# Encyclopædia Inflationaris 

Jérôme Martin ${ }^{\text {a }}$, Christophe Ringeval ${ }^{\text {b,* }}$, Vincent Vennin ${ }^{\text {a }}$<br>${ }^{\mathrm{b}}$ Institut d'Astrophysique de Paris, UMR 7095-CNRS, Université Pierre et Marie Curie, 98 bis boulevard Arago, 75014 Paris, France<br>${ }^{\text {a }}$ Centre for Cosmology, Particle Physics and Phenomenology, Institute of Mathematics and Physics, Louvain University, 2 Chemin du Cyclotron, 1348 Louvain-la-Neuve, Belgium

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#### Abstract

The current flow of high accuracy astrophysical data, among which are the Cosmic Microwave Background (CMB) measurements by the Planck satellite, offers an unprecedented opportunity to constrain the inflationary theory. This is however a challenging project given the size of the inflationary landscape which contains hundreds of different scenarios. Given that there is currently no observational evidence for primordial non-Gaussianities, isocurvature perturbations or any other non-minimal extension of the inflationary paradigm, a reasonable approach is to consider the simplest models first, namely the slow-roll single field models with minimal kinetic terms. This still leaves us with a very populated landscape, the exploration of which requires new and efficient strategies. It has been customary to tackle this problem by means of approximate model independent methods while a more ambitious alternative is to study the inflationary scenarios one by one. We have developed the new publicly available runtime library ASPIC ${ }^{1}$ to implement this last approach. The ASPIC code provides all routines needed to quickly derive reheating consistent observable predictions within this class of scenarios. ASPIC has been designed as an evolutive code which presently supports 74 different models, a number that may be compared with three or four representing the present state of the art. In this paper, for each of the ASPIC models, we present and collect new results in a systematic manner, thereby constituting the first Encyclopædia Inflationaris. Finally, we discuss how this procedure and ASPIC could be used to determine the best model of inflation by means of Bayesian inference.


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## 1. Introduction

The theory of inflation [1-4] represents a cornerstone of the standard model of modern cosmology (the "hot Big-Bang model" of Lemaître and Friedmann) [5-8]. By definition, it is a phase of accelerated expansion which is supposed to take place in the very early universe, at very high energy, between Big-Bang Nucleosynthesis (BBN) and $10^{15} \mathrm{GeV}$. Inflation allows us to understand several puzzles that plagued the pre-inflationary standard model (before 1981) and that could not be understood otherwise. Without inflation, the standard model of cosmology would remain incomplete and highly unsatisfactory. The most spectacular achievement of inflation is that, combined with quantum mechanics, it provides a convincing mechanism for the

[^0]origin of the cosmological fluctuations (the seeds of the galaxies and of the Cosmic Microwave Background - CMB - anisotropies) and predicts that their spectrum should be almost scale invariant (i.e. equal power on all spatial scales) [9-17] which is fully consistent with the observations. Let us notice in passing that this part of the scenario is particularly remarkable since it combines General Relativity and Quantum Mechanics [7,8,18-24]. Given all these spectacular successes and given the fact that, despite many efforts, inflation has not been superseded by its various challengers [25-53], this scenario has gradually become a crucial part of modern cosmology. As can be seen in Fig. 1, the number of papers devoted to this topic and published each year is inflating since the advent of inflation.

In order to produce a phase of inflation within General Relativity, the matter content of the universe has to be dominated by a fluid with negative pressure. At very high energy, the correct description of matter is field theory, the prototypical example being a scalar field since it is compatible with the symmetries implied by the cosmological principle. Quite remarkably, if the potential of this scalar field is sufficiently flat (in fact, more


Fig. 1. Number of articles containing the word "inflation" and its variations (i.e. "inflating", "inflationary", etc...) in its title published each year since the advent of inflation. The total number is estimated to be 4077 papers.
precisely, its logarithm) so that the field moves slowly, then the corresponding pressure is negative. This is why it is believed that inflation is driven by one (or several) scalar field(s). For obvious reasons, this scalar field was given the name "inflaton". However, the physical nature of the inflaton and its relation with the standard model of particle physics and its extensions remain elusive. Moreover the shape of its potential is not known except that it must be sufficiently flat. This is not so surprising since, as mentioned above, the inflationary mechanism is supposed to take place at very high energies in a regime where particle physics is not known and has not been tested in accelerators.

Another crucial aspect of the inflationary scenario is how it ends and how it is connected to the subsequent hot Big-Bang phase. It is believed that, after the slow-roll period, the field oscillates at the bottom of its potential, or undergoes tachyonic preheating, but finally decays into radiation. In this way, inflation is smoothly connected to the radiation-dominated epoch [54-63]. Unfortunately, very little is observationally known on this socalled reheating period. Let us stress that adiabatic initial conditions, as favored from the current CMB measurements, naturally stem from such a setup within single field models. Another constraint is that the reheating temperature, $T_{\text {reh }}$, must be higher than the nucleosynthesis scale (i.e. a few MeV ). If, however, one restricts oneself to specific models, then one can obtain better bounds on $T_{\text {reh }}$, as was recently shown for the first time in Ref. [64]. But, so far, these constraints concern a few models only.

We see that, despite the fact that it has become a cornerstone, the inflationary era is not as observationally known as the other parts of the standard model of Cosmology. However, there is now a flow of increasingly accurate astrophysical data which gives us a unique opportunity to learn more about inflation. In particular, the recently released Planck satellite data [65,71] play a crucial role in this process. The mission complements and improves upon observations made by the NASA WMAP satellite $[72,73]$ and is a major source of information relevant to several cosmological issues including inflation [67,69]. But the flow of new data does not only concern the CMB. The Supernovae projects [74-77] continue to measure the distances to the nearby exploding SN1A stars while the large scale galaxy surveys such as the Sloan Digital Sky Survey (SDSS) $[78,79]$ are providing an unprecedented picture of the structure of the universe. SDSS is planned till 2014 and has recently provided the measure of the so-called Baryonic Acoustic Oscillations (BAO). They are the red-shifted version of the acoustic oscillations observed in the CMB anisotropies which have been transferred to the galaxy power spectrum. The "lever arm" in length scales between CMB and galaxy power spectra increases the sensitivity to the small deviations from scale invariance, and thus should be extremely powerful to constrain inflationary models. For


Fig. 2. Observational predictions for the LFI models, $V(\phi) \propto \phi^{p}$, in the plane ( $n_{S}, r$ ) (i.e. scalar spectral index and gravity wave contribution) compared to the Planck data [65-70]. Each continuous line and each color represent a different value of $p$. Along each line, each point (i.e. each small "cross") denotes a different reheating temperature compatible with the constraint $\rho_{\text {end }}>\rho_{\text {reh }}>\rho_{\text {nuc }}$ (the annotations give the logarithm of the reheating temperature in GeV ). We see that the details of the reheating stage now matter: along a given line, some reheating temperatures are compatible with the observational constraints while others are not. This means that the CMB observations can now put constraints on $T_{\text {reh }}$. The mean equation of state parameter is defined in Eq. (2.38). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
this reason, the future Euclid satellite will be another step forward in our understanding of inflation [80]. Let us also mention the possibility of direct detection of the primordial gravitational waves for high energy inflationary models on large scales [81-87] and also on small scales $[62,88]$.

The CMB small angular scales of Planck are already complemented by ground-based microwave telescopes such as the Atacama Cosmology Telescope (ACT) $[89,90]$ or the South Pole Telescope (SPT) [91,92] while ultra-sensitive polarization dedicated experiments are on their way $[93,94]$. In a foreseeable future, the last bit of yet unexplored length scales are expected to be unveiled by the 21 cm cosmological telescopes. These ones will be sensitive to the red-shifted 21 cm line absorbed by hydrogen clouds before the formation of galaxies [95-101]. With such data, we will have a complete tomography of the universe history from the time of CMB emission at the surface of last scattering to the distribution of galaxies today.

The main goal of this article is to develop methods that will allow us to constrain the inflationary scenario at a level matching the accuracy of these new data. Since we have now entered the era of massive multi-data analysis, the project aims at a change of scale compared to previous approaches. In particular, one way to deal with this question is to perform systematic and "industrial" studies of this issue. Our ability to see through the inflationary window turns the early universe into a laboratory for ultra-high energy physics, at scales entirely inaccessible to conventional experimentation. In other words, this window offers a unique opportunity to learn about the very early universe and about physics in a regime that cannot be tested otherwise, even in accelerators such as the Large Hadron Collider (LHC).

### 1.1. Methodology

Let us now discuss how, in practice, the above described goals can be reached. One issue often raised is that, since there are (literally) a few hundreds different scenarios, it is difficult to falsify inflation. This is, however, not a very convincing argument since different models belong to different classes and usually do differ in their observable predictions. They can thus be observationally distinguished. A natural way to proceed is therefore to test inflationary models step by step, starting with the simplest


Fig. 3. Exact slow-roll predictions for SFI models, $V(\phi) \propto 1-(\phi / \mu)^{4}$, compared to the Planck data [65-70]. Each colored segment represents a different value of $\mu$, the color bar giving the corresponding range of variation. Each segment is made of different points associated with different reheating temperatures. The yellowonly segments on the left represent some extra approximations usually made in the literature on top of slow-roll. We see that both coincide for $\mu / m_{\mathrm{PI}} \ll 1$ but differ in the regime $\mu / m_{\mathrm{PI}} \gg 1$ where the extra approximations become inaccurate. Moreover, these approximations would indicate that this class of models is disfavored while the correct slow-roll predictions show that, on the contrary, they remain compatible with the data. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)


Fig. 4. Predictions of the RCHI model in the plane ( $n_{\mathrm{s}}, r$ ) together with Planck data [65-70]. These predictions depend on one free parameter, $A_{1}$, for details see Section 4.1. The colored segments represent the slow-roll predictions (same conventions as in Fig. 3), obtained when the coefficients $a_{i}=a_{i}\left[\epsilon_{n}\left(\theta_{\text {inf }}\right)\right]$ are numerically evaluated. On the contrary, the thick red dashed line indicates some approximated predictions. We see that there is a significant difference for $A_{\mathrm{I}} \gtrsim 15$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
scenarios. This is consistent with the Occam's razor point of view and the way inference is achieved within Bayesian statistics (see below). With this in mind, we can classify models in three different broad categories: single-field inflation (category I), multiple-field inflation (category II) and models where matter is not described by a scalar field as, for instance, vector inflation [102], chromonatural inflation [103] and/or gauge-flation [104-106] (category III). Within each category, one could further identify various subcategories. For example, within category I, the scalar field can possess a minimal kinetic term and a smooth potential (category IA), a minimal kinetic term and a potential with features (category IB), a non-minimal kinetic term with a smooth potential (category IC) or a non-minimal kinetic term and a potential with features (category ID, see for instance Ref. [107]) (a fifth category could be models of warm inflation [108-111]). The same four subcategories can also be defined within category II [for instance, multiple Dirac Born Infeld (DBI) field inflation [112-114] belongs to category IIC] and so on. As already mentioned, each category
leads to different predictions. For instance, all models of category IA predict a negligible level of non-Gaussianities, $f_{\mathrm{NL}}^{\text {loc }}=5(1-$ $\left.n_{\mathrm{s}}\right) / 12 \simeq 0.017$ [115-123] while, on the contrary, models of categories IB-ID yield non-negligible non-Gaussianities [124139]; models belonging to IB and to IC, or II, may not predict exactly the same type of non-Gaussianities [140,141], etc...In this context, as already mentioned, a crucial step was the recent release of the Planck data [65-68]. Together with the polarization data from WMAP, they are compatible with a negligible running $\mathrm{d} n_{\mathrm{S}} / \mathrm{d} \ln k=-0.0134 \pm 0.009$ and a negligible running of the running $\mathrm{d}^{2} n_{\mathrm{S}} / \mathrm{d} \ln ^{2} k=0.02 \pm 0.016$, with a pivot scale chosen at $k_{*}=0.05 \mathrm{Mpc}^{-1}$. These data are also compatible with adiabaticity at $95 \%$ CL such that there is no evidence for isocurvature modes, although the analysis is done with one isocurvature mode at a time only. The Planck data do not find evidence for primordial nonGaussianity, namely Ref. [67] reports $f_{\mathrm{NL}}^{\text {loc }}=2.7 \pm 5.8, f_{\mathrm{NL}}^{\text {eq }}=$ $-42 \pm 75$ and $f_{\mathrm{NL}}^{\text {ortho }}=-25 \pm 39$. Therefore, at this stage, everything seems to be well described by simplest scenarios of inflation and, as consequence, a reasonable method is to start with the IA-models. Following category IA, if the present observational situation evolves in the future, one should then treat categories IBID, then category II and so on. In this way, one can falsify inflation step by step, in a Bayesian motivated fashion.

Bayesian inference for inflation requires some cosmological data that are sensitive to it, such as the ones enumerated above. For the purpose of illustration, let us consider the CMB angular power spectrum. Cosmological measurements give us a set of numbers, $C_{\ell}^{\text {meas }}$, that we are able to calculate theoretically within an inflationary model. This means that we know the functions $C_{\ell}^{\text {th }} \equiv C_{\ell}^{\text {th }}\left(\theta_{\text {stand }}, \theta_{\text {inf }}\right)$, where $\theta_{\text {stand }}$ represents a set of parameters describing post-inflationary physics, i.e. $\theta_{\text {stand }}=\left(h, \Omega_{\Lambda}, \Omega_{\mathrm{dm}}, \ldots\right)$ and $\theta_{\text {inf }}$ a set of parameters describing inflationary physics. We are interested in constraining the values of those parameters, especially the $\theta_{\text {inf }}$ 's. Within a given experiment, one is given a likelihood, or an effective chi-squared $\chi^{2}\left(\theta_{\text {stand }}, \theta_{\text {inf }}\right)$, encoding all the underlying uncertainties. In a frequentist approach, the searched values of $\theta_{\text {stand }}$ and $\theta_{\text {inf }}$ would be chosen at the best fit, i.e. those verifying $\partial \chi^{2} / \partial \theta=0$. In a Bayesian approach [142], we are interested in determining the posterior distributions of the parameters, using Bayes's theorem

$$
\begin{equation*}
P\left(\theta_{\text {stand }}, \theta_{\text {inf }} \mid C_{\ell}^{\text {meas }}\right)=\frac{1}{\mathcal{N}} \mathscr{L}\left(C_{\ell}^{\text {meas }} \mid \theta_{\text {stand }}, \theta_{\text {inf }}\right) \pi\left(\theta_{\text {stand }}, \theta_{\text {inf }}\right) \tag{1.1}
\end{equation*}
$$

where $\mathcal{L}\left(C_{\ell}^{\text {meas }} \mid \theta_{\text {stand }}, \theta_{\text {inf }}\right)=e^{-\chi^{2}\left(\theta_{\text {stand }}, \theta_{\text {inf }}\right) / 2}$ is the likelihood function, $\pi\left(\theta_{\text {stand }}, \theta_{\text {inf }}\right)$ the prior distribution, describing our prejudices about the values of the parameters before our information is updated, and $\mathcal{N}$ a normalization factor, also called Bayesian evidence. Because we are interested in the inflationary parameters, one has to integrate over the post-inflationary parameters in order to obtain the marginalized probability distribution $P\left(\theta_{\text {inf }} \mid C_{\ell}^{\text {meas }}\right)=\int P\left(\theta_{\text {stand }}, \theta_{\text {inf }} \mid C_{\ell}^{\text {meas }}\right) \mathrm{d} \theta_{\text {stand }}$. CMB physics also tells us that the multipole moment $C_{\ell}^{\text {th }}$ can be written as
$C_{\ell}^{\text {th }}\left(\theta_{\text {stand }}, \theta_{\text {inf }}\right)=\int_{0}^{+\infty} \frac{\mathrm{d} k}{k} j_{\ell}\left(k r_{\ell s \mathrm{~s}}\right) T\left(k ; \theta_{\text {stand }}\right) \mathcal{P}_{\zeta}\left(k ; \theta_{\text {inf }}\right)$,
where $j_{\ell}$ is a spherical Bessel function, $T\left(k ; \theta_{\text {stand }}\right)$ is the transfer function which describes the evolution of cosmological perturbations during the standard Friedmann-Lemaître eras and $\mathcal{P}_{\zeta}$ is the inflationary power spectrum. As a result, the process of constraining inflation from the $C_{\ell}^{\text {meas }}$ reduces to the calculation of $\mathcal{P}_{\zeta}$. The same lines of reasoning could be generalized to any other cosmological observables sourced during inflation, such as higher order correlation functions.

At this stage, there are, a priori, two possibilities (it is also worth noticing that yet another approach is the reconstruction program [143,144]). Either one uses a model-independent, necessarily approximate, shape for $\mathscr{P}_{\zeta}$ or, on the contrary, one scans the inflationary landscape, model by model, and for each of them, calculates $\mathcal{P}_{\zeta}$ exactly.

The advantage of working with a model-independent technique is obvious. However, it often requires an approximation scheme that may not be available for all models. In practice, an approximate method, the slow-roll approach, is known for the category IA and for the category IC, see the recent papers [145150]. In this case, the set of inflationary parameters $\theta_{\text {inf }}$ becomes the Hubble flow functions: $\theta_{\text {inf }}=\left\{\epsilon_{n}\right\}$ where the $\epsilon_{n}$ are defined in Eq. (2.3) and the corresponding expression of $\mathcal{P}_{\zeta}\left(k ; \epsilon_{n}\right)$ is provided in Eqs. (2.18) and (2.20)-(2.22). Assuming some priors $\pi\left(\epsilon_{n}\right)$ on the Hubble flow functions, this method yields the posterior distributions $P\left(\epsilon_{n} \mid C_{\ell}^{\text {meas }}\right)$ for the Hubble flow functions evaluated at the pivot scale. This approach has already been successfully implemented for the WMAP data in Refs. [64,151-154].

The second approach is more ambitious. It consists in treating exactly all the inflationary models that have been proposed so far and in a systematic manner. For each model, the power spectrum is determined exactly by means of a mode by mode numerical integration, for instance using the FieldInf code. ${ }^{2}$ Such an approach can also be used with the higher correlation functions with, for instance, the recent release of the BINGO code calculating the inflationary bispectrum [155].

In this case, the set of parameters $\theta_{\text {inf }}$ differs according to the model considered. For instance, Large Field Inflation (LFI) for which $V(\phi)=M^{4}\left(\phi / M_{\mathrm{PI}}\right)^{p}$, has $\theta_{\text {inf }}=(M, p)$ while Small Field Inflation (SFI) with $V(\phi)=M^{4}\left[1-(\phi / \mu)^{p}\right]$ has $\theta_{\text {inf }}=(M, p, \mu)$. From FieldInf one can then compute $\mathcal{P}_{\zeta}(k ; M, p)$ for LFI and $\mathcal{P}_{\zeta}(k ; M, p, \mu)$ for SFI without any other assumptions than linear perturbation theory and General Relativity. Starting from some priors on the model parameters, e.g. in the case of LFI, $\pi(M, p)$, this method allows us to determine the posterior distributions $P\left(M \mid C_{\ell}^{\text {meas }}\right)$ and $P\left(p \mid C_{\ell}^{\text {meas }}\right)$, thereby providing parameter inference about the corresponding inflationary model. This approach, which was successfully implemented for the first time in Refs. [152,156158], and subsequently used in Ref. [159], has several advantages that we now discuss.

Firstly, the most obvious advantage is that the result is exact. The slow-roll method is an approximation and, for this reason, remains somehow limited. As mentioned before, there are plethora of models, such as single field models with features or multiple field scenarios, for which a numerical integration is mandatory.

A second reason is that a full numerical approach permits a new treatment of reheating. In the standard approach, the influence of the reheating is only marginally taken into account. Any observable predictions depend on the number of $e$-folds associated with a reheating era. From the fact that the reheating must proceed after the end of inflation and before the electroweak scale, one can put an order of magnitude bound on this number of $e$-folds [160]. This causes small uncertainties in the inflationary predictions that were not crucial in the past. However, with the accuracy of the present and future data this question now matters. This is illustrated in Fig. 2 which represents the slow-roll predictions of LFI for which $V(\phi) \propto \phi^{p}$. Each colored segment represents the range of observable predictions for a given value of $p$, each point within a segment corresponding to a given number of $e$-folds for the reheating or, equivalently, to a given reheating temperature $T_{\text {reh }}$. We see that, for relatively small values of $p$, it is necessary to know the number of $e$-folds the Universe reheated to decide whether the

[^1]model is compatible with the data or not. Conversely, the data are becoming so accurate that one can start constraining the reheating epoch. Therefore, instead of viewing the reheating parameters as external source of uncertainties, it is more accurate to include them in the numerical approach and consider they are part of the inflationary model. In its simplest description, the reheating epoch can be modeled as a cosmological fluid with a mean equation of state $\bar{w}_{\text {reh }}>-1 / 3$. Notice that $w_{\text {reh }}$, the instantaneous equation of state parameter, does not need to be constant (see Section 2.2). For a simple quadratic potential, and a parametric reheating, one would have for instance $\bar{w}_{\text {reh }}=0$. In this way, both $\bar{w}_{\text {reh }}$ and $T_{\text {reh }}$ are added to the inflationary parameters, e.g. we now have $\theta_{\text {inf }}=\left(M, p, T_{\text {reh }}, \bar{w}_{\text {reh }}\right)$ for LFI, and FieldInf computes $\mathcal{P}_{\zeta}\left(k ; M, p, T_{\text {reh }}, \bar{w}_{\text {reh }}\right)$. Starting from some priors $\pi\left(T_{\text {reh }}, \bar{w}_{\text {reh }}\right)$ one can then obtain the corresponding posterior distributions $P\left(T_{\text {reh }} \mid C_{\ell}^{\text {meas }}\right)$ and $P\left(\bar{w}_{\text {reh }} \mid C_{\ell}^{\text {meas }}\right)$. The feasibility of this method has already been demonstrated in Refs. [64,152] where constraints on the reheating temperature for LFI and SFI have been derived for the first time (see also Ref. [161]). In view of the expected accuracy of the future data, the preheating/reheating era should become a compulsory element of inflationary model testing. This issue plays an important role in the proposal put forward in this article. In addition, let us also emphasize that a proper treatment of the reheating and preheating stages is mandatory in multiple field inflation because they can affect the evolution of $\mathcal{P}_{\zeta}$ on large scales. Only a numerical approach can presently deal with this problem.

A third advantage of the numerical approach is to address the question of the priors choice in a particularly well-defined way. A crucial aspect of Bayesian statistics is that the result depends on the choice of the priors. Therefore, these ones must be chosen and discussed carefully. In the slow-roll (approximated) approach described before, the priors are chosen on the slow-roll parameters themselves. For instance, a Jeffreys' prior is typically chosen on $\epsilon_{1}$ (i.e. uniform prior on $\log \epsilon_{1}$ ), as appropriate when the order of magnitude of a parameter is not known. However, from a physical point of view, it is better to choose the priors directly on the parameters of the model, e.g. the parameters entering the potential. For instance, several potentials that we will treat are the results of a one-loop calculation, namely a perturbative calculation with the coupling constant playing the role of the small parameter. It is clear that the prior must encode the fact that this parameter is small. With the numerical approach, this is very conveniently done since we directly compute the power spectrum from the potential itself. As another example, let us consider the case of LFI where $\epsilon_{1} \simeq p /\left(4 \Delta N_{*}+p / 4\right)\left(\Delta N_{*}\right.$ is the number of $e$-folds between Hubble exit and the end of inflation, see below). Owing to the non-trivial relation between the first slow-roll parameter and $p$, a Jeffreys' prior $\pi\left(\epsilon_{1}\right)$ on $\epsilon_{1}$ implies a complicated prior $\pi(p)$ on $p$ while a natural choice would be a flat prior. Again, implementing the priors directly on the parameters of the model is a more theoretically justified choice. Conversely, who could dispute that, beside the posterior $P\left(\epsilon_{1} \mid C_{\ell}^{\text {meas }}\right)$, it is theoretically interesting to know the posterior distribution of $p$, i.e. $P\left(p \mid C_{\ell}^{\text {meas }}\right)$. The exact numerical integration is a reliable technique to obtain such distributions.

The numerical approach, however, has also some disadvantages. Firstly, one needs to specify the inflationary scenarios explicitly and, therefore, the constraints obtained are not modelindependent. Although this shortcoming can in fact never be avoided (we always need to make some assumptions even in the slow-roll approach) it may be partially overcome by scanning the complete inflationary landscape. Secondly, and more importantly, it is time consuming since the exact integration of the cosmological perturbations and of the corresponding correlation functions is heavy and can take up to a few minutes for complicated models.


Fig. 5. Higgs Inflation (HI). Top left panel: Higgs potential corresponding to Eq. (3.9). Top right panel: logarithm of the Higgs potential. It is clear from these two plots that inflation proceeds from the right to the left. Bottom left panel: slow-roll parameter $\epsilon_{1}$ as a function of the field $\phi$. The shaded area indicates the breakdown of the slow-roll inflation (strictly speaking when the acceleration stops) and we see that, in this model, the end of inflation occurs by violation of the slow-roll conditions. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) for the same potential.


Fig. 6. Lambert functions $\mathrm{W}_{0}(x)$ (dashed line) and $\mathrm{W}_{-1}(x)$ (solid line). During Higgs inflation, inflation proceeds along the " -1 " branch in the direction specified by the arrow in the figure.

Finally, one should expect multiple degeneracies for models having a high number of inflationary parameters since the data have a limited sensitivity to the shape of the primordial observables.

Based on the previous considerations, we conclude that it would be very interesting to have an intermediate method that would allow us to get most of the results that can be derived using the exact numerical approach while being less time consuming and immune to high parameter degeneracies. This is what we suggest in the following. Our strategy is to use the slow-roll approximation in order to skip the numerical calculation of the power spectrum, while being combined with a systematic scan of the whole inflationary landscape and reheating properties. As argued before, the Planck data drive us towards testing inflation with the simplest models first and such a method would therefore need to be implemented for the class of scenarios IA only. More
precisely, instead of inferring the posterior distributions of the Hubble flow parameters $\epsilon_{n}$ only, as one would naturally do in the approximate approach discussed before, we take advantage of the fact that the $\epsilon_{n}$ 's can be computed in terms of the parameters describing the reheating and $V(\phi)$. In particular, for each model, this permits a quick and efficient extraction of the posterior distributions of those parameters.

In our opinion, this third technique should not be viewed as a competitor of the two others mentioned earlier but rather as complementary and the corresponding results should be compared. Let us also notice that, if, in order to scan all the inflationary scenarios, the full exact numerical approach needs to be carried out at some point, this would by no means render the results derived in the present article useless. Indeed, the slow-roll approach is often a very useful guide of which kind of physics one should expect for a given model (initial conditions, range of the parameters, etc...). In particular it allows us to understand any eventual parameter degeneracies within the primordial observables. In other words, the slow-roll method is an ideal tool to prepare a full numerical study.

At this point, it is worth making the following remark. The method put forward in this article uses an approximate shape for the power spectrum, namely ( $k_{*}$ is the pivot scale)
$\mathcal{P}_{\zeta}(k) \propto a_{0}\left(\epsilon_{n}\right)+a_{1}\left(\epsilon_{n}\right) \ln \left(\frac{k}{k_{*}}\right)+\frac{1}{2} a_{2}\left(\epsilon_{n}\right) \ln ^{2}\left(\frac{k}{k_{*}}\right)+\cdots$,
in order to shortcut a numerical integration of $\mathcal{P}_{\zeta}$ but is otherwise completely self-consistent. In other words, once the slow-roll approximation is accepted, no additional approximation should be made. This may still require some numerical calculations, however, in order to determine the coefficients $a_{i}$, or more precisely the explicit expression, at Hubble crossing, of $a_{i}=a_{i}\left[\epsilon_{n}\left(\theta_{\mathrm{inf}}\right)\right]$. This


Fig. 7. Top left panel: the solid blue line represents the radiatively corrected Higgs potential, see Eq. (4.11), with $A_{\mathrm{I}}=5$. It is compared to the tree level potential given by Eq. (3.9) (dashed green line) and to Eq. (4.11) with $A_{I}=0$ (solid red line) which is supposed to be a good approximation of the tree level potential. It is obvious that this is indeed the case in the regime of interest, where the vev of the Higgs field is not too small. Top right panel: logarithm of potential, the three lines and the color code having the same meaning as in the top left panel. Bottom left panel: slow-roll parameter $\epsilon_{1}$ as a function of the field $\phi$, still with the same convention. As can be seen in this plot, even in presence of radiative corrections, the end of inflation occurs by violation of the slow-roll condition. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid blue line) and $\epsilon_{3}$ (dashed blue line) for $A_{I}=5$ compared to their tree level counter parts (solid and dashed green lines, respectively).


Fig. 8. Predictions of the RCHI model in the plane ( $n_{\mathrm{S}}, r$ ). The exact slow-roll predictions (colored segments starting in black/green at the bottom/left part of the plot and ending in red right slightly on the right of the allowed contours) are compared to various approximations represented by the second collection of colored segments, by the red thick dashed line and by the yellow dotted-dashed line, see the text for a detailed explanation. In the regime $10<A_{I}<100$, the exact predictions significantly differ from the approximate ones. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
is an important issue given the accuracy of the current data as it is illustrated in Fig. 3 (see also Ref. [152]). In this figure, we have represented the slow-roll predictions of a SFI model, $V(\phi) \propto$ $1-(\phi / \mu)^{4}$. Each colored segment represents the exact slow-roll predictions of a model given the parameter $\mu$ and for different numbers of $e$-folds during the reheating. These predictions have been computed by solving numerically the slow-roll equations. But, in the same plot, there are also other segments, on the
left, and represented in yellow only. They are predictions for different values of $\mu$ but based on widespread approximate slowroll formulas used in the literature. We see that, given the accuracy of the data, the approximated formulas are no longer accurate enough: the approximate results would predict that models with $\mu / M_{\mathrm{PI}}>1$ are strongly disfavored while the correct slow-roll results show that they are still compatible with the data. Another textbook example is provided by Higgs inflation with radiative corrections (RCHI) and is presented in Fig. 4. This scenario is studied in detail in Section 4.1 and depends on one free parameter, $A_{\mathrm{I}}$. The colored segments represent the exact predictions for different values of $A_{\mathrm{I}}$ (see the color bar on the side of the plot). The red dashed line indicates predictions based on a commonly used approximate equation for the coefficients $a_{i}=a_{i}\left(\epsilon_{n}\right)$ at Hubble crossing during inflation. We see that this is no longer sufficient in the range $A_{I} \gtrsim 15$. From these two examples, we conclude that it is safer to use the slow-roll approximation (which is usually extremely good) and nothing else, in particular no extra approximation on top of the slow-roll approximation. The fact that we may still need to use numerical calculations to establish the observational predictions of a model does not make our approach useless. Indeed, the numerics needed to estimate $a_{i}=a_{i}\left[\epsilon_{n}\left(\theta_{\text {inf }}\right)\right]$ are, by far, much easier than those needed to exactly compute $\mathcal{P}_{\zeta}$. Therefore, the gain in computational time mentioned above is huge and allows for a fast and reliable method to constrain the inflationary landscape.

### 1.2. The ASPIC library

The project described before contains many different aspects that we intend to publish in several companion articles. We now


Fig. 9. Large Field Inflation (LFI). Top left panel: large field potential for $p=2$. Top right panel: logarithm of the potential for the same value of $p$. The required flatness of the potential becomes obvious on this plot. Bottom left panel: slow-roll parameter $\epsilon_{1}$ for a large field potential with $p=2$. The shaded area indicates where acceleration stops. Bottom right panel: slow-roll parameters $\epsilon_{2}$ and $\epsilon_{3}$ for a large field potential with $p=2$. Only one curve appears because $\epsilon_{2}=\epsilon_{3}$. On this plot, the shaded region signals the breakdown of the slow-roll approximation, which is not necessarily the end of the accelerated phase.
explain the purpose of the present paper and put it in context with the other works that are in preparation. We have coded a public runtime library, named ASPIC for "Accurate Slow-roll Predictions for Inflationary Cosmology", which is supposed to contain all the inflationary models that can be treated with the method described above. ASPIC already has 74 different inflationary scenarios, a number that should be compared to the three or four models that are usually considered. The ASPIC library is an open source evolutive project and, although it already contains all the most popular inflationary scenarios, aims at including more models. In this way, it will converge towards a situation where all the category IA models published since the advent of inflation are implemented thereby allowing us to exhaustively scan this part of the inflationary landscape. This article describes the ASPIC project and presents its first release and others will follow. The list of the 74 ASPIC models, as well as their acronym, is presented in Table 1 at the end of this introduction. If future cosmological data force us to move to more complicated scenarios, the ASPIC library will be upgraded accordingly. It can, moreover, already be interfaced with FieldInf thereby allowing for a full numerical approach, if needed. This would be especially relevant for all the single field models with modified kinetic terms (category IB) such as DBI models, models with features (category IC) such as the Starobinsky model [162] or multiple field inflationary scenarios (category II) such as double inflation [163166], double inflation with an interaction term [167], the different versions of hybrid inflation [57,168,169] and more [156], assisted inflation [170] or Matrix inflation [171-173,173]. However, if the data continue to favor simple models, such as those producing negligible non-Gaussianities and isocurvature perturbations, the ASPIC library in its present form already contains the most relevant inflationary scenarios. The ASPIC library is publicly available at http://cp3.irmp.ucl.ac.be/~ringeval/aspic.html.

The ASPIC library contains the necessary routines to compare the predictions of any of the 74 different models to high-accuracy data. The present article presents the general architecture of the ASPIC project and the calculations needed to understand and write these codes. In practice, for each model, we give the calculation of the three first slow-roll parameters, a discussion of how inflation ends, a discussion of the priors, a calculation of the relevant range of variation of the reheating temperature and an exact integration of the slow-roll trajectory. Then, we work out the theoretical predictions and compare them to the Planck data in the planes $\left(\epsilon_{1}, \epsilon_{2}\right)$ and ( $\left.n_{s}, r\right)$. Let us stress again that, beside slow-roll, no other approximation is used in the numerical codes of ASPIC.

Most of the ASPIC models have already been partially studied in the literature but let us emphasize that, for each of them, this paper contains new results. In other words, it does not aim at being a review and, therefore, the presentation of already derived results have been kept to the minimal. Firstly, for all the models studied, this is the first time that their observational predictions are worked out when the constraints on the reheating phase are accurately taken into account. As explained in Ref. [64], and briefly reviewed in Section 2, it has become too inaccurate to derive the predictions of a model by simply assuming a fixed range for $\Delta N_{*}$. For instance, this could lead to a reheating energy density larger than the energy density at the end of inflation which is physically irrelevant. Therefore, the predictions have been re-worked in such a consistent fashion (except for the LFI and SFI models which had been studied before [64]). This already constitutes a significant result which goes beyond the current state-of-the-art. Secondly, in the Appendix, we present a series of plots which give the predictions of the various ASPIC models in the planes $\left(n_{S}, r\right)$ and $\left(\epsilon_{1}, \epsilon_{2}\right)$ for different values of the free parameters characterizing each potential. Clearly, this is the first time that the predictions of all these models are compared to the Planck data. The only

Table 1
Models contained in the first release of the ASPIC library. For each model, we give the corresponding acronym, the number of free parameters characterizing the potential the number of sub-models and the shape of the potential. The total number of models is 74 .

| Name | Parameters | Sub-models | $V(\phi)$ |
| :---: | :---: | :---: | :---: |
| HI | 0 | 1 | $M^{4}\left(1-e^{-\sqrt{2 / 3} \phi / M_{\mathrm{Pl}}}\right)^{2}$ |
| RCHI | 1 | 1 | $M^{4}\left(1-2 e^{-\sqrt{2 / 3} \phi / M_{\mathrm{Pl}}}+\frac{A_{\mathrm{I}}}{16 \pi^{2}} \frac{\phi}{\sqrt{6} M_{\mathrm{Pl}}}\right)$ |
| LFI | 1 | 1 | $M^{4}\left(\frac{\phi}{M_{\text {Pl }}}\right)^{p}$ |
| MLFI | 1 | 1 | $M^{4} \frac{\phi^{2}}{M_{\mathrm{Pl}}^{2}}\left[1+\alpha \frac{\phi^{2}}{M_{\mathrm{Pl}}^{2}}\right]$ |
| RCMI | 1 | 1 | $M^{4}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{2}\left[1-2 \alpha \frac{\phi^{2}}{M_{\mathrm{Pl}}^{2}} \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right)\right]$ |
| RCQI | 1 | 1 | $M^{4}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{4}\left[1-\alpha \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right)\right]$ |
| NI | 1 | 1 | $M^{4}\left[1+\cos \left(\frac{\phi}{f}\right)\right]$ |
| ESI | 1 | 1 | $M^{4}\left(1-e^{-q \phi / M_{\mathrm{Pl}}}\right)$ |
| PLI | 1 | 1 | $M^{4} e^{-\alpha \phi / M_{\mathrm{Pl}}}$ |
| KMII | 1 | 2 | $M^{4}\left(1-\alpha \frac{\phi}{M_{\mathrm{Pl}}} e^{-\phi / M_{\mathrm{Pl}}}\right)$ |
| HF1I | 1 | 1 | $M^{4}\left(1+A_{1} \frac{\phi}{M_{\mathrm{Pl}}}\right)^{2}\left[1-\frac{2}{3}\left(\frac{A_{1}}{1+A_{1} \phi / M_{\mathrm{Pl}}}\right)^{2}\right]$ |
| CWI | 1 | 1 | $M^{4}\left[1+\alpha\left(\frac{\phi}{Q}\right)^{4} \ln \left(\frac{\phi}{Q}\right)\right]$ |
| LI | 1 | 2 | $M^{4}\left[1+\alpha \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right)\right]$ |
| RpI | 1 | 3 | $M^{4} e^{-2 \sqrt{2 / 3} \phi / M_{\mathrm{Pl}}}\left\|e^{\sqrt{2 / 3} \phi / M_{\mathrm{Pl}}}-1\right\|^{2 p /(2 p-1)}$ |
| DWI | 1 | 1 | $M^{4}\left[\left(\frac{\phi}{\phi_{0}}\right)^{2}-1\right]^{2}$ |
| MHI | 1 | 1 | $M^{4}\left[1-\operatorname{sech}\left(\frac{\phi}{\mu}\right)\right]$ |
| RGI | 1 | 1 | $M^{4} \frac{\left(\phi / M_{P 1}\right)^{2}}{\alpha+\left(\phi / M_{P 1}\right)^{2}}$ |
| MSSMI | 1 | 1 | $M^{4}\left[\left(\frac{\phi}{\phi_{0}}\right)^{2}-\frac{2}{3}\left(\frac{\phi}{\phi_{0}}\right)^{6}+\frac{1}{5}\left(\frac{\phi}{\phi_{0}}\right)^{10}\right]$ |
| RIPI | 1 | 1 | $M^{4}\left[\left(\frac{\phi}{\phi_{0}}\right)^{2}-\frac{4}{3}\left(\frac{\phi}{\phi_{0}}\right)^{3}+\frac{1}{2}\left(\frac{\phi}{\phi_{0}}\right)^{4}\right]$ |
| AI | 1 | 1 | $M^{4}\left[1-\frac{2}{\pi} \arctan \left(\frac{\phi}{\mu}\right)\right]$ |
| CNAI | 1 | 1 | $M^{4}\left[3-\left(3+\alpha^{2}\right) \tanh ^{2}\left(\frac{\alpha}{\sqrt{2}} \frac{\phi}{M_{\mathrm{Pl}}}\right)\right]$ |
| CNBI | 1 | 1 | $M^{4}\left[\left(3-\alpha^{2}\right) \tan ^{2}\left(\frac{\alpha}{\sqrt{2}} \frac{\phi}{M_{\text {Pl }}}\right)-3\right]$ |
| OSTI | 1 | 1 | $-M^{4}\left(\frac{\phi}{\phi_{0}}\right)^{2} \ln \left[\left(\frac{\phi}{\phi_{0}}\right)^{2}\right]$ |
| WRI | 1 | 1 | $M^{4} \ln \left(\frac{\phi}{\phi_{0}}\right)^{2}$ |
| SFI | 2 | 1 | $M^{4}\left[1-\left(\frac{\phi}{\mu}\right)^{p}\right]$ |
| II | 2 | 1 | $M^{4}\left(\frac{\phi-\phi_{0}}{M_{\mathrm{PI}}}\right)^{-\beta}-M^{4} \frac{\beta^{2}}{6}\left(\frac{\phi-\phi_{0}}{M_{\mathrm{PI}}}\right)^{-\beta-2}$ |
| KMIII | 2 | 1 | $M^{4}\left[1-\alpha \frac{\phi}{M_{\mathrm{Pl}}} \exp \left(-\beta \frac{\phi}{M_{\mathrm{Pl}}}\right)\right]$ |
| LMI | 2 | 2 | $M^{4}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{\alpha} \exp \left[-\beta\left(\phi / M_{\mathrm{Pl}}\right)^{\gamma}\right]$ |
| TWI | 2 | 1 | $M^{4}\left[1-A\left(\frac{\phi}{\phi_{0}}\right)^{2} e^{-\phi / \phi_{0}}\right]$ |
| GMSSMI | 2 | 2 | $M^{4}\left[\left(\frac{\phi}{\phi_{0}}\right)^{2}-\frac{2}{3} \alpha\left(\frac{\phi}{\phi_{0}}\right)^{6}+\frac{\alpha}{5}\left(\frac{\phi}{\phi_{0}}\right)^{10}\right]$ |
| GRIPI | 2 | 2 | $M^{4}\left[\left(\frac{\phi}{\phi_{0}}\right)^{2}-\frac{4}{3} \alpha\left(\frac{\phi}{\phi_{0}}\right)^{3}+\frac{\alpha}{2}\left(\frac{\phi}{\phi_{0}}\right)^{4}\right]$ |
| BSUSYBI | 2 | 1 | $M^{4}\left(e^{\sqrt{6} \frac{\phi}{M_{\mathrm{Pl}}}}+e^{\sqrt{6} \gamma \frac{\phi}{M_{P 1}}}\right)$ |
| TI | 2 | 3 | $M^{4}\left(1+\cos \frac{\phi}{\mu}+\alpha \sin ^{2} \frac{\phi}{\mu}\right)$ |
| BEI | 2 | 1 | $M^{4} \exp _{1-\beta}\left(-\lambda \frac{\phi}{M_{\mathrm{Pl}}}\right)$ |
| PSNI | 2 | 1 | $M^{4}\left[1+\alpha \ln \left(\cos \frac{\phi}{f}\right)\right]$ |

Table 1 (continued)

| Name | Parameters | Sub-models | $V(\phi)$ |
| :---: | :---: | :---: | :---: |
| NCKI | 2 | 2 | $M^{4}\left[1+\alpha \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right)+\beta\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{2}\right]$ |
| CSI | 2 | 1 | $\frac{M^{4}}{\left(1-\alpha \frac{\phi}{M_{P I}}\right)^{2}}$ |
| OI | 2 | 1 | $M^{4}\left(\frac{\phi}{\phi_{0}}\right)^{4}\left[\left(\ln \frac{\phi}{\phi_{0}}\right)^{2}-\alpha\right]$ |
| CNCI | 2 | 1 | $M^{4}\left[\left(3+\alpha^{2}\right) \operatorname{coth}^{2}\left(\frac{\alpha}{\sqrt{2}} \frac{\phi}{M_{\mathrm{Pl}}}\right)-3\right]$ |
| SBI | 2 | 2 | $M^{4}\left\{1+\left[-\alpha+\beta \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right)\right]\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{4}\right\}$ |
| SSBI | 2 | 6 | $M^{4}\left[1+\alpha\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{2}+\beta\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{4}\right]$ |
| IMI | 2 | 1 | $M^{4}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{-p}$ |
| BI | 2 | 2 | $M^{4}\left[1-\left(\frac{\phi}{\mu}\right)^{-p}\right]$ |
| RMI | 3 | 4 | $M^{4}\left[1-\frac{c}{2}\left(-\frac{1}{2}+\ln \frac{\phi}{\phi_{0}}\right) \frac{\phi^{2}}{M_{\mathrm{Pl}}^{2}}\right]$ |
| VHI | 3 | 1 | $M^{4}\left[1+\left(\frac{\phi}{\mu}\right)^{p}\right]$ |
| DSI | 3 | 1 | $M^{4}\left[1+\left(\frac{\phi}{\mu}\right)^{-p}\right]$ |
| GMLFI | 3 | 1 | $M^{4}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{p}\left[1+\alpha\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{q}\right]$ |
| LPI | 3 | 3 | $M^{4}\left(\frac{\phi}{\phi_{0}}\right)^{p}\left(\ln \frac{\phi}{\phi_{0}}\right)^{q}$ |
| CNDI | 3 | 3 | $\frac{M^{4}}{\left\{1+\beta \cos \left[\alpha\left(\frac{\phi-\phi_{0}}{M_{\mathrm{PI}}}\right)\right]\right\}^{2}}$ |



Fig. 10. Top left panel: mixed large field (MLFI) potential, see Eq. (4.45), for $\alpha=0.05$. Top right panel: logarithm of the potential for the same value of $\alpha$. The dotted line indicates the potential $V(\phi) \simeq M^{4} \phi^{2} / M_{\mathrm{PI}}^{2}$ which is the limit of the MLFI potential in the regime $\phi / M_{\mathrm{Pl}} \ll 1 / \sqrt{\alpha}$ while the dashed line represents the expression $V(\phi) \simeq M^{4} \alpha \phi^{4} / M_{\mathrm{P}}^{4}$, the limit of $V(\phi)$ when $\phi / M_{\mathrm{PI}} \gg 1 / \sqrt{\alpha}$. For $\alpha=0.05$ the two lines meet at the following value, $1 / \sqrt{\alpha} \simeq 4.5$, as can be directly checked in the figure. The arrow in the top left and right panels indicate in which direction inflation proceeds. Bottom left panel: slow-roll parameter $\epsilon_{1}$ for a mixed large field potential with $\alpha=0.05$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) still for $\alpha=0.05$.
exception is Ref. [174] which studies a very small subset of the ASPIC scenarios (but also studies non-minimal single field models), Ref. [175] which studies the particular case of power law (PLI) and Ref. [176] which studies the particular case of MSSM inflation (MSSMI). Most often, this is also the first time that these predictions are worked out for such a wide range of parameters and, moreover, this is the first time that these predictions are presented in this fashion. In some sense, our paper can be viewed as the first Encyclopædia Inflationaris.

