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QF-3 Rings and Categories of Projective Modules

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In [2] M. Auslander proved that for Artin algebras there is a one-to-one correspondence between Morita equivalence classes of QF-3 maximal quotient rings R with global dimension $\leqslant 2$ and of rings A of finite representation type. Recently C. M. Ringel and the author [11] have remarked that in this result the condition on R and A to be Artin algebras is unnecessary, and have given a generalization for semiprimary rings.

The purpose of this note is to show that our result is in full generality from the view points (i) or (ii) of the following

THEOREM 1. *Let R be a ring with an identity* 1 *and ${}_R\mathcal{M}$ the category of (unital) left R-modules. Denote by $\mathcal{A}$ the full subcategory of ${}_R\mathcal{M}$ consisting of all projective left R-modules. Then, the following conditions* (i)–(v) *are equivalent.*

(i) *$\mathcal{A}$ is a Grothendieck category, i.e., an Abelian category with generators and exact direct limits.*

(ii) *$\mathcal{A}$ is a full subcategory of ${}_R\mathcal{M}$ consisting of all torsion free, divisible left R-modules with respect to the largest torsion theory for which an injective module ${}_RI$ is torsion free (cf., Lambek* [6]).

(iii) *$\mathcal{A}$ is a full subcategory of ${}_R\mathcal{M}$ consisting of all left R-modules with X-dominant dimension $\geqslant 2$, where X is a left R-module of type FI in the sense of Morita* [7].

(iv) *R is a semiprimary QF-3 ring such that* $\text{dom dim}_R\, R \geqslant 2$ *and*

$$\text{gl dim}\ {}_RR \leqslant 2.$$

(v) *R is a semiprimary QF-3 ring with a minimal faithful right ideal fR, $f^2 = f$, such that $R = \text{End}({}_{fRf}fR)$ and fRf is a left Artinian ring having only a finite number of finitely generated indecomposable left fRf-modules (in the sequel such rings will be called of finite representation type), and further* *every finitely generated indecomposable left fRf-module appears as one of the direct summands of* ${}_{fRf}fR$.

It is interesting that we had already many results which play important roles in the proof of Theorem 1.

We shall say that X-dom dim M (respectively, dom dim M) $\geqslant n$ for $X, M \in {}_R\mathscr{M}$ if there is an exact sequence

$$0 \to M \to X_1 \to \cdots \to X_n$$

such that X_i $(1 \leqslant i \leqslant n)$ are direct products of copies of X (respectively, projective and injective). Let $\mathscr{D}(X) = \{M \in {}_R\mathscr{M} \mid X\text{-dom dim } M \geqslant 2\}$. It is well known that when X is injective, $M \in \mathscr{D}(X)$ if and only if M is torsion free and divisible with respect to the torsion theory with the torsion class $\mathscr{T} = \{Y \in {}_R\mathscr{M} \mid \operatorname{Hom}_R(Y, X) = 0\}$, and $\mathscr{D}(X)$ is equivalent to the quotient category ${}_R\mathscr{M}/\mathscr{T}$.

Following R. M. Thrall [14] a ring R is said to be left QF-3 if there exists a (unique) minimal faithful left R-module ${}_RU$. We can identify ${}_RU$ with a left ideal Re $(e^2 = e)$ which is isomorphic to the injective hull of a direct sum of finitely many simple left ideals. Right QF-3 rings are defined similarly, and rings which are both left QF-3 and right QF-3 will simply be called QF-3 rings. For a QF-3 ring, left maximal quotient ring and right maximal quotient ring coincide, and as was proved in [10], R is a QF-3 maximal quotient ring if and only if R is isomorphic to an endomorphism ring of a linear compact generator and cogenerator module M over a ring A. In this case A is a ring which is Morita dual to some ring B with respect to a bimodule ${}_AU_B$, ${}_AM$ is a U-reflexive, generator and cogenerator, and $R \cong \operatorname{End}({}_AM)$.

In case of semiprimary QF-3 ring R, it holds that R is its maximal quotient ring if and only if Re-dom dim ${}_RR$ $(=$ dom dim $R) \geqslant 2$, where Re is a minimal faithful left R-module.

Now, we shall begin the proof of Theorem 1.

Proof. (i) $\Rightarrow$ (ii). Since any direct sum of copies of ${}_RR$ and any epimorphism in ${}_R\mathscr{M}$ mean a direct sum of ${}_RR$ and an epimorphism in $\mathscr{A}$ respectively, ${}_RR$ is a generator in $\mathscr{A}$. Then by Popesco—Gabriel's theorem [9] $\mathscr{A}$ is equivalent to a quotient category of ${}_R\mathscr{M}$ by a localizing subcategory $\mathscr{L}$, because the endomorphism ring of ${}_RR$ is R itself. However, $\mathscr{L}$ is defined by using suitable injective left R-module I and Morita proved in [7] that ${}_R\mathscr{M}/\mathscr{L}$ is equivalent to $\mathscr{D}(I)$.

