}$ the idempotent that generates Δ_{Δ}. From [8] we see that the family of homomorphisms between finitely generated indecomposible left Δ-modules is noetherian if and only if R is left perfect. Similarly, the family of homomorphisms between finitely generated right Δ-modules is conoetherian if and only if $\mathcal{S}$ is left perfect. Since $R/RfR \cong \mathcal{S}/\mathcal{S}h\mathcal{S}$ via the transpose duality, Lemma 5 finishes the proof.

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