### 1.3. New results

In order to be completely clear about the fact that this paper is not a review, we now highlight, in a non-exhaustive way, some of the new results obtained in this paper. In this way, we hope it gives a taste of all the new findings described later and the methods advocated earlier.

In the case of Higgs Inflation (HI), for instance, we have found an exact expression of the slow-roll trajectory and discuss the reheating parameter in the case of scalar-tensor theories of gravity. The exact trajectory is also found for radiatively corrected inflation ( RCHI ) and we show that the exact predictions can differ from the commonly used ones in a certain regime, see also Fig. 4. In the case of Mixed Large Field Inflation (MLFI), the exact expressions of the slow-roll parameters $\epsilon_{2}$ and $\epsilon_{3}$ are new. We also calculate exactly $\phi_{\text {end }}$, the vev at which inflation stops, as well as the exact trajectory $N(\phi)$ and its inverse, $\phi(N)$. Interestingly, since the potential is the sum of a quadratic and a quartic term, one would expect the corresponding predictions to be located between the two lines in the plane ( $n_{\mathrm{s}}, r$ ) representing the quadratic LFI and the quartic LFI models, see for instance Fig. 2. We show that this is not the case. For Natural Inflation (NI), we provide the exact expression of $\phi_{\text {end }}$, of the trajectory and its inverse. In addition, it is often claimed that, in the limit $f / M_{\mathrm{PI}} \gg 1$, the model is indistinguishable from a quadratic one (LFI with $p=2$ ). We show that it is true for $n_{\mathrm{S}}$ and $r$ but is not accurate for $\epsilon_{3}$, that is to say for the running $\alpha_{\mathrm{S}}$. For the Kähler Moduli Inflation I (KMII) and Kähler Moduli Inflation II (KMIII) models, all our results are basically new. We present, for the first time, the exact expressions of the slow-roll parameters, of the trajectories, their inverses, the possible values of $\alpha$, a free parameter characterizing the shape of the potentials (not to be confused with the running). We also emphasize the role played by the running in this model: $n_{\mathrm{S}}$ and $r$ are perfectly compatible with the data while $\alpha_{\mathrm{S}}$ seems to constrain the model more efficiently. However, contrary to what is commonly claimed in the literature, we demonstrate that this does not rule out these models. Within the Logamediate inflation (LMI) scenario, we have derived an analytic expression for the trajectory in terms of hypergeometric functions and exhibited a new inflationary domain LMI2, which is however like almost a pure de Sitter era and currently disfavored. We also have new results for the Coleman Weinberg Inflation (CWI) scenario. We find exact expressions for $\epsilon_{3}$ and an exact determination of the end of inflation. We discuss, for the first time, the predictions of the model in the full parameter space. In the case of Double Well Inflation (DWI), we present a clear slow-roll analysis. The expressions of $\epsilon_{3}, \phi_{\text {end }}$, the slow-roll trajectory, its inverse are all new. Moreover, a detailed comparison with SFI is made and we show that the corresponding predictions actually differ, contrary to what is sometimes written in the literature. In the case of the Minimal Super-Symmetric Model (MSSMI) scenario, we demonstrate several new results. We give the exact expression of the slow-roll parameters $\epsilon_{2}$ and $\epsilon_{3}$, the location and the value of the maximum of the first slowroll parameter $\epsilon_{1}$, an approximated formula for $\phi_{\text {end }}$, the exact slow-roll trajectory and a useful approximated version of it. We also provide a parameter independent treatment of the quantum
diffusion regime: usually this is always done using specific values of the parameters whereas we show that the corresponding conclusions are in fact completely general. We also explain why the model is quite strongly disfavored due to the observational constraints on the spectral index. For the Renormalizable Inflection Point Inflation (RIPI) scenario, the slow roll parameters $\epsilon_{2}$ and $\epsilon_{3}$, the location and the value of the maximum of $\epsilon_{1}$, the approximated determination of $\phi_{\text {end }}$, the exact slow-roll trajectory and a useful approximated version of it are all new. We also discuss the CMB normalization and calculate the energy scale of inflation very accurately. Last but not the least, we show that the model is strongly disfavored by the data. We have also explored the Generalized MSSM Inflation (GMSSMI) scenario. We provide new formulas for $\epsilon_{2}, \epsilon_{3}$ and the trajectory. We also give new bounds on the parameters characterizing the potential from the requirement of having a sufficient number of $e$-folds during inflation. Finally, we show that the model is disfavored by the data. Concerning the Brane Susy Breaking Scenario (BSUSYBI), we have studied the effects coming from the the field value at which inflation ends, in the slow-roll regime. For the ArcTan Inflation (AI) scenario, we work out the slow-roll analysis beyond the approximation of vacuum domination and give an exact expression for $\epsilon_{3}$ and the slow-roll trajectory. For the class of models leading to a constant spectral index, CNAI, CNBI, CNCI and CNDI, we show how to calculate $\phi_{\text {end }}$ and the trajectory exactly. We also demonstrate that the spectral index is in fact constant only in a limited region of the parameter space which turns out to be already disfavored by the data. In the case of Intermediate Inflation (II), we present an analysis which takes into account the two terms of the potential while it is common to keep only the dominant one. We give new expressions for $\epsilon_{3}$, the slow-roll trajectory and its relation with the exact, non-slow-roll, one. In the case of Twisted Inflation (TWI), we study this model for the first time in a regime where it is not equivalent to DSI. We give new expressions for $\epsilon_{3}$, the exact trajectory and the CMB normalization. We also discuss how inflation ends and show, contrary to a naive expectation, that it cannot happen by the end of the slow-rolling phase. For the Pseudo Natural Inflation (PSNI) scenario, we present new formulas for $\epsilon_{2}$, $\epsilon_{3}, \phi_{\text {end }}$ and the trajectory. This is the first time that a slow-roll analysis of Orientifold Inflation (OI) is made. As a consequence, all the corresponding results are new. In particular, we demonstrate that the model is in bad shape because it predicts a too important amount of gravitational waves. The scenario of Spontaneous Symmetry Breaking Inflation (SSBI) is important because it can cover many physically different situations. This model actually contains six different sub-models. The third slow-roll parameter, the trajectory and the CMB normalization are new results obtained for the first time in this paper. In the case of Dynamical Symmetric Inflation (DSI), we present new expressions for $\epsilon_{3}$, the trajectory and the CMB normalization. Another important result is also a careful analysis of the prior space and the limits derived on the parameters of the model which are such that it is disfavored by observations due to its blue tilt. For the Generalized Mixed Large Field Inflation (GMLFI) model, we present new equations for $\epsilon_{2}$ and $\epsilon_{3}$ and the trajectory. Concerning the LPI models, we have exhibited three domains in which inflation could take place, thereafter denoted by LPI1, LPI2 and LPI3. For the Non-Canonical Kähler Inflation model (NCKI), we provide new results for $\epsilon_{2}$ and $\epsilon_{3}$, the trajectory and the CMB normalization. We also analyze the predictions for different values of $\beta$, a parameter characterizing the potential. We show that the case $\beta<0$ is ruled out while $\beta>0$ is disfavored by the observations. We have also studied Loop Inflation (LI). For this model, we give new expressions of $\epsilon_{3}, \phi_{\text {end }}$, the trajectory and its inverse in terms of a Lambert function. Also, the slow-roll analysis is carried out in the case where the correcting term is negative which we could not find elsewhere. In the case of


Fig. 11. Lambert functions $W_{0}(x)$ (dashed line) and $W_{-1}(x)$ (solid line). During Mixed Large Field inflation, inflation proceeds along the " 0 " branch above the line $W=1$ in the direction specified by the arrow.

Tip Inflation (TI), we also give $\epsilon_{3}, \phi_{\text {end }}$ and the trajectory. We also study which amounts of fine-tuning is required by the model and finally show that it is ruled out because its spectrum deviates too strongly from scale invariance. Many other new results are given in this article but, as mentioned above, we do not summarize all of them here due to space limitation. They can be found in the sections devoted to the various models listed in Table 1.

Before concluding this introduction, let us remark that this article and the ASPIC library represent important tools to carry out our final goal which consists in assessing how good is a model and in comparing the various inflationary models. This problem can be dealt within Bayesian inference for model comparison. For this purpose, one has to calculate, for each model, the global likelihood which is obtained by integrating the usual likelihood over all of the model parameter values, weighted by their respective prior probability distribution. The resulting quantity is a number associated with each model which gives the "evidence" that the model explains the data [this is the number $\mathcal{N}$ in Eq. (1.1)]. Their respective ratios give the odds that one model explains all data compared to the others. Bayesian methods have the advantage to automatically incorporate the "Occam's razor": complicated inflationary models will be assigned large probability only if the complexity is required by the data. On the practical side, these two steps can be implemented by the use of Markov-Chains-MonteCarlo (MCMC) methods, which is especially well suited with the exact numerical approach advocated before. These techniques have already been successfully implemented first in Ref. [177], and later on in Ref. [161], and we plan to extent them to all the models of the ASPIC library. As a matter of fact, this will allow us to scan the inflationary landscape in a statistically well-defined way and to address the question of "the best model of inflation" [178-180].

This article is organized as follows. In the next section, Section 2, we briefly summarize slow-roll inflation and give the equations needed for the rest of this article. We also discuss the reheating stage and explains how it can be implemented. Then, in Section 3, we study inflationary models which, up to the potential normalization, do not contain any free parameter (concretely, at this stage, Higgs inflation). In Sections 4-6, we analyze scenarios characterized by one, two and three free parameters, respectively. Finally, in Section 7, we present our conclusions and discuss future works. In the Appendix, we give, in the planes $\left(n_{\mathrm{S}}, r\right)$ and $\left(\epsilon_{1}, \epsilon_{2}\right)$, the predictions of all the 74 ASPIC models.

## 2. Basic equations

In this section, we very briefly recall the theoretical foundations of inflation and we present the main tools and equations that will
be used in the rest of this paper. We start by reviewing the slowroll phase, where the cosmological fluctuations are generated and, then, we describe how the end of inflation and the transition to the standard hot Big Bang phase can be modeled.

### 2.1. The slow-roll phase

Let us consider a single-field inflationary model with a minimal kinetic term and a potential $V(\phi)$. The behavior of the system is controlled by the Friedmann-Lemaître and Klein-Gordon equations, namely
$H^{2}=\frac{1}{3 M_{\mathrm{Pl}}^{2}}\left[\frac{\dot{\phi}^{2}}{2}+V(\phi)\right]$,
$\ddot{\phi}+3 H \dot{\phi}+V_{\phi}=0$,
where $H \equiv \dot{a} / a$ denotes the Hubble parameter, $a(t)$ being the Friedmann-Lemaître-Robertson Walker (FLRW) scale factor and $\dot{a}$ its derivative with respect to cosmic time $t . M_{\mathrm{Pl}}=8 \pi G$ denotes the reduced Planck mass. A subscript $\phi$ means a derivative with respect to the inflaton field. In order to describe the evolution of the background, it is convenient to introduce the Hubble flow functions $\epsilon_{n}$ defined by [181,182]
$\epsilon_{n+1} \equiv \frac{\mathrm{~d} \ln \left|\epsilon_{n}\right|}{\mathrm{d} N}, \quad n \geq 0$,
where $\epsilon_{0} \equiv H_{\text {ini }} / H$ and $N \equiv \ln \left(a / a_{\text {ini }}\right)$ is the number of $e$-folds. By definition, inflation is a phase of accelerated expansion, $\ddot{a} / a>0$, or, equivalently, $\epsilon_{1}<1$. As a consequence, the end of inflation is defined by the condition $\epsilon_{1}=1$. On the other hand, the slow-roll conditions (or slow-roll approximation) refer to a situation where all the $\epsilon_{n}$ 's satisfy $\epsilon_{n} \ll 1$. If this is the case, then the parameters $\epsilon_{n}$ can also be expressed in terms of the successive derivatives of the potential, namely [17]
$\epsilon_{1} \simeq \frac{M_{\mathrm{PI}}^{2}}{2}\left(\frac{V_{\phi}}{V}\right)^{2}$,
$\epsilon_{2} \simeq 2 M_{\mathrm{Pl}}^{2}\left[\left(\frac{V_{\phi}}{V}\right)^{2}-\frac{V_{\phi \phi}}{V}\right]$,
$\epsilon_{2} \epsilon_{3} \simeq 2 M_{\mathrm{Pl}}^{4}\left[\frac{V_{\phi \phi \phi} V_{\phi}}{V^{2}}-3 \frac{V_{\phi \phi}}{V}\left(\frac{V_{\phi}}{V}\right)^{2}+2\left(\frac{V_{\phi}}{V}\right)^{4}\right]$.
Therefore, a measurement of the $\epsilon_{n}$ 's also provides information with regards to the shape of the inflationary potential.

In terms of the number of $e$-folds, one can decouple Eqs. (2.1) and (2.2) to get the field evolution
$\frac{1}{3-\epsilon_{1}} \frac{\mathrm{~d}^{2} \phi}{\mathrm{~d} N^{2}}+\frac{\mathrm{d} \phi}{\mathrm{d} N}=-M_{\mathrm{Pl}}^{2} \frac{\mathrm{~d} \ln V}{\mathrm{~d} \phi}$,
showing that the potential driving the field in FLRW spacetime is $\ln [V(\phi)]$. This equation can be further simplified by using the definition of $\epsilon_{1}$ and $\epsilon_{2}$ to get ride of the second order derivatives. From
$\epsilon_{1}=\frac{1}{2 M_{\mathrm{Pl}}^{2}}\left(\frac{\mathrm{~d} \phi}{\mathrm{~d} N}\right)^{2}$,
one gets
$\left(1+\frac{\epsilon_{2}}{6-2 \epsilon_{1}}\right) \frac{\mathrm{d} \phi}{\mathrm{d} N}=-M_{\mathrm{Pl}}^{2} \frac{\mathrm{~d} \ln V}{\mathrm{~d} \phi}$.
As a result, in the slow-roll approximation, one has
$\frac{\mathrm{d} \phi}{\mathrm{d} N} \simeq-M_{\mathrm{Pl}}^{2} \frac{\mathrm{~d} \ln V}{\mathrm{~d} \phi}$.


Fig. 12. Radiatively Corrected Massive Inflation (RCMI) for $\alpha=0.01$. Top panels: potential (left) and logarithm of the potential (right). Bottom left panel: slow-roll parameter $\epsilon_{1}$ with respect to field values. The shaded area indicates where inflation stops. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line).

This equation can be integrated to give an explicit expression of the classical trajectory. One arrives at
$N-N_{\text {ini }}=-\frac{1}{M_{\mathrm{Pl}}^{2}} \int_{\phi_{\text {ini }}}^{\phi} \frac{V(\chi)}{V_{\chi}(\chi)} \mathrm{d} \chi$.
In this article, for each model, we provide the expressions of the first three Hubble flow parameters, a determination of $\phi_{\text {end }}$, the value of the field at which inflation comes to an end (and the corresponding discussion) and an explicit expression of the slowroll trajectory Eq. (2.11).

Let us now consider the behavior of inflationary cosmological perturbations. The evolution of scalar (density) perturbations can be reduced to the study of a single variable, the so-called Mukhanov-Sasaki variable $v_{\boldsymbol{k}}$. In Fourier space, its equation of motion can be expressed as $[6-8,16]$
$v_{\boldsymbol{k}}^{\prime \prime}+\left[k^{2}-\frac{\left(a \sqrt{\epsilon_{1}}\right)^{\prime \prime}}{a \sqrt{\epsilon_{1}}}\right] v_{\boldsymbol{k}}=0$.
Here, a prime denotes a derivative with respect to conformal time and the quantity $k$ is the comoving wave number of the Fourier mode under consideration. This equation is the equation of a parametric oscillator, i.e. an oscillator with a time-dependent frequency. The time-dependence of the effective frequency is controlled by the dynamics of the background, more precisely by the scale factor and its derivatives (up to fourth order). The quantity $v_{\boldsymbol{k}}$ is related to the curvature perturbation $\zeta_{\boldsymbol{k}}$ through the following expression:
$\zeta_{\boldsymbol{k}}=\frac{1}{M_{\mathrm{PI}}} \frac{v_{\boldsymbol{k}}}{a \sqrt{2 \epsilon_{1}}}$.
The importance of $\zeta_{\boldsymbol{k}}$ lies in the fact that it can be viewed as a "tracer" of the fluctuations on super-Hubble scales, i.e. for all $k \eta \ll 1$, where $\eta$ denotes the conformal time. Indeed, in the
case of single-field inflation, this quantity becomes constant in this limit. Therefore, it can be used to "propagate" the perturbations from inflation to the subsequent cosmological eras. The statistical properties of the fluctuations can be characterized by the $n$-point correlation functions of $\zeta_{\boldsymbol{k}}$. In particular, the two-point correlation function can be written as an integral over wave numbers (in a logarithmic interval) of the power spectrum $\mathcal{P}_{\zeta}(k)$, which can be expressed as
$\mathcal{P}_{\zeta}(k) \equiv \frac{k^{3}}{2 \pi^{2}}\left|\zeta_{\boldsymbol{k}}\right|^{2}=\frac{k^{3}}{4 \pi^{2} M_{\mathrm{Pl}}^{2}}\left|\frac{v_{\boldsymbol{k}}}{a \sqrt{\epsilon_{1}}}\right|^{2}$.
In order to calculate $\mathcal{P}_{\zeta}(k)$, one needs to integrate Eq. (2.12), which requires the knowledge of the initial conditions for the mode function $v_{\boldsymbol{k}}$. Since, at the beginning of inflation, all the modes of cosmological interest today were much smaller than the Hubble radius, the initial conditions are chosen to be the Bunch-Davis vacuum which amounts to
$\lim _{k \eta \rightarrow+\infty} v_{k}=\frac{1}{\sqrt{2 k}} e^{-i k \eta}$,
where $\mathscr{H}=a H$ is the conformal Hubble parameter.
The evolution of tensor perturbations (or primordial gravity waves) can also be reduced to the study of a parametric oscillator. The amplitude of each transverse Fourier mode of the gravity wave, $\mu_{\boldsymbol{k}}(\eta)$, obeys the following equation
$\mu_{\boldsymbol{k}}^{\prime \prime}+\left(k^{2}-\frac{a^{\prime \prime}}{a}\right) \mu_{\boldsymbol{k}}=0$.
We notice that the time-dependence of the effective frequency differs from that of the scalar case and now involves the derivative of the scale factor up to second order only. It is then straightforward to determine the resulting power spectrum. From a calculation of


Fig. 13. Radiatively Corrected Quartic Inflation (RCQI) for $\alpha=0.8$. Top panels: the potential and its logarithm as a function of the field values. Bottom left panel: slow-roll parameter $\epsilon_{1}$. The shaded area indicates where inflation stops. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line). The shaded region for $\epsilon_{2}$ and $\epsilon_{3}$ shows where the slow-roll approximation is violated for that value of $\alpha$.
the two-point correlation function, one obtains
$\mathcal{P}_{h}(k)=\frac{2 k^{3}}{\pi^{2}}\left|\frac{\mu_{\boldsymbol{k}}}{a}\right|^{2}$.
In order to calculate this quantity, the equation of motion Eq. (2.16) needs to be solved. As it is the case for density perturbations, the initial state is chosen to be the Bunch-Davies vacuum.

The power spectra can be computed exactly by means of a mode by mode integration of Eqs. (2.12) and (2.16), which also requires an exact integration of the background, i.e. of Eqs. (2.1) and (2.2). As discussed in the introduction, this can be done with the help of publicly available codes such as FieldInf. We have seen above that the slow-roll approximation can be used to calculate the classical background trajectory. Quite remarkably, the same approximation also permits the derivation of the scalar and tensor power spectra. This involves a double expansion. The power spectra are expanded around a chosen pivot scale $k_{*}$ such that
$\frac{\mathcal{P}(k)}{\mathcal{P}_{0}}=a_{0}+a_{1} \ln \left(\frac{k}{k_{*}}\right)+\frac{a_{2}}{2} \ln ^{2}\left(\frac{k}{k_{*}}\right)+\cdots$,
where
$\mathcal{P}_{\zeta_{0}}=\frac{H^{2}}{8 \pi^{2} \epsilon_{1} M_{\mathrm{Pl}}^{2}}, \quad \mathcal{P}_{h_{0}}=\frac{2 H^{2}}{\pi^{2} M_{\mathrm{Pl}}^{2}}$,
and, then, the coefficients $a_{i}$ are determined in terms of the Hubble flow functions. For scalar perturbations, one gets [145,146,182-

187,187-189]

$$
\begin{align*}
a_{0}^{(S)}= & 1-2(C+1) \epsilon_{1}-C \epsilon_{2}+\left(2 C^{2}+2 C+\frac{\pi^{2}}{2}-f\right) \epsilon_{1}^{2} \\
& +\left(C^{2}-C+\frac{7 \pi^{2}}{12}-g\right) \epsilon_{1} \epsilon_{2}+\left(\frac{1}{2} C^{2}+\frac{\pi^{2}}{8}-1\right) \epsilon_{2}^{2} \\
& +\left(-\frac{1}{2} C^{2}+\frac{\pi^{2}}{24}\right) \epsilon_{2} \epsilon_{3},  \tag{2.20}\\
a_{1}^{(S)}= & -2 \epsilon_{1}-\epsilon_{2}+2(2 C+1) \epsilon_{1}^{2}+(2 C-1) \epsilon_{1} \epsilon_{2}  \tag{2.21}\\
& +C \epsilon_{2}^{2}-C \epsilon_{2} \epsilon_{3},  \tag{2.22}\\
a_{2}^{(S)}= & 4 \epsilon_{1}^{2}+2 \epsilon_{1} \epsilon_{2}+\epsilon_{2}^{2}-\epsilon_{2} \epsilon_{3},
\end{align*}
$$

where $C \equiv \gamma_{\mathrm{E}}+\ln 2-2 \approx-0.7296, \gamma_{\mathrm{E}}$ being the Euler constant, $f=5$ and $g=7$. For the gravitational waves, the coefficients $a_{i}$ read

$$
\begin{align*}
a_{0}^{(\mathrm{T})}= & 1-2(C+1) \epsilon_{1}+\left(2 C^{2}+2 C+\frac{\pi^{2}}{2}-f\right) \epsilon_{1}^{2} \\
& +\left(-C^{2}-2 C+\frac{\pi^{2}}{12}-2\right) \epsilon_{1} \epsilon_{2},  \tag{2.23}\\
a_{1}^{(\mathrm{T})}= & -2 \epsilon_{1}+2(2 C+1) \epsilon_{1}^{2}-2(C+1) \epsilon_{1} \epsilon_{2},  \tag{2.24}\\
a_{2}^{(\mathrm{T})}= & 4 \epsilon_{1}^{2}-2 \epsilon_{1} \epsilon_{2} . \tag{2.25}
\end{align*}
$$

The Hubble flow functions are time-dependent quantities such that in the above expression, it is understood that they should be evaluated at the time at which the pivot scale crosses the Hubble radius during inflation, i.e. at a time $\eta_{*}$ such that $k_{*}=\mathscr{H}\left(\eta_{*}\right)$. Let us notice that setting the pivot at another time affects the previous expression. For instance, setting $\eta_{*}$ such that $k_{*} \eta_{*}=-1$ would set $f=3$ and $g=6$. We will see below that this introduces a dependence in the parameters describing the reheating stage.


Fig. 14. Natural Inflation (NI). Top left panel: potential for $f / M_{\mathrm{Pl}}=1.5$. Top right panel: logarithm of the potential for the same value of $f$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ for a potential with $f / M_{\mathrm{PI}}=1.5$. The shaded area indicates the breakdown of the slow-roll inflation (strictly speaking when the acceleration stops). Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) for a potential with $f / M_{\mathrm{PI}}=1.5$.

The properties of the power spectra can also be characterized by the spectral indices and their "running". They are defined by the coefficients of the Taylor expansions of the power spectra logarithm with respect to $\ln k$, evaluated at the pivot scale $k_{*}$. This gives
$n_{\mathrm{S}}-\left.1 \equiv \frac{\mathrm{~d} \ln \mathcal{P}_{\zeta}}{\mathrm{d} \ln k}\right|_{k_{*}},\left.\quad n_{\mathrm{T}} \equiv \frac{\mathrm{d} \ln \mathcal{P}_{h}}{\mathrm{~d} \ln k}\right|_{k_{*}}$.
For the runnings, one similarly has the two following expressions
$\left.\alpha_{\mathrm{S}} \equiv \frac{\mathrm{d}^{2} \ln \mathcal{P}_{\zeta}}{\mathrm{d}(\ln k)^{2}}\right|_{k_{*}},\left.\quad \alpha_{\mathrm{T}} \equiv \frac{\mathrm{d}^{2} \ln \mathcal{P}_{h}}{\mathrm{~d}(\ln k)^{2}}\right|_{k_{*}}$,
and, in principle, we could also define the running of the running and so on. The slow-roll approximation allows us to calculate the quantities defined above. For instance, we have at first order in the Hubble flow parameters
$n_{\mathrm{S}}=1-2 \epsilon_{1}-\epsilon_{2}, \quad n_{\mathrm{T}}=-2 \epsilon_{1}$.
Let us also notice that the tensor-to-scalar ratio at leading order can be expressed as
$r \equiv \frac{\mathcal{P}_{h}}{\mathcal{P}_{\zeta}}=16 \epsilon_{1}$.
In the rest of this article, we give the observational predictions of each inflationary model of the ASPIC library in the planes $\left(\epsilon_{1}, \epsilon_{2}\right)$ but also ( $n_{\mathrm{S}}, r$ ).

Each inflationary model must also be CMB normalized, that is to say the amplitude of the power spectra, say at $k=k_{*}$, is completely fixed by the amplitude of the CMB anisotropies measured today. On the largest length scales, this is given to a good approximation by the CMB quadrupole $Q_{\text {rms-PS }} / T \equiv \sqrt{5 C_{2} /(4 \pi)} \simeq 6 \times 10^{-6}$, where $T \simeq 2.725 \mathrm{~K}$ is the CMB blackbody temperature. This is achieved
if $\mathcal{P}_{5_{0}} \simeq 60 Q_{\text {rms-PS }}^{2} / T^{2}$. Using the slow-roll approximation of the Friedmann-Lemaître equation and writing the potential as $V(\phi)=$ $M^{4} v(\phi)$, such that the mass scale $M$ is singled out, one arrives at

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=1440 \pi^{2} \frac{\epsilon_{1 *}}{v\left(\phi_{*}\right)} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{2.30}
\end{equation*}
$$

This is a model-depend expression (it depends on $v$ ) in which we have rendered explicit the dependence in the pivot time. On a more robust basis, CMB data are strongly constraining the value of $P_{*} \equiv \mathcal{P}_{\zeta}\left(k_{*}\right)$ and supplementing the Planck CMB temperature likelihood by the WMAP large-scale polarization data, one gets the one-sigma confidence interval
$\ln \left(10^{10} P_{*}\right)=3.092 \pm 0.026$,
at $k_{*}=0.05 \mathrm{Mpc}^{-1}$. This constraint and the one- and two-sigma contours in the planes $\left(\epsilon_{1}, \epsilon_{2}\right)$ and ( $\left.n_{\mathrm{S}}, r\right)$ represented in all the figures have been obtained from a slow-roll analysis of the Planck data. Since the analysis is in all point identical to the one of the WMAP seven years data performed in Ref. [64], we do not repeat it here. The interested reader can find all the details in the Appendix B of Ref. [64]. Moreover, in order to get a robust inference, we have used the second order expression for the power spectra. Therefore, all the results presented below are marginalized over the second order slow-roll parameters.

Since at leading order in the slow-roll expansion we have $P_{*} \simeq$ $H_{*}^{2} /\left(8 \pi^{2} \epsilon_{1 *} M_{\mathrm{PI}}^{2}\right)$, the Friedmann-Lemaître equation allows us to derive the relation

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=24 \pi^{2} \frac{\epsilon_{1 *}}{v\left(\phi_{*}\right)} P_{*} \tag{2.32}
\end{equation*}
$$

which is, as expected, formally identical to Eq. (2.30) with
$\frac{Q_{\mathrm{rms}}^{2} \mathrm{PS}}{T^{2}}=60 P_{*}$.


Fig. 15. Exponential SUSY Inflation (ESI) for $q=\sqrt{2}$. Top panels: the potential and its logarithm. Bottom left panel: slow-roll parameter $\epsilon_{1}$. The shaded area indicates where acceleration stops. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line). For those, the shaded region signals the breakdown of the slow-roll approximation but not necessarily the end of the accelerated expansion.

It has however the advantage of using $P_{*}$ which is a well inferred quantity because it is fitted against all the $C_{\ell}$. In the following we will make no-distinction between the so-called COBE normalization and the CMB normalization, both being identical provided the above equation is used. For each inflationary model, these expressions will completely fix the allowed values for $M$.

We have shown how to calculate the two point correlation functions in the slow-roll approximation. The next logical step would be to determine the higher correlation functions. However, for the type of models considered here (i.e. category IA models), it is well-known that the corresponding signal is so small that it will stay out of reach for a while [115-119]. Therefore, we now consider the question of how to calculate the values of $\epsilon_{1}$ and $\epsilon_{2}$ when the pivot scale exits the Hubble radius and how this result depends on the details of the reheating period.

### 2.2. The reheating phase

In the last subsection, we have seen that the power spectrum (2.18) can be calculated with the help of the slow-roll approximation and expressed in terms of the Hubble flow parameters evaluated at Hubble radius crossing. Here, we briefly explain how these Hubble flow parameters can be determined. It is easy to calculate $\epsilon_{1}, \epsilon_{2}$ and $\epsilon_{3}$ as a function of $\phi$ from Eqs. (2.4)-(2.6). Then, from the trajectory (2.11), one can calculate $N_{\text {end }}$, the total number of $e$-folds during inflation and $N_{*}$, the number of $e$-folds at the point when the pivot scale crosses the Hubble radius. If we denote by $\ell$ the following primitive
$\ell(\phi)=\int^{\phi} \frac{V(\psi)}{V_{\psi}(\psi)} \mathrm{d} \psi$,
which is also the slow-roll trajectory of Eq. (2.11), then we have
$N_{\text {end }}=-\frac{1}{M_{\mathrm{Pl}}^{2}}\left[\ell\left(\phi_{\text {end }}\right)-\ell\left(\phi_{\text {ini }}\right)\right]$,
$N_{*}=-\frac{1}{M_{\mathrm{Pl}}^{2}}\left[\ell\left(\phi_{*}\right)-\ell\left(\phi_{\mathrm{ini}}\right)\right]$,
where $\phi_{*}$ is the vacuum expectation value of the field, again evaluated when the pivot scale crosses the Hubble radius. From these two expressions, it follows that
$\phi_{*}=\ell^{-1}\left[\ell\left(\phi_{\text {end }}\right)+M_{\text {Pl }}^{2} \Delta N_{*}\right]$,
where $\Delta N_{*} \equiv N_{\text {end }}-N_{*}$. Inserting this formula into the expressions of the Hubble flow parameters allows us to find $\epsilon_{n *}$ and, therefore, $r$ and $n_{s}$.

However, in order to make the above-described calculation concrete, we need to say something about the quantity $\Delta N_{*}$. As was explained in details in Ref. [64], this requires to take into account the reheating stage. Let $\rho$ and $P$ be the energy density and pressure of the effective fluid dominating the Universe during reheating. Conservation of energy implies that
$\rho(N)=\rho_{\text {end }} \exp \left\{-3 \int_{N_{\text {end }}}^{N}\left[1+w_{\text {reh }}(n)\right] \mathrm{d} n\right\}$,
where $w_{\text {reh }} \equiv P / \rho$ is the "instantaneous" equation of state during reheating. One can also define the mean equation of state
parameter, $\bar{w}_{\text {reh }}$, by ${ }^{3}$
$\bar{w}_{\text {reh }} \equiv \frac{1}{\Delta N} \int_{N_{\text {end }}}^{N_{\text {reh }}} w_{\text {reh }}(n) \mathrm{d} n$,
where
$\Delta N \equiv N_{\text {reh }}-N_{\text {end }}$,
is the total number of $e$-folds during reheating, $N_{\text {reh }}$ being the number of $e$-folds at which reheating is completed and the radiation dominated era begins. Then, one introduces a new parameter
$R_{\mathrm{rad}} \equiv \frac{a_{\mathrm{end}}}{a_{\mathrm{reh}}}\left(\frac{\rho_{\mathrm{end}}}{\rho_{\mathrm{reh}}}\right)^{4}$,
where $\rho_{\text {reh }}$ has to be understood as the energy density at the end of the reheating era, i.e. $\rho\left(N_{\text {reh }}\right)$. This definition shows that $R_{\text {rad }}$ encodes any deviations the reheating may have compared to a pure radiation era. In fact, $R_{\text {rad }}$ completely characterizes the reheating stage and can be expressed in terms of
$\ln R_{\mathrm{rad}} \equiv \frac{\Delta N}{4}\left(-1+3 \bar{w}_{\mathrm{reh}}\right)$,
which renders explicit that if $\bar{w}_{\text {reh }}=1 / 3$, i.e. the effective fluid during reheating is equivalent to radiation, then reheating cannot be distinguished from the subsequent radiation dominated era. In this case, one simply has $R_{\mathrm{rad}}=1$. Let us notice that it is also possible to express (or define) $\ln R_{\mathrm{rad}}$ as
$\ln R_{\mathrm{rad}}=\frac{1-3 \bar{w}_{\text {reh }}}{12\left(1+\bar{w}_{\text {reh }}\right)} \ln \left(\frac{\rho_{\mathrm{reh}}}{\rho_{\mathrm{end}}}\right)$.
Using entropy conservation till the beginning of the radiation era, the redshift at which inflation ended can be expressed in terms of $R_{\mathrm{rad}}$ as
$1+z_{\mathrm{end}}=\frac{1}{R_{\mathrm{rad}}}\left(\frac{\rho_{\mathrm{end}}}{\tilde{\rho}_{\gamma}}\right)^{1 / 4}, \quad \tilde{\rho}_{\gamma} \equiv \mathcal{Q}_{\mathrm{reh}} \rho_{\gamma}$.
The quantity $\rho_{\gamma}=3 H_{0}^{2} M_{\mathrm{Pl}}^{2} \Omega_{\gamma}$ is the total energy density of radiation today ( $\Omega_{\gamma} \simeq 2.471 \times 10^{-5} h^{-2}$ ) while $\mathcal{Q}_{\text {reh }} \equiv$ $q_{0}^{4 / 3} g_{\text {reh }} /\left(q_{\text {reh }}^{4 / 3} g_{0}\right)$ is the measure of the change of relativistic degrees of freedom between the reheating epoch and today. In this expression $q$ and $g$ respectively denotes the number of entropy and energetic relativistic degrees of freedom. In view of the current CMB data, the precise value for $\mathcal{Q}_{\text {reh }}$ is unimportant as this factor has only a minimal effect. At most it can shift the values of $\ln R_{\mathrm{rad}}$ by a $\mathcal{O}(1)$ number.

Then, straightforward considerations $[64,190]$ show that the quantities $\Delta N_{*}$ and $R_{\mathrm{rad}}$ are related by

$$
\begin{align*}
\Delta N_{*}= & \ln R_{\mathrm{rad}}-N_{0}-\frac{1}{4} \ln \left[\frac{9}{\epsilon_{1 *}\left(3-\epsilon_{1 \mathrm{end}}\right)} \frac{V_{\mathrm{end}}}{V_{*}}\right] \\
& +\frac{1}{4} \ln \left(8 \pi^{2} P_{*}\right), \tag{2.44}
\end{align*}
$$

[^2]

Fig. 16. Lambert functions $\mathrm{W}_{0}(x)$ (dashed line) and $\mathrm{W}_{-1}(x)$ (solid line). During Exponential SUSY inflation, inflation proceeds along the " -1 " branch in the direction specified by the arrow on the figure.
where we have defined ${ }^{4}$
$N_{0} \equiv \ln \left(\frac{k_{*} / a_{0}}{\tilde{\rho}_{\gamma}^{1 / 4}}\right)$,
which roughly measures the number of $e$-folds of deceleration of the Friedmann-Lemaître model. From Eq. (2.42), we see that the quantity $\ln R_{\text {rad }}$ is not arbitrary since $-1 / 3<\bar{w}_{\text {reh }}<1$ and $\rho_{\text {nuc }}<\rho_{\text {reh }}<\rho_{\text {end }}$. Notice that the range allowed for $\bar{w}_{\text {reh }}$ might be extended to smaller values if one allows a phase of acceleration to take place at lower energy than $\rho_{\text {end }}$, such as in thermal or multistage inflation [191,192]. The quantity $\Delta N_{*}$ is also constrained to vary in a given range, i.e. $\Delta N_{*} \in\left[\Delta N_{*}^{\text {nuc }}, \Delta N_{*}^{\text {end }}\right]$. Moreover, this range is model-dependent since $\rho_{\text {end }}$ or $V_{\text {end }} / V_{*}$ differ for different inflationary scenarios. In fact, for each allowed value of $\ln R_{\mathrm{rad}}$, Eq. (2.44) must be viewed as an algebraic equation allowing us to determine the corresponding $\phi_{*}$. Explicitly, using Eq. (2.35), this equation reads

$$
\begin{align*}
- & \frac{1}{M_{\mathrm{Pl}}^{2}}\left[\ell\left(\phi_{*}\right)-\ell\left(\phi_{\mathrm{end}}\right)\right]=\ln R_{\mathrm{rad}}-N_{0}-\frac{1}{4} \ln \\
& \times\left\{\frac{9}{\epsilon_{1}\left(\phi_{*}\right)\left[3-\epsilon_{1}\left(\phi_{\mathrm{end}}\right)\right]} \frac{V\left(\phi_{\mathrm{end}}\right)}{V\left(\phi_{*}\right)}\right\}+\frac{1}{4} \ln \left(8 \pi^{2} P_{*}\right) . \tag{2.47}
\end{align*}
$$

In general, this equation can not be solved explicitly (except for LFI models, see Ref. [64]) and we have to rely on numerical calculations. Solving for each allowed value of $\ln R_{\text {rad }}$, one can determine the range of variation of $\phi_{*} \in\left[\phi_{*}^{\text {nuc }}, \phi_{*}^{\text {end }}\right]$ and, therefore, find the corresponding dispersion in $r$ and $n_{\mathrm{S}}$. In this paper, this task is carried out for all the models of the ASPIC library. Let us notice that it is compulsory to do so otherwise, assuming blindly say $\Delta N_{*} \in[40,60]$, would lead to inconsistent reheating energy densities, either larger than $\rho_{\text {end }}$ or smaller than $\rho_{\text {nuc }}$. Clearly, this method also allows us to put model-dependent constraints on the reheating temperature. Indeed, for some values of $\rho_{\text {reh }}$, the corresponding $\epsilon_{n *}$ will turn out to be outside the $1 \sigma$ or $2 \sigma$ contours (depending on the criterion one wishes to adopt)

[^3]

Fig. 17. Power Law Inflation (PLI) for $\alpha=0.3$. Top panels: power law potential (left) and its logarithm (right). Bottom left panel: slow-roll parameter $\epsilon_{1}$. Bottom right panel: slow-roll parameters $\epsilon_{2}=\epsilon_{3}=0$. On these plots, the shaded area indicates the region where slow-roll is violated.
thus signaling some tension with the data, see the discussion in the Introduction and Fig. 2.

Let us emphasize that the parametrization presented in this section is independent on the microphysics of reheating and we do not need to specify explicitely the couplings of the inflaton field with the rest of the world. In particular, preheating effects on the background evolution are already taken into account with the present framework. Furthermore, at the perturbed level, they cannot influence the shape of the large scale power spectrum for the class of models considered here [58].

Before closing this section, let us remind that, for each inflationary model, ASPIC gives the expression of the first three Hubble flow parameters, a discussion of the mechanism that ends inflation and the value of $\phi_{\text {end }}$, the classical trajectory $\ell(\phi)$, the CMB normalization $M / M_{\mathrm{Pl}}$ and a determination of the exact range [ $\left.\phi_{*}^{\text {nuc }}, \phi_{*}^{\text {end }}\right]$. Then all these information are compared to CMB data in the planes $\left(\epsilon_{1}, \epsilon_{2}\right)$ and ( $\left.n_{\mathrm{S}}, r\right)$. This provides a powerful tool to systematically derive the predictions for the ASPIC models and, therefore, to scan the inflationary landscape. In the next section, we start the systematic exploration of the category IA models that have been studied in the literature since the advent of inflation.

## 3. Zero parameter models

### 3.1. Higgs inflation (HI)

### 3.1.1. Theoretical justifications

This model postulates that the inflaton field is the Higgs field $h$ (recently discovered at the Large Hadron Collider, see Refs. [193, 194]) non-minimally coupled to gravity, see Refs. [195-198]. Indeed, one can argue that, in curved spacetime, the simplest model compatible with our knowledge of particle physics is described by a Lagrangian which is the standard model Lagrangian plus an extra term of the form $\xi H^{\dagger} H R$. This last term is compulsory
since, in curved spacetime, it will automatically be generated by quantum corrections [199]. In the Jordan frame, the action of the model can be written as
$S=\frac{\bar{M}^{2}}{2} \int \mathrm{~d}^{4} \boldsymbol{x} \sqrt{-\bar{g}}\left[F(h) \bar{R}-Z(h) \bar{g}^{\mu \nu} \partial_{\mu} h \partial_{\nu} h-2 U(h)\right]$.
The quantity $\bar{M}$ is a mass scale that, for the moment, is not identified with the Planck scale and the tensor $\bar{g}_{\mu \nu}$ denotes the metric in the Jordan frame (in what follows, all the quantities with a bar denote quantities evaluated in the Jordan frame; quantities without a bar are quantities evaluated in the Einstein frame). The three functions $F(h), Z(h)$ and $U(h)$ completely characterize the model and are chosen to be
$F(h)=1+\xi h^{2}, \quad Z(h)=1$,
$U(h)=\bar{M}^{2} \frac{\lambda}{4}\left(h^{2}-\frac{v^{2}}{\bar{M}^{2}}\right)^{2}$,
where $\xi$ is a new dimensionless parameter and $U(h)$ is the standard Higgs boson potential with $v$ the Higgs (current) vacuum expectation value and $\lambda$ the self-interacting coupling constant. Here, the field $h$ is dimensionless (as the functions $F$ and $Z$ ) while the potential $U$ is of dimension two. The effective gravitational constant (measured in Cavendish-type experiments) is given by Ref. [200]

$$
\begin{equation*}
\frac{1}{M_{\mathrm{Pl}}^{2}}=\frac{1}{\bar{M}^{2}} \frac{2\left(1+\xi h^{2}\right)+16 \xi^{2} h^{2}}{\left(1+\xi h^{2}\right)\left[2\left(1+\xi h^{2}\right)+12 \xi^{2} h^{2}\right]} \tag{3.3}
\end{equation*}
$$

Since, today, one has $h \simeq v / \bar{M} \ll 1$, it follows that $\bar{M} \simeq M_{P 1}$ with very good accuracy and, from now on, we will always consider that this identification is valid.