(ii) $\Rightarrow$ (i). This was proved in [7]. (ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (ii). Let us denote $B = \operatorname{End}({}_RI)$ and $C = \operatorname{End}(I_B)$. Since ${}_RI$ is of type FI, it holds that

(1) ${}_CI$ is finitely cogenerating and injective.

(2) ${}_C[\mathrm{Hom}_R({}_RC_C\,,\,{}_RI_B)]_B \cong {}_CI_B\,.$

On the other hand, by the assumption ${}_RR \in \mathscr{D}(I)$.

Hence by Morita [9, Theorem 5.1] or Suzuki [12, Theorem 8] (2) implies $C \cong R$

It follows that ${}_RI$ is injective.

(ii) $\Rightarrow$ (iv). Since ${}_RI$, ${}_RR \in \mathscr{D}(I)$, ${}_RI$ is projective, (injective) and faithful.

The fact $R \in \mathscr{D}(I)$ implies also that $\prod_{\Gamma} R \in \mathscr{D}(I)$ for any index set Γ (cf., [8]) and hence $\prod_{\Gamma} R$ is projective. Then, it follows by Chase [4, Theorem 3.3] that R is left perfect.

So, without loss of generality we can assume that I is a left ideal Re, with $e^2 = e$, and Re is $\prod$-projective, because any product of copies of $Re\,(=I)$ belongs to $\mathscr{D}(I)$ and is projective. Now, it is clear that

$$\text{dom dim}\ {}_RR \geqslant 2.$$

Then by Colby—Rutter [5, 1.2 Theorem and 1.3 Theorem] R is QF-3 and semiprimary, and Re is Σ-injective. Let $S = \bigoplus_{\alpha\in\Lambda} S_\alpha$, S_α simple, be an essential socle of $\prod Re$. Since each S_α is isomorphic to a submodule of Re, the injective hull $E(S_\alpha)$ of S_α is a direct summand of Re. So $E(S_\alpha)$ is Σ-injective. Since there are only finitely many isomorphism types of simple modules, $\bigoplus_{\alpha\in\Lambda} E(S_\alpha)$ is the injective hull of S. Hence

$$\prod Re = \bigoplus_{\alpha\in\Lambda} E(S_\alpha) \subseteq \bigoplus_{\Lambda} Re.$$

Now ${}_RR \in \mathscr{D}(Re)$ insures us the existence of an exact sequence

$$0 \longrightarrow R \xrightarrow{\ \sigma\ } \bigoplus_{i=1}^{r} Re \xrightarrow{\ \tau\ } \bigoplus_{i=1}^{s} Re$$

with positive integers r and s.

Finally, in order to prove that gl dim ${}_RR \leqslant 2$ we shall show that for any homomorphism ρ of a free R-module $\bigoplus_J R$ into R, $\mathrm{Ker}\,\rho$ is projective. For any finite subset F of J, let ρ_F be the restriction of ρ on $\bigoplus_F R$. Since $(\bigoplus_{i=1}^{r} Re)$ is injective, we have a homomorphism ψ to make the following diagram commutative:

$$\begin{array}{ccccccc}
0 \longrightarrow & \bigoplus_F R & \xrightarrow{\oplus_F\sigma} & \bigoplus_F\left(\bigoplus_{i=1}^{r} Re\right) & \xrightarrow{\oplus_F\tau} & \bigoplus_F\left(\bigoplus_{j=1}^{s} Re\right) \\
 & \downarrow{\scriptstyle\rho_F} & & \downarrow{\scriptstyle\psi} & & \\
0 \longrightarrow & R & \xrightarrow{\quad\sigma\quad} & \bigoplus_{i=1}^{r} Re. & &
\end{array}$$

Then, $\mathrm{Ker}\,\rho_F \cong \mathrm{Ker}(\bigoplus_F \tau) \cap \mathrm{Ker}\,\psi$.

On the other hand, $\mathrm{Ker}(\oplus_F \tau) \cap \mathrm{Ker}\, \psi$ becomes a kernel of the homomorphism

$$\bigoplus_F \left(\bigoplus_{i=1}^{r} Re\right) \xrightarrow{(\oplus_F \tau, \psi)} \bigoplus_F \left(\bigoplus_{j=1}^{s} Re\right) \oplus \left(\bigoplus_{i=1}^{r} Re\right).$$

Hence $\mathrm{Ker}\, \rho_F \in \mathscr{D}(Re)$ and is projective. Since $\mathrm{Ker}\, \rho = \bigcup_{\rho_F} \mathrm{Ker}\, \rho_F$, where F goes through all finite subsets of J, and R is left perfect, $\mathrm{Ker}\, \rho$ is projective and $\mathrm{gl\,dim}\, R \leqslant 2$.