The above-described model can also be written in the Einstein frame where the corresponding slow-roll analysis is easier.


Fig. 18. Top left panel: Kähler moduli inflation (KMII) potential for $\alpha=1.5$. The two arrows indicate the two regions of the potential where inflation can take place. Top right panel: logarithm of the potential for the same value of $\alpha$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ for $\alpha=0.5$ (solid green line), $\alpha=1.5$ (solid blue line) and $\alpha=2.5$ (solid pink line). Obviously, the number of solutions of the equation $\epsilon_{1}=1$ depends on the value of $\alpha$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) for $\alpha=1.5$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Denoting the metric tensor in this frame by $g_{\mu \nu}$, the action now takes the form
$S=2 M_{\mathrm{Pl}}^{2} \int \mathrm{~d}^{4} \boldsymbol{x} \sqrt{-g}\left[\frac{R}{4}-\frac{1}{2} g^{\mu \nu} \partial_{\mu} \chi \partial_{\nu} \chi-W(\chi)\right]$,
where the fields $h$ and $\chi$ are related by
$\frac{\mathrm{d} \chi}{\mathrm{d} h}=\frac{\sqrt{1+\xi(1+6 \xi) h^{2}}}{\sqrt{2}\left(1+\xi h^{2}\right)}$,
and the potential is given $V \equiv 2 M_{\mathrm{PI}}^{2} W=M_{\mathrm{PI}}^{2} U / F^{2}$. Notice also that the canonically normalized field in the Einstein frame can be expressed as $\phi \equiv \sqrt{2} M_{\mathrm{PI}} \chi$. It is also important to recall that, in the Einstein frame, matter is now explicitly coupled to the scalar field $\phi$. This has of course important consequences for the description of the reheating period, see Refs. [201-203] and below. The differential equation (3.5) can be integrated exactly and the result reads

$$
\begin{align*}
\chi= & \sqrt{\frac{1+6 \xi}{2 \xi}} \operatorname{arcsinh}[h \sqrt{\xi(1+6 \xi)}] \\
& -\sqrt{3} \operatorname{arctanh}\left[\frac{\xi \sqrt{6} h}{\sqrt{1+\xi(1+6 \xi) h^{2}}}\right] . \tag{3.6}
\end{align*}
$$

The inverse hyperbolic tangent is always well-defined since its argument is always smaller than one. This exact formula between the Einstein and Jordan frame fields was also derived in Ref. [201]. In fact, we are interested in the regime $\xi \gg 1$ and $\xi h \gg 1$. In this case, one can derive an approximated expression for $\chi$. Notice that this limit must be carefully calculated because if one just replaces $1+6 \xi$ with $\xi$ in the above expression, one finds that $\chi=0$ !.

Using the identity $\operatorname{arcsinh} x=\ln \left(x+\sqrt{1+x^{2}}\right)$, the first term in Eq. (3.6) can be approximated as $\sqrt{3} \ln (2 \xi \sqrt{6} h)$. Then, one can use the identity $\operatorname{arctanh} x=1 / 2 \ln [(1+x) /(1-x)]$ and expand the argument of this logarithm in $1 / \xi$ and $1 /(\xi h)^{2}$. One finds that the latter reduces to $24 \xi^{2} h^{2} /\left(1+\xi h^{2}\right)$. Finally, combining the two terms in Eq. (3.6), one arrives at
$\chi \simeq \frac{\sqrt{3}}{2} \ln \left(1+\xi h^{2}\right)$.
The same expression can also be directly derived from Eq. (3.5) which, in the regime studied here, can be approximated as
$\frac{\mathrm{d} \chi}{\mathrm{d} h} \simeq \frac{\sqrt{6} \xi h}{\sqrt{2}\left(1+\xi h^{2}\right)}$.
The solution to this equation is exactly Eq. (3.7). The last step consists in inserting the expression of $h$ in terms of $\chi$ (and, therefore, in terms of $\phi$ ) into the definition of the potential $V$ in the Einstein frame. This leads to the following expression
$V(\phi)=\frac{M_{\mathrm{Pl}}^{4} \lambda}{4 \xi^{2}}\left(1-e^{-\sqrt{2 / 3} \phi / M_{\mathrm{Pl}}}\right)^{2}$.
Interestingly enough, the parameters $\xi$ and $\lambda$ enter the potential only through its overall amplitude. In the following, we define $M$ by $M^{4} \equiv M_{\mathrm{PI}}^{4} \lambda /\left(4 \xi^{2}\right)$. In this sense, Higgs inflation is a "zero parameter model" since the scale $M$ is entirely determined by the amplitude of the CMB anisotropies.

More recently, in Ref. [204], a supergravity realization of this model was presented. We now briefly review how this can be achieved. The model is based on no-scale supergravity and has


Fig. 19. Lambert functions $\mathrm{W}_{0}(x)$ (dashed line) and $\mathrm{W}_{-1}(x)$ (solid line). During Kähler moduli inflation, inflation proceeds along the " -1 " branch in the direction specified by the arrow.
two fields, a modulus $T$ and the inflaton $\phi$. The Kähler and superpotentials are given by $K=-3 \ln \left(T+T^{\dagger}-|\phi|^{2} / 3\right)$ and $W=$ $\hat{\mu} \phi^{2}-\lambda \phi^{3} / 3$, respectively. The quantities $\hat{\mu}$ and $\lambda$ are constants characterizing the model. It follows that the Kähler matrix and its inverse can be written as
$K_{i \bar{j}}=\frac{3}{\left(T+T^{\dagger}-|\phi|^{2} / 3\right)^{2}}\left[\begin{array}{cc}\left(T+T^{\dagger}\right) / 3 & -\phi^{\dagger} / 3 \\ -\phi / 3 & 1\end{array}\right]$,
$K^{k \bar{j}}=\left(T+T^{\dagger}-\frac{|\phi|^{2}}{3}\right)\left[\begin{array}{cc}1 & \phi / 3 \\ \phi^{\dagger} / 3 & \left(T+T^{\dagger}\right) / 3\end{array}\right]$.
Then, assuming that the modulus is stabilized such $\left\langle T+T^{\dagger}\right\rangle=$ $c$ and $\left\langle T-T^{\dagger}\right\rangle=0$, one obtains the following Lagrangian: $-c\left|\partial_{\mu} \phi\right|^{2} / \Delta^{2}-|\partial W / \partial \phi|^{2} / \Delta^{2}$ where $\Delta \equiv c-|\phi|^{2} / 3$. The next step consists in introducing the fields $x$ and $y$ defined by $\phi=$ $\sqrt{3 c} \tanh [(x+i y) / \sqrt{3}]$. Expressed in terms of these two fields, the previous Lagrangian takes the following form

$$
\begin{align*}
\mathcal{L}_{\text {eff }}= & -\frac{1}{2 \cos ^{2}(\sqrt{2 / 3} y)}\left[\left(\partial_{\mu} x\right)^{2}+\left(\partial_{\mu} y\right)^{2}\right] \\
& -\frac{\mu^{2}}{2} \frac{1}{2 \cos ^{2}(\sqrt{2 / 3} y)} \\
& \times e^{-\sqrt{2 / 3} x}\left[\cosh \left(\sqrt{\frac{2}{3}} x\right)-\cos \left(\sqrt{\frac{2}{3}} y\right)\right], \tag{3.12}
\end{align*}
$$

where $\mu \equiv \hat{\mu} \sqrt{3 / c}$. In order to obtain this formula, we have crucially assumed that
$\lambda=\frac{\mu}{3}$.
The form of the effective Lagrangian has also been studied in Ref. [204] in the case where this relation is no longer valid. The last step consists in remarking that $y=0$ during inflation. If we expand the above Lagrangian about $y=0$, then the field $x$ is canonically normalized and the potential becomes precisely the one of Eq. (3.9). Therefore, it constitutes another scenario where this potential arises. Let us also notice that other approaches based on superconformal D-term inflation also lead to the Starobinsky model [205]. Various multifield extensions have also been studied in which the inflationary phase can still be described by the onefield Higgs potential [206-208].

### 3.1.2. Slow-roll analysis

Having established the shape of the potential, namely
$V(\phi)=M^{4}\left(1-e^{-\sqrt{2 / 3} \phi / M_{\mathrm{Pl}}}\right)^{2}$,
we can now proceed to the slow-roll analysis. For convenience, let us define in the following $x \equiv \phi / M_{\mathrm{Pl}}$. Then, the first three slow-roll parameters are given by
$\epsilon_{1}=\frac{4}{3}\left(1-e^{\sqrt{2 / 3 x}}\right)^{-2}, \quad \epsilon_{2}=\frac{2}{3}\left[\sinh \left(\frac{x}{\sqrt{6}}\right)\right]^{-2}$,
$\epsilon_{3}=\frac{2}{3}\left[\operatorname{coth}\left(\frac{x}{\sqrt{6}}\right)-1\right] \operatorname{coth}\left(\frac{x}{\sqrt{6}}\right)$.
These quantities are represented in Fig. 5 (left and right bottom panels) together with the potential.

In this model, as can be noticed on these plots, inflation stops by violation of the slow-roll conditions. The condition $\epsilon_{1}=1$ occurs for $x=x_{\text {end }}$ where $x_{\text {end }}$ can be expressed as
$x_{\text {end }}=\sqrt{\frac{3}{2}} \ln \left(1+\frac{2}{\sqrt{3}}\right) \simeq 0.94$.
In fact, before the end of inflation, the slow-roll approximation breaks down when $\epsilon_{2}$ becomes greater than 1 . This happens for $x=x_{\epsilon_{2}=1}$ where
$x_{\epsilon_{2}=1}=\sqrt{6} \operatorname{arcsinh}\left(\sqrt{\frac{2}{3}}\right) \simeq 1.83$.
The third slow-roll parameter $\epsilon_{3}$ also becomes greater than one before the end of inflation (but after the second slowroll parameter has become unity). The corresponding vacuum expectation value can be written as
$x_{\epsilon_{3}=1}=\sqrt{6} \operatorname{arctanh}\left(\frac{2}{1+\sqrt{7}}\right) \simeq 1.51$.
In the case where the inflaton field is interpreted as the Higgs field, these three vacuum expectation values do not depend on the parameter $\xi$ since this parameter is "hidden" in the mass scale $M$.

We are now in a position where one can calculate the slow-roll trajectory. Using Eq. (3.14), it can be integrated exactly and yields to
$N-N_{\text {ini }}=\frac{1}{2} \sqrt{\frac{3}{2}}\left(x-x_{\text {ini }}\right)-\frac{3}{4}\left(e^{\sqrt{\frac{2}{3}} x}-e^{\sqrt{\frac{2}{3}} x_{\text {ini }}}\right)$.
In the regime where $x \gg 1$, the last term is dominant and this is the one usually considered in the literature, see Ref. [195]. The trajectory can be inverted and expressed in term of the "-1branch" of the Lambert function $\mathrm{W}_{-1}$, leading to

$$
\begin{align*}
x= & \sqrt{\frac{3}{2}}\left\{\frac{4}{3} N+\sqrt{\frac{2}{3}} x_{\text {ini }}-e^{\sqrt{\frac{2}{3}} x_{\text {ini }}}\right. \\
& \left.-\mathrm{W}_{-1}\left[-\exp \left(\frac{4}{3} N+\sqrt{\frac{2}{3}} x_{\text {ini }}-e^{\sqrt{\frac{2}{3}} x_{\text {ini }}}\right)\right]\right\} . \tag{3.20}
\end{align*}
$$

The fact that inflation proceeds on the -1 branch of the Lambert function $\mathrm{W}_{-1}$, as can be seen in Fig. 6, can be justified by the following considerations. When $N=0$, the value taken by the Lambert function is $-\exp \left(\sqrt{2 / 3} x_{\mathrm{ini}}\right)$, which is smaller than -1 . On the other hand, if $x=0$, the value given for $N$ by Eq. (3.19) can be inserted in Eq. (3.20) and one finds that the argument of the Lambert function is -1 , i.e. the connection point between the -1 branch and the 0 branch. Therefore inflation takes place between these two points.

Finally, the value of the inflaton field, $x_{*}$, calculated $\Delta N_{*}=$ $N_{\text {end }}-N_{*} e$-folds before the end of inflation reads
$x_{*}=\sqrt{\frac{3}{2}}\left(-\frac{4}{3} \Delta N_{*}+\ln \left(1+\frac{2}{\sqrt{3}}\right)-\left(1+\frac{2}{\sqrt{3}}\right)\right.$


Fig. 20. Top left panel: Horizon Flow Inflation at first order potential for $A_{1}=0.1$. Top panels: the potential and its logarithm with respect to the field values. Bottom left panel: the first Hubble flow function $\epsilon_{1}$ (exact) and the corresponding shaded area where inflation stops. Bottom right panel: Hubble flow functions $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) for the same potential. These two functions are equal to $2 \epsilon_{1}$.


Fig. 21. Colemann-Weinberg Inflation (CWI) for $\alpha=4 e$. Top left panel: Colemann-Weinberg Inflation potential as a function of $\phi / Q$. Top right panel: logarithm of the potential for the same value of $\alpha$. Bottom left panel: normalized first slow-roll parameter $Q^{2} / M_{\mathrm{P}}^{2} \epsilon_{1}$. The shaded area indicates the where inflation stops if $Q=M_{P 1}$. Bottom right panel: normalized second and third slow-roll parameters $Q^{2} / M_{P 1}^{2} \epsilon_{2}$ (solid line) and $Q^{2} / M_{P 1}^{2} \epsilon_{3}$ (dotted line) for the same potential.


Fig. 22. End of inflation in Coleman-Weinberg inflation. The approximated formula of Eq. (4.170) for $x_{\text {end }}$ (red dashed line) is compared with the exact numerical solution of $\epsilon_{1}=1$ (blue solid line), for $\alpha=4 e$, in the physically relevant range of values for $Q / M_{\mathrm{PI}}$. The agreement is obviously excellent.

$$
\begin{align*}
& -\mathrm{W}_{-1}\left\{-\exp \left[-\frac{4}{3} \Delta N_{*}+\ln \left(1+\frac{2}{\sqrt{3}}\right)\right.\right. \\
& \left.\left.\left.-\left(1+\frac{2}{\sqrt{3}}\right)\right]\right\}\right) \tag{3.21}
\end{align*}
$$

In principle, inserting this formula into the expressions of the slowroll parameters (3.15) allows us to determine the observational predictions of the model.

At this stage, however, a comment is in order about reheating in the case where the inflaton field is the Higgs field (these remarks do not apply to the supergravity realization of the model). As explained above, all the previous considerations are derived in the Einstein frame. In this frame, matter is not universally coupled to the metric tensor and, therefore, it is compulsory to re-consider the parametrization presented in Section 2.2. In the Einstein frame, the matter action is given by $S_{\text {mat }}\left[\psi, A^{2}(\phi) g_{\mu \nu}\right]$, where $\psi$ denotes some generic matter field and $g_{\mu \nu} \equiv F(h) \bar{g}_{\mu \nu}$ with $A \equiv F^{-1 / 2}$, see Ref. [200] (quantities in the Jordan frame are denoted with a bar). In the Jordan frame, the energy density of a (conserved) fluid with a constant equation of state $w=\bar{p} / \bar{\rho}$ scales as $\bar{\rho} \propto$ $\bar{a}^{-3(1+w)}$ while, in the Einstein frame, $\rho \propto A^{4} \bar{\rho} \propto A^{1-3 w} a^{-3(1+w)}$ since the scale factors in the two frames are related by $\bar{a}=$ Aa. As explained in Ref. [64] and briefly reviewed in Section 2.2, the dependence of the observational predictions on reheating originates from the gradient term $k / \mathscr{H}$ present in the MukhanovSasaki variable equation of motion. In order to evaluate concretely this term, one must relate the comoving wave-number $k$ during inflation with physical scales measured now. Clearly, this depends on the whole history of the Universe and, therefore, explains why the final result depends on the reheating duration. In the Einstein frame, one can show that the gradient term takes the standard form, namely

$$
\begin{equation*}
\frac{k}{\mathscr{H}}=\frac{e^{N_{\mathrm{end}}-N}}{H} \frac{k}{a_{0}}\left(\frac{\rho_{\mathrm{end}}}{\rho_{\gamma}}\right)^{1 / 4} \frac{1}{R_{\mathrm{rad}}} \tag{3.22}
\end{equation*}
$$

with

$$
\begin{align*}
\ln R_{\mathrm{rad}}= & \frac{1-3 w_{\mathrm{reh}}}{12\left(1+w_{\mathrm{reh}}\right)} \ln \left(\frac{\rho_{\mathrm{reh}}}{\rho_{\mathrm{end}}}\right) \\
& -\frac{1-3 w_{\mathrm{reh}}}{3\left(1+w_{\mathrm{reh}}\right)} \ln \left(\frac{A_{\mathrm{reh}}}{A_{\mathrm{end}}}\right), \tag{3.23}
\end{align*}
$$

where $w_{\text {reh }}$ is the equation of state of the effective dominant fluid during reheating. In the above expressions, it is important to emphasize that all the quantities are defined in the Einstein frame and that the non-standard scaling of the various energy densities
(pressure-less matter and radiation) has been systematically taken into account. All the extra terms cancel out except in the definition of the parameter $R_{\mathrm{rad}}$ where there is an additional term depending on the function $A$. Remarkably, this additional term is exactly such that the parameter $R_{\mathrm{rad}}$ in the Einstein frame can be re-expressed in terms of the energy densities in the Jordan frame only, namely

$$
\begin{equation*}
\ln R_{\mathrm{rad}}=\frac{1-3 w_{\mathrm{reh}}}{12\left(1+w_{\mathrm{reh}}\right)} \ln \left(\frac{\bar{\rho}_{\mathrm{reh}}}{\bar{\rho}_{\mathrm{end}}}\right) \tag{3.24}
\end{equation*}
$$

Let us stress again that the above equation has an unusual form: it is a quantity in the Einstein frame expressed in terms of quantities defined in the Jordan frame.

It is also important to notice an additional limitation compared to the standard case: in presence of non-minimal coupling to gravity, our parametrization of the reheating stage works only for a constant equation of state $w_{\text {reh }}$ while in Ref. [64] it was valid for any $w_{\text {reh }}$. We now explain the origin of this limitation. In the Einstein frame, the general expression of the parameter $R_{\mathrm{rad}}$ is given by
$\frac{1}{R_{\mathrm{rad}}}=\left(\frac{\rho_{\mathrm{reh}}}{\rho_{\mathrm{end}}}\right)^{1 / 4} \frac{a_{\mathrm{reh}}}{a_{\mathrm{end}}}$.
In order to obtain Eq. (3.23) from that formula, one should express the Einstein frame scale factor in term of the energy density $\rho$. If the equation of state $w_{\text {reh }}$ is a constant, then $a \propto$ $A^{\left(1-3 w_{\text {reh }}\right) /\left(3+3 w_{\text {reh }}\right)} a^{-1 /\left(3+3 w_{\text {reh }}\right)}$. This is what has been used above and this led to Eqs. (3.23) and (3.24). But let us now assume that $w_{\text {reh }}$ is not a constant (notice that one always has $w=\bar{w}$ since the energy density and the pressure scales with the same power of the function $A$ in the Einstein frame). Then, $\rho$ and $a$ are related by
$\frac{\mathrm{d} \rho}{\rho}=\left(1-3 w_{\text {reh }}\right) \frac{\mathrm{d} A}{A}-3\left(1+w_{\text {reh }}\right) \frac{\mathrm{d} a}{a}$.
If $A$ is a constant, one can always write [64]
$\frac{a_{\mathrm{reh}}}{a_{\mathrm{end}}}=\left(\frac{\rho_{\mathrm{reh}}}{\rho_{\mathrm{end}}}\right)^{-1 /\left(3+3 \bar{w}_{\mathrm{reh}}\right)}$,
where $\bar{w}_{\text {reh }}$ is the mean equation of state during reheating, namely
$\bar{w}_{\text {reh }} \equiv \frac{1}{N_{\text {reh }}-N_{\text {end }}} \int_{N_{\text {end }}}^{N_{\text {reh }}} w_{\text {reh }}(n) \mathrm{d} n$.
If $A$ and $w_{\text {reh }}$, however, are not constant, it is no longer possible to express the final formula in terms of $\bar{w}_{\text {reh }}$. In particular, we do not obtain a term $A^{1-3 \bar{w}_{\text {reh }}}$ as desired. Therefore, in what follows, we restrict our considerations to the case where the effective fluid dominating the matter content of the Universe has a constant equation of state.

Then, from Eq. (3.22), one can re-express $R_{\mathrm{rad}}$ in terms of quantities defined at Hubble radius crossing. One obtains

$$
\begin{align*}
\Delta N_{*}= & \ln R_{\mathrm{rad}}-\ln \left(\frac{k / a_{0}}{\rho_{\gamma}^{1 / 4}}\right)+\frac{1}{4} \ln \left(\frac{H_{*}^{2}}{M_{\mathrm{Pl}}^{2} \epsilon_{1 *}}\right) \\
& -\frac{1}{4} \ln \left(\frac{3}{\epsilon_{1 *}} \frac{V_{\text {end }}}{V_{*}} \frac{3-\epsilon_{1 *}}{3-\epsilon_{\text {1end }}}\right) . \tag{3.29}
\end{align*}
$$

Of course, this equation resembles a lot Eq. (2.44) but one has to realize that it involves quantities defined in the Einstein frame only. The term $\ln \left[\left(k / a_{0}\right) / \rho_{\gamma}^{1 / 4}\right]=\ln \left[\left(k / \bar{a}_{0}\right) / \bar{\rho}_{\gamma}^{1 / 4}\right]$ and, therefore, its numerical value remains unchanged. The other quantities appearing in this equation are obtained using our standard procedures since they refer to the inflaton sector only. Then, the range of variation of $\Delta N_{*}$ in Eq. (3.29) is determined by putting limits on $\ln R_{\mathrm{rad}}$ coming from the fact that reheating must proceed between the end of inflation and the BBN. This means that


Fig. 23. Loop Inflation (LI). Top left panel: Loop Inflation potential for $\alpha= \pm 0.5$, the case $\alpha=0.5$ being displayed in blue and the case $\alpha=-0.5$ being displayed in pink. Top right panel: logarithm of the potential for the same values of $\alpha$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ with the same values of $\alpha$. The shaded area indicates where inflation stops. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) for the same values of $\alpha$.
the physical value of the energy density, that is to say $\bar{\rho}_{\text {reh }}$, must be such that $\bar{\rho}_{\text {nuc }} \equiv(10 \mathrm{MeV})^{4}<\bar{\rho}_{\text {reh }}<\bar{\rho}_{\text {end }}$. We emphasize that physical limits must of course refer to quantities defined in the Jordan frame. But, precisely, we have shown that $\ln R_{\mathrm{rad}}$ in the Einstein frame can be expressed according to the standard formula, provided the energy densities in the argument of the logarithm are Jordan frame energy densities. Therefore, in practice, we have $\Delta N_{*} \in\left[\Delta N_{*}^{\text {nuc }}, \Delta N_{*}^{\text {end }}\right]$ with
$\Delta N_{*}^{\text {end }}=-N_{0}+\ln \left(\frac{H_{*}}{M_{\mathrm{Pl}}}\right)-\frac{1}{4} \ln \left(\frac{\rho_{\mathrm{end}}}{M_{\mathrm{Pl}}^{4}}\right)$,
where all the quantities in the above equation are calculated in the Einstein frame and, hence, are directly available since they are, by definition, the outcomes of the ASPIC library code. The other limit can be expressed as

$$
\begin{align*}
\Delta N_{*}^{\mathrm{nuc}}= & -N_{0}+\ln \left(\frac{H_{*}}{M_{\mathrm{Pl}}}\right)-\frac{1}{3(1+w)} \ln \left(\frac{\bar{\rho}_{\mathrm{end}}}{M_{\mathrm{Pl}}^{4}}\right) \\
& -\frac{1-3 w}{12(1+w)} \ln \left(\frac{\bar{\rho}_{\mathrm{nuc}}}{M_{\mathrm{Pl}}^{4}}\right) . \tag{3.31}
\end{align*}
$$

The quantity $\bar{\rho}_{\text {nuc }}$ is defined in the Jordan frame but its value is explicitly known, see above. On the other hand, we need to evaluate $\bar{\rho}_{\text {end }}$ since the code only delivers $\rho_{\text {end }}$. By definition, we have
$\bar{\rho}_{\text {end }}=\frac{\rho_{\text {end }}}{A_{\text {end }}^{4}}=F_{\text {end }}^{2} \rho_{\text {end }}=\left(1+\xi h_{\text {end }}^{2}\right)^{2} \rho_{\text {end }}$.
But $1+\xi h_{\text {end }}^{2}=e^{2 \chi_{\text {end }} / \sqrt{3}}$ and $\chi_{\text {end }}=\phi_{\text {end }} /\left(\sqrt{2} M_{\text {PI }}\right)=\sqrt{3} / 2$ $\ln (1+2 / \sqrt{3})$. As a consequence, the relation between the two
final energy densities in the two frames can be written as
$\bar{\rho}_{\text {end }}=\left(1+\frac{2}{\sqrt{3}}\right)^{2} \rho_{\text {end }} \simeq 2.15 \rho_{\text {end }}$.
Therefore, the lower bound is only slightly modified (recall that $\bar{\rho}_{\text {end }}$ appears in a logarithmic term). Anyway, given the uncertainty in the definition of $\bar{\rho}_{\text {nuc }}$, it is irrelevant to include this tiny correction in our determination of $\Delta N_{*}$. Consequently, we conclude that the range of variation of $\Delta N_{*}$ can be obtained without modifying anything to our usual way to calculate it and one can use the ASPIC code without introducing these negligible corrections. Of course, if one considers that the potential studied here comes from supergravity, the above considerations just not apply and one can work with the standard approach.

The reheating consistent observational predictions of Higgs inflation are represented in Fig. 81 where we have displayed their dependence in the reheating temperature defined in the Jordan frame by $g_{*}^{1 / 4} \bar{T}_{\text {reh }}=\left(30 \bar{\rho}_{\text {reh }} / \pi^{2}\right)^{1 / 4}$. Notice that, a priori, the reheating temperature can be calculated exactly in Higgs inflation since all the couplings between the Higgs and the other fields in the standard model are known [201]. This gives a spectral index which is in good agreement with the data and a small contribution of gravity waves. At this stage, in the Higgs case, we do not have constraints on the parameter $\xi$ since it is hidden in the mass scale $M$. Its observational value therefore comes from the amplitude of the CMB anisotropies and reads
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=1920 \pi^{2}\left(1-e^{\sqrt{\frac{2}{3}} x_{*}}\right)^{-4} e^{2 \sqrt{\frac{2}{3}} x_{*}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
Upon using the trajectory given by Eq. (3.21), the mass scale $M$ can be written as $M / M_{\mathrm{PI}} \simeq 0.02\left(\Delta N_{*}\right)^{-3 / 2}$, which for the fiducial value $\Delta N_{*}=55$, implies that $M \simeq 4 \times 10^{-5} M_{\text {Pl }}$, i.e., roughly


Fig. 24. Left panel: Lambert functions $\mathrm{W}_{0}(x)$ (dashed line) and $\mathrm{W}_{-1}(x)$ (solid line). During loop inflation, inflation proceeds along the " 0 " branch in the direction specified by the green arrow on the figure if $\alpha>0$, and along the " -1 " branch in the direction specified by the pink arrow on the figure if $\alpha<0$. Right panel: Maximal number of $e$-folds $\Delta N_{\max }$ one can realize when $\alpha<0$, between $x_{\epsilon_{1}=1}^{-}$and $x_{\epsilon_{1}=1}^{+}$, as a function of $\alpha$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
speaking, inflation takes place at the GUT scale in this model. Then, using this expression of $M$, one obtains the following numerical value for the parameter $\xi$,
$\xi \simeq 49000 \sqrt{\lambda}$,
where we have considered $\lambda=m_{\mathrm{H}} / v$, with $v \simeq 175 \mathrm{GeV}$ and $m_{\mathrm{H}} \simeq 125 \mathrm{GeV}$ (see Refs. [193,194]). These considerations are in agreement with the conclusions obtained in Refs. [195-197]. Notice such a large value for the coupling constant $\xi$ has been considered problematic [209]. If we now consider the supergravity realization of the model, one obtains a constraint on the parameter $\hat{\mu}$, that is to say if one takes $c=1$ on $\mu$ and $\lambda$, see Ref. [204].

## 4. One parameter models

### 4.1. Radiatively corrected higgs inflation (RCHI)

### 4.1.1. Theoretical justifications

Let us consider again the model given by Eq. (3.1). The three functions describing this action are modified when quantum corrections are taken into account. As a consequence, the potential which supports inflation is also modified and this leads to a new inflationary scenario that we call Radiatively Corrected Higgs Inflation (RCHI). This scenario has been studied in Refs. [210-215]. At first order, the corrections to the function $Z(h)$ can be neglected while the corrections to $F(h)$ and to $U(h)$ read
$F(h)=1+\xi h^{2}+\frac{C}{16 \pi^{2}} h^{2} \ln \left(\frac{M_{\mathrm{P}}^{2} h^{2}}{\mu^{2}}\right)$,
$U(h)=M_{\mathrm{Pl}}^{2} \frac{\lambda}{4}\left(h^{2}-\frac{v^{2}}{M_{\mathrm{Pl}}^{2}}\right)^{2}+\frac{\lambda A}{128 \pi^{2}} M_{\mathrm{Pl}}^{2} h^{4} \ln \left(\frac{M_{\mathrm{Pl}}^{2} h^{2}}{\mu^{2}}\right)$,
where $\mu$ is the renormalization scale and $A$ and $C$ are two new constants given by
$A=\frac{3}{8 \lambda}\left[2 g^{4}+\left(g^{2}+g^{\prime 2}\right)-16 y_{\mathrm{t}}^{4}\right]+6 \lambda+\mathcal{O}\left(\xi^{-2}\right)$,
$C=3 \xi \lambda+\mathcal{O}\left(\xi^{0}\right)$,
$y_{\mathrm{t}}$ being the Yukawa coupling of the top quark and $g$ and $g^{\prime}$ the coupling constants of the $S U(2)_{\mathrm{L}}$ and $\mathrm{U}(1)_{\mathrm{Y}}$ groups. The presence of quantum corrections modifies the relation between the Jordan and the Einstein frames and changes the shape of the potential in the Einstein frame. Assuming the smallness of $A /\left(32 \pi^{2}\right) \ll 1$ and $C /\left(8 \pi^{2} \xi\right) \ll 1$, which is necessary for the consistence of the one-loop calculation (the second condition is in fact equivalent to
$C \lambda /\left(8 \pi^{2}\right) \ll 1$ because $C$ is proportional to $\left.\xi\right)$, one obtains the following expression

$$
\begin{align*}
V \simeq & \frac{M_{\mathrm{Pl}}^{4} \lambda}{4 \xi^{2}} \frac{\xi^{2} h^{4}}{\left(1+\xi h^{2}\right)^{2}}\left[1-\frac{\xi h^{2}}{1+\xi h^{2}} \frac{C}{8 \pi^{2} \xi} \ln \left(\frac{M_{\mathrm{Pl}}^{2} h^{2}}{\mu^{2}}\right)\right. \\
& \left.+\frac{A}{32 \pi^{2}} \ln \left(\frac{M_{\mathrm{Pl}}^{2} h^{2}}{\mu^{2}}\right)\right] . \tag{4.5}
\end{align*}
$$

Of course, if $A=C=0$, one checks that this potential reduces to the potential of the previous section. Notice that, at this stage, we have not assumed that $\xi h^{2} \gg 1$. If we further postulate that $\xi h^{2} \gg 1$ and approximate $\xi^{2} h^{4} /\left(1+\xi h^{2}\right)^{2} \simeq 1-2 /\left(\xi h^{2}\right)$, then the above formula reduces to
$V \simeq \frac{M_{\mathrm{Pl}}^{4} \lambda}{4 \xi^{2}}\left[1-\frac{2}{\xi h^{2}}+\frac{A_{\mathrm{I}}}{16 \pi^{2}} \ln \left(\frac{M_{\mathrm{Pl}} h}{\mu}\right)\right]$,
where $A_{I} \equiv A-12 \lambda$ is the inflationary anomalous scaling. This formula coincides with Eq. (6) of Ref. [212] and Eq. (9) of Ref. [214]. Although the above formulas give $V$ in the Einstein frame, it is still expressed in term of $h$. The expression for the field in the Einstein frame, $\chi$, remains to be established. Assuming the smallness of the loop corrections (but, here, we do not yet assume that $\xi h^{2} \gg 1$ ), we obtain
$\frac{\mathrm{d} \chi}{\mathrm{d} h} \simeq \frac{\sqrt{3} h \xi}{\left(1+\xi h^{2}\right)}\left[1+\frac{C}{16 \pi^{2} \xi}+\frac{C}{8 \pi^{2} \xi} \frac{1}{1+\xi h^{2}} \ln \left(\frac{M_{\mathrm{P} 1} h}{\mu}\right)\right]$.

Notice that, in order to obtain this equation, we have neglected a term proportional to $1 /(\xi h)^{2} \ll 1$. Contrary to the assumption $\xi h^{2} \gg 1$, the condition $(\xi h)^{2} \gg 1$ was also used in Section 3.1. Then, the integration of this differential equation leads to
$\chi \simeq \frac{\sqrt{3}}{2} \ln \left(1+\xi h^{2}\right)+\frac{\sqrt{3} C}{16 \pi^{2} \xi}\left[\ln h-\frac{1}{1+\xi h^{2}} \ln \left(\frac{M_{\mathrm{Pl}} h}{\mu}\right)\right]$.

Using only now the limit $\xi h^{2} \gg 1$, this expression reduces to
$\chi \simeq \frac{\sqrt{3}}{2} \ln \left(\xi h^{2}\right)+\frac{\sqrt{3} C}{16 \pi^{2} \xi} \ln h$.
As expected the relation between the Jordan frame field $h$ and the Einstein frame field $\chi$ is modified by the quantum corrections. Inverting the above formula gives
$\xi^{1 / 2} h \simeq e^{\chi / \sqrt{3}}-\frac{C}{16 \pi^{2} \xi} e^{\chi / \sqrt{3}}\left(\frac{\chi}{\sqrt{3}}-\frac{1}{2} \ln \xi\right)$.


Fig. 25. $\left(R+R^{2 p}\right)$ Inflation (RpI) in the Einstein frame for $p=2$ (RpI1 and RpI2), and $p=0.9$ (RpI3). Top panels: the potential and its logarithm. Bottom left panel: slow-roll parameter $\epsilon_{1}$ with the region in which inflation stops (shaded area). In the RpI2 regime, inflation never stops and one has to consider an extra-mechanism to end inflation. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line).

This equation allows us to find the expression of the potential in the Einstein frame. Inserting Eq. (4.10) into Eq. (4.6) and introducing the canonically normalized field $\phi \equiv \sqrt{2} M_{\mathrm{PI}} \chi$, one obtains

$$
\begin{align*}
V(\phi) \simeq & \frac{M_{\mathrm{Pl}}^{4} \lambda}{4 \xi^{2}}\left[1-2 e^{-2 \phi /\left(\sqrt{6} M_{\mathrm{Pl}}\right)}-\frac{C}{4 \pi^{2} \xi}\right. \\
& \times e^{-2 \phi /\left(\sqrt{6} M_{\mathrm{Pl}}\right)}\left(\frac{\phi}{\sqrt{6} M_{\mathrm{Pl}}}-\frac{1}{2} \ln \xi\right) \\
& \left.+\frac{A_{\mathrm{I}}}{16 \pi^{2}} \ln \left(\frac{M_{\mathrm{PI}}}{\mu \sqrt{\xi}}\right)+\frac{A_{\mathrm{I}}}{16 \pi^{2}} \frac{\phi}{\sqrt{6} M_{\mathrm{PI}}}\right] \\
\simeq & \frac{M_{\mathrm{Pl}}^{4} \lambda}{4 \xi^{2}}\left[1-2 e^{-2 \phi /\left(\sqrt{6} M_{\mathrm{Pl}}\right)}+\frac{A_{\mathrm{I}}}{16 \pi^{2}} \frac{\phi}{\sqrt{6} M_{\mathrm{PI}}}\right] . \tag{4.11}
\end{align*}
$$

We see that we now deal with a "one parameter model", $A_{\mathrm{I}}$, since the mass scale $M^{4} \equiv M_{\mathrm{PI}}^{4} \lambda /\left(4 \xi^{2}\right)$ is determined by the COBE normalization. In the case $A_{I}=0$, it is also interesting to compare the above potential with the one given by Eq. (3.9). We see that this corresponds to assuming that the exponential $e^{-2 \phi /\left(\sqrt{6} M_{\mathrm{PI}}\right)} \ll 1$ (or, equivalently, $\phi / M_{\mathrm{Pl}} \gg 1$ ) and to expand the corresponding expression at first order in this small parameter. This leads to the following formula: $V \simeq M^{4}\left[1-2 e^{-2 \phi /\left(\sqrt{6} M_{P 1}\right.}\right]$, i.e. exactly Eq. (4.11) for $A_{I}=0$. It is worth remarking that this approximation is not very good towards the end of inflation. Indeed, it is easy to show that (see below), for the potential (4.11) with $A_{\mathrm{I}}=$ $0, \phi_{\text {end }} / M_{\mathrm{Pl}}=\sqrt{3 / 2} \ln (2+2 / \sqrt{3}) \simeq 1.4$ which should be compared with Eq. (3.16) for the potential (3.9) according to which $\phi_{\text {end }} / M_{\text {Pl }} \simeq 0.94$.

### 4.1.2. Slow-roll analysis

Given the potential (4.11), namely
$V(\phi)=M^{4}\left[1-2 e^{-2 \phi /\left(\sqrt{6} M_{\mathrm{Pl}}\right)}+\frac{A_{\mathrm{I}}}{16 \pi^{2}} \frac{\phi}{\sqrt{6} M_{\mathrm{PI}}}\right]$,
we can now proceed to the slow-roll analysis. The potential (4.12) is represented and compared with its tree level counterpart in Fig. 7. Defining $x \equiv \phi / M_{\mathrm{PI}}$, the three first slow-roll parameters can be written as

$$
\begin{align*}
& \epsilon_{1}=\frac{1}{12}\left[\frac{4 e^{-\sqrt{2 / 3 x}}+A_{\mathrm{I}} /\left(16 \pi^{2}\right)}{1-2 e^{-\sqrt{2 / 3 x}}+A_{\mathrm{I}} /\left(32 \pi^{2}\right) \sqrt{2 / 3 x}}\right]^{2},  \tag{4.13}\\
& \epsilon_{2}=\frac{1}{3} \frac{8 e^{-\sqrt{2 / 3 x}}\left[1+A_{\mathrm{I}} /\left(16 \pi^{2}\right)+A_{\mathrm{I}} /\left(32 \pi^{2}\right) \sqrt{2 / 3 x}\right]+A_{\mathrm{I}}^{2} /\left(256 \pi^{4}\right)}{\left[1-2 e^{-\sqrt{2 / 3 x}}+A_{\mathrm{I}} /\left(32 \pi^{2}\right) \sqrt{2 / 3 x}\right]^{2}}, \tag{4.14}
\end{align*}
$$

and

$$
\begin{align*}
\epsilon_{3}= & 12\left(4+\frac{A_{\mathrm{I}}}{16 \pi^{2}} e^{\sqrt{2 / 3} x}\right)\left\{48+8 \frac{A_{\mathrm{I}}}{16 \pi^{2}}(9+\sqrt{6} x)\right. \\
& +3 \frac{A_{\mathrm{I}}^{3}}{4096 \pi^{6}} e^{2 \sqrt{2 / 3 x}}+2 e^{\sqrt{2 / 3 x}}\left[12+18 \frac{A_{\mathrm{I}}}{16 \pi^{2}}\left(1+\frac{A_{\mathrm{I}}}{16 \pi^{2}}\right)\right. \\
& \left.\left.+\sqrt{6} \frac{A_{\mathrm{I}}}{16 \pi^{2}}\left(4+3 \frac{A_{\mathrm{I}}}{16 \pi^{2}}\right) x+2 \frac{A_{\mathrm{I}}^{2}}{256 \pi^{4}} x^{2}\right]\right\} \\
& \times\left[24+\frac{A_{\mathrm{I}}}{16 \pi^{2}}\left(24+4 \sqrt{6} x+3 \frac{A_{\mathrm{I}}}{16 \pi^{2}} e^{\sqrt{2 / 3 x}}\right)\right]^{-1} \\
& \times\left[-12+e^{\sqrt{2 / 3 x}}\left(6+\sqrt{6} \frac{A_{\mathrm{I}}}{16 \pi^{2}} x\right)\right]^{-2} . \tag{4.15}
\end{align*}
$$



Fig. 26. Top left panel: Double Well Inflation (DWI) potential as a function of $\phi / \phi_{0}$. Only the $\phi>0$ region is displayed since the potential is symmetric under $\phi \rightarrow-\phi$. Top right panel: logarithm of the potential. The arrow indicates in which direction inflation can proceed. Bottom left panel: slow-roll parameter $\epsilon_{1}$, rescaled by the quantity $M_{\mathrm{Pl}}^{2} / \phi_{0}^{2}$, such that the corresponding expression becomes universal, i.e. independent of $\phi_{0}$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line), rescaled by $M_{\mathrm{Pl}}^{2} / \phi_{0}^{2}$ for the same reason as mentioned before.

These three slow-roll parameters are represented in Fig. 7 (bottom panels). It is interesting to compare these formulas with the expressions derived in Ref. [210] [see Eqs. (22) and (23) of that paper]. An approximate equation for the first slow-roll parameter is obtained by neglecting the second and third terms in the denominator of Eq. (4.13), which, as a matter of fact, consists in writing $V(\phi) \simeq M^{4}$. Then, it follows that

$$
\begin{align*}
\epsilon_{1} & \simeq \frac{4}{3} e^{-2 \sqrt{2 / 3 x}}\left(1+\frac{A_{\mathrm{I}}}{64 \pi^{2}} e^{\sqrt{2 / 3 x}}\right)^{2} \\
& \simeq \frac{4}{3} \frac{1}{\xi^{2} h^{4}}\left(1+\frac{h^{2}}{h_{\mathrm{I}}^{2}}\right)^{2} \tag{4.16}
\end{align*}
$$

where we have defined $h_{\mathrm{I}}^{2} \equiv 64 \pi^{2} /\left(\xi A_{\mathrm{I}}\right)$ in agreement with Ref. [210]. The same approximation is made for the second slowroll parameter (except that Ref. [210] calculates $\hat{\eta} \equiv M_{P 1}^{2} V_{\phi \phi} / V$ rather than $\epsilon_{2}$ ). The second field derivative of the potential can be written as $V_{\phi \phi}=-4 M^{4} e^{-\sqrt{2 / 3 x}} /\left(3 M_{\mathrm{Pl}}^{2}\right)$ and, therefore, if one considers that $V(\phi) \simeq M^{4}$, then $\hat{\eta} \simeq-4 /\left(3 \xi h^{2}\right)$. We conclude that our expressions of $\epsilon_{1}$ and $\epsilon_{2}$ reproduce Eqs. (22) and (23) of Ref. [210] in the limit where $V(\phi) \simeq M^{4}$.

Let us now study how inflation ends in this model. From Fig. 7, it is clear that this occurs by violation of the slow-roll conditions. Working out the condition $\epsilon_{1}=1$, it follows that

$$
\begin{align*}
x_{\text {end }}= & \frac{1}{\sqrt{2}}-\sqrt{\frac{3}{2}} \frac{32 \pi^{2}}{A_{\mathrm{I}}} \\
& +\sqrt{\frac{3}{2}} \mathrm{~W}_{-1}\left[\frac{64 \pi^{2}}{A_{\mathrm{I}}}\left(1+\frac{1}{\sqrt{3}}\right) e^{32 \pi^{2} / A_{I}-1 / \sqrt{3}}\right] \tag{4.17}
\end{align*}
$$

where, if $A_{\mathrm{I}}>0, \mathrm{~W}_{-1}=\mathrm{W}_{0}$ while, if $A_{\mathrm{I}}<0, \mathrm{~W}_{-1}=\mathrm{W}_{-1}$.
We now turn to the slow-roll trajectory. It can be integrated exactly and straightforward manipulations lead to the following expression

$$
\begin{align*}
N & -N_{\text {ini }}=\sqrt{\frac{3}{2}} x-\frac{48 \pi^{2}}{A_{\mathrm{I}}}\left[1+\frac{A_{\mathrm{I}}}{32 \pi^{2}}\left(1+\sqrt{\frac{2}{3}} x\right)\right] \\
& \times \ln \left(1+\frac{A_{\mathrm{I}}}{64 \pi^{2}} e^{\sqrt{2 / 3} x}\right)-\frac{3}{2} \operatorname{Li}_{2}\left(-\frac{A_{\mathrm{I}}}{64 \pi^{2}} e^{\sqrt{2 / 3} x}\right)-\sqrt{\frac{3}{2}} x_{\mathrm{ini}} \\
& +\frac{48 \pi^{2}}{A_{\mathrm{I}}}\left[1+\frac{A_{\mathrm{I}}}{32 \pi^{2}}\left(1+\sqrt{\frac{2}{3}} x_{\mathrm{ini}}\right)\right] \\
& \times \ln \left(1+\frac{A_{\mathrm{I}}}{64 \pi^{2}} e^{\sqrt{2 / 3} x_{\mathrm{ini}}}\right)+\frac{3}{2} \mathrm{Li}_{2}\left(-\frac{A_{\mathrm{I}}}{64 \pi^{2}} e^{\sqrt{2 / 3} x_{\text {ini }}}\right) \tag{4.18}
\end{align*}
$$

where $\mathrm{Li}_{2}$ denotes the dilogarithm function [216,217]. Let us also notice that if we use the approximation $V(\phi) \simeq M^{4}$ already discussed before, then one can obtain a much simpler formula, namely

$$
\begin{align*}
N-N_{\text {ini }}= & -\frac{48 \pi^{2}}{A_{\mathrm{I}}} \ln \left(1+\frac{A_{\mathrm{I}}}{64 \pi^{2}} e^{\sqrt{2 / 3} x}\right) \\
& +\frac{48 \pi^{2}}{A_{\mathrm{I}}} \ln \left(1+\frac{A_{\mathrm{I}}}{64 \pi^{2}} e^{\sqrt{2 / 3} x_{\text {ini }}}\right) . \tag{4.19}
\end{align*}
$$

This expression is in agreement with Eq. (24) of Ref. [210]. In this case, the trajectory can even be inverted and the corresponding


Fig. 27. Lambert functions $\mathrm{W}_{0}(x)$ (dashed line) and $\mathrm{W}_{-1}(x)$ (solid line). In DWI, inflation proceeds along the negative part of the " 0 " branch in the direction specified by the arrow.
expression for the field $\phi$ reads
$x=\sqrt{\frac{3}{2}} \ln \left[\left(\frac{64 \pi^{2}}{A_{\mathrm{I}}}+e^{\sqrt{2 / 3} x_{\text {ini }}}\right) e^{A_{\mathrm{I}}\left(N-N_{\text {ini }}\right) /\left(48 \pi^{2}\right)}-\frac{64 \pi^{2}}{A_{\mathrm{I}}}\right]$.

We are now in a position where the predictions of the models can be calculated. They are presented in Fig. 81. We see that very negative values of $A_{\mathrm{I}}$ are incompatible with the CMB while large values of $A_{I}$ remain close to the allowed contours. Of course $\left|A_{I}\right|$ cannot be too large since we have required $A_{\mathrm{I}} /\left(64 \pi^{2}\right) \ll 1$. We have chosen the upper bound in Fig. 81 to be $A_{\mathrm{I}}=100$ for which $A_{1} /\left(64 \pi^{2}\right) \simeq 0.16$, i.e. still a reasonable number. It is interesting to compare these findings with the existing literature. Using the approximate trajectory (4.19) and neglecting the contribution originating from the end of inflation, one obtains
$x_{*}=\sqrt{\frac{3}{2}} \ln \left[\frac{64 \pi^{2}}{A_{\mathrm{I}}}\left(e^{x_{\mathrm{BKS}}}-1\right)\right]$,
where $x_{\text {BKS }} \equiv A_{\mathrm{I}} \Delta N_{*} /\left(48 \pi^{2}\right)$ ( $x_{\text {BKS }}$ is denoted $x$ in Ref. [210]). Then, from Eq. (4.16) and the fact that $\epsilon_{2}=4 \epsilon_{1}-2 \hat{\eta}$, it follows that
$\epsilon_{1}=\frac{4}{3}\left(\frac{A_{I}}{64 \pi^{2}}\right)^{2}\left(\frac{e^{x_{\mathrm{BKS}}}}{e^{x_{\mathrm{BKS}}}-1}\right)^{2}=\frac{3}{4 \Delta N_{*}^{2}}\left(\frac{x_{\mathrm{BKS}} e^{x_{\mathrm{BKS}}}}{e^{x_{\mathrm{BKS}}}-1}\right)^{2}$,
$\epsilon_{2}=4 \epsilon_{1}+\frac{8}{3} \frac{A_{\mathrm{I}}}{64 \pi^{2}} \frac{1}{e^{x_{\mathrm{BKS}}}-1}=4 \epsilon_{1}+\frac{2}{\Delta N_{*}} \frac{x_{\mathrm{BKS}}}{e^{x_{\mathrm{BSS}}}-1}$.
From these two expressions, one deduces that
$n_{\mathrm{S}}=1-\frac{2}{\Delta N_{*}} \frac{x_{\mathrm{BKS}}}{e^{x_{\mathrm{BKS}}}-1}, \quad r=\frac{12}{\Delta N_{*}^{2}}\left(\frac{x_{\mathrm{BKS}} e^{x_{\mathrm{BKS}}}}{e^{x_{\mathrm{BKS}}}-1}\right)^{2}$.
Notice that, in the formula giving the spectral index, the contribution originating from $\epsilon_{1}$ has been neglected since it scales $\propto 1 / \Delta N_{*}^{2}$. These approximate expressions match Eqs. (32) and (34) of Ref. [210]. For $\Delta N_{*}=60$, they can be represented as a line $r=$ $r\left(n_{\mathrm{S}}\right)$ in the plane ( $\left.n_{\mathrm{S}}, r\right)$, the parameter along the curve being $A_{\mathrm{I}}$. This line has been plotted in Fig. 8 for $-30<A_{\mathrm{I}}<100$ (red dashed line). Requiring $0.9457<n_{\mathrm{S}}<0.9749$ which is the $2 \sigma$ Planck range [70] (or $0.934<n_{\mathrm{S}}<0.988$, which is the $2 \sigma$ range coming from combining the WMAP 9th year data, the Baryon Acoustic Oscillations (BAO) data and the Supernovae measurements), one obtains the solid thick red segment. It follows that $-8 \lesssim A_{\mathrm{I}} \lesssim 4$ (or $-12 \lesssim A_{\mathrm{I}} \lesssim 14$ with WMAP, again in agreement with Ref. [210]). These predictions are compared to the exact slow-roll predictions
of Fig. 81. As before, the slow-roll predictions are represented by a collection of segments, each segment corresponding to different values of $A_{I}$ and each point of a given segment being in one-toone correspondence with a given reheating temperature. The exact slow-roll predictions are such that, for $A_{\mathrm{I}}<0$, the green segments go to the bottom left side of the figure while for $A_{I} \rightarrow 100$, the pink/red segments remain close to the allowed contours (see also Fig. 81). In the limit of "large" positive values of $A_{\mathrm{I}}$, the exact slowroll predictions and the predictions based on Eq.(4.24) significantly differ. While, in order to remain close to the allowed contours, Eq. (4.24) tell us that $A_{I} \lesssim 4$, the exact slow-roll predictions show that the model is still viable for any positive values of $A_{I} \lesssim 100$. We conclude that the upper bound $A_{\mathrm{I}} \lesssim 4$ (with the WMAP data, $\left.A_{\mathrm{I}} \lesssim 14\right)$ is inaccurate and is just an artifact due to the inaccurate nature of the "approximation to the slow-roll approximation".

Let us try to identify the origin of this discrepancy more precisely. In order to investigate this issue, we have also represented in Fig. 8, the predictions obtained when the approximate trajectory (4.19), the approximate expression of the first slow-roll parameter (4.16) and the relation $\epsilon_{2}=4 \epsilon_{1}-2 \hat{\eta}$ but, now, without neglecting $\epsilon_{1}$, are used together with an exact expression for $\phi_{\text {end }}$. They are represented by the second collections of segments in Fig. 8. We see that for $A_{I} \gtrsim 0$, they differ from the red thick solid line and bend toward the upper left part of the plot which is also the direction taken by the exact predictions. This suggests that neglecting the term $4 \epsilon_{1}$ in the expression of $\epsilon_{2}$ causes a nonnegligible error. This is confirmed if, instead of using Eq. (4.24) for $n_{s}$, we now take
$n_{\mathrm{S}}=1-\frac{3}{2 \Delta N_{*}^{2}}\left(\frac{x_{\mathrm{BKS}} e^{x_{\mathrm{BKS}}}}{e^{x_{\mathrm{BKS}}}-1}\right)^{2}-\frac{2}{\Delta N_{*}} \frac{x_{\mathrm{BKS}}}{e^{x_{\mathrm{BKS}}}-1}$,
and plot again the line $r=r\left(n_{\mathrm{S}}\right)$. This gives the yellow dotted-dashed curve which follows the second collection of segments. If, however, we compare the red segments, namely those with $A_{I}$ "large", corresponding the exact predictions to the approximate red ones, we see that including the term $4 \epsilon_{1}$ is not sufficient. For $A_{I} \simeq 60$, the exact predictions are roughly compatible with the data while the segments corresponding to the approximate formulas are not. We conclude that RCHI represents a textbook case for ASPIC. It illustrates that, sometimes, "approximating the slow-roll approximation" can lead to too drastic conclusions, especially given the current accuracy of the data. It is an additional motivation to use the slow-roll method without any other scheme of approximations and this is the essence of the ASPIC project presented in this article.

A last word is in order concerning the constraints on the parameter $A_{\mathrm{I}}$. Particle physics implies that $-48 \lesssim A_{\mathrm{I}} \lesssim-20$ and the previously discussed inaccuracies were concerning only a weaker upper limit on $A_{1}$. On the contrary, we see in Fig. 8 that the bound $A_{I} \gtrsim-8$ is accurate whatever the approximation scheme chosen. Therefore, when particle physics and cosmological data are simultaneously taken into account, the conclusions of Ref. [210] are unchanged and RCHI remains disfavored.

Finally, the scale $M$ can be determined from the CMB normalization and this leads to the following expression
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=120 \pi^{2} \frac{\mathrm{Q}_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \frac{\left[4 e^{-\sqrt{2 / 3} x_{*}}+A_{\mathrm{I}} /\left(16 \pi^{2}\right)\right]^{2}}{\left[1-2 e^{-\sqrt{2 / 3} x_{*}}+A_{\mathrm{I}} /\left(32 \pi^{2}\right) \sqrt{2 / 3} x_{*}\right]^{3}}$.

The knowledge of $\phi_{*}$ allows us to find the posterior distribution of $M$, that is to say of $\lambda / \xi^{2}$ or $\xi$, since the Higgs self coupling, $\lambda=m_{\mathrm{H}} / v$, is now known.


Fig. 28. Mutated Hilltop Inflation (MHI). The top panels show the potential and its logarithm as a function of $x=\phi / \mu$. Bottom left panel: Rescaled slow-roll parameter $\epsilon_{1}$ (divided by $M_{\mathrm{Pl}}^{2} / \mu^{2}$ ). The shaded area represents the region in which inflation stops if $\mu=M_{\mathrm{PI}}$. It should be accordingly rescaled for other values of $\mu$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line), again rescaled by $M_{\mathrm{PI}}^{2} / \mu^{2}$ together with the region of slow-roll violation for $\mu=M_{\mathrm{PI}}$.

### 4.2. Large field inflation (LFI)

### 4.2.1. Theoretical justifications

Large fields models, also referred to as chaotic inflation [218], are characterized by the monomial potential [219-223] $V(\phi) \propto$ $M^{4} \phi^{p}$. The number $p$ is the only model parameter, in addition to the normalization $M$ of the potential. The index $p$ is usually a positive integer (and it was recently realized in Ref. [224] that this type of scenario can emerge in the context of supergravity) but various models have been proposed in which it can also be a rational number [225-230]. It is interesting to briefly discuss concrete models where this is actually the case. Here, we follow Refs. [229,230]. These models are supergravity models where one assumes that the Kähler potential is invariant under a generalization of the shift symmetry (usually needed in order to avoid the so called $\eta$ problem). In the present case, the transformation is taken to be $\chi^{n} \rightarrow \chi^{n}+\alpha$ where $\alpha$ is a real number and $\chi$ a chiral superfield. This means that the Kähler potential should be a function of $\chi^{n}-\chi^{\dagger n}$ only. In addition, we allow the presence of a small breaking term in the Kähler potential of the form $b \chi \chi^{\dagger}$ where $b \ll 1$. We also assume that the superpotential breaks the generalized shift symmetry. Summarizing, we assume that

$$
\begin{align*}
K= & b \chi \chi^{\dagger}+c_{1} \kappa^{(n-1) / 2}\left(\chi^{n}-\chi^{\dagger n}\right) \\
& -\frac{\kappa^{n-1}}{2}\left(\chi^{n}-\chi^{\dagger n}\right)^{2}+X X^{\dagger} \tag{4.27}
\end{align*}
$$

$W=\lambda X \chi^{m}$,
where $X$ is another superfield and $\lambda$ and $c_{1}$ (notice that it is pure imaginary) are constant. The model is parametrized by the quantities $n$ and $m$ and $\kappa \equiv 1 / M_{\mathrm{Pl}}^{2}$. If, during inflation, $X$ acquires a large mass compared to the Hubble parameter and is stabilized at the origin, $\langle X\rangle=0$, then it is not difficult to show that this
supergravity model can be described by the following effective Lagrangian

$$
\begin{align*}
\mathcal{L}= & -\left[b+n^{2} \kappa^{n-1}\left(\chi \chi^{\dagger}\right)^{n-1}\right] \partial_{\mu} \chi \partial^{\mu} \chi^{\dagger} \\
& -\exp \left[b \kappa|\chi|^{2}+c_{1} \kappa^{n / 2}\left(\chi^{n}-\chi^{\dagger n}\right)\right. \\
& \left.-\frac{\kappa^{n}}{2}\left(\chi^{n}-\chi^{\dagger n}\right)^{2}\right] \lambda^{2}\left(\chi \chi^{\dagger}\right)^{m} \tag{4.29}
\end{align*}
$$

Then, one can write the field $\chi$ in polar form, $\chi \equiv \alpha e^{i \beta}$ ( $\alpha$ is of dimension one and $\beta$ dimensionless) and the above potential takes the form

$$
\begin{align*}
V= & \lambda^{2} \alpha^{2 m} \exp \left[b \kappa \alpha^{2}+2 i c_{1} \kappa^{n / 2} \alpha^{n} \sin (n \beta)\right. \\
& \left.+2 \kappa^{n} \alpha^{2 n} \sin ^{2}(n \beta)\right] \tag{4.30}
\end{align*}
$$

Writing $\partial V / \partial \beta=0$, one obtains the condition $2 i \kappa^{n / 2} \alpha^{n} \sin (n \beta)=$ $-i c_{1}$ or $\kappa^{n / 2}\left(\chi^{n}-\chi^{\dagger n}\right)=c_{1}$. It is thus natural to assume that the inflaton field rolls along that direction. As a consequence, the effective Lagrangian takes the form

$$
\begin{align*}
\mathcal{L}= & -\left[b+n^{2} \kappa^{n-1}\left(\chi \chi^{\dagger}\right)^{n-1}\right] \partial_{\mu} \chi \partial^{\mu} \chi^{\dagger} \\
& -e^{b \kappa|x|^{2}+c_{1}^{2} / 2} \lambda^{2}\left(\chi \chi^{\dagger}\right)^{m} . \tag{4.31}
\end{align*}
$$

Now, in the regime $b \kappa|\chi|^{2} \ll 1$, the exponential becomes essentially independent of the field $\chi$ and the coefficient $b$ in the kinetic term becomes negligible. It is therefore natural to define a new quantity $\theta \equiv \kappa^{(n-1) / 2} \chi^{n}$ for which one obtains the Lagrangian of a canonically normalized field, namely
$\mathcal{L}=-\partial_{\mu} \theta \partial^{\mu} \theta^{\dagger}-e^{c_{1}^{2} / 2} \lambda^{2}\left(\theta \theta^{\dagger}\right)^{m / n}$.
Finally, we take the imaginary part of $\theta$ to be stabilized to $c_{1}$ in order to satisfy the condition discussed above and we define the


Fig. 29. Lambert functions $\mathrm{W}_{0}(x)$ (dashed line) and $\mathrm{W}_{-1}(x)$ (solid line). During Mutated Hilltop inflation, inflation proceeds along the " -1 " branch in the direction specified by the arrow on the figure.
real field $\phi$ by $\theta=\phi / \sqrt{2}+c_{1} / 2$. As a consequence, it follows
$\mathcal{L} \simeq-\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi^{\dagger}-e^{c_{1}^{2} / 2} \lambda^{2} \phi^{2 m / n}$.
Therefore, we have obtained a LFI model with $p=2 \mathrm{~m} / \mathrm{n}$ (neglecting a term $\left|c_{1}\right|^{2}$ in $V$ ). In Ref. [229], the case $n=2$ and $m=1$ was considered and we see that this leads to a linear potential. In Ref. [230], the generalized case considered before was introduced and studied. It is worth mentioning that, when the condition $b \kappa|\chi|^{2} \ll 1$ is not satisfied, the potential remains of the LFI form but with a different $p$, see Ref. [230]. For instance, as shown in Ref. [229], if $n=2$ and $m=1$, the potential is in fact quadratic at the origin. This means that the standard relation between $p$ (in the inflationary regime) and the mean equation of state during reheating namely, $\bar{w}_{\text {reh }}=(p-2) /(p+2)$ [54], is no longer valid in that case.

### 4.2.2. Slow-roll analysis

Having studied how the LFI model can be implemented in high energy physics, we now turn to the inflationary analysis. In the following, we write $V(\phi)$ as
$V(\phi)=M^{4}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{p}$.
This potential is represented in Fig. 9 for $p=2$. The three Hubble flow functions are straightforwardly obtained from Eqs. (2.4)(2.6). Defining $x \equiv \phi / M_{\mathrm{Pl}}$, one gets
$\epsilon_{1}=\frac{p^{2}}{2 x^{2}}, \quad \epsilon_{2}=\frac{2 p}{x^{2}}, \quad \epsilon_{3}=\epsilon_{2}$.
These functions are represented in the two bottom panels of Fig. 9. They are monotonic decreasing functions of $\phi$. One can immediately deduce that, for a given $p$, the model in the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ is contained in the line $\epsilon_{1}=(p / 4) \epsilon_{2}$.

The slow-roll trajectory is completely explicit and obtained by quadrature from Eq. (2.11)

$$
\begin{align*}
N-N_{\mathrm{end}} & =-\frac{1}{M_{\mathrm{Pl}}^{2}} \int_{\phi_{\text {end }}}^{\phi} \frac{V(\chi)}{V^{\prime}(\chi)} \mathrm{d} \chi=-\frac{1}{p} \int_{\phi_{\text {end }} / M_{\mathrm{Pl}}}^{\phi / M_{\mathrm{Pl}}} x \mathrm{~d} x \\
& =\frac{1}{2 p}\left(x_{\mathrm{end}}^{2}-x^{2}\right) . \tag{4.36}
\end{align*}
$$

This expression can be inverted and reads
$x=\sqrt{x_{\mathrm{end}}^{2}-2 p\left(N-N_{\mathrm{end}}\right)}$.

For the large field models, inflation ends naturally when $\epsilon_{1}=1$ (see Section 1). Along the $\phi>0$ branch of the potential, this leads to
$x_{\text {end }}=\frac{p}{\sqrt{2}}$.
This expression also allows us to obtain the total number of $e$-folds. Plugging Eq. (4.38) into Eq. (4.36), one arrives at
$N_{\text {end }}-N_{\text {ini }}=\frac{1}{2 p} x_{\text {ini }}^{2}-\frac{p}{4}$,
which can be very large if the initial field value is super-Planckian. Notice that this does not imply that the energy density is close to the Planck scale as this one is typically given by the potential and proportional to $M^{4}$. In fact, the model remains under control only if the initial energy density is smaller than $M_{\mathrm{Pl}}^{4}$ and this imposes a constraint on both $\phi_{\text {ini }}$ and $M$ which reads
$x_{\text {ini }}=\frac{\phi_{\text {ini }}}{M_{\mathrm{PI}}} \lesssim\left(\frac{M_{\mathrm{PI}}}{M}\right)^{4 / p}$.
Let us notice that, when the inflaton energy density approaches the Planck energy density, quantum effects become important. In this case, the stochastic inflation formalism must be used [231-237].

We now turn to the explicit determination of the slow-roll parameters. We have seen that the model is represented by the trajectory $\epsilon_{1}=(p / 4) \epsilon_{2}$ but observable models only lie in a limited portion of this straight line. Indeed, the Hubble flow parameters should be evaluated when the scales of astrophysical interest today left the Hubble radius during inflation. Following the discussion of Section 2.2, we assume the pivot mode crossed the Hubble radius for $\phi=\phi_{*}$ at the $e$-fold number $N_{*}$. From the trajectory, we have
$x_{*}^{2}=2 p\left(\Delta N_{*}+\frac{p}{4}\right)$,
and the slow-roll parameters read
$\epsilon_{1 *}=\frac{p}{4\left(\Delta N_{*}+p / 4\right)}, \quad \epsilon_{2 *}=\frac{1}{\Delta N_{*}+p / 4}$,
$\epsilon_{3 *}=\epsilon_{2 *}$.
Solving Eq. (2.47) for $\phi_{*}$ yields the slow-roll predictions represented in Fig. 83. As expected, the whole family lies in the region $\epsilon_{2}>0$ and verifies $\epsilon_{1}=p / 4 \epsilon_{2}$. From Fig. 83, we see that all the models with $p \gtrsim 3$ lie outside the $2 \sigma$ contour. The quadratic (or massive) model is under great pressure since it predicts quite a high contribution of gravitational waves, up to $r \simeq 15 \%$ level.

Finally, the parameter $M$ can be determined from the amplitude of the CMB anisotropies, and one gets
$\frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}=\frac{1}{480 \pi^{2} \epsilon_{1 *}} \frac{H_{*}^{2}}{M_{\mathrm{Pl}}^{2}}=\frac{1}{1440 \pi^{2} \epsilon_{1 *}} \frac{V_{*}}{M_{\mathrm{Pl}}^{4}}$.
In the case of large fields model, this implies
$\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=\frac{720 \pi^{2} p^{2}}{\left(x_{*}^{2}\right)^{p / 2+1}} \frac{Q_{\text {rms-PS }}^{2}}{T^{2}}$,
and given the constraints on $p$ and $\Delta N_{*}$, this leads to $M / M_{\mathrm{PI}} \simeq$ $3 \times 10^{-3}$. We recover the conclusion that, for large field models, inflation takes place close to the Grand Unified Theory (GUT) scale.


Fig. 30. Radion Inflation (RGI) for $\alpha=10^{-4}$. Top frames: the potential and its logarithm. Bottom left panel: slow-roll parameter $\epsilon_{1}$ and the shaded area in which inflation stops ( $\epsilon_{1}>1$ ). Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line).


Fig. 31. MSSM Inflation (MSSMI). Top left panel: MSSM Inflation potential Eq. (4.246) as a function of $\phi / \phi_{0}$. Top right panel: logarithm of the potential. Bottom left panel: slow-roll parameter $\epsilon_{1}$ scaled by $\phi_{0}^{2} / M_{\mathrm{Pl}}^{2}$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) scaled by $\phi_{0}^{2} / M_{\mathrm{Pl}}^{2}$.


Fig. 32. Location of the slow-roll violation induced end of inflation $x_{\text {end }}=\phi_{\text {end }} / \phi$ for the MSSM inflation models, as a function of $\phi_{0} / M_{P l}$. The blue solid curve represents a numerical solution of $\epsilon_{1}=1$, while the red dotted curve corresponds to the approximated analytic solution Eq. (4.254). For physical values $\phi_{0} \simeq 10^{-4} M_{P 1}$, the agreement is obviously excellent.

### 4.3. Mixed large field inflation (MLFI)

This model is a generalization of the LFI model $V(\phi) \propto \phi^{p}$, see Section 4.2 , where two monomials $\propto \phi^{2}$ and $\propto \phi^{4}$ are added. The MLFI potential reads
$V(\phi)=M^{4} \frac{\phi^{2}}{M_{\mathrm{Pl}}^{2}}\left(1+\alpha \frac{\phi^{2}}{M_{\mathrm{Pl}}^{2}}\right)$,
where $\alpha$ is a positive dimensionless parameter. If $\phi / M_{\mathrm{PI}} \ll 1 / \sqrt{\alpha}$, then the potential is of the LFI type with $p=$ 2, i.e. $V(\phi) \simeq$ $M^{4} \phi^{2} / M_{\mathrm{Pl}}^{2}$, whereas if $\phi / M_{\mathrm{PI}} \gg 1 / \sqrt{\alpha}$, the potential is of the LFI type with $p=4$, i.e. $V(\phi) \simeq M^{4} \alpha \phi^{4} / M_{\mathrm{Pl}}^{4}$. Clearly, the interesting regime is when $\phi / M_{\mathrm{PI}} \simeq 1 / \sqrt{\alpha}$, where the two terms are of equal importance. The potential and its logarithm are displayed in Fig. 10. We notice that $V(\phi)$ is an increasing function of the field vev and, as a consequence, that inflation proceeds from the right to the left.

This model has been investigated in different contexts. Of course, the shape of the potential appears to be natural and wellmotivated since it just represents a free theory (with particles of mass $2 M^{4} / M_{\mathrm{PI}}^{2}$ ) corrected by the usual self-interacting quartic term. Therefore, it does not come as a surprise that this potential has been used in many different works. In Ref. [238], this model is studied in the case where a bulk scalar field is driving inflation in large extra dimensions. In Ref. [239], it is considered in a situation where inflation is driven by highly excited quantum states. In Refs. [240-242], the MLFI potential is utilized in the context of "fresh inflation". The same potential was again considered in Ref. [243] where the role of inflaton is played by the Higgs triplet in a model where the type II seesaw mechanism is used to generate the small masses of left-handed neutrinos. Finally, it is also studied in Ref. [244] where supersymmetric hybrid inflation (in the framework of the Randall-Sundrum type II Braneworld model) is considered. The only constraint on the parameters of the model that is (sometimes) required is that the self-interacting term should be sub-dominant. This leads to the condition $\alpha M^{4} / M_{\mathrm{Pl}}^{4} \ll$ 1. Given the typical values imposed by CMB normalization, i.e. $M / M_{\mathrm{PI}} \simeq 10^{-3}$ [see Eq. (4.44)], this is not very stringent and $\alpha$ can in fact vary in a quite large range of values.

Defining $x \equiv \phi / M_{\mathrm{PI}}$, the three first slow-roll parameters can be expressed as
$\epsilon_{1}=\frac{2}{x^{2}}\left(\frac{1+2 \alpha x^{2}}{1+\alpha x^{2}}\right)^{2}, \quad \epsilon_{2}=\frac{4}{x^{2}} \frac{1+\alpha x^{2}+2 \alpha^{2} x^{4}}{\left(1+\alpha x^{2}\right)^{2}}$,
and
$\epsilon_{3}=\frac{M_{P \mathrm{Pl}}^{2}}{x^{2}} \frac{1+2 \alpha x^{2}}{\left(1+\alpha x^{2}\right)^{2}} \frac{4+12 \alpha x^{2}+8 \alpha^{3} x^{6}}{1+\alpha x^{2}+2 \alpha^{2} x^{4}}$.
They are displayed in Fig. 10. We see that the three slowroll parameters are decreasing functions of the field vev, which means that they are all increasing functions during inflation. As a consequence, inflation can stop by violation of the slow-roll conditions at $x_{\text {end }}$ given by $\epsilon_{1}=1$ (see below). We also notice that $\epsilon_{2}$ and $\epsilon_{3}$ are larger than one at $x_{\text {end }}$. This means that the slow-roll approximation breaks down slightly before the end of inflation and that the last few $e$-folds of inflation may be not properly described by the slow-roll approximation.

Let us now study the slow-roll trajectory. It is given by

$$
\begin{align*}
N_{\mathrm{end}}-N= & -\frac{1}{8}\left[x_{\mathrm{end}}^{2}+\frac{1}{2 \alpha} \ln \left(1+2 \alpha x_{\mathrm{end}}^{2}\right)\right. \\
& \left.-x^{2}-\frac{1}{2 \alpha} \ln \left(1+2 \alpha x^{2}\right)\right], \tag{4.48}
\end{align*}
$$

where $N_{\text {end }}$ is the number of $e$-folds at the end of inflation. One can check that this expression is asymptotically correct. Indeed, when $\alpha \ll 1$, the slow-roll trajectory reduces to
$x_{\text {end }}^{2}=x^{2}-4\left(N_{\text {end }}-N\right)$,
which is the trajectory in the massive case, i.e. LFI with $p=2$, see Eq. (4.36). On the other hand, in the limit $\alpha \rightarrow \infty$, one obtains
$x_{\text {end }}^{2}=x^{2}-8\left(N_{\text {end }}-N\right)$,
which is, as expected, the slow-roll trajectory in the quartic case, i.e. LFI with $p=4$. In general, the trajectory can be inverted and expressed in terms of the Lambert function. Straightforward manipulations lead to
$x=\frac{1}{\sqrt{2 \alpha}} \sqrt{-1+\mathrm{W}_{0}\left[e^{1+2 \alpha x_{\mathrm{end}}^{2}}\left(1+2 \alpha x_{\mathrm{end}}^{2}\right) e^{-16 \alpha\left(N-N_{\mathrm{end}}\right)}\right]}$.

The corresponding Lambert function is displayed in Fig. 11, together with the region where inflation proceeds. (see Fig. 12)

We have seen that, in MLFI, inflation stops by violation of the slow-roll condition. Let us therefore determine the corresponding vev of the field. The condition $\epsilon_{1}=1$ leads to
$\alpha x_{\text {end }}^{3}-2 \sqrt{2} \alpha x_{\text {end }}^{2}+x_{\text {end }}-\sqrt{2}=0$.
This is a cubic algebraic equation that can be solved exactly. In the limit $\alpha \gg 1$, the solution reads $x_{\text {end }} \simeq 2 \sqrt{2}$ which is indeed the solution for the quartic case, see Eq. (4.38). On the other hand, if $\alpha \ll 1$, then $x_{\text {end }} \simeq \sqrt{2}$ which is also the correct result for the quadratic case. The general solution is

$$
\begin{align*}
x_{\mathrm{end}}= & \frac{2 \sqrt{2}}{3}+\frac{1}{3 \alpha}\left\{\frac { 1 } { 4 \sqrt { 2 } } \left[4 \alpha^{2}(32 \alpha+9)\right.\right. \\
& \left.\left.+2 \alpha \sqrt{4 \alpha^{2}(32 \alpha+9)^{2}-8 \alpha(8 \alpha-3)^{3}}\right]\right\}^{1 / 3} \\
& +\frac{1}{3}(8 \alpha-3)\left\{\frac { 1 } { 4 \sqrt { 2 } } \left[4 \alpha^{2}(32 \alpha+9)\right.\right. \\
& \left.\left.+2 \alpha \sqrt{4 \alpha^{2}(32 \alpha+9)^{2}-8 \alpha(8 \alpha-3)^{3}}\right]\right\}^{-1 / 3}, \tag{4.53}
\end{align*}
$$

which is the one used in the ASPIC library.


Fig. 33. Renormalizable Inflection Point Inflation (RIPI). Top left panel: renormalizable inflection point inflation potential as a function of $\phi / \phi_{0}$. Top right panel: logarithm of the potential, the required flatness of the potential close to its inflection point becomes obvious on this plot. Bottom left panel: slow-roll parameter $\epsilon_{1}$ normalized by $M_{\mathrm{Pl}}^{2} / \phi_{0}^{2}$. The shaded area indicates the region in which $\epsilon_{1}>1$ and thus where inflation stops (this has to be rescaled for $\phi_{0} \neq M_{\mathrm{PI}}$ ). Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line), normalized by $M_{\mathrm{Pl}}^{2} / \phi_{0}^{2}$.


Fig. 34. Top left panel: Arctan Inflation (AI) potential as a function of $\phi / \mu$. Top right panel: logarithm of the potential. Bottom left panel: slow-roll parameter $\epsilon_{1}$ rescaled by $M_{\mathrm{Pl}}^{2} / \mu^{2}$ which renders the corresponding expression "universal", i.e. independent of the free parameter $\mu$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) rescaled by $M_{\mathrm{Pl}}^{2} / \mu^{2}$ (for the same reason as mentioned before).

$$
\begin{align*}
\epsilon_{3}= & \frac{4}{x^{2}} \frac{1-\alpha x^{2}-4 \alpha x^{2} \ln x}{\left(1-2 \alpha x^{2} \ln x\right)^{2}} \\
& \times \frac{1-\alpha x^{2}\left[\alpha x^{2}\left(4 \alpha x^{2}+9\right)+1\right]-\alpha x^{2} \ln x\left[4 \alpha^{2} x^{4} \ln x(4 \ln x+1)+\left(\alpha x^{2}+3\right)\left(6 \alpha x^{2}+2\right)\right]}{\left(1+\alpha x^{2}\right)\left(1+2 \alpha x^{2}\right)-2 \alpha x^{2} \ln x\left(1-\alpha x^{2}-4 \alpha x^{2} \ln x\right)} \tag{4.60}
\end{align*}
$$

Box I.

Finally, the parameter $M$ can be determined from the amplitude of the CMB anisotropies, and one gets

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=\frac{2880 \pi^{2}}{x^{4}} \frac{\left(1+2 \alpha x_{*}^{2}\right)^{2}}{\left(1+\alpha x_{*}^{2}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{4.54}
\end{equation*}
$$

Similarly to LFI (see Section 4.2), this gives rise to $M / M_{P 1} \simeq 10^{-3}$. The reheating consistent slow-roll predictions for the MLFI models are displayed in Fig. 84. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 which is consistent with the fact that the potential is quadratic close to its minimum. As expected, when $\alpha \ll 1$ the predictions of the model match those of LFI with $p=2$ and are aligned along the $\epsilon_{1}=\epsilon_{2} / 2$ line. On the other hand, if $\alpha \gg$ 1 , then the predictions are consistent with those of LFI with $p=4$ and are aligned along the $\epsilon_{1}=\epsilon_{2}$ line. In the intermediate regime, it is interesting to notice that the MLFI predictions continuously interpolate between these two asymptotic solutions but do not remain inside the domain delimited by those two lines. Indeed, when $\alpha$ is larger than some value, one has $\epsilon_{1}>\epsilon_{2}$. This means that, if one starts from a pure quartic potential (LFI with $p=4$ ) and adds a small quadratic term, this extra term has the effect of increasing the "effective value" of $p$, which is quite counter intuitive. On the other hand, since the quadratic model fits better the data than the quartic one, small values for the parameter $\alpha$ are favored (all the models with $\alpha>10^{-3}$ lie outside the $2 \sigma$ contour of the Planck data). High reheating temperatures are also preferred.

### 4.4. Radiatively corrected massive inflation (RCMI)

This model is based on Ref. [245] and implements radiative corrections due to fermion couplings over the massive ( $p=2$ ) large field model (see Section 4.2). With an appropriate choice of the renormalization scale $\mu=g M_{\mathrm{Pl}}, g$ denoting the Yukawa coupling, the potential is given by

$$
\begin{align*}
V(\phi) & =\frac{1}{2} m^{2} \phi^{2}-\frac{g^{4}}{16 \pi^{2}} \phi^{4} \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right) \\
& =M^{4}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{2}\left[1-2 \alpha \frac{\phi^{2}}{M_{\mathrm{Pl}}^{2}} \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right)\right], \tag{4.55}
\end{align*}
$$

where
$M^{4} \equiv \frac{1}{2} m^{2} M_{\mathrm{Pl}}^{2}, \quad \alpha \equiv \frac{g^{4} M_{\mathrm{Pl}}^{2}}{16 \pi^{2} m^{2}}$.
This expression is obtained in the large field regime $\phi \gg$ $\mathrm{m} / \mathrm{g}$ (this condition coming from the requirement that the fermion loop contribution dominates over the self-interaction loop contribution), i.e. assuming that the inflationary regime takes place under the condition
$\frac{\phi^{4}}{M_{\mathrm{Pl}}^{4}} \gg \frac{1}{8 \pi^{2} \alpha} \frac{M^{4}}{M_{\mathrm{Pl}}^{4}}$.
Defining $x \equiv \phi / M_{\mathrm{Pl}}$, the Hubble flow functions are given by
$\epsilon_{1}=\frac{2}{x^{2}}\left(\frac{1-\alpha x^{2}-4 \alpha x^{2} \ln x}{1-2 \alpha x^{2} \ln x}\right)^{2}$,
$\epsilon_{2}=\frac{4}{x^{2}} \frac{\left(1+\alpha x^{2}\right)\left(1+2 \alpha x^{2}\right)-2 \alpha x^{2} \ln x\left(1-\alpha x^{2}-4 \alpha x^{2} \ln x\right)}{\left(1-2 \alpha x^{2} \ln x\right)^{2}}$,
and See the equation in Box I. If $\alpha=0$, one recovers the slowroll parameters of the massive case (namely LFI with $p=2$, see Section 4.2) as expected.

Let us now discuss the field domains in which inflation can take place. It is clear that the above potential is not positive definite for all field values. It becomes negative at the point
$x_{V=0}=\frac{\phi_{V=0}}{M_{\mathrm{Pl}}}=\sqrt{\frac{1}{\alpha \mathrm{~W}_{0}(1 / \alpha)}}$,
where $\mathrm{W}_{0}$ is the 0 -branch of the Lambert function. The model is defined only in the regime $\phi<\phi_{V=0}$. On the other hand, the top of the potential, where $V^{\prime}=0$ (or equivalently $\epsilon_{1}=0$ ), is given by
$x_{\mathrm{top}}=\frac{\phi_{\mathrm{top}}}{M_{\mathrm{Pl}}}=\sqrt{\frac{1}{2 \alpha \mathrm{~W}_{0}\left(\frac{\sqrt{e}}{2 \alpha}\right)}}$.
As the model makes sense only if the logarithmic terms do not dominate the potential, the acceptable regime is $\phi<\phi_{\text {top }}<\phi_{V=0}$, and a large field region only exists for $\phi_{\text {top }} / M_{\text {Pl }} \gg 1$. From the above expression, this means that we must be in the regime $\alpha \ll 1$. For $\phi<\phi_{\text {top }}$ one can check from Eqs. (4.55) and (4.62) that the loop corrections never exceed $\alpha / e$.

Let us now turn to the slow-roll trajectory. It is given by
$N-N_{\text {end }}=-\frac{1}{2} \int_{\phi_{\text {end }} / M_{\mathrm{Pl}}}^{\phi / M_{\mathrm{Pl}}} \frac{x-2 \alpha x^{3} \ln x}{1-\alpha x^{2}-4 \alpha x^{2} \ln x} \mathrm{~d} x$,
an integral that cannot be performed analytically and which is numerically evaluated in ASPIC. For the purpose of this section, we can nevertheless make an expansion in $\alpha$ to obtain an approximate expression

$$
\begin{align*}
N-N_{\mathrm{end}}= & -\frac{x^{2}}{4}\left[1+\alpha \frac{x^{2}}{4}(1+4 \ln x)\right] \\
& +\frac{x_{\text {end }}^{2}}{4}\left[1+\alpha \frac{x_{\text {end }}^{2}}{4}\left(1+4 \ln x_{\mathrm{end}}\right)\right]+\mathcal{O}\left(\alpha^{2}\right) . \tag{4.64}
\end{align*}
$$

Inflation stops close to the minimum of the potential when $\epsilon_{1}=1$. This last equation cannot be solved analytically but we can also perform an expansion at first order in $\alpha$ and one gets
$x_{\mathrm{end}}=\frac{\phi_{\mathrm{end}}}{M_{\mathrm{Pl}}} \simeq \frac{1}{\sqrt{2 \alpha \mathrm{~W}_{0}\left[\frac{e^{1+1 /(4 \alpha)}}{2 \alpha}\right]}} \simeq \sqrt{2}-2 \sqrt{2} \alpha$.
In the limit $\alpha \rightarrow 0$, we recover the large field result for $p=2$, i.e. $x_{\text {end }} \rightarrow \sqrt{2}$. The maximum total number of $e$-folds one can realize between $\phi=\phi_{\text {top }}$ and $\phi=\phi_{\text {end }}$ can be calculated from the previous expressions. It reads

$$
\begin{align*}
\Delta N_{\max }= & N_{\mathrm{end}}-N_{\mathrm{top}}=\frac{5}{32 \alpha \mathrm{~W}_{0}\left(\frac{\sqrt{e}}{2 \alpha}\right)} \\
& +\frac{1+2 \alpha-20 \alpha \mathrm{~W}_{0}\left[\frac{e^{1+1 /(4 \alpha)}}{2 \alpha}\right]}{128 \alpha^{2} \mathrm{~W}_{0}^{2}\left[\frac{\mathrm{e}^{1+1 /(4 \alpha)}}{2 \alpha}\right]} \\
\simeq & -\frac{5}{32 \alpha \ln (\alpha)} . \tag{4.66}
\end{align*}
$$



Fig. 35. Constant $n_{S}$ A Inflation (CNAI) potential and slow-roll parameters versus the vacuum expectation value of the inflaton field. Top left panel: Constant $n_{S} A$ Inflation potential for $\alpha=1$. Top right panel: logarithm of the potential for the same value of $\alpha$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ (same value of $\alpha$ ): inflation stops when $\epsilon_{1}=1$ in this model. Bottom right panel: slow-roll parameters $\epsilon_{2}$ and $\epsilon_{3}(\alpha=1)$.

This is a decreasing function of $\alpha$, so that $\alpha$ has to be small enough if one wants a sufficiently high number of $e$-folds to take place. Indeed, if one wants at least $\Delta N_{\text {min }} e$-folds to occur, one needs to work with
$\alpha<\frac{5}{32 \Delta N_{\min }} \frac{1}{\ln \left(\frac{32 \Delta N_{\min }}{10}\right)}$.
For example, $\Delta N_{\min }=50$ imposes $\alpha<6 \times 10^{-4}$. The fact that $\alpha$ is bounded from above can be directly checked in Fig. 85. The field $\phi_{*}$ value at which the pivot mode crossed the Hubble radius during inflation is obtained from Eq. (2.47) whereas the corresponding $e$ fold number can be obtained from the trajectory.

Finally, the parameter $M$ can be determined from the amplitude of the CMB anisotropies, and one gets

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=\frac{2880 \pi^{2}}{x_{*}^{4}} \frac{\left(1-2 \alpha x_{*}^{2} \ln x_{*}\right)^{3}}{\left(1-\alpha x_{*}^{2}-4 \alpha x_{*}^{2} \ln x_{*}\right)^{2}} \frac{Q_{\text {rms }-\mathrm{PS}}^{2}}{T^{2}} \tag{4.68}
\end{equation*}
$$

The reheating consistent slow-roll predictions for the RCMI models are represented in Fig. 85. As expected, the LFI quadratic model case is properly recovered for $\alpha \rightarrow 0$. From this figure, we see that all models having $\alpha \gtrsim 10^{-3.7}$ lie outside the $2 \sigma$ contour. Let us emphasize that the value of $\alpha$ cannot be infinitely small due to Eq. (4.57). At zero order, one has $\phi>\phi_{\mathrm{end}} \simeq \sqrt{2} M_{\mathrm{Pl}}$ such that Eq. (4.57) can be recast into
$\alpha>\frac{M^{4}}{8 \pi^{2} M_{\mathrm{Pl}}^{4}}=\frac{m^{2}}{16 \pi^{2} M_{\mathrm{Pl}}^{2}}$.
From the COBE normalization, and in the limit of small $\alpha$, one gets $M / M_{\mathrm{Pl}} \gtrsim 10^{-3}$ and the lower bound reads $\alpha>10^{-15}$.