(iv) $\Leftrightarrow$ (v). This was proved in [11]. So we shall omit the proof.

(iv) $\Rightarrow$ (ii). Let R be semiprimary QF-3 with a minimal faithful left module (ideal) Re. By the assumption we have an exact sequence:

$0 \to R \to^{\sigma} \prod_{J_1} Re \to \prod_{J_2} Re$ and R is maximal left quotient ring. Assume K is any index set. Then

$$0 \longrightarrow \bigoplus_K R \xrightarrow{\oplus_K \sigma} \bigoplus_K \left(\prod_{J_1} Re\right) \longrightarrow \bigoplus_K \left(\prod_{J_1} Re/\sigma(R)\right) \longrightarrow 0$$

is exact. Since R is semiprimary QF-3, Re and $\prod_{J_1} Re$ are $\sum$-injective. Hence, $\oplus_K (\prod_{J_1} Re)$ is injective. Then

$$E\left(\bigoplus_K R\right) \subseteq \bigoplus_K \left(\prod_{J_1} Re\right) \subseteq \prod_K \left(\prod_{J_1} Re\right).$$

Putting

$$\prod_K \left(\prod_{J_1} Re\right) = E\left(\bigoplus_K R\right) \oplus Y,$$

we have

$$\prod_K \left(\prod_{J_1} Re\right) \Big/ \bigoplus_K R \cong E\left(\bigoplus_K R\right) \Big/ \bigoplus_K R \oplus Y.$$

Therefore

$$\prod_K \left(\prod_{J_1} Re\right) \Big/ \bigoplus_K R \subseteq \prod_K \left(\prod_{J_2} Re\right) \oplus \prod_K \left(\prod_{J_1} Re\right).$$

This implies

$$Re\text{-dom dim} \bigoplus_K R \geqslant 2, \qquad \text{i.e.,} \quad \bigoplus_K R \in \mathscr{D}(Re).$$

Hence, if ${}_R X \in \mathscr{A}$, then $X \in \mathscr{D}(Re)$.

Conversely, let ${}_RX \in \mathscr{D}(Re)$. Then there is an exact sequence:

$$0 \to X \to \prod_{I_1} Re \to \prod_{I_2} Re.$$

Since dom dim $R \geqslant 2$ and R is semi-primary QF-3, R is its maximal quotient ring. Hence $R = \mathrm{End}(Re_{eRe})$ and eRe possesses fRe-duality.

Then, since eRe is semiprimary, eRe is right Artinian. Thus Re is $\sum$-injective, for Re_{eRe} is finitely generated.

By [5, 1.3 Theorem] Re is $\prod$-projective, and $\prod_{I_1} Re$ and $\prod_{I_2} Re$ are both projective. Therefore it follows from gl dim ${}_RR \leqslant 2$ that $X \in \mathscr{A}$.

This completes the proof.

Following F. W. Anderson and K. R. Fuller [1], a decomposition

$$M = \bigoplus_A M_\alpha$$

of a module M as a direct sum of nonzero submodules $(M_\alpha)_{\alpha \in A}$ is said to complement direct summands in case for each direct summand L of M there is a subset $B \subset A$ with

$$M = L \oplus \Big(\bigoplus_{\beta \in B} M_\beta\Big).$$

They proved that a ring R is left perfect if and only if every projective left R-module has a decomposition that complements direct summands.

Now, let A be a ring of finite representation type and $M_1, \ldots, M_n$ left A-modules representing all isomorphism classes of finitely generated indecomposable left A-modules. Let M be $\oplus_{i=1}^n M_i$ and $R = \mathrm{End}({}_AM)$.

Then, as was proved in [11] R is a semiprimary QF-3 maximal quotient ring with a minimal faithful right ideal $fR_R \simeq M_R$, and l gl dim $R \leqslant 2$. In this case we may identify A with fRf, ${}_{fRf}fR$ is a generator and ${}_{fRf}fRe$ is an injective cogenerator provided Re is a minimal faithful left ideal. It follows by [13, Theorem 3] that $\mathscr{D}(Re)$ and ${}_{fRf}\mathscr{M}$ are equivalent to each other by functors $S = (fR \otimes_R -)$ and $T = \mathrm{Hom}_{fRf}(fR, -)$. But by Theorem 1 $\mathscr{D}(Re)$ is the full subcategory of ${}_R\mathscr{M}$ consisting of all projective left R-modules.

As R is semiprimary (of course, left perfect), the result of Anderson–Fuller quoted above gives us easily

COROLLARY 2. *If a ring A is of finite representation type, then every left A-module has an indecomposable direct sum decomposition that complements direct summands.*

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