Fig. 36. Lambert functions $\mathrm{W}_{0}(x)$ (dashed line) and $\mathrm{W}_{-1}(x)$ (solid line). During CNAI inflation, inflation proceeds along the " 0 " branch in the direction specified by the arrow on the figure.

### 4.5. Radiatively corrected quartic inflation (RCQI)

This model is similar to RCMI discussed in Section 6.1 but implements radiative corrections due to fermion couplings over a quartic $(p=4)$ large field model [245] (see Section 4.2). The potential is given by

$$
\begin{align*}
V & =\lambda \phi^{4}-\frac{g^{4}}{16 \pi^{2}} \phi^{4} \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right) \\
& =M^{4}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{4}\left[1-\alpha \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right)\right], \tag{4.70}
\end{align*}
$$



Fig. 37. Top left panel: constant $n_{\text {S }}$ B Inflation (CNBI) potential for $\alpha=0.1$, see Eq. (4.308). Top right panel: logarithm of this potential (for the same value of $\alpha$ ). Bottom left panel: slow-roll parameter $\epsilon_{1}$ still for $\alpha=0.1$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ and $\epsilon_{3}$ again for $\alpha=0.1$.
where
$M^{4}=\lambda M_{\mathrm{Pl}}^{4}, \quad \alpha \equiv \frac{g^{4}}{16 \pi^{2} \lambda}$.
Defining $x=\phi / M_{\mathrm{Pl}}$, the Hubble flow functions in the slow-roll approximation read
$\epsilon_{1}=\frac{8}{x^{2}}\left(\frac{1-\frac{\alpha}{4}-\alpha \ln x}{1-\alpha \ln x}\right)^{2}$,
$\epsilon_{2}=\frac{8}{x^{2}} \frac{1+\frac{\alpha}{4}(\alpha-1)+\alpha\left(\frac{\alpha}{4}-2\right) \ln x+\alpha^{2} \ln ^{2} x}{(1-\alpha \ln x)^{2}}$,
and
$\epsilon_{3}=\frac{8}{x^{2}}$
$\times \frac{\left(1-\frac{\alpha}{2}-\alpha \ln x\right)\left(1-\frac{\alpha}{4}-\alpha \ln x\right)\left[1+\frac{\alpha^{2}}{2}+\frac{\alpha}{4}-\alpha\left(2+\frac{\alpha}{4}-\alpha \ln x\right) \ln x\right]}{(1-\alpha \ln x)^{2}\left[1+\frac{\alpha}{4}(\alpha-1)-\alpha\left(2-\frac{\alpha}{4}-\alpha \ln x\right) \ln x\right]}$.

The shape of the potential and the Hubble flow functions are very similar to the ones of the RCMI model and have been represented in Fig. 13. In particular, the potential is vanishing and maximal at the field values
$x_{V=0}=\frac{\phi_{V=0}}{M_{\mathrm{Pl}}}=e^{1 / \alpha}, \quad x_{\mathrm{top}}=\frac{\phi_{\mathrm{top}}}{M_{\mathrm{Pl}}}=e^{1 / \alpha-1 / 4}$,
respectively. As the model makes sense only if the corrections are small compared to the quartic term, one should consider $\alpha \ll 1$ and not too large super-Planckian field values.

The slow-roll trajectory can integrated analytically from Eqs. (2.11) and (4.70) and one gets
$N-N_{\text {end }}=-\frac{1}{16}\left[2 x^{2}-e^{-1 / 2+2 / \alpha} \operatorname{Ei}\left(\frac{1}{2}-\frac{2}{\alpha}+2 \ln x\right)\right.$

$$
\begin{equation*}
\left.-2 x_{\mathrm{end}}^{2}+e^{-1 / 2+2 / \alpha} \operatorname{Ei}\left(\frac{1}{2}-\frac{2}{\alpha}+2 \ln x_{\mathrm{end}}\right)\right], \tag{4.75}
\end{equation*}
$$

where the exponential integral function is defined by
$\operatorname{Ei}(x) \equiv-\int_{-x}^{+\infty} \frac{e^{-t}}{t} \mathrm{~d} t$.
The quartic limit $\alpha \rightarrow 0$ is recovered by noticing that
$\operatorname{Ei}(-2 / \alpha) \underset{\alpha \rightarrow 0}{\sim}-\frac{\alpha}{2} e^{-2 / \alpha}$.
Contrary to the RCMI model, the top of the potential is flat enough to support inflation. Indeed, one sees from Eq. (4.74) that the argument of the exponential integral function vanishes at $x=x_{\text {top }}$. Since for $y \rightarrow 0$, one has $\operatorname{Ei}(y) \sim \gamma+\ln y$, whatever the value of $x_{\text {end }}$ the total number of $e$-folds is divergent. This means that it is always possible to realize the required $\Delta N_{*}$ number of $e$-folds provided inflation starts close enough to the top of the potential.

As for RCMI, inflation stops at $\epsilon_{1}=1$ but this equation can only be solved numerically. For illustrative purpose, one can nevertheless solve it at first order in $\alpha$ to get
$x_{\text {end }}=\frac{\phi_{\text {end }}}{M_{\mathrm{Pl}}} \simeq 2 \sqrt{2}-\frac{\sqrt{2}}{2} \alpha$.
The link between $\phi_{*}$ and $\Delta N_{*}$ is given by the slow-roll trajectory with $\phi_{*}$ given by Eq. (2.47).

Finally, the parameter $M$ can be determined from the amplitude of the CMB anisotropies, and one gets
$\lambda=\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=\frac{11520 \pi^{2}}{x_{*}^{6}} \frac{\left(1-\frac{\alpha}{4}-\alpha \ln x_{*}\right)^{2}}{\left(1-\alpha \ln X_{*}\right)^{3}} \frac{Q_{\text {rms-PS }}^{2}}{T^{2}}$.
The slow-roll predictions for RCQI are represented in Figs. 86 and 87. As expected, the quartic model case is properly recovered in


Fig. 38. Lambert functions $\mathrm{W}_{0}(x)$ (dashed line) and $\mathrm{W}_{-1}(x)$ (solid line). During Constant $n_{\mathrm{S}}$ B Inflation, inflation proceeds along the " -1 " branch in the direction specified by the arrow.
the limit $\alpha \rightarrow 0$. From Fig. 86, we see that all the models seem to lie outside the $2 \sigma$ contour for $\bar{w}_{\text {reh }}=0$. As the reheating phase takes place at the bottom of a quartic-like potential, we have also represented the prediction for $\bar{w}_{\text {reh }}=1 / 3$ in Fig. 87. For a radiation-dominated reheating, $\Delta N_{*}$ is fixed and for each value of $\alpha$ one has only a single point. In that situation, all these models are still disfavored at the two-sigma level.

### 4.6. Natural inflation (NI)

### 4.6.1. Theoretical justifications

Natural inflation was first proposed as an attempt to solve the so-called "fine-tuning" problem of inflation. In particular, in order to obtain sufficient inflation and the correct normalization for the microwave background anisotropies, the potential $V(\phi)$ of the inflaton must be sufficiently flat. It is usually argued that, on general grounds, such a flatness is not robust under radiative corrections, unless it is protected by some symmetry. This is the reason that has motivated Refs. $[246,247]$ to put forward Natural Inflation, in which the inflaton potential is flat due to shift symmetries. The model makes use of Nambu-Goldstone bosons [248,249] which arise whenever a global symmetry is spontaneously broken. The main idea can be very simply illustrated with the following action

$$
\begin{align*}
S= & -\int \mathrm{d} \boldsymbol{x} \sqrt{-g}\left[g^{\mu \nu} \partial_{\mu} \Phi^{\dagger} \partial_{\nu} \Phi+i \bar{\Psi} \gamma^{\mu} \partial_{\mu} \Psi\right. \\
& \left.+\lambda\left(\Phi^{\dagger} \Phi-\frac{f^{2}}{2}\right)^{2}+g_{\mathrm{f}} \bar{\Psi}_{\mathrm{L}} \Phi \Psi_{\mathrm{R}}+g_{\mathrm{f}} \bar{\Psi}_{\mathrm{R}} \Phi^{\dagger} \Psi_{\mathrm{L}}\right] \tag{4.80}
\end{align*}
$$

where $\Phi$ is a complex scalar field, $\Psi$ a Dirac spinor and $\Psi_{\mathrm{LR}}=$ $\left(1 \pm \gamma_{5}\right) / 2 \Psi$. The quantity $f$ is the energy scale at which the symmetry is spontaneously broken, $\lambda$ is a dimensionless coupling constant and $g_{f}$ a dimensionless Yukawa coupling. This action is invariant under the $\mathrm{U}(1)$ transformation: $\Phi \rightarrow e^{i \alpha} \Phi, \Psi_{\mathrm{L}} \rightarrow$ $e^{i \alpha / 2} \Psi_{\mathrm{L}}$ and $\Psi_{\mathrm{R}} \rightarrow e^{-i \alpha / 2} \Psi_{\mathrm{R}}$, where $\alpha$ is an arbitrary constant. Due to the "Mexican hat" potential for the scalar field, this symmetry is spontaneously broken below the scale $f$ and the scalar field acquires the vev $\langle\Phi\rangle=f / \sqrt{2} e^{i \phi / f}$. The field $\phi$ corresponds to an "angular variable" and is a Goldstone boson. Below the scale of broken symmetry, the effective Lagrangian can be expressed as

$$
\begin{align*}
\mathcal{L}= & \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi+i \bar{\Psi} \gamma^{\mu} \partial_{\mu} \Psi \\
& +g_{\mathrm{f}} \frac{f}{\sqrt{2}}\left(\bar{\Psi}_{\mathrm{L}} \Psi_{\mathrm{R}} e^{i \phi / f}+\bar{\Psi}_{\mathrm{R}} \Psi_{\mathrm{L}} e^{-i \phi / f}\right) . \tag{4.81}
\end{align*}
$$

It is now invariant under $\phi \rightarrow \phi+2 \pi f, \Psi_{\mathrm{L}} \rightarrow e^{i \alpha / 2} \Psi_{\mathrm{L}}$ and $\Psi_{\mathrm{R}} \rightarrow e^{-i \alpha / 2} \Psi_{\mathrm{R}}$. Then, we assume that an explicit symmetry breaking takes place, for instance through the appearance of a fermion condensate for which $\langle\bar{\Psi} \Psi\rangle \simeq M_{\mathrm{s}}^{3}$ where $M_{\mathrm{s}}<f$ is the scale at which this symmetry breaking occurs. As a consequence, the effective Lagrangian takes the form
$\mathcal{L}=\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi+2 g_{\mathrm{f}} M_{\mathrm{s}}^{3} \frac{f}{\sqrt{2}} \cos \left(\frac{\phi}{f}\right)$.
We see that the Nambu-Goldstone boson has acquired a cosine potential and the overall scale of the potential is given by $M^{4} \simeq$ $g_{f} M_{s}^{3} f$. Therefore, if one takes $f \simeq M_{\mathrm{Pl}}, M_{\mathrm{s}}$ slightly below the GUT scale and a Yukawa coupling of order one, one can "naturally" generate a small ratio $M / f$. A last remark is in order on this model. Suppose that quantum gravity effects generate nonrenormalizable higher order terms in the action (4.80) like
$\Delta V=a_{m n} \frac{|\Phi|^{2 m}}{M_{\mathrm{Pl}}^{2 m+n-4}}\left(\Phi^{n}+\Phi^{\dagger n}\right)$,
where $a_{m n}$ are a priori unknown coefficients. After symmetry breaking, one would therefore obtain a correction of the form
$\Delta V=a_{m n} M_{\mathrm{Pl}}^{4}\left(\frac{f}{M_{\mathrm{Pl}}}\right)^{2 m+n} \cos \left(n \frac{\phi}{f}\right)$.
If $f \gtrsim M_{\mathrm{Pl}}$, as favored by current cosmological data (see below) these terms should dominate unless the coefficients $a_{m n}$ are finetuned to very small values. Notice that the overall scale of the potential is now given by $a_{m n} M_{\mathrm{Pl}}^{4}$, which also demands that $a_{m n} \lesssim$ $10^{-15}$ in order to have the correct CMB normalization. These terms are therefore dangerous for the consistency and the natural character of the model. This model has been studied in more details in Refs. [250-264].

Many other types of candidates have subsequently been explored in order to produce scenarios similar to that of Natural Inflation. For example, in Ref. [265], it was suggested to use a pseudoNambu Goldstone boson as the rolling field in double field inflation. Then, NI potentials generated by radiative corrections in models with explicitly broken Abelian [266] and non-Abelian [267,268] symmetries were considered, showing that NI models with $f \simeq$ $M_{\mathrm{PI}}$ and $f \ll M_{\mathrm{PI}}$ can both be generated. In Refs. [269], the field $\phi$ is considered to be a Polonyi field [270] and the model predicts that $f=M_{\mathrm{PI}}$. Refs. [271,272] have examined natural inflation in the context of extra dimensions and Ref. [273] has used pseudoNambu Goldstone bosons from little Higgs models to drive hybrid inflation. Also, Refs. [274,275] have used the natural inflation idea of pseudo-Nambu Goldstone bosons in the context of braneworld scenarios to drive inflation, Ref. [276] has studied the model in 5-D warped backgrounds. The same potential has also been obtained and studied in Ref. [277] when studying instantons in non-linear sigma models, and in Ref. [278] as providing quintessential inflation. In some of these references the potential is sometimes found with the minus sign in front of the cosine term, which is, up to a shift in the field vev $\phi / f \rightarrow \phi / f+\pi$, the same potential as already studied before. This last model has also been derived and studied in Refs. [271,272,279] in the context of orbifold GUT inflation, where the potential is given by
$V(\phi)=M^{4}\left[F\left(\frac{\phi}{\phi_{0}}\right)+F\left(2 \frac{\phi}{\phi_{0}}\right)+\frac{F(0)}{2}\right]$,
with
$F(x)=-\sum_{n=1}^{\infty} \frac{\cos (n \pi x)}{n^{5}}$.


Fig. 39. Top left panel: Open String Tachyonic Inflation (OSTI) potential as a function of $\phi / \phi_{0}$. Top right panel: logarithm of the potential. The arrow indicates in which direction inflation proceeds. Bottom left panel: slow-roll parameter $\epsilon_{1}$, rescaled by the quantity $M_{\mathrm{Pl}}^{2} / \phi_{0}^{2}$, such that the corresponding expression becomes universal, i.e. independent of $\phi_{0}$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line), rescaled by $M_{P l}^{2} / \phi_{0}^{2}$ for the same reason as mentioned before.

This potential must be studied in its increasing branch, and in the small field limit. At leading order, one recovers the cosine potential.

Finally, an important question is whether a situation where $f>M_{\mathrm{PI}}$ makes sense from the high energy physics and effective field theory point of view. In fact, it was shown in Refs. [280-282] that $f \lesssim 10^{12} \mathrm{GeV}$ in order for the corresponding energy density not to exceed the critical energy density. But this constraint applies to the post inflationary Universe and, during inflation, Ref. [283] has argued that it is not relevant. However, it remains the question of whether $f>M_{\text {PI }}$ makes sense or not. To address this issue, an interesting mechanism has been proposed in Ref. [284] (see also Ref. [285]) which shows that two axion fields at sub-Planckian scales can have an effective dynamics similar to the one field Natural Inflation model with $f>M_{\mathrm{Pl}}$.

Let us consider a model with two axions, $\theta$ and $\rho$ the effective Lagrangian of which is given by

$$
\begin{align*}
\mathscr{L}= & \frac{1}{2} \partial_{\mu} \theta \partial^{\mu} \theta+\frac{1}{2} \partial_{\mu} \rho \partial^{\mu} \rho+M_{1}^{4}\left[1-\cos \left(\frac{\theta}{f}+\frac{\rho}{g_{1}}\right)\right] \\
& +M_{2}^{4}\left[1-\cos \left(\frac{\theta}{f}+\frac{\rho}{g_{2}}\right)\right], \tag{4.87}
\end{align*}
$$

where $M_{1}$ and $M_{2}, f, g_{1}$ and $g_{2}$ are constant, a priori, arbitrary scales. The same model can be re-written in terms of the fields $\psi$ and $\xi$ defined by
$\psi=\frac{f g_{1}}{\sqrt{f^{2}+g_{1}^{2}}}\left(\frac{\theta}{f}+\frac{\rho}{g_{1}}\right)$,
$\xi=\frac{f g_{1}}{\sqrt{f^{2}+g_{1}^{2}}}\left(-\frac{\theta}{g_{1}}+\frac{\rho}{f}\right)$.

It is easy to show that this leads to

$$
\begin{align*}
\mathcal{L} & =\frac{1}{2} \partial_{\mu} \psi \partial^{\mu} \psi+\frac{1}{2} \partial_{\mu} \xi \partial^{\mu} \xi+M_{1}^{4}\left[1-\cos \left(\frac{\sqrt{f^{2}+g_{1}^{2}}}{f g_{1}} \psi\right)\right] \\
& +M_{2}^{4}\left[1-\cos \left(\frac{f^{2}+g_{1} g_{2}}{f g_{2} \sqrt{f^{2}+g_{1}^{2}}} \psi+\frac{g_{1}-g_{2}}{g_{2} \sqrt{f^{2}+g_{1}^{2}}} \xi\right)\right] . \tag{4.89}
\end{align*}
$$

Moreover, the mass of the two fields $\psi$ and $\xi$ can be expressed as
$m_{\psi}^{2}=\left(\frac{1}{f^{2}}+\frac{1}{g_{1}^{2}}\right) M_{1}^{4}, \quad m_{\xi}^{2}=\frac{\left(g_{1}-g_{2}\right)^{2}}{g_{2}^{2}\left(f^{2}+g_{1}^{2}\right)} M_{2}^{4}$.
If $g_{1}$ is very close to $g_{2}$, then the field $\xi$ will be light and, therefore, will have a non-trivial dynamics. In addition, if the field $\psi$ is sufficiently heavy (compared to the Hubble parameter), then its $v e v$ will be frozen at $\psi=0$. In this case, we see that the original two fields model effectively reduces to a one field NI model with a scale $f_{\xi}$ given by
$f_{\xi}=\frac{g_{2} \sqrt{f^{2}+g_{1}^{2}}}{g_{1}-g_{2}}$.
But, since, $g_{1}$ is close to $g_{2}$, the scale $f_{\xi}$ will be large even if the fundamental scales $f, g_{1}$ and/or $g_{2}$ are sub-Planckian. In this way, one can generate super-Planckian values for the scale $f$ and, at the same time, have a theory which can be consistent from the effective field theory point of view.


 Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line), rescaled by $M_{\mathrm{PI}}^{2} / \phi_{0}^{2}$ for the same reason as mentioned before.

### 4.6.2. Slow-roll analysis

Summarizing the above discussion, the model that we consider in this section makes use of a potential that can be written as
$V(\phi)=M^{4}\left[1+\cos \left(\frac{\phi}{f}\right)\right]$.
The scale $M$ is determined by the CMB normalization and the potential depends on one parameter: the a priori unknown scale $f$. The potential of Eq. (4.92) is displayed with its logarithm in Fig. 14. Since it is a periodic and even function of the field vev $\phi$, it is enough to study it in the range $\phi \in[0, \pi f]$ where inflation proceeds from the left to the right. If one lets $x \equiv \phi / f$, the slow-roll parameters can be expressed as
$\epsilon_{1}=\frac{M_{\mathrm{Pl}}^{2}}{2 f^{2}} \frac{\sin ^{2} x}{(1+\cos x)^{2}}, \quad \epsilon_{2}=\frac{2 M_{\mathrm{Pl}}^{2}}{f^{2}} \frac{1}{1+\cos x}$,
$\epsilon_{3}=2 \epsilon_{1}$.
They are displayed in Fig. 14, where one can see that they are all increasing functions of the field vev, which means that they all increase during inflation. Inflation stops at the position $x_{\text {end }}$ given by $\epsilon_{1}=1$ (see below), and one can see that $\epsilon_{2}$ and $\epsilon_{3}$ are already greater than one at this point. This means that the slow-roll approximation stops being valid slightly before the end of inflation, and the few last $e$-folds may not be properly described in this frame of approximations. Another remark to be made is the fact that one generically has
$\epsilon_{2}>\frac{M_{P l}^{2}}{f^{2}}$.
This means that in order for the slow-roll approximation to be valid, one must require $f / M_{\mathrm{Pl}} \gg 1$ which is not necessarily
problematic from a high energy physics point of view (see the above discussion).

The end of inflation occurs when $\epsilon_{1}=1$, i.e. at a position given by
$x_{\text {end }}=\arccos \left(\frac{1-2 f^{2} / M_{\mathrm{Pl}}^{2}}{1+2 f^{2} / M_{\mathrm{Pl}}^{2}}\right)$.
From this expression, one can calculate the value of the other slow roll parameters at the end of inflation, namely $\epsilon_{2}^{\text {end }}=2+M_{\mathrm{Pl}}^{2} / f^{2}$ and $\epsilon_{3}^{\text {end }}=2 \epsilon_{2}^{\text {end }}$, which confirms that the last few $e$-folds may not be described properly in the slow-roll approximation.

Let us now calculate the slow-roll trajectory. It is given by
$N_{\text {end }}-N=\frac{f^{2}}{M_{\mathrm{Pl}}^{2}} \ln \left(\frac{1-\cos x_{\text {end }}}{1-\cos x}\right)$,
where $N_{\text {end }}$ is the number of $e$-folds at the end of inflation, and $N$ is the number of $e$-folds at some point when the scaled field vev is $x$. This trajectory can be inverted and one obtains
$x=\arccos \left\{1-\left(1-\cos x_{\text {end }}\right) \exp \left[-\frac{M_{\mathrm{PI}}^{2}}{f^{2}}\left(N_{\text {end }}-N\right)\right]\right\}$.
Replacing $x_{\text {end }}$ by its value [see Eq. (4.95)] gives
$x=\arccos \left\{1-\frac{4 f^{2}}{M_{\mathrm{Pl}}^{2}+2 f^{2}} \exp \left[-\frac{M_{\mathrm{Pl}}^{2}}{f^{2}}\left(N_{\mathrm{end}}-N\right)\right]\right\}$.
Finally, the amplitude of the CMB anisotropies fixes the parameter $M$ to
$\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 \pi^{2} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \frac{M_{\mathrm{Pl}}^{2}}{f^{2}} \frac{\sin ^{2} x_{*}}{\left(1+\cos x_{*}\right)^{3}}$.


Fig. 41. Lambert functions $\mathrm{W}_{0}(x)$ (dashed line) and $\mathrm{W}_{-1}(x)$ (solid line). During Witten-O'Raifeartaigh inflation, inflation proceeds along the " 0 " branch in the direction specified by the arrow.

If $f / M_{\mathrm{PI}}=\mathcal{O}(1)$, this expression simplifies to

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4} \simeq 720 \pi^{2} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \frac{e^{-2 M_{\mathrm{Pl}}^{2} / f^{2} \Delta N_{*}}}{1+2 f^{2} / M_{\mathrm{Pl}}^{2}} \tag{4.100}
\end{equation*}
$$

which gives rise to $M / M_{\mathrm{PI}} \simeq 10^{-13}$. On the contrary, if $f / M_{\mathrm{PI}} \gg 1$ one has
$\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4} \simeq 360 \pi^{2} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}\left(\frac{f}{M_{\mathrm{Pl}}}\right)^{2} \frac{1}{\Delta N_{*}^{2}}$,
and the potential energy scale goes up. For instance, if $f / M_{\mathrm{PI}}=10^{2}$ one has $M / M_{\mathrm{PI}} \simeq 10^{-2}$.

The reheating consistent slow-roll predictions for the natural inflation models are displayed in Fig. 88. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 since the potential is quadratic close to its minimum. In the limit $f / M_{\mathrm{Pl}} \rightarrow \infty$, the quadratic model predictions (LFI with $p=2$, see Section 4.2) seem to be recovered. Indeed, from the above formula, one can check that in this limit both $x_{\text {end }}$ and $x_{*}$ approach $\pi$ and the potential is, at leading order, a parabola. More precisely, one can check from Eq. (4.98) that in the limit $f / M_{\mathrm{Pl}} \rightarrow \infty$, one has $\cos x_{*} \simeq-1+\left(1+2 \Delta N_{*}\right) M_{\mathrm{Pl}}^{2} / f^{2}$, from which one deduces that $\epsilon_{1 *} \simeq 1 /\left(1+2 \Delta N_{*}\right)$ and $\epsilon_{2 *} \simeq 2 /\left(1+2 \Delta N_{*}\right) \simeq 2 \epsilon_{1 *}$. These relations are characteristic of the LFI quadratic models, see Eq. (4.42). However, one has $\epsilon_{3 *}=2 \epsilon_{2 *}$ which differs from the LFI quadratic relationship $\epsilon_{3 *}=\epsilon_{2 *}$, and therefore quantities sensitive to $\epsilon_{3}$, such as the running $\alpha_{\mathrm{S}}$, would break the degeneracy between NI and the LFI quadratic model. As expected, large values of $f / M_{\text {PI }}$ seem to be favored by the data (as well as high reheating temperatures), and in practice, $f / M_{\mathrm{Pl}}<4$ appears to be disfavored at the $2 \sigma$ level by the Planck data.

### 4.7. Exponential SUSY Inflation (ESI)

### 4.7.1. Theoretical justifications

This model has been discussed in Ref. [286] in the context of spin-driven inflation and derived in Ref. [287] in the context of supergravity and superstrings. The potential is given by $V(\phi) \propto$ $\left(1-e^{-q \phi / M \mathrm{Pl}}\right)$. The same potential also appears in Ref. [288] in the context of brane inflation, in Ref. [289] in the context of type IIB string compactification as fiber inflation and more recently in Ref. [290] as unitarized Higgs inflation models. This type of models can be obtained under very general considerations. Suppose that one has a supergravity model with a Kähler potential depending
on one field $\psi$ given by $K=-\beta / \kappa \ln \left(1-\alpha \kappa \psi \psi^{\dagger}\right)$, where $\alpha$ and $\beta$ are two free parameters. This model leads to a scalar potential but for a field which is not canonically normalized. The canonically normalized field $\theta$ is given by
$\kappa^{1 / 2} \theta \simeq \frac{1}{\sqrt{\alpha}}\left(1-2 e^{-\sqrt{2 / \beta} \kappa^{1 / 2} \psi}\right)$,
where we have assumed that inflation takes place at relatively large $\psi v e v$ 's. Then, suppose that the superpotential leads to a given function $V=f(\theta)$. One can always expand $f$ such that
$V(\phi) \simeq V_{0}\left(1-e^{-\sqrt{2 / \beta} \kappa^{1 / 2} \phi}\right)+\cdots$,
where $\kappa^{1 / 2} \phi \equiv \kappa^{1 / 2} \theta+\sqrt{\beta / 2} \ln \left(\left[2 f_{\theta} /(\sqrt{\alpha} f)\right]\right)$ and $V_{0}$ is just the function $f$ evaluated at $1 / \sqrt{\alpha}$. We see that one obtains exactly the ESI potential with $q=\sqrt{2 / \beta}$. Preferred choices for $\beta$ are $\beta=1$ or $\beta=3$ leading to $q=\sqrt{2}$ or $q=\sqrt{2 / 3}$. In absence of any more further guidance, it seems reasonable to assume that $\beta$, and hence $q$, is just a number of order one.

### 4.7.2. Slow-roll analysis

Based on the previous considerations, we now study the following potential
$V(\phi)=M^{4}\left(1-e^{-q \phi / M_{\mathrm{Pl}}}\right)$,
where $q$ is a positive dimensionless parameter and inflation proceeds at decreasing field values in the region where $\phi / M_{\mathrm{PI}}>0$. Defining $x \equiv \phi / M_{\mathrm{Pl}}$, the Hubble flow functions in the slow-roll approximation read
$\epsilon_{1}=\frac{q^{2}}{2} \frac{e^{-2 q x}}{\left(1-e^{-q x}\right)^{2}}, \quad \epsilon_{2}=2 q^{2} \frac{e^{-q x}}{\left(1-e^{-q x}\right)^{2}}$,
$\epsilon_{3}=q^{2} \frac{e^{-q x}\left(1+e^{-q x}\right)}{\left(1-e^{-q x}\right)^{2}}$.
The potential and the Hubble flow functions with respect to the field values are represented in Fig. 15.

The slow-roll trajectory can be integrated analytically from Eq. (2.11) and one finds
$N-N_{\text {end }}=-\frac{e^{q x}-q x}{q^{2}}+\frac{e^{q x_{\text {end }}}-q x_{\text {end }}}{q^{2}}$.
This equation can also be inverted in terms of the Lambert function to get the field value in terms of the number of $e$-folds:

$$
\begin{align*}
x= & q\left(N-N_{\mathrm{end}}\right)-\frac{e^{q x_{\mathrm{end}}}-q x_{\mathrm{end}}}{q} \\
& -\frac{1}{q} \mathrm{~W}_{-1}\left\{-\exp \left[q^{2}\left(N-N_{\mathrm{end}}\right)-\left(e^{q x_{\mathrm{end}}}-q x_{\mathrm{end}}\right)\right]\right\} \tag{4.107}
\end{align*}
$$

The fact that one should choose the branch $\mathrm{W}_{-1}$ is justified below. The argument of the Lambert function is always negative as the exponential is always positive. Moreover, since $x_{\text {end }}>0$ and $N<$ $N_{\text {end }}$, the maximal value of exponential argument is saturated for $x_{\text {end }} \rightarrow 0$, i.e. for a Lambert function argument equals to $-1 / e$. As the result the Lambert function argument varies, at most, in [ $-1 / e, 0]$. Finally, since $x>0$, we see directly from Eq. (4.107) that the Lambert function values have to be negative thereby ensuring that inflation proceeds only along the " -1 "-branch (see Fig. 16).

With such a potential, inflation ends naturally at $\epsilon_{1}=1$, i.e. at the field value
$x_{\text {end }}=\frac{1}{q} \ln \left(1+\frac{q}{\sqrt{2}}\right)$.


Fig. 42. Small Field Inflation (SFI) for $p=4$ and $\mu=M_{\mathrm{PI}}$. Upper panels: the potential and its logarithm as a function of $\phi / \mu$. Bottom left panel: slow-roll parameter $\epsilon_{1}$, the shaded area indicates where inflation stops. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line).

From this equation and the trajectory, we have an explicit relation between the field value $\phi_{*}$ at which the pivot mode crossed the Hubble radius during inflation and the corresponding $e$-fold number $\Delta N_{*}$.

Finally, the parameter $M$ can be determined from the amplitude of the CMB anisotropies, and one gets

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 q^{2} \pi^{2} \frac{e^{-2 q \alpha_{*}}}{\left(1-e^{-q x_{*}}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{4.109}
\end{equation*}
$$

where the value of $\phi_{*}$ (or $\Delta N_{*}$ ) is obtained from Eq. (2.47). The reheating consistent slow-roll prediction for the exponential Susy models are represented in Figs. 89 and 90. In the limit $q \rightarrow 0$, we recover the same prediction as a linear large field model. From Fig. 89, we see that all the models remains compatible with the current data. These figures correspond to $\bar{w}_{\text {reh }}=0$, but one could argue that $\bar{w}_{\text {reh }} \gtrsim-1 / 3$ make more sense if a parametric reheating would feel the linear shape of the potential. This quite extreme situation is represented in Fig. 90. In that case, the low reheating temperatures are clearly disfavored.

### 4.8. Power law inflation (PLI)

These models refer to inflationary potentials of the form

$$
\begin{equation*}
V(\phi)=M^{4} e^{-\alpha \phi / M_{\mathrm{Pl}}} \tag{4.110}
\end{equation*}
$$

where $\alpha$ is a dimensionless parameter. They have been intensively studied since they lead to an exact inflationary dynamics, of the power law form, hence their name. Moreover, the power spectrum can also be determined exactly in this case. The background solution reads $a \propto\left(t / t_{0}\right)^{2 / \alpha^{2}}$ and $\phi=\phi_{0}+2 M_{\mathrm{PI}} / \alpha \ln \left(t / t_{0}\right)$ with $t_{0}^{2}=2 M_{\mathrm{Pl}}^{2} /\left(\alpha^{2} M^{4}\right)\left(6 / \alpha^{2}-1\right) e^{\alpha \phi_{0} / M_{\mathrm{Pl}}}$. We see that we have inflation provided $\alpha \in[0, \sqrt{2}]$.

This scenario was introduced in Ref. [291] where the two point correlation function of the cosmological fluctuations was calculated for the first time (see also Refs. [292,293]). The predictions of this model were recently compared to the Planck data in Ref. [175]. Soon after Ref. [291], it was also considered in Refs. [294,295] but in the context of quintessence, i.e. for models of dark energy in which the energy density of the scalar field redshifts as a power law of the scale factor $\rho \propto a^{-q}$. In that case, one has $\alpha=\sqrt{q / 2}$. The same potential also arises in the case where large field inflation is considered (LFI, see Section 4.2) but with a non-minimal coupling of the inflaton to the gravity sector, see Refs. [296,297] (the exponential potential appears after the transformation to the Einstein frame). In Ref. [298], a cosmic nohair theorem for Bianchi models was proven assuming that the potential of the inflaton is of type (4.110). It was shown that one must have $0<\alpha<\sqrt{2 / 3}$ so that the isotropic power law solution is the unique attractor for any initially expanding Bianchi type model (except type IX). In Ref. [299], the potential (4.110) has been studied in the Kantowski-Sachs metric, and it was found that the production of particles by the scalar field acts as viscous forces which enlarges the range of initial conditions leading to successful inflation. In Ref. [300], the nature of the potential $V(\phi)$ relevant to having inflation in presence of a minimally coupled scalar field together with a causal viscous fluid was investigated. It was shown that this leads to an exponential potential. In Refs. [301-303], the exponential potential was used to describe the dynamics of a tachyonic matter field (i.e. with a non-minimal kinetic term). In Ref. [304], the general transformations that leave unchanged the form of the field equations for Bianchi $V$ cosmologies were investigated, and it was found that they admit asymptotic stable points that lead to power law solutions of the type (4.110). In Ref. [305], inflation was studied in the context of M-theory on $S^{1} / \mathbb{Z}_{2}$ via the non-perturbative dynamics of M5-branes. The open membrane instanton interactions between the branes give rise to potentials of the type (4.110). Within the same framework,


Fig. 43. Intermediate Inflation (II). Upper panels: the potential and its logarithm for $\beta=2.5$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ for a potential with $\beta=2.5$ and $\beta=12$. The position of the maximum of $\epsilon_{1}$ with respect to one depends on $\beta$. The shaded area indicates where inflation stops.. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) for a potential with $\beta=2.5$.

Ref. [306] has dicussed a realization of cascade inflation as assisted inflation built upon a succession of power law inflationary eras. Ref. [307] has used the exponential potential (4.110) in the context of Randall-Sandrum type II Braneworld model. Finally, the general dynamics of power law inflation was studied in detail in Refs. [308-317], where various aspects of its phenomenology were highlighted.

The potential and its logarithm are displayed in Fig. 17. They are decreasing functions of the field, hence inflation proceeds from the left to the right. The slow-roll parameters take a simple form given by
$\epsilon_{1}=\frac{\alpha^{2}}{2}, \quad \epsilon_{i>1}=0$.
Since the first slow-roll parameter is constant, inflation cannot stop by slow-roll violation and one has to assume that, at some vev $\phi_{\text {end }}$, a tachyonic instability is triggered. A priori, this means that the model has in fact an additional new free parameter. However, because the slow-roll parameters do not depend on $\phi$, as well as all the other properties of the inflationary dynamics (even when the slow-roll approximation is not satisfied, see below), the observational predictions of the model cannot depend on $\phi_{\text {end }}$ and this parameter turns out to be irrelevant.

The Hubble flow hierarchy being almost trivial, the exact dynamics of the model can be worked out even if the slow-roll approximation is violated. Indeed, let us first notice that the slowroll trajectory can be explicitly integrated, and gives
$\frac{\phi}{M_{\mathrm{Pl}}}=\frac{\phi_{\mathrm{end}}}{M_{\mathrm{Pl}}}+\alpha\left(N_{\mathrm{end}}-N\right)$.
Then, one can remark that this trajectory is also a solution of the exact Klein-Gordon equation of motion, which reads in terms of
the number of $e$-folds $N$,
$H^{2} \frac{\partial^{2} \phi}{\partial N^{2}}+\left(3 H^{2}+H \partial \frac{\partial H}{\partial N}\right) \frac{\partial \phi}{\partial N}+\frac{\mathrm{d} V}{\mathrm{~d} \phi}=0$.
Indeed, the first term vanishes, and the second term requires
$H^{2}=\frac{V+\dot{\phi}^{2} / 2}{3 M_{\mathrm{Pl}}^{2}}=\frac{V+\frac{H^{2}}{2}\left(\frac{\partial \phi}{\partial N}\right)^{2}}{3 M_{\mathrm{Pl}}^{2}}=\frac{V+\frac{H^{2}}{2} \alpha^{2} M_{\mathrm{Pl}}^{2}}{3 M_{\mathrm{Pl}}^{2}}$,
from which one gets
$H^{2}=\frac{V}{3 M_{\mathrm{Pl}}^{2}} \frac{1}{1-\alpha^{2} / 6}$.
From there, one can evaluate all terms in the Klein-Gordon equation, and verify that Eq. (4.112) is indeed a solution of Eq. (4.113). Since it is a second order differential equation, other solutions exist, but it can be shown $[294,295]$ that the exact solution is an attractor. Let us also notice that combining Eq. (4.115) with Eq. (4.112) gives rise to
$H=H_{\text {end }}\left(\frac{a_{\text {end }}}{a}\right)^{\alpha^{2} / 2}$,
which can be integrated and gives
$a(t)=a_{\text {end }}\left(\frac{t}{t_{\text {end }}}\right)^{2 / \alpha^{2}}$.
One recovers the solution mentioned at the beginning of this section. Finally, the equation of state $w=P / \rho$ can also be worked out exactly and one gets
$w=-1+\frac{\alpha^{2}}{3}$.


Fig. 44. Prior space on $x_{\text {end }}$ derived from Eq. (5.34) with $N_{\text {end }}-N_{\text {ini }}=60$, as a function of $\beta>9 / 2(1+\sqrt{2})$ (black solid line). The black dotted line corresponds to $x_{V^{\prime}=0}$. For $\beta<9 / 2(1+\sqrt{2})$, provided some fine-tuning on the initial conditions, $x_{\text {end }}$ can take any values. The dashed area corresponds to parameters for the model which produce at least the required number of $e$-folds.

Again, all the previous expressions are valid even if the slow-roll approximation is not satisfied. One can see that pure de Sitter corresponds to $\alpha=0$. In this case the potential is constant, the equation of state is -1 and the scale factor expands exponentially.

Another nice feature of power-law inflation is that the spectrum of the perturbations can be computed exactly without relying on any approximation. Defining the parameter $\beta \leq-2$ from $\alpha^{2} / 2=$ $(\beta+2) /(\beta+1)$, the primordial scalar power spectrum is given by
$\mathcal{P}_{\zeta}=\frac{H_{*}^{2}}{\pi \epsilon_{1}\left(8 \pi M_{\mathrm{Pl}}^{2}\right)} f(\beta)\left(\frac{k}{k_{*}}\right)^{2 \beta+4}$,
where
$f(\beta) \equiv \frac{1}{\pi}\left[\frac{(1+\beta)^{1+\beta}}{2^{1+\beta}} \Gamma\left(\frac{1}{2}+\beta\right)\right]^{2}$.
In particular, $f(\beta=-2)=1$. The power spectrum of gravitational waves can also be obtained remarking that we have $\mu_{\mathrm{S}}=\mu_{\mathrm{T}}$ for power law inflation. From
$\mathcal{P}_{\zeta}=\frac{k^{3}}{8 \pi^{2}}\left|\frac{\mu_{\mathrm{S}}}{a \sqrt{\epsilon_{1}}}\right|^{2}, \quad \mathcal{P}_{h}=\frac{2 k^{3}}{\pi^{2}}\left|\frac{\mu_{\mathrm{T}}}{a}\right|^{2}$,
one gets
$r \equiv \frac{\mathcal{P}_{h}}{\mathcal{P}_{\zeta}}=16 \epsilon_{1}=\frac{16 n_{\mathrm{T}}}{n_{\mathrm{T}}-2}$,
since $n_{T}=n_{S}-1=2 \beta+4$.
Finally, the overall amplitude of the CMB anisotropies leads to a determination of the scale $M$, namely

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 \pi^{2} \alpha^{2} e^{\alpha \phi_{*} / M_{\mathrm{Pl}}} \frac{Q_{\mathrm{mls}-\mathrm{PS}}^{2}}{T^{2}} . \tag{4.123}
\end{equation*}
$$

Obviously, this normalization depends on the value of $\phi_{\text {end }}$, and it is more relevant to express it in terms of the potential energy, say, at the end of inflation:
$\frac{V_{\mathrm{end}}}{M_{\mathrm{Pl}}^{4}}=720 \pi^{2} \alpha^{2} e^{-\alpha^{2} \Delta N_{*}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$,
from which one typically gets $V_{\text {end }}^{1 / 4} / M_{\mathrm{Pl}} \simeq 10^{-4}$.
The reheating consistent slow-roll predictions for the power law inflation models are displayed in Fig. 91. Because the slowroll parameters are constant during inflation, one can check that
the predictions of the models do not depend on the energy scale at which the power law reheating ends. One has $n_{\mathrm{S}}=1-\alpha^{2}$ and $r=8 \alpha^{2}$, and from the Planck constraints, all the models are disfavored at more than two-sigma confidence level.

### 4.9. Kähler moduli inflation I (KMII)

These models are stringy models and arise when type IIB string theories via Calabi-Yau flux compactification are used. KMII scenarios have been derived and studied in Refs. [318-324]. More specifically, when internal spaces are weighted projective spaces, one of the Kähler moduli can play the role of an inflaton field and its potential, in the large field limit, reads
$V(\phi)=M^{4}\left(1-\alpha \frac{\phi}{M_{\mathrm{Pl}}} e^{-\phi / M_{\mathrm{Pl}}}\right)$,
$\alpha$ being a positive dimensionless parameter. Actually, since we deal with a modulus, $\phi$ usually possesses a non-minimal kinetic term. Then, once the inflaton field has been canonically normalized, $\phi$ has to be replaced with $\propto \phi^{4 / 3}$. The corresponding corrected potential is studied as "Kähler Moduli Inflation II" (KMIII) in Section 5.3. However, sometimes, the potential (4.125) (with $\phi$ already canonically normalized) is also studied as a toy model (notably in Ref. [324]), the hope being that it can give a simpler description of the physics that naturally appears in the context of moduli inflation. Therefore, in this section, we also consider this scenario.

The potential in Eq. (4.125) depends on one free parameter, $\alpha$. A priori, there does not exist any bound on its value. However, as explained below, in order for slow-roll inflation to occur, one must restrict the range of possible values for $\alpha$. Within this range, we will show that the predictions of the model turn out to be almost independent of $\alpha$ (in fact, they logarithmically depend on $\alpha$ ). The potential (4.125) and its logarithm are displayed in Fig. 18. It decreases from $\phi=0$ (where it blows up), reaches a minimum at $\phi=M_{\mathrm{Pl}}$, and then increases to the asymptotic value $V=M^{4}$ when $\phi \rightarrow+\infty$. Therefore, two regimes of inflation may a priori exist: either inflation proceeds from the left to the right in the decreasing $\phi<M_{\mathrm{PI}}$ branch of the potential (in this branch the vev $\phi$ increases during inflation) or it proceeds from the right to the left in the increasing $\phi>M_{\text {PI }}$ branch of the potential (and the vev decreases during inflation). However, one should keep in mind that the potential is derived under the large field assumption and, consequently, only the second regime is in fact meaningful. As a toy model, one might nevertheless want to study both regimes but it turns out that, in the first one, inflation could not stop by violation of the slow-roll conditions. This is why we will mainly focus on the second regime in the rest of this section. Let us also notice that the minimum value of the potential is located at $\phi=M_{\mathrm{PI}}$ and is $V_{\min }=M^{4}(1-\alpha / e)$. Therefore, if one requires the potential to be positive definite everywhere, then one must have $0<\alpha<e \simeq$ 2.72. However, this condition may also be ignored if one considers that the potential (4.125) is in any case not valid at $\phi / M_{\mathrm{PI}} \lesssim 1$.

Defining $x \equiv \phi / M_{\mathrm{PI}}$, the three first slow-roll parameters can be expressed as
$\epsilon_{1}=\frac{\alpha^{2}}{2} e^{-2 x} \frac{(1-x)^{2}}{\left(1-\alpha e^{-x} x\right)^{2}}$,
$\epsilon_{2}=\frac{2 \alpha e^{-x}}{\left(1-\alpha e^{-x} x\right)^{2}}\left(\alpha e^{-x}+x-2\right)$,
and

$$
\begin{align*}
\epsilon_{3}= & \frac{\alpha e^{-x}(x-1)}{\left(1-\alpha e^{-x} x\right)^{2}\left(\alpha e^{-x}+x-2\right)} \\
& \times\left[x-3+\alpha e^{-x}\left(x^{2}-3 x+6\right)-2 \alpha^{2} e^{-2 x}\right] \tag{4.127}
\end{align*}
$$



Fig. 45. Top left panel: Kähler moduli inflation II (KMIII) potential for $\alpha=4$ and $\beta=1$. These parameters are not physical but they are used for display convenience. Top right panel: logarithm of the potential for the same value of $\alpha$ and $\beta$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ for a potential with $\alpha=4$ and $\beta=1$. The shaded area indicates the breakdown of the slow-roll inflation (strictly speaking when the acceleration stops). Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) for $\alpha=4$ and $\beta=1$.

Let us now study in more detail how inflation stops in this model. As can be seen in Fig. 18, the number of solutions of $\epsilon_{1}=1$ depends on the value of $\alpha$. We now define the numbers $\alpha_{1}$ and $\alpha_{2}$ by
$\alpha_{1} \equiv \frac{\sqrt{2}}{\sqrt{2}-1} e^{\frac{2-\sqrt{2}}{1-\sqrt{2}}} \simeq 0.83$,
$\alpha_{2} \equiv \frac{\sqrt{2}}{\sqrt{2}+1} e^{\frac{2+\sqrt{2}}{1+\sqrt{2}}} \simeq 2.41$.
If $0<\alpha<\alpha_{1}$, then there is no solution (this corresponds to the green line in the bottom left panel in Fig. 18). The inflaton field eventually oscillates around the minimum of its potential but remains in a region where inflation continues forever. In this case, in order to stop inflation, one must add an auxiliary field to the model such that a tachyonic instability is triggered at some value $x_{\text {end }}$. This of course increases the number of parameters of this model. If $\alpha_{1}<\alpha<\alpha_{2}$ (which corresponds to the blue line in Fig. 18), then two solutions appear:

$$
\begin{align*}
\left.x_{\epsilon_{1}=1}^{-}\right|_{x<1} & =\left.x_{\text {end }}\right|_{x<1}=\frac{1}{1-\sqrt{2}}-\mathrm{W}_{0}\left(\frac{\sqrt{2}}{1-\sqrt{2}} \frac{e^{\frac{1}{1-\sqrt{2}}}}{\alpha}\right) \\
& \simeq-2.4-\mathrm{W}_{0}\left(-\frac{0.3}{\alpha}\right),  \tag{4.129}\\
\left.x_{\epsilon_{1}=1}^{+}\right|_{x<1} & =\frac{1}{1-\sqrt{2}}-\mathrm{W}_{-1}\left(\frac{\sqrt{2}}{1-\sqrt{2}} \frac{e^{\frac{1}{1-\sqrt{2}}}}{\alpha}\right) \\
& \simeq-2.4-\mathrm{W}_{-1}\left(-\frac{0.3}{\alpha}\right), \tag{4.130}
\end{align*}
$$

where $\mathrm{W}_{0}$ and $\mathrm{W}_{-1}$ denotes the " 0 -branch" and the " -1 -branch" of the Lambert function respectively. These two solutions are both smaller than one so that they both lie in the decreasing branch of the potential. Correspondingly, two regimes of inflation exist. The first one proceeds from the left to the right and stops at $\left.x_{\text {end }}\right|_{x<1}$. However, using the expression for the slow-roll parameters (4.126), it is easy to see that $\epsilon_{1}$ is always larger than $1 / 2$ in this domain. Therefore, the slow-roll approximation breaks down in this case. The second regime takes place in the $\phi / M_{\mathrm{Pl}}>$ 1 branch of the potential but inflation cannot stop by slow-roll violation. Finally, if $\alpha_{2}<\alpha$ (this situation corresponds to the pink line in the bottom left panel in Fig. 18), then four solutions exist: two were already given in Eq. (4.129), (4.130) and the two new ones read

$$
\begin{align*}
\left.x_{\epsilon_{1}=1}^{-}\right|_{x>1} & =\frac{1}{1+\sqrt{2}}-\mathrm{W}_{0}\left(-\frac{\sqrt{2}}{1+\sqrt{2}} \frac{e^{\frac{1}{1+\sqrt{2}}}}{\alpha}\right) \\
& \simeq 0.4-\mathrm{W}_{0}\left(\frac{-0.9}{\alpha}\right),  \tag{4.131}\\
\left.x_{\epsilon_{1}=1}^{+}\right|_{x>1} & =\left.x_{\text {end }}\right|_{x>1}=\frac{1}{1+\sqrt{2}}-\mathrm{W}_{-1}\left(-\frac{\sqrt{2}}{1+\sqrt{2}} \frac{e^{\frac{1}{1+\sqrt{2}}}}{\alpha}\right) \\
& \simeq 0.4-\mathrm{W}_{-1}\left(\frac{-0.9}{\alpha}\right) . \tag{4.132}
\end{align*}
$$

The two new solutions are greater than one and therefore lie in the increasing branch of the potential. Thus two regimes exist in this situation. The first one is the same as before, proceeds again from the left to right, stops at $\left.x_{\text {end }}\right|_{x<1}$ and suffers from the fact that $\epsilon_{1}$ is always larger than $1 / 2$. The second one proceeds from the right to the left and ends at $\left.x_{\text {end }}\right|_{x>1}$. We conclude that this regime is the


Fig. 46. Comparison between the exact numerical value of $x_{\text {end }}(\alpha, \beta)$ (blue solid line), and the approximated formula given by Eq. (5.55) (red dotted line) for $\alpha=$ $\mathcal{V}^{5 / 3}$ and $\beta=\mathcal{V}^{2 / 3}$. The agreement is excellent but a numerical calculation is used in ASPIC anyway.
regime of interest for the KMII model and that we must therefore require $\alpha>\alpha_{2}$.

Let us now study the slow-roll trajectory. It can be integrated exactly and its expression can be written as

$$
\begin{align*}
N_{\mathrm{end}}-N= & x_{\mathrm{end}}-\frac{e}{\alpha} \operatorname{Ei}\left(x_{\mathrm{end}}-1\right)+\ln \left(x_{\mathrm{end}}-1\right) \\
& -x+\frac{e}{\alpha} \operatorname{Ei}(x-1)-\ln (x-1), \tag{4.133}
\end{align*}
$$

where Ei is the exponential integral function [216,217]. At this point, a few remarks are in order. Firstly, let us notice that $N$ goes to $\infty$ when $x$ tends to 1 . This means that, in the slowroll approximation, the field can never cross the minimum of its potential. In particular, if $\alpha<\alpha_{2}$, that is to say if one starts from the $\phi / M_{\mathrm{Pl}}<1$ branch and rolls down from the left to the right, then one can never reach the physical $\phi / M_{\mathrm{PI}}>1$ branch of the potential and inflation can never come to an end. Secondly, when $x \gg 1$, the trajectory can be approximated by
$N_{\text {end }}-N \simeq \frac{e}{\alpha}\left(\frac{e^{x}}{x}-\frac{e^{x_{\text {end }}}}{x_{\text {end }}}\right)$.
Moreover, in this approximation, it can be inverted exactly and one obtains
$x \simeq-\mathrm{W}_{-1}\left[-\frac{1}{\alpha\left(N_{\text {end }}-N\right) / e+e^{x_{\text {end }}} / x_{\text {end }}}\right]$,
in agreement with what was obtained in Ref. [324]. In the above expression, $\mathrm{W}_{-1}$ is the -1 branch of the Lambert function. Let us also notice that, in Ref. [324], the branch of the Lambert function was in fact incorrectly chosen. The fact that the -1 branch of the Lambert function has to be considered comes from the following argument. When $N_{\text {end }}-N \rightarrow \infty$, the argument of the Lambert function goes to $0^{-}$and, therefore, since $x$ must tend towards $+\infty$ in this limit, the -1 branch must be chosen. In addition, if $N_{\text {end }}-N \rightarrow 0$, then one must have $x \rightarrow x_{\text {end }}>1$ which is also the case if the -1 branch is retained. This is represented in Fig. 19 where the arrow indicates the direction along which inflation proceeds. In the third place, since, when $x \rightarrow \infty$, one has $N_{\text {end }}-N \rightarrow \infty$, a sufficient number of $e$-folds can always be realized in this model. Finally, it is inaccurate to assume that $x_{\text {end }} \gg 1$ and, therefore, the above approximated trajectory is not so useful. However, if one only assumes that $x \gg 1$ (which can be checked to be a good approximation, especially at $x=x_{*}$ ) but not $x_{\text {end }} \gg 1$, then one can write
$N_{\mathrm{end}}-N \simeq \frac{e}{\alpha} \frac{e^{x}}{x}+x_{\mathrm{end}}-\frac{e}{\alpha} \operatorname{Ei}\left(x_{\mathrm{end}}-1\right)$,


Fig. 47. Lambert functions $\mathrm{W}_{0}(x)$ (dashed line) and $\mathrm{W}_{-1}(x)$ (solid line). During Kähler moduli inflation II, inflation proceeds along the " -1 " branch in the direction specified by the arrow.
which, moreover, can be inverted into
$x \simeq-\mathrm{W}_{-1}\left[-\frac{1}{\alpha\left(N_{\text {end }}-N\right) e+\operatorname{Ei}\left(x_{\text {end }}-1\right)-\alpha x_{\text {end }} / e}\right]$,
and which is valid whenever $x \gg 1$. However, one should keep in mind that, now, and contrary to the former approximated trajectory, taking the limit $N \rightarrow N_{\text {end }}$ in the above expression is meaningless.

The energy scale $M$ is, as before, given by the CMB normalization and one obtains the following expression
$\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 \pi^{2} \alpha^{2} \frac{\left(1-x_{*}\right)^{2}}{\left(1-\alpha x_{*} e^{-x_{*}}\right)^{3}} e^{-2 x_{*}} \frac{Q_{\text {rms-PS }}^{2}}{T^{2}}$.
If one uses the $x_{*} \gg 1$ approximation, then Eq. (4.137) tells us that $x_{*} \simeq \ln \left(\alpha \Delta N_{*}\right)$ and Eq. (4.138) can be re-written as
$\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=\mathcal{O}(1) 720 \frac{\pi^{2}}{\Delta N_{*}^{2}} \frac{Q_{\mathrm{m} \mathrm{m}-\mathrm{PS}}^{2}}{T^{2}}$.
It is remarkable that this equation does not depend on $\alpha$. Using a fiducial value for $\Delta N_{*}$, one typically gets $M / M_{\mathrm{PI}} \sim 10^{-3}$.

The predictions of KMII models are displayed in Fig. 92, for $\alpha>$ $\alpha_{2}$. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 since the potential is quadratic close to its minimum [but, it should be reminded that, in principle, the potential Eq. (4.125) cannot be trusted close to its minimum]. One can see that, as announced at the beginning of this section, the predictions depend on $\alpha$ in a very mild way, a conclusion which is in agreement with Refs. [318,324]. This can be understood as follows. If one assumes that $x_{*} \gg 1$, then we have already noticed that Eq. (4.137) implies that $x_{*} \simeq \ln \left(\alpha \Delta N_{*}\right)$. From this result, one obtains that
$\epsilon_{1 *} \simeq \frac{1}{2 \Delta N_{*}^{2}} \ln ^{2}\left(\alpha \Delta N_{*}\right), \quad \epsilon_{2 *} \simeq \frac{2}{\Delta N_{*}} \ln \left(\alpha \Delta N_{*}\right)$,
$\epsilon_{3 *} \simeq \frac{1}{\Delta N_{*}} \ln \left(\alpha \Delta N_{*}\right)$.
In these expressions, we notice that the slow-roll parameters (at Hubble crossing) logarithmically depend on $\alpha$. This explains the weak $\alpha$ dependence observed in Fig. 92. Of course, one can also calculate the corresponding expressions of the spectral index, tensor to scalar ratio and running. One arrives at
$n_{\mathrm{S}} \simeq 1-2 \frac{\ln \left(\alpha \Delta N_{*}\right)}{\Delta N_{*}}, \quad r \simeq 8 \frac{\ln ^{2}\left(\alpha \Delta N_{*}\right)}{\Delta N_{*}^{2}}$,
$\alpha_{S} \simeq-2 \frac{\ln ^{2}\left(\alpha \Delta N_{*}\right)}{\Delta N_{*}^{2}}$.



 colour in this figure legend, the reader is referred to the web version of this article.)

These expressions are in accordance with the estimates derived in Refs. [318,324]. However, contrary to what is claimed in Refs. [324], the predicted value of the running is not excluded by the CMB observations since, according to the Planck results [70], one has $\alpha_{S}=-0.013 \pm 0.009$.

### 4.10. Horizon flow inflation at first order (HF1I)

The horizon flow models have been introduced in Ref. [325] and consist into designing field potentials to exactly produce a truncated Taylor expansion of the Hubble parameter with respect to the field. As such they constitute a whole class of phenomenological inflationary models. Here, we are considering a potential designed such that $H(\phi)=H_{0}\left(1+A_{1} \phi / M_{\text {PI }}\right)$, where $A_{1}$ is a free dimensionless parameter. The shape of the potential reads [325]
$V(\phi)=M^{4}\left(1+A_{1} \frac{\phi}{M_{\mathrm{Pl}}}\right)^{2}\left[1-\frac{2}{3}\left(\frac{A_{1}}{1+A_{1} \frac{\phi}{M_{\mathrm{Pl}}}}\right)^{2}\right]$.
Denoting $x \equiv \phi / M_{\mathrm{Pl}}$, the potential admits a global minimum at $x_{V \text { min }}=-1 / A_{1}$, which is negative
$V_{\text {min }}=V\left(\phi_{V_{\text {min }}}\right)=-\frac{2}{3} M^{4} A_{1}^{2}<0$.
As a result, there are two disconnected field domains in which the potential remains definite positive, either $x>x_{V=0}^{+}$or $x<x_{V=0}^{-}$ where $x_{V=0}^{ \pm}$are the two roots of $V\left(x_{V=0}^{ \pm}\right)=0$, i.e.
$x_{V=0}^{+}=\sqrt{\frac{2}{3}}-\frac{1}{A_{1}}$,

An interesting consequence of the horizon flow approach is that the Hubble flow functions can be calculated exactly, i.e. without the slow-roll approximation because $H(\phi)$ is exactly known. As discussed in Refs. [17,326], one could compare them with the other hierarchy of parameters, $\epsilon_{i}^{V}$, that are defined by the successive logarithmic derivatives of the potential. In the slowroll approximation, one precisely uses the potential derivatives to approximate the Hubble flow functions. From $H \propto 1+A_{1} x$, one gets the exact Hubble flow functions
$\epsilon_{1}=2\left(\frac{A_{1}}{1+A_{1} x}\right)^{2}, \quad \epsilon_{2}=\epsilon_{3}=2 \epsilon_{1}$,
whereas the slow-roll functions associated with the potential are
$\epsilon_{1}^{V}=\frac{18 A_{1}^{2}\left(A_{1} x+1\right)^{2}}{\left[3+6 A_{1} x+A^{2}\left(3 x^{2}-2\right)\right]^{2}}$,
$\epsilon_{2}^{V}=\frac{12 A_{1}^{2}\left[3+6 A_{1} x+A_{1}^{2}\left(3 x^{2}+2\right)\right]}{\left[3+6 A_{1} x+A_{1}^{2}\left(3 x^{2}-2\right)\right]^{2}}$,
and
$\epsilon_{3}^{V}=\frac{108 A_{1}^{2}\left(A_{1} x+1\right)^{2}\left[1+2 A_{1} x+A_{1}^{2}\left(x^{2}+2\right)\right]}{\left[3+6 A_{1} x+A_{1}^{2}\left(3 x^{2}-2\right)\right]^{2}\left[3+6 A_{1} x+A_{1}^{2}\left(3 x^{2}+2\right)\right]}$.
As shown in Ref. [17], the link between the two hierarchies can be made explicit and one has
$\epsilon_{1}^{V}=\epsilon_{1}\left(\frac{1-\eta / 3}{1-\epsilon_{1} / 3}\right)^{2}$.
The $\eta$ parameter is defined as
$\eta \equiv \frac{2}{H} \frac{\mathrm{~d}^{2} H}{\mathrm{~d} x^{2}}$,


Fig. 49. Top left panel: Twisted Potential Inflation (TWI) for $\phi_{0}=0.02 \mathrm{M}_{\mathrm{Pl}}$. Top right panel: logarithm of the potential for the same value of $\phi_{0}$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ with $\phi_{0}=0.02 M_{\mathrm{Pl}}$ (solid blue line) and $\phi_{0}=0.05 M_{\mathrm{PI}}$ (solid green line). The shaded area indicates the non-inflationary region. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) with $\phi_{0}=0.02 M_{\mathrm{PI}}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
and vanishes in our case. As a result, provided $\epsilon_{1} \ll 1$, i.e. we are in the slow-roll approximation, both hierarchies give the same results at first order. In order to establish Eq. (4.148), one has to show first that
$\eta=\epsilon_{1}+\frac{1}{\sqrt{2 \epsilon_{1}}} \frac{\mathrm{~d} \epsilon_{1}}{\mathrm{~d} x}$,
and then that ${ }^{5}$
$\frac{\mathrm{d} \epsilon_{1}}{\mathrm{~d} x}=\left(\epsilon_{1}-3\right)\left(\frac{\mathrm{d} \ln V}{\mathrm{~d} x}-\sqrt{2 \epsilon_{1}}\right)$.
The potential and the exact Hubble flow functions have been represented in Fig. 20.

Inflation can take place inside the two positive definite domains of the potential, i.e. at negative or positive field values. However, the Hubble parameter has to be positive such that $H_{0}$ has to be chosen negative if $1+A_{1} x<0$ along the field trajectory. Since the potential is completely symmetric with respect to its minimum $x_{V \min }$, we can study in full generality only the $x>x_{V=0}^{+}$branch. In particular, as the Hubble flow functions are exact, we can also derive the exact field trajectory
$N-N_{\text {end }}=-\frac{1}{2 A_{1}}\left(x+\frac{1}{2} A_{1} x^{2}-x_{\text {end }}-\frac{1}{2} A_{1} x_{\text {end }}^{2}\right)$.
Let us notice that, in the slow-roll approximation, one would have derived the trajectory from $\epsilon_{1}^{V}$. Doing so, one would have obtained
$N-N_{\text {end }}=-\frac{1}{2 A_{1}}\left(x+\frac{1}{2} A_{1} x^{2}-x_{\text {end }}-\frac{1}{2} A_{1} x_{\text {end }}^{2}\right.$

[^4]\[

$$
\begin{equation*}
\left.-\frac{2}{3} A_{1} \ln \left|\frac{1+A_{1} x}{1+A_{1} x_{\mathrm{end}}}\right|\right) \tag{4.153}
\end{equation*}
$$

\]

It is amusing to remark that here, the simplest formula is not given by the slow-roll derived one, but rather by the exact one. From this remark one should keep in mind that, in order to simplify trajectories integration, one can always add factors of order $\mathcal{O}\left(\epsilon_{1}\right)$. The exact trajectory (4.152) can be inverted and one finds
$x=-\frac{1}{A_{1}}+\frac{1}{A_{1}} \sqrt{1+2 A_{1} x_{\text {end }}+A_{1}^{2}\left[x_{\text {end }}^{2}-4\left(N-N_{\text {end }}\right)\right]}$.
Along both the positive and negative branch of the potential, inflation ends naturally at $\epsilon_{1}=1$, that is at
$x_{\epsilon_{1}=1}^{ \pm}=\frac{-1 \pm \sqrt{2} A_{1}}{A_{1}}$.
Along the positive branch we are interested in, we therefore have
$x_{\text {end }}=x_{\epsilon_{1}=1}^{+}=\frac{-1+\sqrt{2} A_{1}}{A_{1}}$.
Plugging this expression into Eq. (4.154) gives the field value $x_{*}$ at which the pivot mode crossed the Hubble radius during inflation in terms of the $e$-fold number $\Delta N_{*}=N_{\text {end }}-N_{*}$. Let us remember that solving for $x_{*}$ (or $\Delta N_{*}$ ) is made through Eq. (2.47). From Eq. (4.145), one gets
$\epsilon_{1 *}=\frac{1}{1+2 \Delta N_{*}}$
which, together with $\epsilon_{2}=2 \epsilon_{1}$, yields
$n_{\mathrm{S}}-1=2 n_{\mathrm{T}}, \quad r=4\left(1-n_{\mathrm{S}}\right)$.






 predictions. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Notice that this relation is different from the power law case and consistent with Ref. [327]. In that reference, the authors mention that the horizon flow models predicts $r \simeq 4.8\left(1-n_{\mathrm{S}}\right)$ as a result of Monte-Carlo simulations.

Finally, the potential parameter $M$ can be determined from the CMB normalization

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=960 \pi^{2} \frac{A_{1}^{2}}{\left(1+A_{1} x_{*}\right)^{4}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{4.159}
\end{equation*}
$$

It is interesting to notice that the typical energy scale of inflation in these models does not depend on $A_{1}$. The previous equation indeed leads to
$\frac{V\left(x_{*}\right)}{M_{\mathrm{Pl}}^{4}}=\frac{480 \pi^{2}}{1+2 \Delta N_{*}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}\left(1-\frac{1}{3+6 \Delta N_{*}}\right) \simeq 10^{-9}$.
The reheating consistent (exact) predictions for the horizon flow inflation I models are represented in Fig. 93. As expected, the relation $\epsilon_{2}=2 \epsilon_{1}$, which is the same as for the LFI quadratic case, is properly recovered. The predictions do not depend much on the potential parameter $A_{1}$.

### 4.11. Colemann-Weinberg inflation (CWI)

### 4.11.1. Theoretical justifications

The potential of this model was first introduced by Coleman and Weinberg in Ref. [328], in the context of spontaneous symmetry breaking generated by radiative corrections. The starting point of this work is to calculate the effective potential for a massless charged meson minimally coupled to the electrodynamic field.

In that reference, the effective action is explicitly constructed from a Legendre transform of the partition function, and expanded into one-particle-irreducible Feynman diagrams with $n$ external lines (and summing up over $n$ ). The exact knowledge of the effective potential requires an infinite summation of all these Feynman diagrams, which is in practice intractable. It is thus made use of the one loop expansion method where all diagrams with no closed loops are first summed, then all diagrams with one closed loop are added, and all higher loops diagrams neglected. Starting with a quartic interacting scalar field, and requiring that the renormalized mass vanishes, one obtains a potential of the form
$V(\phi) \propto 1+\alpha\left(\frac{\phi}{Q}\right)^{4} \ln \left(\frac{\phi}{Q}\right)$.

Let us emphasize that another useful frame of approximation is the Gaussian effective potential method. The Gaussian effective potential is a non-perturbative approach to quantum field theory [329-337], originally developed in the context of quantum mechanics, and generalized to field theory afterwards. In quantum mechanics, when studying systems governed by Hamiltonians of the form $H=p^{2} / 2+V(\phi)$, the idea is to calculate en effective potential $V_{\text {GEP }}$ defined as

$$
\begin{align*}
V_{\mathrm{GEP}}\left(\phi_{0}\right) & =\min _{\Omega}[\langle\psi| H|\psi\rangle, \\
\psi(\phi) & \left.=\left(\frac{\Omega}{\hbar \pi}\right)^{1 / 4} e^{-\Omega\left(\phi-\phi_{0}\right)^{2} /(2 \hbar)}\right], \tag{4.162}
\end{align*}
$$

i.e. the minimum possible quantum mean energy of a Gaussian wavefunction centered over $\phi_{0}$. Such an object turns out to be a powerful tool to addressing the effects of quantum fluctuations on the physical behavior of a system in a non-perturbative way. It can be easily generalized to quantum field theories, expanding the field operator $\Phi$ only over $\Omega$-massive excitations around the classical value $\phi_{0}$ in dimensions,

$$
\begin{align*}
\Phi(t, \boldsymbol{x})= & \phi_{0}+(2 \pi)^{(1-d) / 2} \int \frac{\mathrm{~d}^{d-1} \boldsymbol{k}}{\sqrt{2 \sqrt{k^{2}+\Omega^{2}}}} \\
& \times\left(a_{\boldsymbol{k}} e^{-i \sqrt{k^{2}+\Omega^{2}} t+i \boldsymbol{k} \cdot \boldsymbol{x}}+a_{\boldsymbol{k}}^{\dagger} e^{i \sqrt{k^{2}+\Omega^{2}} t-i \boldsymbol{k} \cdot \boldsymbol{x}}\right), \tag{4.163}
\end{align*}
$$

where $a_{\boldsymbol{k}}^{\dagger}$ and $a_{\boldsymbol{k}}$ are the usual creation and annihilation operators, and minimizing the quantum mean value of the Hamiltonian density over $\Omega$. In Ref. [330], the quartic interacting scalar field has been worked out with this method, i.e. starting from $V(\phi)=$ $m^{2} \phi^{2} / 2+\lambda \phi^{4}$. The Gaussian effective potential $V_{\text {GEP }}$ obtained in this way can expanded in power of $\hbar$ to show that the first order terms match with the potential of Coleman and Weinberg. This is not surprising as this is equivalent of performing a one loop expansion over the effective action. However, it should be stressed that the Gaussian effective potential method provides a much more general expression for the potential, that is valid beyond this perturbative limit and that can address regimes where quantum diffusion dominates the dynamics of the scalar field.

The model is defined such that inflation ends by violation of the slow-roll conditions, and is followed by a preheating stage in which


Fig. 51. GMSSM Inflation (GMSSMI). Top left panel: GMSSM Inflation potential Eq. (5.102) for $\alpha=0.1,0.7,1.5,2.5$, as a function of $\phi / \phi_{0}$. Top right panel: logarithm of the potentials for the same value of $\alpha$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ for a potential with the same values of $\alpha$. Bottom right panel: slow-roll parameter $\epsilon_{2}$ for a potential with the same values of $\alpha$. See discussion in the text body.


Fig. 52. GMSSM Inflation (GMSSMI). Left panel: $x_{\epsilon_{2}=0}^{ \pm}$defined in Eq. (5.108) and $x_{V^{\prime}=0}^{ \pm}$defined in Eq. (5.104) together with $x_{\epsilon_{1}}^{\min }$ [see Eq. (5.110)] as a function of $\alpha$. Right panel: minimal value of the slow-roll parameter $\epsilon_{1}$ (rescaled by $\phi_{0}^{2} / M_{\mathrm{Pl}}^{2}$ ) as a function of $\alpha$. When it is greater than unity, inflation cannot occur.
the inflaton field oscillates at the bottom of its potential. Therefore this potential minimum must be set to zero, which implies
$\alpha=4 e$.
One is thus left with one mass parameter, $Q$, which sets the typical $v e v$ at which inflation takes place. On the other hand, the value taken for $Q$ also depends on the underlying high energy model from which the CW potential emerges.

The CWI potential appears in various other contexts and, in fact, historically, it was the first model of inflation ever proposed [1] (also known as "old inflation"). The idea was that inflation occurs while the field is trapped in a false vacuum state $\langle\phi\rangle=0$. Then, inflation comes to an end when the field tunnels from this state to the symmetry breaking true minimum. Unfortunately, this models was quickly realized to be ruled out since the above mentioned process is accompanied by bubble formation and these
bubbles, while colliding, produce too large inhomogeneities. Then, this problem was solved by a modification of the old inflation scenario called "new inflation" $[2,3]$. The main idea is that inflation does not occur while the field is trapped but when the field is rolling down from the origin to its true minimum. Bubbles are also formed but there are so big that our entire universe is contained in one of them. As a consequence, we do not observe bubble collisions and our universe is extremely homogeneous as indicated by the observations. This new inflationary scenario was explicitly implemented in Ref. [2] where the $\mathrm{SU}(5) \rightarrow \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ phase transition in GUTs is investigated. The model makes use of a CWI potential that can be described by
$V(\phi)=\frac{5625}{512 \pi^{2}} g^{4}\left[\phi^{4} \ln \left(\frac{\phi}{\phi_{0}}\right)-\frac{\phi^{4}}{4}+\frac{\phi_{0}^{4}}{4}\right]$,


 right panel: slow-roll parameter $\epsilon_{2}$, rescaled by $M_{\mathrm{PI}}^{2} / \phi_{0}^{2}$. A description of these various quantities can be found in the text.


Fig. 54. Left panel: $x_{\epsilon_{2}=0}^{ \pm}$and $x_{V^{\prime}=0}^{ \pm}$[defined in Eq. (5.121)] together with $x_{\epsilon_{1}^{\min }}$ [see Eq. (5.126)] as a function of $\alpha$. Right panel: minimal value of the slow-roll parameter $\epsilon_{1}$, i.e. $\epsilon_{1}\left(x_{\epsilon_{1}^{\min }}\right)$, rescaled by $\phi_{0}^{2} / M_{\mathrm{Pl}}^{2}$, as a function of $\alpha$. When it is greater than unity, inflation cannot occur.
where $\phi_{0} \simeq 10^{14}-10^{15} \mathrm{GeV}$, representing the GUT symmetry breaking scale, and $g^{2} \simeq 1 / 3$ is the $\operatorname{SU}(5)$ gauge coupling constant. However, as noticed afterwards in Refs. [338-342], this model has also a fatal flaw. Indeed, one sees in Eq. (4.165) that the overall normalization of the potential reads $M^{4}=5625 g^{4} \phi_{0}^{4} /\left(2048 \pi^{2}\right)$ and that, therefore, the amplitude of the fluctuations is in fact already fixed. Using the value of the $\operatorname{SU}(5)$ coupling constant and $Q / M_{\mathrm{Pl}}=e^{1 / 4} \phi_{0} / M_{\mathrm{Pl}} \simeq 5 \times 10^{-5}-5 \times 10^{-4}$, one arrives at $M^{4} \simeq\left(10^{-13}-10^{-17}\right) M_{\mathrm{Pl}}^{4}$. This turns out to be incompatible with the CMB normalization [see Eq. (4.173) below]. However, the same model was re-considered in Refs. [341,343] (see also Ref. [344]), but with additional fields and couplings. It was then shown that the scale $M$ acquires a different form and can scale as the inverse of the coupling constants. Since these ones are small, it becomes possible to obtain a higher value for $M$ and to correctly CMB normalize the
model. In what follows, we will therefore consider the scale $M$ as a free parameter fixed by the overall amplitude of the cosmological fluctuations.

We also notice that, in Ref. [345], the CWI potential is obtained in the context of Kaluza-Klein inflation, i.e. in higher dimensions and with higher derivative terms and logarithmic dependence on the curvature scalar. Again, the typical value for $Q \simeq 10^{15} \mathrm{GeV}$. The CWI potential appears also in Ref. [346], but the value used for $Q$ is rather different, $Q=0.223 M_{\mathrm{Pl}}$, and is fine-tuned in order to have two phases of inflation, a "chaotic inflationary" phase followed by a "new inflationary" phase. Finally, in Ref. [347], the Coleman-Weinberg potential is studied in the framework of Einstein-Brans-Dicke gravity, with the same typical value for $Q \simeq$ $10^{15} \mathrm{GeV}$ and the same typical value for $M^{4} / M_{\mathrm{Pl}}^{4} \simeq 10^{-15}$ as in the original paper.


Fig. 55. Brane SUSY breaking Inflation (BSUSYBI) for $\gamma=0.1$. Upper panels: the potential and its logarithm. Bottom left panel: the first slow-roll parameter $\epsilon_{1}$ as a function of the field value, the shaded area indicates where inflation stops. Bottom right panel: slow-roll parameter $\epsilon_{2}$ and $\epsilon_{3}$.

### 4.11.2. Slow-roll analysis

Considering the previous considerations, we take the potential to be
$V(\phi)=M^{4}\left[1+\alpha\left(\frac{\phi}{Q}\right)^{4} \ln \left(\frac{\phi}{Q}\right)\right]$,
with a parameter $Q / M_{P I}$ in the range $\left[10^{-5}, 10^{-3}\right]$ and $\alpha=4 e$. As already mentioned, the mass parameter $M$ will be viewed as free and fixed by the normalization to the amplitude of the CMB anisotropies. The potential is displayed Fig. 21. It starts decreasing with the inflaton vev at $\phi=0$, reaches a minimum at $\phi / Q=e^{-1 / 4}$ where it vanishes, and then increases and diverges as $\phi$ goes to $\infty$. As mentioned above, inflation proceeds along the decreasing branch of the potential, in the direction specified by the arrow in the figure.

Let us now derive the first slow-roll parameters. Defining $x \equiv$ $\phi / Q$, they are given by
$\epsilon_{1}=\frac{M_{\mathrm{PI}}^{2}}{Q^{2}} \frac{\alpha^{2}}{2} x^{6}\left(\frac{1+4 \ln x}{1+\alpha x^{4} \ln x}\right)^{2}$,
while

$$
\begin{equation*}
\epsilon_{2}=2 \frac{M_{\mathrm{Pl}}^{2}}{Q^{2}} \alpha x^{2} \frac{-7-12 \ln x+\alpha x^{4}+\alpha x^{4} \ln x+4 \alpha x^{4} \ln ^{2} x}{\left(1+\alpha x^{4} \ln x\right)^{2}} \tag{4.168}
\end{equation*}
$$

and finally

$$
\begin{align*}
\epsilon_{3}= & \frac{M_{\mathrm{Pl}}^{2}}{Q^{2}}\left(-26 \alpha x^{2}+21 \alpha^{2} x^{6}-2 \alpha^{3} x^{10}-128 \alpha x^{2} \ln x\right. \\
& +152 \alpha^{2} x^{6} \ln x-11 \alpha^{3} x^{10} \ln x-96 \alpha x^{2} \ln ^{2} x \\
& +368 \alpha^{2} x^{6} \ln ^{2} x-14 \alpha^{3} x^{10} \ln ^{2} x+384 \alpha^{2} x^{6} \ln ^{3} x \\
& \left.-16 \alpha^{3} x^{10} \ln ^{3} x-32 \alpha^{3} x^{10} \ln ^{4} x\right)\left(1+\alpha x^{4} \ln x\right)^{-2} \\
& \times\left(7-\alpha x^{4}+12 \ln x-\alpha x^{4} \ln x-4 \alpha x^{4} \ln ^{2} x\right)^{-1} \tag{4.169}
\end{align*}
$$



Fig. 56. Maximum value of $x_{\text {end }}$ in order to realize $N e$-folds of inflation between $x_{\epsilon_{1}=1}$ and $x_{\text {end }}$ as a function of $0<\gamma<1 / \sqrt{3}$. This condition defines a prior for the model parameter $x_{\text {end }}$, which is the region lying under the curves on the figure.

The three of them have the same general behavior. They vanish at $x=0$, increase with $x$ in the decreasing branch of the potential and diverge at the minimum of the potential. Then they decrease from infinity in the increasing branch of the potential, and reach asymptotically vanishing values when the field vev goes to infinity. Inflation stops by slow-roll violation when $\epsilon_{1}=1$. The value of $x$ at which this happens needs to be determined numerically, but in the limit $Q / M_{\mathrm{PI}} \ll 1$ (remember that $Q / M_{\mathrm{PI}} \simeq 10^{-4}$ ) where one expects $x_{\text {end }} \ll 1$, one can derive an analytic approximated formula, namely
$x_{\text {end }} \simeq e^{-1 / 4} \exp \left[\mathrm{~W}_{-1}\left(-\frac{3 \sqrt{2}}{4 \alpha} \frac{Q}{M_{\mathrm{Pl}}} e^{3 / 4}\right)\right]$,


Fig. 57. Tip Inflation (TI). Upper panels: Tip Inflation potential and its logarithm for $\alpha=0.1$ (blue line) and $\alpha=1$ (pink line), as a function of $\phi / \mu$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ normalized by $M_{\mathrm{PI}}^{2} / \mu^{2}$. The shaded area indicates the breakdown of the slow-roll inflation if $\mu=M_{\mathrm{PI}}$ (strictly speaking when the acceleration stops). Bottom right panel: slow-roll parameter $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line), again rescaled by $M_{\mathrm{Pl}}^{2} / \mu^{2}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)


Fig. 58. $\beta$ exponential inflation (BEI) for $\beta=0.1$. Upper panels: the potential and its logarithm. Bottom left panel: slow-roll parameter $\epsilon_{1}$ with respect to the field values. The shaded area indicates where inflation stops if $\lambda=1$. Bottom right panel: slow-roll parameters $\epsilon_{2}=\epsilon_{3}$.
where $\mathrm{W}_{-1}$ is the -1 branch of the Lambert function. A comparison between this approximated formula and the numerical solution for $x_{\text {end }}$ is displayed in Fig. 22. The agreement is excellent.

Let us now calculate the slow-roll trajectory from Eq. (2.11). It is given by

$$
\begin{align*}
N_{\text {end }}-N= & \frac{Q^{2}}{M_{\mathrm{Pl}}^{2}} \frac{\sqrt{e}}{4 \alpha}\left[\operatorname{Ei}\left(-\frac{1}{2}-2 \ln x\right)-\operatorname{Ei}\left(-\frac{1}{2}-2 \ln x_{\mathrm{end}}\right)\right] \\
& +\frac{Q^{2}}{M_{\mathrm{Pl}}^{2}} \frac{1}{16 \sqrt{e}}\left[\operatorname{Ei}\left(\frac{1}{2}+2 \ln x_{\mathrm{end}}\right)\right. \\
& \left.-\operatorname{Ei}\left(\frac{1}{2}+2 \ln x\right)\right]+\frac{1}{8} \frac{Q^{2}}{M_{\mathrm{Pl}}^{2}}\left(x^{2}-x_{\mathrm{end}}^{2}\right), \tag{4.171}
\end{align*}
$$

where Ei is the exponential integral function, $N_{\text {end }}$ is the number of $e$-folds at the end of inflation and $N$ is the number of $e$-folds corresponding to the scaled field vev $x$. In the $Q / M_{P I} \ll 1$ limit where $x \ll 1$, the first term of this expression dominates. Since $\alpha=4 e$, the previous expression can be slightly simplified:

$$
\begin{align*}
N_{\mathrm{end}}-N= & \frac{Q^{2}}{M_{\mathrm{Pl}}^{2}} \frac{1}{16 \sqrt{e}}\left[\operatorname{Ei}\left(-\frac{1}{2}-2 \ln x\right)-\operatorname{Ei}\left(-\frac{1}{2}-2 \ln x_{\mathrm{end}}\right)\right. \\
& \left.+\operatorname{Ei}\left(\frac{1}{2}+2 \ln x_{\mathrm{end}}\right)-\operatorname{Ei}\left(\frac{1}{2}+2 \ln x\right)\right] \\
& +\frac{1}{8} \frac{Q^{2}}{M_{\mathrm{Pl}}^{2}}\left(x_{\mathrm{end}}^{2}-x^{2}\right) . \tag{4.172}
\end{align*}
$$

After having solved the above equation for $x_{*}$, the field value at which the pivot scale crossed the Hubble radius during inflation, $M$ is fixed by the amplitude of the CMB anisotropies to

$$
\begin{align*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}= & 720 \pi^{2} \alpha^{2} \frac{M_{\mathrm{Pl}}^{2}}{Q^{2}} x_{*}^{6}\left(1+4 \ln x_{*}\right)^{2}\left(1+\alpha x_{*}^{4} \ln x_{*}\right)^{-3} \\
& \times \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{4.173}
\end{align*}
$$

The reheating consistent slow-roll predictions of the Cole-man-Weinberg models are displayed Fig. 94 in the physical range $Q / M_{\mathrm{Pl}} \in\left[10^{-5}, 10^{-3}\right]$. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 since the potential is quadratic close to its minimum $V(x) \simeq 2 \alpha M^{4} e^{-1 / 2}\left(x-e^{-1 / 4}\right)^{2}$. The typical predicted amount of gravitational waves is extremely small, and a nonnegligible deviation from $n_{S}=1$ is noticed. Also, one could choose to relax the constraint on the parameter $Q$ and study the Cole-man-Weinberg potential in general. This was done for instance in Ref. [343] where the Coleman-Weinberg potential predictions are compared with the WMAP observations on general grounds. It is found that the potential normalization should be of the order $M \simeq 10^{16} \mathrm{GeV}$, and that $Q \simeq 10 M_{\mathrm{PI}}$ in order to match $n_{\mathrm{S}} \simeq 0.96$. For this reason the reheating consistent slow-roll predictions are displayed in Fig. 95 in the extended range $Q / M_{P I} \in[1,100]$. In the limit $Q / M_{P 1} \gg 1$, the model is well approximated by a quadratic potential around its minimum, and one asymptotically approaches the LFI predictions with $p=2$ (see Section 4.2).

### 4.12. Loop inflation (LI)

### 4.12.1. Theoretical justifications

The flatness of an inflationary potential is in general altered by radiative corrections. One loop order corrections generically take the form of a logarithmic function, $\ln (\phi / \mu)$, where $\mu$ is a renormalization scale. Starting from a perfectly flat potential, one obtains a potential of the form $V(\phi)=M^{4}\left[1+\alpha \ln \left(\phi / M_{\mathrm{PI}}\right)\right]$ where $\alpha$ is a dimensionless parameter that tunes the strength of
the radiative effects. Studying such potentials is therefore a simple way to discuss in which cases the quantum correction "spoil" the flatness of a potential, and how this happens.

In fact, this type of scenarios were invented in the context of $F$ and $D$-term inflation in Refs. [348-351]. The original motivation was to build an inflationary model in supersymmetry but without the $\eta$-problem that appears in the $F$-term approach. Indeed, if one considers a simple superpotential $W=f / 2 X \phi^{2}-\mu^{2} X$ where $\phi$ and $X$ are two superfields, then it is easy to obtain the supersymmetric potential assuming a minimal Kähler potential: $V=\left|f \phi^{2} / 2-\mu^{2}\right|^{2}+f^{2}|X|^{2}|\phi|^{2}$. There is a flat direction for $\phi=0$ along the $X$ direction with $V=\mu^{4}$. Lifting this direction with a one loop correction leads to the LI potential which is suitable for inflation. However, considering non-minimal term in the Kähler potential destroys the flatness of $V$. The $D$-term approach was shown to be a viable alternative. The idea is to consider a theory with a $\mathrm{U}(1)$ symmetry and three chiral superfields, $X, \phi_{+}$and $\phi_{-}$ with charges $0,+1$ and -1 respectively. It then follows that the superpotential has the form $W=\lambda X \phi_{+} \phi_{-}$. If we compute the corresponding potential in global supersymmetry, one arrives at

$$
\begin{align*}
V= & \lambda^{2}|X|^{2}\left(\left|\phi_{-}\right|^{2}+\left|\phi_{+}\right|^{2}\right)+\lambda^{2}\left|\phi_{+} \phi_{-}\right|^{2} \\
& +\frac{g^{2}}{2}\left(\left|\phi_{+}\right|^{2}-\left|\phi_{-}\right|^{2}+\xi\right)^{2}, \tag{4.174}
\end{align*}
$$

where the part proportional to $g$ ( $g$ being the gauge coupling) represents the $D$-part of $V$. In this expression $\xi$ is a Fayet-Iliopoulos term. There is a unique supersymmetric vacuum at $X=\phi_{+}=0$ and $\left|\phi_{-}\right|=\sqrt{\xi}$ and a flat direction along the $X$ direction with $\phi_{+}=\phi_{-}=0$ where the potential $V=g^{2} \xi^{2} / 2$ can drive inflation. Since supersymmetry is broken along the flat direction, this produces one loop corrections and we obtain
$V=\frac{g^{2}}{2} \xi^{2}\left[1+\frac{g^{2}}{16 \pi^{2}} \ln \left(\frac{\lambda^{2}|X|^{2}}{\mu^{2}}\right)\right]$,
where $\mu$ is a renormalization scale. We see that this potential has exactly the form of an LI potential where the scale $M$ is related to the Fayet-Iliopoulos term $\xi$ and where $\alpha$ is in fact the square of the gauge coupling. In particular, this implies that $\alpha>0$ in this context. One can also reproduce the above calculation in supergravity (with minimal Kähler potentials) and show that the $D$-part of the theory leads to the same potential which is free of the $\eta$ problem.

After these initial works on $D$-term inflation, many other papers addressing different issues were published. Observational constraints on this type of scenarios were discussed in Refs. [352,353]. Ref. [354] has discussed how to produce $D$-term inflation and to stabilize the moduli at the same time. Then, in Refs. [355-357], it was shown that the stringy implementation of $D$-term inflation is problematic. We have seen that the scale $M$ is essentially controlled by the value of the Fayet-Iliopoulos term $\xi$. Therefore, the CMB normalization allows us to calculate the value of $\xi$. Anticipating the calculation at the end of this section, if one uses the equation after Eq. (4.187) with $M^{4}=g^{2} \xi^{2} / 2$ and $\alpha=g^{2} /\left(8 \pi^{2}\right)$ [from Eq. (4.175)], then one arrives at

$$
\begin{align*}
\xi & \simeq\left[\left(\frac{90}{\Delta N_{*}}\right)^{1 / 4}\left(\frac{Q_{\mathrm{rms}-\mathrm{PS}}}{T}\right)^{1 / 2} M_{\mathrm{Pl}}\right]^{2} \\
& \simeq\left(6.9 \times 10^{15} \mathrm{GeV}\right)^{2} \tag{4.176}
\end{align*}
$$

where we have taken the fiducial value $\Delta N_{*} \simeq 50$. As noticed in Refs. [355-357], in string theory, one typically obtains $\xi=$ $(\operatorname{TrQ}) M_{\mathrm{s}}^{2} /\left(192 \pi^{2}\right)$ where $M_{\mathrm{s}}$ is the string scale and $\operatorname{TrQ} \simeq 100$ sums the $U(1)$ charges of all massless states. This leads to $\xi \simeq$ (few $\times 10^{17} \mathrm{GeV}$ ) ${ }^{2}$ and, therefore, does not match the CMB nor-

$$
\begin{equation*}
x^{2}=\frac{4\left(N_{\text {end }}-N\right)-x_{\text {end }}^{2}\left[1-\frac{2}{\alpha}-\ln \left(x_{\text {end }}^{2}\right)\right]}{\mathrm{W}_{-1}\left\{4\left(N_{\text {end }}-N\right) e^{-(1-2 / \alpha)}-\left[1-\frac{2}{\alpha}-\ln \left(x_{\text {end }}^{2}\right)\right] \exp \left[-1+\frac{2}{\alpha}+\ln \left(x_{\text {end }}^{2}\right)\right]\right\}} \tag{4.184}
\end{equation*}
$$

Box II.
malization (4.176). Then, Refs. [358,359] studied more complicated models in the supersymmetric context in order to fix the problem we have just discussed. Other scenarios were also investigated in Refs. [360-363]. D-term inflation in the context of string theory and brane inflation was also discussed in Refs. [244,364-369]. The same topic was also addressed in Refs. [370,371] but in the context where the Friedmann equations receives quadratic corrections. Finally, Ref. [372] studied LI potentials in the case of Wess-Zumino models. Let us emphasize again that, in all these models, the constant $\alpha$ is positive and given in terms of the square of a gauge coupling.

The LI potential was also derived in a different framework in Ref. [373]. This article uses the O'Raifeartaigh-Witten model that will be studied in more detail in Section 4.23. Therefore, we do not give the details here and only quote results that will be reviewed in that section. In particular, we will see in Eq. (4.338) that the only difference is that the parameter $\alpha$ is now given in terms of three coupling constants and has a rather involved form which allows for negative $\alpha$ values. For this reason we will not fix the sign of $\alpha$ in the following.

### 4.12.2. Slow-roll analysis

Let us now turn to the slow-roll study of loop inflation. We recall that the potential takes the following form
$V(\phi)=M^{4}\left[1+\alpha \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right)\right]$,
where $\alpha$ is a dimensionless parameter, that can a priori be either positive or negative (see the above discussion). Let us define the quantity $x \equiv \phi / M_{\mathrm{PI}}$. The potential Eq. (4.177), as well as its logarithm, is displayed in Fig. 23. If $\alpha>0$, it is an increasing function of the field vev, and vanishes at
$x_{V=0}=e^{-1 / \alpha}$.
Hence inflation proceeds from the right to the left at $x>x_{V=0}$ in that case. If $\alpha<0$ however, the potential is a decreasing function of the field, which vanishes at $x_{V=0}$, still given by Eq. (4.178), hence inflation proceeds from the left to the right at $x<x_{V=0}$.

The three first Hubble flow functions in the slow-roll approximation are given by
$\epsilon_{1}=\frac{\alpha^{2}}{2} \frac{1}{x^{2}}(1+\alpha \ln x)^{-2}, \quad \epsilon_{2}=2 \alpha \frac{1}{x^{2}} \frac{1+\alpha+\alpha \ln x}{(1+\alpha \ln x)^{2}}$,
and

$$
\begin{align*}
\epsilon_{3}= & 2 \alpha \frac{1}{x^{2}}(1+\alpha \ln x)^{-2}(1+\alpha+\alpha \ln x)^{-1} \\
& \times\left[1+\frac{3 \alpha}{2}+\alpha^{2}+\left(2 \alpha+\frac{3}{2} \alpha^{2}\right) \ln x+\alpha^{2} \ln ^{2} x\right] . \tag{4.180}
\end{align*}
$$

If $\alpha>0$, the first slow-roll parameter is a decreasing function of the field vev, which diverges at $x_{V=0}$ and vanishes when $x \rightarrow \infty$. Therefore inflation stops by slow-roll violation, at the point $x_{\text {end }}$ satisfying $\epsilon_{1}=1$ and given by
$x_{\text {end }}=\frac{1}{\sqrt{2}}\left[\mathrm{~W}_{0}\left(\frac{e^{1 / \alpha}}{\sqrt{2}}\right)\right]^{-1}$,
where $\mathrm{W}_{0}$ is the 0 -branch of the Lambert function. One can check that since $\mathrm{W}_{0}(y)<y$ for any $y$, one always has $x_{\text {end }}>x_{V=0}$, as required. When $\alpha \ll 1$, one has $x_{\text {end }} \simeq \alpha / \sqrt{2}$. If $\alpha<0$ on the other hand, the first slow-roll parameter diverges at $x=0$, decreases with $x$, reaches a minimum at $x_{\epsilon_{2}=0}=\exp (-1-1 / \alpha)$, then increases with $x$ and diverges at $x_{V=0}$. The minimum value of $\epsilon_{1}$ equals $\epsilon_{1}\left(x_{\epsilon_{2}=0}\right)=\exp (2+2 / \alpha) / 2$ which is smaller than unity only if $\alpha>2 /(\ln 2-2) \simeq-1.53$. Otherwise $\epsilon_{1}(x)>1$ all over the domain and inflation cannot take place. If $\alpha>2 /(\ln 2-2)$, the inflationary domain lies between $x_{\epsilon_{1}=1}^{-}$and $x_{\text {end }}=x_{\epsilon_{1}=1}^{+}$, with
$x_{\epsilon_{1}=1}^{-}=-\frac{1}{\sqrt{2}}\left[\mathrm{~W}_{-1}\left(\frac{-e^{1 / \alpha}}{\sqrt{2}}\right)\right]^{-1}$,
$x_{\text {end }}=x_{\epsilon_{1}=1}^{+}=-\frac{1}{\sqrt{2}}\left[\mathrm{~W}_{0}\left(\frac{-e^{1 / \alpha}}{\sqrt{2}}\right)\right]^{-1}$,
and where $\mathrm{W}_{-1}$ is the -1 -branch of the Lambert function. When $|\alpha| \ll 1$, one has $x_{\text {end }} \simeq e^{-1 / \alpha}-1 / \sqrt{2} \gg 1$. Let us notice that the end of inflation occurs in the region $\phi \gg M_{\mathrm{Pl}}$, where Eq. (4.177) may not be well defined. Therefore, depending on the underlying theoretical setting, the end of inflation by slow-roll violation may not be meaningful.

Let us now turn to the slow-roll trajectory. It can be integrated, giving rise to

$$
\begin{align*}
N_{\mathrm{end}}-N= & \frac{x^{2}}{2}\left(\ln x+\frac{1}{\alpha}-\frac{1}{2}\right) \\
& -\frac{x_{\mathrm{end}}^{2}}{2}\left(\ln x_{\mathrm{end}}+\frac{1}{\alpha}-\frac{1}{2}\right) . \tag{4.183}
\end{align*}
$$

When $|\alpha| \ll 1$, it approximately takes the form $2 \alpha\left(N_{\text {end }}-N\right)=$ $x^{2}-x_{\text {end }}^{2}$. The trajectory Eq. (4.183) can be inverted making use of the Lambert function, and one obtains see the equation in Box II where the 0 branch of the Lambert function must be chosen if $\alpha>0$, while the -1 branch must be chosen if $\alpha<0$. The Lambert function is displayed in the left panel of Fig. 24, together with the regions in which inflation proceeds. Let us now comment and check that this expression is valid. Firstly, if $N=N_{\text {end }}$, the Lambert function is of the form $\mathrm{W}\left(-z_{\text {end }} e^{-z_{\text {end }}}\right)=-z_{\text {end }}$, where $z \equiv(1-$ $2 / \alpha)-\ln \left(x^{2}\right)$, and this automatically cancels the numerator such that one has indeed $x=x_{\text {end }}$. Secondly, if $\alpha>0$, the condition $x_{\text {end }}>x_{V=0}$ implies that $z_{\text {end }}<1$, and the Lambert function at $N_{\text {end }}$ is equal to $-z_{\text {end }}>-1$. Therefore, at the end of inflation, one should use the zero branch of the Lambert function. Finally, as inflation is under way, the argument of the Lambert function is decreasing which implies that the whole inflationary stage takes place on the zero branch. On the other hand, if $\alpha<0$ using similar arguments, the whole inflationary stage can be shown to take place on the -1 branch.

In this later case $(\alpha<0)$, it is also interesting to notice that the total number of $e$-folds is bounded, since inflation can only proceed between $x_{\epsilon_{1}=1}^{-}$and $x_{\epsilon}^{+}=1$. The corresponding maximal number of $e$-folds $\Delta N_{\text {max }}^{\epsilon_{1}=1}$ is displayed, as a function of $\alpha$, in the right panel of Fig. 24. One can see that when $\alpha \lesssim-0.35$, not a sufficient number of $e$-folds can be realized. For such values of $\alpha$, one already has $x_{\text {end }}>10$. Since inflation is supposed to take place at subPlanckian vevs, it means that this regime of inflation is a priori


Fig. 59. Top left panel: Pseudo Natural Inflation (PSNI) potential, for $\alpha=0.1$, as a function of $\phi / f$. Top right panel: logarithm of the potential for the same value of $\alpha$. Bottom left panel: slow-roll parameter $\epsilon_{1}$, rescaled by the quantity $M_{\mathrm{Pl}}^{2} / f^{2}$ such that it acquires a universal form, for the same value of $\alpha$. Bottom right panel: slow-roll parameter $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line), rescaled by the quantity $M_{\mathrm{Pl}}^{2} / f^{2}$, still for the same value of $\alpha$.


Fig. 60. Top left panel: Non Canonical Kähler Inflation (NCKI) potential for $\alpha=0.1$ and $\beta= \pm 1$. The solid blue line represents the case $\beta=-1$ while the solid pink line represents the case $\beta=1$. Top right panel: logarithm of the potential for the same values of $\alpha$ and $\beta$. Bottom left panel: slow-roll parameter $\epsilon_{1}$, for a potential with the same values of $\alpha$ and $\beta$ and the same color code. The shaded area indicates the region where inflation is not possible. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid blue and pink lines) and $\epsilon_{3}$ (dotted blue and pink lines), for a potential with the values of $\alpha$ and $\beta$ already considered in the other panels. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)


Fig. 61. Constant Spectrum Inflation (CSI) for $\alpha=0.1$. Upper panels: the potential and its logarithm along the branch $x<1 / \alpha$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ together with the region in which it is larger than unity and in which inflation cannot occur (shaded). Bottom right panel: slow-roll parameter $\epsilon_{2}=\epsilon_{3}$ along the same branch $x<1 / \alpha$.
forbidden. If one allows slightly super-Planckian field vevs, up to $x \simeq 100$ or $x \simeq 1000$, this implies that $\alpha<-0.1$. Therefore even in this case, $\alpha$ must lie in the rather narrow range $-0.3<\alpha<-0.1$.

Making use of the approximated trajectories and expressions for $x_{\text {end }}$, some analytic predictions can be derived in the case $\alpha>0$. The observable field value $x_{*}$, and its associated number of $e$-folds $\Delta N_{*}=N_{\text {end }}-N_{*}$ at which the pivot mode crossed the Hubble radius during inflation are obtained from the above equations together with Eq. (2.47). In the limit $\alpha \ll 1$, one obtains the approximate expressions
$\epsilon_{1 *} \simeq \frac{\alpha}{4 \Delta N_{*}}, \quad \epsilon_{2 *} \simeq \epsilon_{3 *} \simeq \frac{1}{\Delta N_{*}}$,
hence
$r \simeq \frac{\alpha}{64 \Delta N_{*}}, \quad n_{\mathrm{S}}-1 \simeq-\frac{1}{\Delta N_{*}}, \quad \alpha_{\mathrm{S}} \simeq \frac{1}{\Delta N_{*}^{2}}$.
Finally, the parameter $M$ can be determined from the amplitude of the CMB anisotropies, and one gets
$\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 \pi^{2} \frac{\alpha^{2}}{x_{*}^{2}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}\left(1+\alpha \ln x_{*}\right)^{-3}$.
In the small $|\alpha|$ limit, one obtains $M^{4} / M_{\mathrm{Pl}}^{4} \simeq 360 \pi^{2} \alpha / \Delta N_{*} Q_{\mathrm{rms}-\mathrm{PS}}^{2} /$ $T^{2}$ for $\alpha>0$, and $M^{4} / M_{\mathrm{Pl}}^{4} \simeq 720 \pi^{2} \alpha^{2} e^{2 / \alpha} Q_{\text {rms-PS }}^{2} / T^{2}$ for negative values of $\alpha$.

The reheating consistent slow-roll predictions of the loop inflation models are displayed in Fig. 96 for $\alpha>0$, and in Fig. 97 for $\alpha<0$. For $\alpha>0$ and $\alpha \ll 1$, the approximations in Eq. (4.185) give a good description of what is numerically obtained, namely a deviation from scale invariance which almost does not depend on $\alpha$, and an amount of gravitational waves which grows linearly with $\alpha$. For $\alpha<0$, the predictions blow out of the observational
one- and two-sigma contours when $\alpha$ approaches the upper bound derived above, as expected. Correspondingly, the parameter $\alpha$ does not seem to be much constrained when it is positive, whereas close-to-zero values are favored when it is negative.

### 4.13. $\left(R+R^{2 p}\right)$ inflation (RpI)

This model is the Einstein frame description of a scalar-tensor theory equivalent to $f(R)=R+\epsilon R^{2 p} / \mu^{4 p-2}$, where $\mu$ is a mass scale, $\epsilon= \pm 1$, and $p>1 / 2$ (otherwise the expansion is meaningless). It generalizes the original Starobinsky model [374] obtained for $p=1$. Such theories are quite generic and appear as limiting cases of more general modified gravity theories [375-379] (see Ref. [380] for a review).

Following Refs. [377,380], one can introduce the scalar degree of freedom $\phi$ defined by
$\frac{\phi}{M_{\mathrm{Pl}}}=\sqrt{\frac{3}{2}} \ln (|F(R)|)$,
where $F(R) \equiv \partial f / \partial R$. The quantity $F \equiv \Omega^{2}$ is also the square of the conformal factor inducing the transformation from the Jordan frame to the Einstein frame. In the Einstein frame, the field $\phi$ evolves in a potential given by
$V(\phi)=\frac{M_{\mathrm{Pl}}^{2}}{2} \frac{|F|}{F} \frac{R F-f}{F^{2}}$.
In the present case, one has
$F(R)=1+2 \epsilon p\left(\frac{R}{\mu^{2}}\right)^{2 p-1}$,
which, for small departures with respect to the Einstein-Hilbert action $R \ll \mu^{2}$, implies that $F(R)>0$ as needed. Let us notice


Fig. 62. Orientifold Inflation (OI) for $\alpha=0.1$. Upper panels: the potential and its logarithm. Bottom left panel: slow-roll parameter $\epsilon_{1}$, rescaled by the factor $\phi_{0}^{2} / M_{\mathrm{Pl}}^{2}$. The shaded area indicates where inflation cannot occur (for $\phi_{0}=M_{\mathrm{PI}}$ ). Bottom right panel: rescaled slow-roll parameter $\epsilon_{2}$.
that in the opposite situation, accelerated (and super-accelerated) solutions have been shown to exist [380]. Defining the quantity $y$ by
$y \equiv \sqrt{\frac{2}{3}} \frac{\phi}{M_{\mathrm{Pl}}}$,
and inserting Eq. (4.190) into Eq. (4.189) one obtains the Einstein frame potential
$V=M^{4} e^{-2 y}\left|e^{y}-1\right|^{2 p /(2 p-1)}$.
The normalization constant $M^{4}$ is related to the modified gravity scale $\mu$ through the following expression
$M^{4}=\frac{2 p-1}{4 p} \frac{M_{\mathrm{P}}^{2} \mu^{2}}{(2 p)^{1 /(2 p-1)}}$.
For $F(R)>0$, Eq. (4.188) implies that for $\epsilon=1$, the model is defined in the domain $y>0$, whereas for $\epsilon=-1$ one should consider the domain $y<0$ only. Such a potential has also been studied in Ref. [381] for $p=1$, in Refs. [377,382] for $p=4$ and in Ref. [383] for $p=2$. Let us notice that the case $p=1$ corresponds to the Higgs inflation potential studied in Section 3.1. The case $p=1 / 2$ is singular since one recovers $f(R) \propto R$. Taking the limit $p \rightarrow \infty$, the potential asymptotes $V \rightarrow M^{4} e^{-2 y}\left|e^{y}-1\right|$ and varying $p$ allows us to explore different potential shapes.

Let us first consider the case $y>0(\epsilon=1)$. If $p>1$, the potential admits a maximum at
$y_{\text {max }}=\ln \left(\frac{2 p-1}{p-1}\right)$,
such that inflation can proceed either for $0<y<y_{\text {max }}$ or $y>y_{\text {max }}$. We respectively call these regimes RpI1 and RpI2. If $p<1$, the potential is an increasing function of $y$, hence inflation proceeds from the right to the left. We call this regime RpI3. The case $p=1$
is singular and again, it corresponds to the Higgs inflation potential studied in Section 3.1.

The Hubble flow functions in the slow-roll approximation read
$\epsilon_{1}=\frac{4}{3} \frac{\left[1+(p-1) e^{y}-2 p\right]^{2}}{(2 p-1)^{2}\left(e^{y}-1\right)^{2}}$,
$\epsilon_{2}=\frac{8}{3} \frac{p e^{y}}{(2 p-1)\left(e^{y}-1\right)^{2}}$,
and
$\epsilon_{3}=-\frac{4}{3} \frac{\left(e^{y}+1\right)\left[1+(p-1) e^{y}-2 p\right]}{(2 p-1)\left(e^{y}-1\right)^{2}}$.
The potential and the Hubble flow functions for $y>0$ have been represented in Fig. 25. As one can check on these figures, inflation never stops in the Rpl2 regime and one needs to complement the model with a mechanism that can end inflation, as for instance with an extra-field and a tachyonic instability. This adds one additional parameter $y_{\text {end }}$ to the model. When this parameter is large, all the three Hubble flow functions admit asymptotically constant values:
$\lim _{y \rightarrow \infty} \epsilon_{1}=\frac{4}{3}\left(\frac{p-1}{2 p-1}\right)^{2}, \quad \lim _{y \rightarrow \infty} \epsilon_{2}=0$,
$\lim _{y \rightarrow \infty} \epsilon_{3}=-\frac{4}{3} \frac{p-1}{2 p-1}$.
If $p$ is an integer, except for the special case $p=1$ (see Section 3.1), these values are always smaller that unity, but not particularly small. As such, all these models predict large deviation from scale invariance. Indeed, the spectral index at first order is given by
$n_{S}-1 \simeq-\frac{8}{3}\left(\frac{p-1}{2 p-1}\right)^{2}$,


Fig. 63. Top left panel: Constant $n_{S} C$ inflaton potential for $\alpha=0.1$. Inflation proceeds from the left to the right as indicated by the arrow. Top right panel: logarithm of the potential for the same value of $\alpha$. Bottom left panel: the first slow-roll parameter $\epsilon_{1}$ for $\alpha=0.1$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ and $\epsilon_{3}$, still for $\alpha=0.1$.
which, for $p \geq 2$, remains always smaller than $-8 / 27 \simeq$ -0.3 . This is strongly disfavored by current CMB measurements. Therefore, only the models such that $p$ is close enough to 1 are to be considered (i.e. non integer values of $p$ ).

If inflation proceeds in the RpI1 regime, then inflation stops naturally when $\epsilon_{1}=1$, i.e. at the field value
$y_{\text {end }}=\ln \left[(2 p-1) \frac{1+2 p(\sqrt{3}+1)}{8 p^{2}-4 p-1}\right]$.
However, the second Hubble flow function can only take relatively large value. From Eq. (4.195), since $y<y_{\text {max }}$, one gets
$\epsilon_{2}>\epsilon_{2}\left(y_{\max }\right)=\frac{8}{3} \frac{p-1}{p}$.
For $p \geq 2$, we are in a situation where $\epsilon_{2}>4 / 3$ and again, the models are ruled out by a simple slow roll analysis. Therefore, as already noticed before, $p$ must take (non integer) close enough to 1 values for the models to be viable.

Finally, in the RpI3 regime, inflation stops naturally when $\epsilon_{1}=$ 1, with $y_{\text {end }}$ still given by Eq. (4.199). This expression is defined only if $p>(1+\sqrt{3}) / 2 \simeq 0.68$ but the first slow roll parameter continuously decreases with $y$, and its asymptotic value is again given by Eq. (4.197). Therefore, this regime is viable only when $p$ is close enough to unity.

Let us now turn to the slow-roll trajectory. It is given by
$N-N_{\text {end }}=\frac{3}{4}\left\{\frac{p}{p-1} \ln \left[\frac{(p-1) e^{y}+1-2 p}{(p-1) e^{y_{\mathrm{end}}}+1-2 p}\right]+y-y_{\mathrm{end}}\right\}$.

This expression is not properly defined for $p=1$ but this case has already been considered in the section on the Higgs inflation model. When $p>1$, if $y=y_{\text {max }}$, the argument of the logarithm vanishes and the total number of $e$-folds diverges. As a result,
provided inflation starts close enough to the top of the potential, it is always possible to find a long enough inflationary period. For $p<1$, the number of $e$-folds diverges when $y \rightarrow \infty$. The slowroll trajectory cannot be analytically inverted, but using the same reheating model as in Section 3.1, one can solve for the field value $y_{*}$ at which the pivot mode crossed out the Hubble radius. The corresponding number of $e$-fold $\Delta N_{*}=N_{\text {end }}-N_{*}$ being given by Eq. (4.201).

Concerning the case $\epsilon=-1$, i.e. the domain $y<0$, all of the previous formula still apply but the potential is now a monotonic decreasing function of the field $v e v$ which is too steep to support inflation. In particular, over the whole negative domain, Eq. (4.195) implies that $\epsilon_{1}(y<0)>\epsilon_{1}(y \rightarrow-\infty)=4 / 3$, independently on whether $p>1$ or $p<1$.

Finally, the constant $M$ can be determined from the amplitude of the CMB anisotropies. It follows that
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=1920 \pi^{2} \frac{\left[1+(p-1) e^{y_{*}}-2 p\right]^{2} e^{2 y_{*}}}{(2 p-1)^{2}\left(e^{y_{*}}-1\right)^{\frac{6 p-2}{2 p-1}}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
The reheating consistent slow-roll predictions of the RpI models are displayed in Fig. 98 for the RpI1 regime, in Fig. 99 for the RpI2 regime, and in Fig. 100 for the RpI3 regime. In the RpI1 regime, the Higgs inflation model predictions (see Fig. 81) are recovered when $p \rightarrow 1$, and one can see that $p<1.02$ is a necessary condition for the spectral index not to be too red. For RpI2 the limit $p \rightarrow 1$ is such that one does not reproduce the Higgs inflation results and for $y_{\text {end }} \rightarrow \infty$ the predictions lie on the line $\epsilon_{2 *}=0$. Moreover, one can see that when $p>1.1$, the models predict too much gravity waves to be compatible with the CMB data. Finally for the RpI3 regimes, the Higgs inflation model predictions (see Fig. 81) are recovered when $p \rightarrow 1$, and they remain compatible with the data within the two-sigma contours provided $p>0.99$.


Fig. 64. Supergravity Brane Inflation (SBI) for $\beta=0.7$ and $\alpha=0.13>\alpha_{\min }(\beta), \alpha=\alpha_{\min }(\beta)$, and $\alpha=0.09<\alpha_{\min }(\beta)$ (where $\alpha_{\min }$ is defined in Eq. (5.270)). Upper panels: the potential and its logarithm. Inflation proceeds in the place and direction labeled by the arrow. Bottom left panel: slow-roll parameter $\epsilon_{1}$. The shaded area indicates where inflation stops. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line), only displayed in the branch of the potential where inflation proceeds.

### 4.14. Double-Well inflation (DWI)

In this section, we study the famous "Mexican hat" potential given by
$V(\phi)=M^{4}\left[\left(\frac{\phi}{\phi_{0}}\right)^{2}-1\right]^{2}$.
Except for the mass $M$ determined by the CMB normalization, it depends on one parameter, the vev $\phi_{0}$. Historically, this potential was first introduced by Goldstone in Ref. [384] as a toy model for dynamical symmetry breaking. In cosmology, it is of course utilized to investigate the formation and the microscopic structure of topological defects [385-391]. In the context of inflation, it was first used to construct scenarios of topological inflation [392, 393]. In this case, it is made use of the fact that the discrete $\mathbb{Z}_{2}$ symmetry, $\phi \rightarrow-\phi$, makes the state $\phi=0$ unstable. Therefore, the Universe will split into two different regions separated by a domain wall. One can then show that inflation takes place within this topological defect. More precisely, the potential is usually written as $V=\lambda / 4\left(\phi^{2}-\eta^{2}\right)^{2}$ where $\eta$ represents the position of the minima of the potential. Then, Refs. [392,393] show that topological inflation occurs if $\eta>M_{\mathrm{Pl}}$. On the other hand, if one writes Eq. (4.203) as $V=M^{4} / \phi_{0}^{4}\left(\phi^{2}-\phi_{\rho}^{2}\right)^{2}$, one sees that one can identify $\eta$ with $\phi_{0}$. And we will precisely show that agreement with the CMB observations requires $\phi_{0}>M_{\mathrm{Pl}}$. The potential (4.203) was also used in Refs. [394,395] in the context of open inflation. In a rather different theoretical framework, Eq. (4.203) was studied in Refs. [396,397] where it was derived in $N=1$ supergravity coupled to matter. It is also interesting to notice that it was obtained using various stringy constructions as early as the 80's, see Refs. [398,399]. More recently, this potential was found to be relevant in a large number of different physical situations [343,

400-410]. Let us also mention that this model is sometimes viewed as a realistic version of Small Field Inflation (SFI) with $p=2$ (see Section 5.1), the extra quartic term preventing the potential from becoming negative. However, as will be shown in the following, these two classes of models should actually be described separately since their predictions differ in the relevant range of parameters.

The parameter $\phi_{0}$ sets the typical vev at which inflation proceeds and depends on the symmetry breaking scale one considers. In principle, it could vary over a wide range of values, from $\phi_{0} \sim 10^{15} \mathrm{GeV}$ for GUT symmetry breaking schemes to super-Planckian vev in a stringy or supergravity context. As will be shown in the following, it is in fact constrained to be large (super-Planckian) in order for the predictions of the model to be compatible with the CMB data. The DWI potential is displayed in Fig. 26 together with its logarithm. One has represented the region $\phi>0$ only because the potential is symmetric under $\phi \rightarrow-\phi$. We see that it decreases for $\phi<\phi_{0}$, vanishes at $\phi_{0}$ and then increases for $\phi>\phi_{0}$. As was already mentioned before, this potential is used to describe dynamical symmetry breaking and, as a consequence, inflation should proceed from the left to the right at $\phi<\phi_{0}$, in the direction specified by the arrow in Fig. 26.

Let us now calculate the slow-roll parameters. If one defines $x \equiv \phi / \phi_{0}$ they are given by
$\epsilon_{1}=\left(\frac{M_{\mathrm{PI}}}{\phi_{0}}\right)^{2} \frac{8 x^{2}}{\left(x^{2}-1\right)^{2}}, \quad \epsilon_{2}=\left(\frac{M_{\mathrm{PI}}}{\phi_{0}}\right)^{2} \frac{8\left(1+x^{2}\right)}{\left(x^{2}-1\right)^{2}}$,
$\epsilon_{3}=\left(\frac{M_{\mathrm{PI}}}{\phi_{0}}\right)^{2} \frac{8\left(x^{4}+3 x^{2}\right)}{\left(x^{2}-1\right)^{2}\left(x^{2}+1\right)}$.
The behavior of these parameters is represented in Fig. 26. The first slow-roll parameter $\epsilon_{1}$ is an increasing function of $\phi$ in the range $x \in[0,1]$. It vanishes at $x=0$ and blows up at $x=1$. Then, for $x>1$, it becomes a decreasing function going to zero when $x$


Fig. 65. Spontaneous Symmetry Breaking Inflation (SSBI) potential and the corresponding Hubble flow parameter $\epsilon_{1}$ for the two cases $\alpha>0, \beta>0$ (SSBI1), and $\alpha<0$, $\beta<0$ (SSBI2). The values of the parameters are chosen to be $\alpha, \beta= \pm 1$. The four other possibilities, namely SSBI3, SSBI4, SSBI5, SSBI6 are displayed in Fig. 66 .
goes to infinity. We see in Fig. 26 that inflation stops by violation of the slow-roll conditions. The slow roll parameters $\epsilon_{2}$ and $\epsilon_{3}$ have similar behaviors, except that $\epsilon_{2}$ does not vanish when $x=0$ but is equal to $\epsilon_{2}(x=0)=8\left(M_{\mathrm{Pl}} / \phi_{0}\right)^{2}$. Therefore, in order for slow-roll to be valid, this last value should be less than one, which amounts to
$\frac{\phi_{0}}{M_{\mathrm{Pl}}}>2 \sqrt{2}$.
This constraint on the parameter $\phi_{0}$ shows that the symmetry breaking scale needs to be super-Planckian. If this last condition is verified, then $\epsilon_{2}$ becomes greater than one during inflation at $\phi_{\epsilon_{2}=1}$ defined by
$x_{\epsilon_{2}=1}=\sqrt{1+4\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2}\left[1-\sqrt{1+\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2}}\right]}$.
This happens before the end of inflation $\left(\epsilon_{1}=1\right)$ which occurs at the following value of the field
$x_{\mathrm{end}}=\sqrt{2+\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2}}-\sqrt{2}$.
Let us now turn to the slow-roll trajectory. It can be integrated exactly and yields the following formula
$N_{\text {end }}-N=\frac{1}{4}\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2}\left[\ln \left(\frac{x_{\text {end }}}{x}\right)-\frac{1}{2}\left(x_{\text {end }}^{2}-x^{2}\right)\right]$,
where $N_{\text {end }}$ is the number of $e$-folds at the end of inflation. Using the 0 -branch of the Lambert function $\mathrm{W}_{0}$, this trajectory can be
inverted. One obtains
$x=\sqrt{-\mathrm{W}_{0}\left[-x_{\text {end }}^{2} e^{-x_{\text {end }}^{2}} e^{8\left(\frac{M_{p}}{\phi_{0}}\right)^{2}\left(N-N_{\text {end }}\right)}\right]}$.
The fact that the 0 -branch of the Lambert function should be chosen comes from the requirement that $x<1$. The corresponding "trajectory" along the Lambert curve is displayed in Fig. 27, the arrow indicating in which direction inflation proceeds. This trajectory is remarkably similar to the one of SFI with $p=$ 2, see Section 5.1 and Eq. (5.6), the only difference being that the factor 8 in front of $N-N_{\text {end }}$ is just 4 in the case of SFI. Therefore not only these two potentials coincide at small fields, but they also give rise to the same kind of slow-roll trajectory. This is why these two models are sometimes identified, DWI being considered as a realistic realization of SFI. However, as shown below, the observations favors super-Planckian values of $\phi_{0}$ and, in this limit, the two models are not equivalent (of course, this also has something to do with the debate about whether having super-Planckian vev is meaningful or not). In fact, in the regime $\phi_{0} / M_{\mathrm{Pl}} \gg 1$, one can write

$$
\begin{align*}
x_{*} \simeq & 1-\sqrt{2} \frac{M_{\mathrm{Pl}}}{\phi_{0}} \sqrt{1+2 \Delta N_{*}} \\
& +\frac{1}{3}\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2}\left(1+2 \Delta N_{*}+\frac{2}{\sqrt{1+2 \Delta N_{*}}}\right)+\cdots . \tag{4.210}
\end{align*}
$$

From this expression it is clear that, for super-Planckian values of $\phi_{0}, \phi_{*}$ is close to the minimum of the potential where the quartic term plays an important role and, consequently, where the SFI potential is not a good approximation. A calculation of the Hubble flow parameters at Hubble crossing confirms this conclusion. They


Fig. 66. Spontaneous Symmetry Breaking Inflation (SSBI) potential and the corresponding Hubble flow parameter $\epsilon_{1}$ for the two cases $\alpha>0, \beta<0$ (corresponding to SSBI3 to SSBI4) and $\alpha<0, \beta>0$ (corresponding to SSBI5 and to SSBI6). In each of these cases, the direction in which inflation proceeds is indicated by the arrow.
are given by
$\epsilon_{1 *} \simeq \frac{1}{1+2 \Delta N_{*}}, \quad \epsilon_{2 *} \simeq \frac{2}{1+2 \Delta N_{*}}$,
$\epsilon_{3 *} \simeq \frac{2}{1+2 \Delta N_{*}}$.
This allows us to establish the corresponding expressions of the tensor to scalar ratio, spectral index and running. One obtains
$r \simeq \frac{16}{1+2 \Delta N_{*}}, \quad n_{\mathrm{S}}-1 \simeq-\frac{4}{1+2 \Delta N_{*}}$,
$\alpha_{\mathrm{S}} \simeq-\frac{8}{1+2 \Delta N_{*}}$.
These expressions should be compared with Eq. (5.17). We see that the first Hubble flow parameter for SFI and DWI differ by a factor close to 4 and that the $\epsilon_{2}$ roughly differ by a factor of 2 . As a consequence, as can be checked in Fig. 101, the DWI predictions are such that $\epsilon_{2 *}=2 \epsilon_{1 *}$ [or equivalently, $r=4\left(1-n_{\mathrm{S}}\right)$ ], whereas, as can be checked in Fig. 112, we have $\epsilon_{2 *}=4 \epsilon_{1 *}$ for SFI [or equivalently, $r=8 / 3\left(1-n_{S}\right)$ ]. This explains why the two models can in fact lead to quite different predictions and why DWI cannot be simply viewed as a mere realistic continuation of SFI.

Finally, it is also interesting to constrain the energy scale $M$. For this purpose, we use the CMB normalization which gives
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=11520 \pi^{2}\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} \frac{x_{*}^{2}}{\left(x_{*}^{2}-1\right)^{4}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
Then, using the approximated trajectory $x_{*} \simeq 1-\sqrt{2+4 \Delta N_{*}}$ $M_{\mathrm{Pl}} / \phi_{0}$ in the above formula, one obtains the following expression
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}} \simeq 1440 \pi^{2}\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2} \frac{1}{\left(1+2 \Delta N_{*}\right)^{2}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.

Then, requiring that $M<M_{\mathrm{Pl}}$ leads to the following upper bound on the value of $\phi_{0}, \phi_{0} / M_{\mathrm{Pl}} \lesssim 1.5 \times 10^{5}$. Combined with the lower limit (4.205), we see that the possible range of variation of $\phi_{0}$ is quite large.

The reheating consistent slow-roll predictions for the DWI models are displayed in Fig. 101. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been chosen to be 0 since the potential is quadratic close to its minimum $V(\phi) \simeq 4 M^{4} / \phi_{0}^{2}\left(\phi-\phi_{0}\right)^{2}$. As claimed before, one can check that only super-Planckian values of the symmetry breaking scale $\phi_{0}$ are compatible with the data. Actually, this is also true for the SFI models, see Section 5.1 and Fig. 112. As already mentioned before, in this regime, the two models differ while, as expected, they are very similar for subPlanckian values of the field vev.

### 4.15. Mutated hilltop inflation (MHI)

This model belongs to the class of hilltop models [411,412]. In this type of scenarios, inflation is supposed to occur at the top of the potential. In particular, it was shown in Refs. [411,412] that, by adding the contributions coming from higher order operators, $F$ or $D$ term inflation can be turned into hilltop models. Here, we consider mutated hilltop inflation which was first introduced and discussed in Refs. [413,414]. The potential is phenomenological only and given by
$V=M^{4}\left[1-\operatorname{sech}\left(\frac{\phi}{\mu}\right)\right]$,
with sech $x=1 / \cosh x$. As argued in Refs. [413,414], it can be viewed as small field inflation (hilltop inflation) completed by an infinite number of higher order operators, these operators giving rise to a power series responsible for the appearance of the sech function. From an effective field theory point of view, reasonable values of the parameter $\mu$ seem to be such that $\mu<M_{\mathrm{PI}}$ but in


Fig. 67. Second slow-roll parameter $\epsilon_{2}$ (solid line) and third slow-roll parameter $\epsilon_{3}$ (dotted line), for the six SSBI models studied in this section. The free parameters of the models are chosen to be $\alpha, \beta= \pm 1$.


Fig. 68. The black solid line gives the minimum value of $|\alpha|$, denoted here by $\alpha_{\min }$, as a function of $\beta$ in order for inflation to stop by slow-roll violation for SSBI1 (top left panel), SSBI5 (bottom left panel) and SSBI6 (bottom right panel). For SSBI3 (top right panel), the green dotted line denotes the minimum value of $\alpha$ for inflation to stop by slow-roll violation, and the cyan and red dotted line restrict the values of $\alpha$ for which $\epsilon_{2}^{\text {top }}>1$ (defined only for $\beta<-1 / 64$ ). In the bottom panels, the dotted lines correspond to $\alpha^{2}=4 \beta$, see the discussion in the text. In all the panels, the region above the black solid curve (shaded region) represents the allowed region (i.e. the one where a slow roll regime of inflation stops because $\epsilon_{1}$ reaches one). For SSBI1, when $\beta \gtrsim 0.25$, this is always the case. For SSBI1 and SSBI3, $\alpha_{\min }$ approaches the asymptotic value $\alpha_{\min }=2$ when $|\beta| \ll 1$. For SSBI5 and SSBI6, inflation stops by slow-roll violation when $\alpha<-\left|\alpha_{\min }\right|$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)


Fig. 69. Top left panel: Inverse Monomial Inflation (IMI) potential for $p=2$. Top right panel: logarithm of the potential for the same value of $p$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ for $p=2$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ and $\epsilon_{3}$ for $p=2$. Only one line appears because $\epsilon_{2}=\epsilon_{3}$. On these plots, the shaded region represents the region where the slow-roll approximation breaks down.
other contexts such a restriction may not be necessary. This is why although the model is studied for any value of $\mu$, approximated formula will also be derived in the $\mu \ll M_{\mathrm{PI}}$ approximation.

Defining $x \equiv \phi / \mu$, the three first Hubble flow functions in the slow-roll approximation are given by
$\epsilon_{1}=\frac{M_{\mathrm{Pl}}^{2}}{2 \mu^{2}} \operatorname{coth}^{2}\left(\frac{x}{2}\right) \operatorname{sech}^{2} x$,
$\epsilon_{2}=\frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}}\left[\operatorname{csch}^{2}\left(\frac{x}{2}\right)+2 \operatorname{sech}^{2} x\right]$,
$\epsilon_{3}=\frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \frac{\cosh x \operatorname{coth}^{2}\left(\frac{x}{2}\right)+2 \tanh ^{2} x}{\cosh x+\sinh ^{2} x}$
where $\operatorname{csch} x=1 / \sinh x$. These three quantities are monotonically decreasing functions of the field values and inflation proceeds from large field values towards small field values. Together with the potential, they are represented as a function of $x$ in Fig. 28.

The slow-roll trajectory can be integrated exactly from Eq. (2.11) and reads

$$
\begin{align*}
N-N_{\mathrm{end}}= & \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\left\{2 \ln \left[\frac{\cosh (x / 2)}{\cosh \left(x_{\mathrm{end}} / 2\right)}\right]\right. \\
& \left.-\cosh x+\cosh x_{\mathrm{end}}\right\} . \tag{4.218}
\end{align*}
$$

It can also be inverted analytically to give the field values in terms of the number of $e$-folds using the Lambert function $\mathrm{W}_{-1}$. One obtains

$$
\begin{align*}
x= & \operatorname{arccosh}\left(-1-\mathrm{W}_{-1}\left\{-\left(1+\cosh x_{\mathrm{end}}\right)\right.\right. \\
& \left.\left.\times \exp \left[\frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}}\left(N-N_{\mathrm{end}}\right)-1-\cosh x_{\mathrm{end}}\right]\right\}\right) . \tag{4.219}
\end{align*}
$$

Since $N-N_{\text {end }}<0$ and the function $y e^{-y}$ has a global maximum equals to $1 / e$, inflation proceeds along the -1 branch of the Lambert function as represented in Fig. 29. Note that in the $\mu \ll M_{\mathrm{Pl}}$ limit, this trajectory simply becomes $N-N_{\text {end }} \simeq$ $\mu^{2} /\left(2 M_{\mathrm{PI}}^{2}\right)\left(e^{x_{\text {end }}}-e^{x}\right)$.

For MHI, inflation naturally stops when $\epsilon_{1}=1$, which has an unique solution given by

$$
\begin{align*}
& x_{\mathrm{end}}=\operatorname{arcsech}\left[-\frac{1}{3}+\frac{1}{3}\left(1-6 \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\right)\right. \\
& \quad \times\left(-1+36 \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}+3 \sqrt{6} \frac{\mu}{M_{\mathrm{Pl}}} \sqrt{\left.4 \frac{\mu^{4}}{M_{\mathrm{Pl}}^{4}}+22 \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}-1\right)^{-1 / 3}}\right. \\
& \left.\quad+\frac{1}{3}\left(-1+36 \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}+3 \sqrt{6} \frac{\mu}{M_{\mathrm{Pl}}} \sqrt{4 \frac{\mu^{4}}{M_{\mathrm{Pl}}^{4}}+22 \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}-1}\right)^{1 / 3}\right] \tag{4.220}
\end{align*}
$$

and with $\operatorname{arcsech} x=\operatorname{arccosh}(1 / x)$. One should note that the previous equation is always well defined, regardless of the sign of the square root argument by analytic continuation. Let us notice that from Eq. (4.216) one has
$\epsilon_{2}-\epsilon_{1}=\frac{1}{2} \operatorname{csch}^{2}\left(\frac{x}{2}\right)+\operatorname{sech} x+\frac{5}{2} \operatorname{sech}^{2} x>0$.
Consequently, the slow-roll approximation may become inaccurate before the end of inflation because $\epsilon_{2}>1$ occurs just before $\epsilon_{1}=1$. However, one can check that this happens during a negligible number of $e$-folds and the observable predictions for MHI remain mostly unaffected. Also, in the limit $\mu \ll M_{\mathrm{Pl}}$, Eq. (4.220) gives $x_{\text {end }} \simeq \ln \left(\sqrt{2} M_{\mathrm{Pl}} / \mu\right)$.


Fig. 70. Brane Inflation (BI) for $p=2$. Upper panels: the potential and its logarithm as a function of $\phi / \mu$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ rescaled by $M_{\mathrm{PI}}^{2} / \mu^{2}$. The shaded area indicates the region in which inflation cannot occur for $\mu=M_{\mathrm{Pl}}$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line), rescaled by $M_{\mathrm{Pl}}^{2} / \mu^{2}$.

The value $x_{*}=\phi_{*} / \mu$ at which the pivot mode crossed the Hubble radius during inflation is obtained by solving Eq. (2.47) for a given reheating energy. In terms of $\Delta N_{*}$, and in the limit $\mu \ll M_{\mathrm{P}}$, one has $x_{*} \simeq \ln \left(2 \Delta N_{*} M_{\mathrm{Pl}}^{2} / \mu^{2}\right)$. This enables to give estimates for the slow-roll parameters at Hubble crossing, namely
$\epsilon_{1 *} \simeq \frac{1}{2 \Delta N_{*}^{2}}\left(\frac{\mu}{M_{\mathrm{Pl}}}\right)^{2}, \quad \epsilon_{2 *} \simeq \frac{2}{\Delta N_{*}}, \quad \epsilon_{3 *} \simeq \frac{1}{\Delta N_{*}}$,
hence, at first order in slow-roll
$r \simeq \frac{8}{\Delta N_{*}^{2}}\left(\frac{\mu}{M_{\mathrm{Pl}}}\right)^{2}, \quad n_{\mathrm{S}}-1 \simeq-\frac{2}{\Delta N_{*}}$,
$\alpha_{\mathrm{S}} \simeq-\frac{2}{\Delta N_{*}^{2}}$.
One can see that for $\mu / M_{\mathrm{PI}} \ll 1$, the typical predicted amount of gravitational waves is very small, and the deviation from scale invariance almost does not depend on $\mu$.

Finally, the constant $M$ can be determined from the amplitude of the CMB anisotropies
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=90 \pi^{2} \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \operatorname{csch}^{6}\left(\frac{x_{*}}{2}\right) \sinh x_{*} \tanh x_{*} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
In the $\mu / M_{\mathrm{Pl}} \ll 1$ limit, one obtains
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}} \simeq \frac{720 \pi^{2}}{\Delta N_{*}^{2}} \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}} \frac{\mathrm{Q}_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
Typically, for $\mu / M_{\mathrm{PI}} \simeq 10^{-2}$, one has $M / M_{\mathrm{PI}} \simeq 10^{-4}$.
The reheating consistent slow-roll predictions for MHI have been represented in Fig. 102. As expected, for small values of $\mu / M_{\mathrm{Pl}}$, the predicted amount of gravitational waves is extremely small and the deviation from scale invariance almost does not depend on $\mu$.


Fig. 71. Theoretical prior space for the stringy scenario of brane inflation [158] in the plane of the "universal" coordinates $(y, \bar{v})$. The solid blue line is the frontier above which inflation ends by tachyonic pre-heating triggered by brane annihilation (light green region). Only in the region enclosed by this curve (light blue region), inflation ends by slow-roll violation. The upper thick red line is the volume bound of Eq. (5.343). The lower black straight line is the "UV" limit given by ( 5.346 ) and is relevant only if inflation stops by slow-roll violation. The solid green curve is given by (5.350) and also represents the "UV" limit but, this time, in the regime where inflation stops when the two branes collide. As a consequence, the admissible region is the one shaded in light black. We see that, even in this allowed region, inflation can either end by tachyonic instability or slow-roll violation depending on the string parameter values. In principle, the blue, black and green lines should cross at a single point. Due to the approximations used here, we see that this is true only approximately. In order to give a more faithful description of the allowed region, the light black area has been slightly deformed around the crossing point (see Ref. [158] for an exact determination of these frontiers). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

### 4.16. Radion gauge inflation (RGI)

This model was studied in Ref. [415]. It is an extension of the gauge inflation scenario in which the radius modulus field around which the Wilson loop is wrapped assists inflation as it shrinks [246]. Assuming that the radion field value is such that the potential energy is minimal, for each value of the inflaton field $\phi$, one can derive an effective potential
$V(\phi)=M^{4} \frac{\left(\phi / M_{\mathrm{PI}}\right)^{2}}{\alpha+\left(\phi / M_{\mathrm{PI}}\right)^{2}}$,
where $\alpha$ is a dimensionless positive parameter. In the context of Ref. [415], the model is natural for $\alpha<1$ but larger than unity values are not forbidden. The same potential has been obtained in Ref. [416] in the context of S-dual superstring models. In that case, $\alpha$ represents a typical vev for the inflaton, in Planck units. Defining $x=\phi / M_{\mathrm{PI}}$, the first three slow-roll parameters read
$\epsilon_{1}=\frac{2 \alpha^{2}}{x^{2}\left(\alpha+x^{2}\right)^{2}}, \quad \epsilon_{2}=4 \alpha \frac{\alpha+3 x^{2}}{x^{2}\left(\alpha+x^{2}\right)^{2}}$,
$\epsilon_{3}=4 \alpha \frac{\alpha^{2}+3 \alpha x^{2}+6 x^{4}}{x^{2}\left(\alpha+x^{2}\right)^{2}\left(\alpha+3 x^{2}\right)}$.
The potential, its logarithm, and the Hubble flow functions are represented in Fig. 30.

The slow-roll trajectory can be integrated analytically from Eq. (2.11) to obtain
$N-N_{\text {end }}=\frac{x_{\text {end }}^{2}}{4}+\frac{x_{\text {end }}^{4}}{8 \alpha}-\frac{x^{2}}{4}-\frac{x^{4}}{8 \alpha}$.
Moreover, it can be inverted explicitly to give the field values in terms of the number of $e$-folds as
$x=\sqrt{-\alpha+\sqrt{-8 \alpha\left(N-N_{\text {end }}\right)+\left(\alpha+x_{\text {end }}^{2}\right)^{2}}}$.
The end of inflation naturally occurs for $\epsilon_{1}=1$, i.e., from Eq. (4.227), at the field value $x_{\text {end }}$ given by
$x_{\text {end }}=\frac{-\sqrt[3]{6} \alpha+\left[9 \alpha+\sqrt{3 \alpha^{2}(2 \alpha+27)}\right]^{2 / 3}}{162^{1 / 6}\left[9 \alpha+\sqrt{3 \alpha^{2}(2 \alpha+27)}\right]^{1 / 3}}$.
As for the MHI models, one should pay attention that
$\epsilon_{2}-\epsilon_{1}=2 \alpha \frac{\alpha+6 x^{2}}{x^{2}\left(\alpha+x^{2}\right)^{2}}>0$,
for any positive values of $\alpha$. As a result, slow-roll violation, i.e. $\epsilon_{2}>$ 1, occurs in RGI before inflation ends. However, since the first Hubble flow function is monotonic, this is not very problematic as it happens only during a negligible number of $e$-folds and only around $N_{\text {end }}$. The slow-roll observable predictions therefore remain accurate.

As before, the observable field value $x_{*}$ is obtained by solving Eq. (2.47) for a given reheating model and allows the determination of the parameter $M$ from the amplitude of the CMB anisotropies. One gets
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=\frac{2880 \pi^{2} \alpha^{2}}{x_{*}^{4}\left(\alpha+x_{*}^{2}\right)} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
The reheating consistent slow-roll predictions for these models are displayed in Fig. 103. Large values of $\alpha$ give back the same predictions as the large field models with $p=2$ (see Section 4.2) having $\epsilon_{2 *}=2 \epsilon_{1 *}$.

### 4.17. MSSM inflation (MSSMI)

### 4.17.1. Theoretical justifications

The Minimal Supersymmetric Standard Model (MSSM) is an extension of the Standard Model of particle physics. Its Lagrangian is characterized by the following super potential

$$
\begin{align*}
W_{\mathrm{MSSM}}= & \lambda_{u}^{i j} Q_{i} \cdot H_{u} U_{j}^{\mathrm{c}} \\
& +\lambda_{d}^{i j} Q_{i} \cdot H_{d} D_{j}^{\mathrm{c}}+\lambda_{e}^{i j} L_{i} \cdot H_{d} E_{j}^{\mathrm{c}}+\mu H_{u} \cdot H_{d} \tag{4.233}
\end{align*}
$$

The quantity $Q_{i}$ denotes a doublet of left handed quarks super fields where $i$ is a family index. In practice this means that
$Q_{1}=\binom{U}{D}, \quad Q_{2}=\binom{C}{S}, \quad Q_{3}=\binom{T}{B}$,
where the components of the doublets are super fields. For instance, the scalar part of $U$ is the $\tilde{u}$ squark and its fermionic part is the ordinary $u$ quark. Of course, there is also a color index $a=$ $1,2,3$ and, in fact, one should write the corresponding doublet as $Q_{i a}$. Moreover, one can also introduce a third $\operatorname{SU}(2)_{\mathrm{L}}$ index $\alpha=1,2$ and write $Q_{i a \alpha}$ with, for instance, $Q_{1 a 1}=U$ and $Q_{1 a 2}=D$. On the other hand, the quantities $U_{j}^{\mathrm{c}}$ and $D_{j}^{c}$ denotes the right handed super fields where $j$ is the family index (and the color index has been ignored in order to simplify the notation): for instance, $U_{2}^{c}$ means the right handed charm quark super field which is a singlet under $\mathrm{SU}(2)_{\mathrm{L}}$.

In the same fashion, $L_{i}$ denotes a doublet of left handed lepton superfields
$L_{1}=\binom{N_{e}}{E_{e}}, \quad Q_{2}=\binom{N_{\mu}}{E_{\mu}}, \quad Q_{3}=\binom{N_{\tau}}{E_{\tau}}$,
where, for instance, $N_{e}$ denotes the electronic neutrino superfield (the scalar part being the neutralino and the fermionic part the electronic neutrino itself) while $E_{e}$ denotes the electron superfield. On the other hand, the quantities $E_{j}^{c}$ denote the right handed superfields that are singlet under $\operatorname{SU}(2)_{\mathrm{L}}$ (for instance, $E_{2}^{c}$ is the right handed muonic superfield). In the superpotential (4.233), there are two terms involving the quarks and only one involving the leptons because, as well-known, there is no right handed neutrinos in the standard model.

The last term in Eq. (4.233) describes the Higgs sector with two Higgs doublet $H_{u}$ and $H_{d}$. The quantity $\mu$ is a new dimensionful (of dimension one) parameter of the model. The dot indicates an $\operatorname{SU}(2)$ invariant product. Finally, $\lambda_{u}, \lambda_{d}, \lambda_{e}$ are the $3 \times 3$ Yukawa matrices.

From the superpotential (4.233), one can determine the scalar potential of the theory by means of the usual supersymmetric machinery. As is well-known, the scalar potential is made of two pieces, the $F$-term part and the $D$-term part. Clearly, given the number of fields in the theory, the scalar potential is a complicated object. For inflation, we are especially interested in the flat directions of this potential. A flat direction is a direction such that the $F$ and $D$-terms vanish, that is to say such that $V_{F}=0$, $V_{D}=0$ and, therefore, $V \equiv V_{F}+V_{D}=0$. It was shown that the MSSM scalar potential contains nearly 300 gauge invariant flat directions $[57,417,418]$. Finding these directions is a non-trivial task and we now very briefly explain how this can be done. Usually, it consists in putting all the fields to zero except a few ones, these few ones being carefully chosen such that cancellations occur in such a way that the potential exactly vanishes. We now illustrate this method on a particular case. Let us first recall that the general formula giving the $D$-term potential is
$V_{D}=\frac{1}{2} \sum_{a} g_{a}^{2} D^{a} D^{a}$,
where $D^{a}=\phi^{\dagger} T^{a} \phi, T^{a}$ being the generator of the group and $\phi$ denoting a generic field (of course, the index $a$ should not be confused with the color index discussed above). For the standard model, we have the group $S U(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}}$ and, therefore, the explicit expression of the $D$-term reads
$V_{D}=\frac{g^{2}}{2}\left(D_{1}^{2}+D_{2}^{2}+D_{3}^{2}\right)+\frac{g_{Y}}{2} D_{Y}^{2}$,
$g$ and $g_{Y}$ being the coupling constants of the two groups. For the $\mathrm{SU}(2)$ group, the generators $T^{a}$ are nothing but the Pauli matrices and, therefore, $T^{a}=\sigma^{a} / 2$. Following Refs. [417,419], let us consider a situation where all the fields in the MSSM are assumed to have a vanishing vev except $L_{i}$ and $E_{j}^{\mathrm{c}}$ where we remind that $i$ and $j$ are family indices. If we write $L_{i}^{\uparrow}$ and $L_{i}^{\downarrow}$ as respectively the upper and lower component of the doublet $L_{i}$, then one has (i.e. we put $\phi=L_{i}$ in the general formula expressing $D^{a}$ )
$D_{1}=\frac{1}{2} \sum_{i=1}^{3}\left(L_{i}^{\uparrow *} L_{i}^{\downarrow}+L_{i}^{\downarrow *} L_{i}^{\uparrow}\right)$,
$D_{2}=\frac{i}{2} \sum_{i=1}^{3}\left(L_{i}^{\uparrow *} L_{i}^{\downarrow}-L_{i}^{\downarrow *} L_{i}^{\uparrow}\right)$,
$D_{3}=\frac{1}{2} \sum_{i=1}^{3}\left(\left|L_{i}^{\uparrow}\right|^{2}-\left|L_{i}^{\downarrow}\right|^{2}\right)$.
The quantity $E^{c}$ being a $S U(2)$ singlet does not participate to the above expression. On the other hand, the contribution from the $\mathrm{U}(1)$ group reads
$D_{Y}=\frac{1}{2} \sum_{i=1}^{3}\left(2\left|e_{i}\right|^{2}-\left|L_{i}^{\uparrow}\right|^{2}-\left|L_{i}^{\downarrow}\right|^{2}\right)$,
where $e_{i}$ denotes the scalar field of the $E_{i}^{c}$ supersymmetric multiplet. We see that, if we take
$L_{i}=\binom{\phi}{0}, \quad L_{j}=\binom{0}{\phi}, \quad e_{k}=\phi$,
then we have $V_{D}=0$.
The next step consists in calculating the $F$-term for the choice (4.241). It is easy to check that $V_{F}=0$. Therefore, we have identified a flat direction. It is denoted $L_{i} L_{j} e_{k}$ or $\mathbf{L L e}$ to recall that all family combination are possible. This direction is represented by a "composite operator $X_{m}$ " formed by the product of the superfields making up the flat direction. In our case $X_{3}=L_{i} L_{j} e_{k}=\phi^{3}$ and $m=3$ since we have three operators participating to the definition of $X_{3}$. This direction has been proposed in Ref. [420] as a possible candidate for the inflaton field. Let us also remark that another choice put forward in that reference was udd.

We have just seen how to identify flat directions in the MSSM potential. However, this flatness is usually spoiled by the presence of higher order non-renormalizable operators appearing in the MSSM (viewed here as a low energy effective field) and by supersymmetry breaking [57,417,418]. Higher order operators are described by the following superpotential
$W=\frac{\lambda_{n}}{n} \frac{X_{m}^{k}}{M_{\mathrm{Pl}}^{m k-3}}$,
where $\lambda_{n}$ is a coupling constant, $n \equiv m k$ and $k=1$ or $k=2$ depending on whether the flat direction is even or odd under R parity. Recall that $Q, L, U^{\mathrm{c}}, D^{\mathrm{c}}$ and $E^{\mathrm{c}}$ have R-parity -1 and $H_{u}, H_{d}$ have R-parity +1 . It follows that LLe (for instance) has odd R-parity and, therefore, that $k=2$. For the directions LLe (this is also true for uud), this means that
$n \equiv m k=6$.

The above superpotential (4.242) will produce a term $|\partial W / \partial \phi|^{2} \propto$ $\phi^{2(k m-1)}$ in the scalar potential. Then, we have the contributions originating from supersymmetry breaking. They can be easily calculated if, for instance, we assume that we have an independent hidden sector where supersymmetry is broken and that this breaking is mediated by gravity only. This gives two types of soft terms, one proportional to $\phi^{2}$ and another, the so-called " $A$-term", proportional to ( $\phi \partial W / \partial \phi+c c$ ) that is to say, given Eq. (4.242), proportional to $\phi^{m k}$.

More generally, if one starts from a flat direction with a given $n$, then the superpotential has the form $W=\lambda_{n} / n \Phi^{n} M_{\mathrm{Pl}}^{3-n}$, where $\Phi=\phi e^{i \theta}$ is the superfield which contains the flat direction. Then, the scalar potential takes the form
$V(\phi)=\frac{1}{2} m_{\phi}^{2} \phi^{2}+A \cos \left(n \theta+\theta_{0}\right) \frac{\lambda_{n}}{n} \frac{\phi^{n}}{M_{\mathrm{Pl}}^{n-3}}+\lambda_{n}^{2} \frac{\phi^{2(n-1)}}{M_{\mathrm{Pl}}^{2(n-3)}}$,
where the the second term involves the angular part of the superfield via a term $\cos \left(n \theta+\theta_{0}\right)$, which in practice is fixed at -1 to maximize its contribution. As explained below, the fact that the second term appears with a negative coefficient plays a crucial role in making this scenario a credible inflationary one.

Together with the global minimum at $\phi=0$, under the condition $A^{2} \geq 8(n-1) m_{\phi}^{2}$, the potential has a secondary minimum at $\phi_{0} \simeq\left(m_{\phi} M_{\mathrm{Pl}}^{n-3}\right)^{1 /(n-2)}$. If $A^{2} \gg 8(n-1) m_{\phi}^{2}$, this secondary minimum becomes the deepest one and thus the true one. The curvature of the potential at this minimum is of the order $m_{\phi}^{2}$. If inflation occurs there, one gets $H \simeq m_{\phi}\left(m_{\phi} / M_{P 1}\right)^{1 /(n-2)}$, which is much smaller than the potential curvature for $m_{\phi} \ll M_{\mathrm{Pl}}$. This implies that the potential is too steep for quantum effects during inflaton to kick $\phi$ out of the false minimum. Such a situation is similar to the old inflationary scenario. However, this barrier disappears if one saturates the previous inequality and takes
$A^{2}=8(n-1) m_{\phi}^{2}$.
In that case, the potential has a flat inflection point at $\phi_{0}$ and inflation can proceed between this plateau and $\phi=0$. This is the case we study in this section. This model (and its generalizations) has also been studied in Refs. [421-431]. Its generalizations will be investigated in more details in Sections 5.6 and 5.7. Let also us notice that when $n=3$, the same potential appears in Refs. [432, 433] as "Generalized Chaotic Inflation", and later in Refs. [434436] as "Punctuated Inflation". In these references, it is shown that slow-roll inflation is briefly interrupted when the inflaton crosses the flat inflection point and this can produce step-like features in the primordial power spectra. These effects are outside the scope of the following slow-roll analysis as we will be dealing with the last slow-roll inflationary stage within this scenario.

### 4.17.2. Slow-roll analysis

We now turn to the slow-roll analysis of MSSM inflation. As discussed before, we assume that the inflaton is the flat direction LLe or uud. This implies that $n=6$ in Eq. (4.244). Then, rewriting the potential (4.244) in a more convenient fashion, one arrives at
$V(\phi)=M^{4}\left[\left(\frac{\phi}{\phi_{0}}\right)^{2}-\frac{2}{3}\left(\frac{\phi}{\phi_{0}}\right)^{6}+\frac{1}{5}\left(\frac{\phi}{\phi_{0}}\right)^{10}\right]$,
where we have defined new parameters according to
$M^{8}=\frac{M_{\mathrm{Pl}}^{3} m_{\phi}^{5}}{4 \sqrt{10} \lambda_{6}}, \quad \phi_{0}^{8}=\frac{M_{\mathrm{Pl}}^{6} m_{\phi}^{2}}{10 \lambda_{6}^{2}}$.
These definitions and the value of the coefficients ensure that $\phi_{0}$ is the location of a flat inflection point. Since $m_{\phi}^{2} \phi^{2}$ is a soft SUSY
breaking term, we typically expect that $m_{\phi} \simeq 1 \mathrm{TeV}$ and this is the reason why, in what follows, typical values of the field are taken to be
$\phi_{0} \simeq 10^{14} \mathrm{GeV}$,
in agreement with the second of Eq. (4.247) (the coupling constant $\lambda_{6}$ is taken to be of order one). An interesting feature of this model is that it provides inflation at sub-Planckian vev and at low scale $V \simeq\left(10^{9} \mathrm{GeV}\right)^{4}$. As noticed in Ref. [420], higher values than $n=6$ would produce too small amplitude for the scalar perturbations. This is why the model is commonly studied with $n=6$ (with $n=3$, this is RIPI, see Section 4.18).

The potential in Eq. (4.246) is displayed in Fig. 31, together with its logarithm. It is an increasing function of the field, the derivative of which vanishes at $\phi=0$ and at its second inflection point $\phi=\phi_{0}$, the position of the first inflection point being given by $\phi_{V^{\prime \prime}=0}^{-}=\phi_{0} / \sqrt{3}$. Inflation proceeds in the region $\phi \in\left[0, \phi_{0}\right]$, in the direction specified by the arrow in Fig. 31.

Defining the dimensionless quantity $x$ by
$x \equiv \frac{\phi}{\phi_{0}}$,
the first three Hubble flow functions in the slow-roll approximation are given by
$\epsilon_{1}=450 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}} \frac{\left(x^{4}-1\right)^{4}}{x^{2}\left(3 x^{8}-10 x^{4}+15\right)^{2}}$,
$\epsilon_{2}=60 \frac{M_{P l}^{2}}{\phi_{0}^{2}} \frac{3 x^{16}-58 x^{8}+40 x^{4}+15}{x^{2}\left(3 x^{8}-10 x^{4}+15\right)^{2}}$,
and

$$
\begin{align*}
\epsilon_{3} & =\frac{M_{\mathrm{PI}}^{2}}{\phi_{0}^{2}} \frac{60}{x^{2}}\left(-225+1575 x^{4}-3165 x^{8}+395 x^{12}\right. \\
& \left.+2605 x^{16}-1275 x^{20}+81 x^{24}+9 x^{28}\right)\left(3 x^{8}-10 x^{4}+15\right)^{-2} \\
& \times\left(-15-55 x^{4}+3 x^{8}+3 x^{12}\right)^{-1} \tag{4.251}
\end{align*}
$$

These two slow-roll parameters diverge when the field vev goes to 0 , and vanish when the field vev goes to infinity. The first slow roll parameter $\epsilon_{1}$ first decreases, vanishes at the flat inflection point where $\epsilon_{2}$ vanishes too, then increases to reach a local maximum where $\epsilon_{2}$ vanishes again, and eventually decreases again, to vanish at infinity where $\epsilon_{2}$ also goes to zero. Denoting by $x_{\epsilon_{2}=0}^{+}$the position of the second extremum, one has

$$
\begin{align*}
& x_{\epsilon_{2}=0}^{+}=\left(\frac{1}{3}\right)^{1 / 4}\left[2^{4 / 3}(i \sqrt{685}-1)^{1 / 3}\right. \\
& \left.\quad+14 \times 2^{2 / 3}(i \sqrt{685}-1)^{-1 / 3}-1\right]^{1 / 4} \simeq 1.41022 . \tag{4.252}
\end{align*}
$$

In between the two local extrema of $\epsilon_{1}$, the second slow-roll parameter $\epsilon_{2}$ is negative whereas it is positive elsewhere. The value of $\epsilon_{1}$ at its local maximum is given by
$\epsilon_{1}^{\max }=\epsilon_{1}\left(x_{\epsilon_{2}=0}^{+}\right) \simeq 34.459 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}}$.
With the typical above-mentioned value for $\phi_{0} \simeq 10^{14} \mathrm{GeV}$, one has $M_{\mathrm{Pl}}^{2} / \phi_{0}^{2} \simeq 10^{8}$ and $\epsilon_{1}^{\max }>1$. This means that if inflation proceeds for vev's larger than that of the flat inflection point, it can naturally stop by slow-roll violation. However, if this happens, inflation proceeds at $x \gg 1$ and the potential is effectively very close to a large field model one (LFI, see Section 4.2) with $p=10$.

For this reason, we will be focused to the case in which inflation occurs for vev's smaller than that of the flat inflection point. In this case, the value of $x_{\text {end }}$ at which inflation stops by slow-roll violation must be determined numerically. In the limit $\phi_{0} / M_{\mathrm{Pl}} \ll 1$ however, one has $x_{\text {end }} \simeq 1$ and an approximate analytic formula can be derived
$x_{\text {end }} \simeq 1-\frac{1}{2^{3 / 4} \sqrt{15}} \sqrt{\frac{\phi_{0}}{M_{\mathrm{Pl}}}}$.
A comparison between this expression and the numerical solution of $\epsilon_{1}=1$ is displayed in Fig. 32. For physical values $\phi_{0} \simeq 10^{-4} M_{\mathrm{Pl}}$, the agreement is excellent.

Let us now turn to the slow-roll trajectory. It can be integrated from Eq. (2.11) and leads to

$$
\begin{aligned}
N_{\text {end }}-N= & \left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2}\left\{\frac{x^{2}-x_{\text {end }}^{2}}{20}+\frac{1}{15}\left(\frac{x_{\text {end }}^{2}}{x_{\text {end }}^{4}-1}-\frac{x^{2}}{x^{4}-1}\right)\right. \\
& \left.-\frac{2}{15}\left[\operatorname{arctanh}\left(x_{\text {end }}^{2}\right)-\operatorname{arctanh}\left(x^{2}\right)\right]\right\},
\end{aligned}
$$

where $N_{\text {end }}$ is the number of $e$-folds at the end of inflation and $N$ is the number of $e$-folds at some point when the scaled field $v e v$ is $x$. A few remarks are in order. Firstly, when $x \simeq 1$, the second term of the previous expression dominates, and one has $N_{\text {end }}-N \simeq 1 / 15\left(\phi_{0} / M_{\mathrm{Pl}}\right)^{2}\left[1 /\left(x_{\text {end }}^{4}-1\right)-1 /\left(x^{4}-1\right)\right]$, which can be inverted and gives
$x \simeq 1-\frac{1}{4}\left[2^{-5 / 4} \sqrt{15} \sqrt{\frac{M_{\mathrm{Pl}}}{\phi_{0}}}+15 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}}\left(N_{\mathrm{end}}-N\right)\right]^{-1}$.
Secondly, one could wonder if a sufficient number of e-folds can be realized in the regime studied here. When $x \rightarrow 1$, the corresponding number of e-folds diverges, but in practice, the inflationary dynamics close to the flat inflection point is governed by the quantum diffusion and the classical equation of motion can not be trusted in this domain.

If one introduces the ratio $\eta$ between the quantum kicks amplitude $H /(2 \pi)$ and the classical drift $M_{\mathrm{Pl}}^{2} V_{\phi} / V$, when $x \simeq 1$, one has

$$
\begin{align*}
\eta & \simeq \frac{1}{90 \sqrt{30} \pi} M^{2} \phi_{0} M_{\mathrm{Pl}}^{-3}(x-1)^{-2} \\
& \simeq \frac{4 \sqrt{10}}{\pi \sqrt{3}} M^{2} M_{\mathrm{Pl}} \phi_{0}^{-3}\left(N_{\mathrm{end}}-N\right)^{2} \tag{4.256}
\end{align*}
$$

where the last equality comes from the approximate trajectory. In order to estimate the value of $\eta$, one needs the value of $M$ which is fixed by the amplitude of the CMB anisotropies. With $x_{*}$ the observable field value associated with $\Delta N_{*}=N_{\text {end }}-N_{*}$, one gets

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=2880 \pi^{2} \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}} \frac{\left(1-x_{*}^{4}\right)^{4}}{x_{*}^{4}\left(1-\frac{2}{3} x_{*}^{4}+\frac{1}{5} x_{*}^{8}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{4.257}
\end{equation*}
$$

In the $x_{*} \simeq 1$ approximation, this gives
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}} \simeq \frac{3}{8} \pi^{2} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \frac{\phi_{0}^{6}}{M_{\mathrm{Pl}}^{6}\left(N_{\text {end }}-N_{*}\right)^{4}}$,
and thus
$\eta \simeq \sqrt{20 \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}}\left(\frac{N_{\mathrm{end}}-N}{\Delta N_{*}}\right)^{2}$.
It is quite remarkable that this formula does not depend on $\phi_{0}$ anymore but only on the ratio $\left(N_{\text {end }}-N\right) / \Delta N_{*}$. From $Q_{\text {rms-PS }} / T \simeq$ $6 \times 10^{-6}$, one has $N_{\text {end }}-N_{\min } \simeq 10^{4}$ in the classical regime [420].


Fig. 72. Top left panel: running mass potential for $c=0.8$ (blue line) or $c=-0.8$ (green line) and $\phi_{0}=0.5 M_{\mathrm{PI}}$. Top right panel: logarithm of the potentials for the same values of $c$ and $\phi_{0}$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ for a potential with $c= \pm 0.8$ and $\phi_{0}=0.5 M_{\mathrm{Pl}}$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line) for $c= \pm 0.8$ and $\phi_{0}=0.5 M_{\mathrm{Pl}}$. The value $c= \pm 0.8$ may not be physical and was chosen only in order to produce a clear plot. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)


Fig. 73. Valley Hybrid inflation (VHI) for $p=1 / 2$ (red line) and $p=2$ (blue line). Upper panels: the potential and its logarithm for $\mu=0.6 M_{\mathrm{PI}}$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ for $p=1 / 2, \mu=0.6 M_{\mathrm{PI}}$ (red line), $p=2, \mu=0.6 M_{\mathrm{PI}}$ (blue line) and $p=2, \mu=0.9 M_{\mathrm{PI}}$ (green line). For small values of $\mu$ and $p>1$, the inflationary regions are separated into a large field one and the vacuum dominated one. The latter may not exist due to slow-roll violations if the field first rolls down the potential in the large field domain (see the text for a detailed discussion). The shaded area indicates the regions in which acceleration cannot occur. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon 3$ (dotted line) for $\mu=0.6 M_{\mathrm{Pl}}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

For $\phi_{0} \simeq 10^{14} \mathrm{GeV}$, one obtains $M \simeq 10^{8} \mathrm{GeV}$, in agreement with what was announced earlier.

Finally, it can be interesting to write down the approximated slow-roll parameters at Hubble crossing and in the limit $\phi_{0} / M_{\mathrm{PI}} \ll$ 1. One obtains
$\epsilon_{1 *} \simeq\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{6} \frac{1}{7200 \Delta N_{*}^{4}}, \quad \epsilon_{2 *} \simeq \frac{4}{\Delta N_{*}}$,
$\epsilon_{3 *} \simeq \frac{1}{\Delta N_{*}}$,
hence
$r \simeq\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{6} \frac{1}{450 \Delta N_{*}^{4}}, \quad n_{\mathrm{S}} \simeq 1-\frac{4}{\Delta N_{*}}$,
$\alpha_{\mathrm{S}} \simeq-\frac{4}{\Delta N_{*}^{2}}$.
They are similar with the typical predictions of the RIPI models [see Eq. (4.277)].

The reheating consistent slow-roll predictions of the MSSMI models are displayed in Fig. 104. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 since the potential is quadratic in the vicinity of its minimum. One can check that, in the limit $\phi_{0} / M_{\mathrm{PI}} \ll 1$, the first slow-roll parameter is indeed extremely small, while the second slow-roll parameter does not depend much on $\phi_{0}$. Remembering that $\phi_{0} / M_{\mathrm{Pl}} \simeq 10^{-4}$, one can see that these models seem to be disfavored by the data since they predict a too large deviation from scale invariance. In order to better reproduce the constraints on the spectral index, these models should be such that $\phi_{0} / M_{\mathrm{PI}} \gg 1$, for which they become similar to large field models (LFI, see Section 4.2). This can be seen from the previous formulas in the limit $x \gg 1$. Unfortunately, such values for $\phi_{0}$ are not compatible with the MSSM. Finally, comparing Fig. 104 with Fig. 105, one can see that the general features of MSSMI are very similar to the RIPI ones, and that the conclusions drawn here are rather robust against a change in $n$ appearing in Eq. (4.244).

### 4.18. Renormalizable inflection point inflation (RIPI)

### 4.18.1. Theoretical justifications

In Section 4.17 inflation is implemented within the Minimal Supersymmetric Standard Model (MSSM) around a flat inflection point. Here, we consider a similar model but with $n=3$ instead of $n=6$. Such a scenario can emerge in the following situation, see Refs. [437,438]. Let us consider the MSSM with three additional superfields $N_{i}$ representing three right-handed neutrinos. These fields are singlet under the standard model gauge group but this one can be extended to $\mathrm{SU}(3)_{\mathrm{c}} \times \operatorname{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{Y}} \times \mathrm{U}(1)_{\mathrm{B}-\mathrm{L}}$ and the $N_{i}$ are assumed to be charged under the extra $\mathrm{U}(1)_{\mathrm{B}-\mathrm{L}}$. Then, we postulate the following superpotential
$W=W_{\text {MSSM }}+h N H_{u} L$,
where $h \lesssim 10^{-12}$ in order to explain the neutrino mass, $m_{v} \simeq$ $\mathcal{O}(0.1) \mathrm{eV}$. It follows that $N H_{u} L$ is a $D$-flat direction of the potential and we parametrize this direction by $\phi$. As a consequence, if one now calculates the corresponding potential, one finds that
$V=\frac{1}{2} m_{\phi}^{2} \phi^{2}-\frac{A h}{6 \sqrt{3}} \phi^{3}+\frac{h^{2}}{12} \phi^{4}$,
where, as usual, we have included the soft supersymmetry breaking terms (since $W \propto \phi^{3}$, the $A$-term, proportional to $\phi \partial W / \partial \phi$ is, this time, cubic) and have minimized $V$ along the angular direction. If $A$ is chosen such that $A=4 m_{\phi}$, then we have a flat inflection point at $\phi_{0}=\sqrt{3} m_{\phi} / h$. A discussion on the fine-tuning required to get a flat inflection point can be found in Section 5.7.

### 4.18.2. Slow-roll analysis

We now turn to the slow-roll analysis of the potential given in Eq. (4.263). For this purpose, it is more convenient to re-write it as
$V(\phi)=M^{4}\left[\left(\frac{\phi}{\phi_{0}}\right)^{2}-\frac{4}{3}\left(\frac{\phi}{\phi_{0}}\right)^{3}+\frac{1}{2}\left(\frac{\phi}{\phi_{0}}\right)^{4}\right]$,
where we have defined the quantities $M$ and $\phi_{0}$ by
$M^{4}=\frac{1}{2} m_{\phi}^{2} \phi_{0}^{2}, \quad \phi_{0}=\sqrt{3} \frac{m_{\phi}}{h}$.
Relevant values of $m_{\phi}$ range from 100 GeV to 10 TeV and $h \simeq$ $10^{-12}$. This means that $[437,438]$
$\phi_{0} \simeq 10^{14} \mathrm{GeV}$,
a value that turns out to be similar to the one considered in the MSSMI case (see Section 4.17).

Let us now define the quantity $x$ by the following expression
$x \equiv \frac{\phi}{\phi_{0}}$.
The potential is an increasing function of the field vev, hence inflation proceeds from the right to the left. It has two inflection points $x_{V^{\prime \prime}=0}^{ \pm}$, given by
$x_{V^{\prime \prime}=0}^{-}=\frac{1}{3} \quad$ and $\quad x_{V^{\prime \prime}=0}^{+}=1$,
the second one being a flat inflection point [i.e. $V^{\prime}\left(x_{V^{\prime \prime}}^{+}=0\right)=0$ ], close to which inflation takes place. This potential is displayed in Fig. 33, together with its logarithm.

Let us now turn to the slow-roll parameters. The first three Hubble flow functions in the slow-roll approximation are given by

$$
\begin{align*}
& \epsilon_{1}=72 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}} \frac{(x-1)^{4}}{\left(3 x^{3}-8 x^{2}+6 x\right)^{2}}  \tag{4.269}\\
& \epsilon_{2}=24 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}}(x-1) \frac{3 x^{3}-9 x^{2}+10 x-6}{\left(3 x^{3}-8 x^{2}+6 x\right)^{2}}
\end{align*}
$$

and

$$
\begin{aligned}
\epsilon_{3} & =24 \frac{M_{P l}^{2}}{\phi_{0}^{2}}(x-1)\left(36-144 x+246 x^{2}-236 x^{3}+144 x^{4}\right. \\
& \left.-54 x^{5}+9 x^{6}\right)\left(6 x-8 x^{2}+3 x^{3}\right)^{-2}\left(10 x-9 x^{2}+3 x^{3}-6\right)^{-1}
\end{aligned}
$$

Both $\epsilon_{1}(x)$ and $\epsilon_{2}(x)$ diverge when the field vev goes to 0 , and vanish when the field vev goes to infinity. The first slow-roll parameter $\epsilon_{1}$ first decreases, vanishes at $x_{V^{\prime \prime}=0}^{+}$where $\epsilon_{2}$ vanishes too, $x_{\epsilon_{2}=0}^{-}=x_{V^{\prime \prime}=0}^{+}$, then increases to reach a local maximum at $x_{\epsilon_{2}=0}^{+}$where $\epsilon_{2}$ vanishes again, and eventually decreases again. The value of $x_{\epsilon_{2}=0}^{+}$is given by
$x_{\epsilon_{2}=0}^{+}=1-\frac{1}{3(9+\sqrt{82})^{1 / 3}}+\frac{1}{3}(9+\sqrt{82})^{1 / 3} \simeq 1.75$.
In between these two local extrema of $\epsilon_{1}$, the second slow roll parameter $\epsilon_{2}$ is negative, and it is positive elsewhere. The value of $\epsilon_{1}$ at its local maximum, $\epsilon_{1}^{\max }$, is given by
$\epsilon_{1}^{\max } \simeq 5.2753 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}}$.
Therefore, if $\phi_{0} / M_{\mathrm{Pl}} \lesssim 2.3$, inflation can stop by slow-roll violation in the region corresponding to $v e v$ 's larger than that of


Fig. 74. Dynamical Supersymmetric Inflation (DSI) for $p=2$. Upper panels: the potential and its logarithm as a function of $\phi / \mu$. Bottom left panel: slow-roll parameter $\epsilon_{1}$ rescaled by $M_{\mathrm{PI}}^{2} / \mu^{2}$. The shaded area indicates the region in which inflation cannot occur for $\mu=M_{\mathrm{Pl}}$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line), rescaled by $M_{\mathrm{Pl}}^{2} / \mu^{2}$.
the second inflection point $x_{\epsilon_{2}=0}^{+}$. Remembering that typically $\phi_{0} \simeq$ $10^{14} \mathrm{GeV} \simeq 4 \times 10^{-5} M_{\mathrm{PI}}$, this condition is easily satisfied. In that case, an expression for the vev at which inflation ends, $x_{\epsilon_{1}=1}^{+}$, can be obtained but is does not add much to the discussion since for reasonable values of $\phi_{0}$, it is extremely far from the flat inflection point (e.g. for $\phi_{0} / M_{P 1}=10^{-4}$, one has $x_{\epsilon_{1}=1}^{+} \simeq 28,285$ ). Since the potential is introduced in order to study inflation in the vicinity of the flat inflection point, it should be studied in the other regime, as it is the case for MSSM inflation (see Section 4.17), i.e. when inflation takes place between $x=0$ and the second inflection point $x_{\epsilon_{2}=0}^{-}$. In that situation, it ends at

$$
\begin{align*}
x_{\text {end }} & =x_{\epsilon_{1}=1}^{-}=\frac{1}{9} \frac{M_{\mathrm{Pl}}}{\phi_{0}}\left[6 \sqrt{2}+8 \frac{\phi_{0}}{M_{\mathrm{Pl}}}\right. \\
& +2\left(-36+6 \sqrt{2} \frac{\phi_{0}}{M_{\mathrm{Pl}}}-5 \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}}\right) \\
& \times\left(216 \frac{\phi_{0}}{M_{\mathrm{Pl}}}-99 \sqrt{2} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}}+136 \frac{\phi_{0}^{3}}{M_{\mathrm{Pl}}^{3}}-432 \sqrt{2}\right. \\
& \left.+27 \sqrt{2} \sqrt{-72 \sqrt{2} \frac{\phi_{0}^{3}}{M_{\mathrm{Pl}}^{3}}+33 \frac{\phi_{0}^{4}}{M_{\mathrm{Pl}}^{4}}-16 \sqrt{2} \frac{\phi_{0}^{5}}{M_{\mathrm{Pl}}^{5}}+12 \frac{\phi_{0}^{6}}{M_{\mathrm{Pl}}^{6}}}\right)^{-1 / 3} \\
& -\left(216 \frac{\phi_{0}}{M_{\mathrm{Pl}}}-99 \sqrt{2} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}}+136 \frac{\phi_{0}^{3}}{M_{\mathrm{Pl}}^{3}}-432 \sqrt{2}\right. \\
+ & \left.\left.27 \sqrt{2} \sqrt{-72 \sqrt{2} \frac{\phi_{0}^{3}}{M_{\mathrm{Pl}}^{3}}+33 \frac{\phi_{0}^{4}}{M_{\mathrm{Pl}}^{4}}-16 \sqrt{2} \frac{\phi_{0}^{5}}{M_{\mathrm{Pl}}^{5}}+12 \frac{\phi_{0}^{6}}{M_{\mathrm{Pl}}^{6}}}\right)^{1 / 3}\right] . \tag{4.272}
\end{align*}
$$



Fig. 75. Dynamical Supersymmetric Inflation. Maximal value of $\mu / M_{P I}$ with respect to $p$, and for different values of $q$, such that the condition $x_{\text {end }}^{\min }<x_{\text {end }}^{\max }$ is satisfied. We have fixed $\Delta N_{\max }=50$. The black dotted line show a typical value for $\mu / M_{\mathrm{PI}} \simeq$ $10^{10} \mathrm{GeV}$ [439].

For $\phi_{0} / M_{\mathrm{Pl}} \ll 1$, one can numerically check that this expression is very close to the flat inflection point location $x_{\epsilon_{2}=0}^{-}$, namely
$x_{\mathrm{end}} \simeq 1-\sqrt{6 \sqrt{2} \frac{\phi_{0}}{M_{\mathrm{Pl}}}}$.
The whole inflationary stage therefore proceeds in the vicinity of this point.

The slow-roll trajectory is obtained from Eq. (2.11) and reads

$$
\begin{gather*}
N_{\mathrm{end}}-N=\frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}}\left[-\frac{x}{6}+\frac{x^{2}}{8}+\frac{1}{12(1-x)}-\frac{\ln (1-x)}{12}\right. \\
\left.\quad+\frac{x_{\mathrm{end}}}{6}-\frac{x_{\mathrm{end}}^{2}}{8}-\frac{1}{12\left(1-x_{\mathrm{end}}\right)}+\frac{\ln \left(1-x_{\mathrm{end}}\right)}{12}\right] \tag{4.274}
\end{gather*}
$$

Several remarks are in order. Firstly, from this expression, one can see that the number of $e$-folds diverges when the field approaches the inflection point of the potential. This means that this point is never crossed and that, if inflation proceeds for vev's larger than that of this inflection point, then the field approaches it asymptotically but never actually reaches it. However, an exact numerical integration of the equations of motion reveals that, if the field approaches the inflection point in such a way that the slow-roll conditions are not satisfied, then it can cross it. This is typically the case if its speed is large enough. On the other hand, the field dynamics at the exact location of the inflection point is dominated by quantum diffusion, and a more careful study must be carried out to describe what exactly happens there. Following the considerations of Section 4.17, we focus on the inflationary regime only in the region where the vev of $\phi$ is smaller than that of the flat inflection and where deviations from slow-roll and quantum diffusion plays a negligible role. Since for $\phi_{0} / M_{\mathrm{PI}} \ll 1$ inflation takes place relatively close to the inflection point, the two last terms of Eq. (4.274) dominate over the two first ones. In this limit, the trajectory can be inverted to get

$$
\begin{align*}
x_{*} \simeq & 1-W_{0}^{-1}\left\{\operatorname { e x p } \left[12\left(\frac{M_{\mathrm{PI}}}{\phi_{0}}\right)^{2} \Delta N_{*}+\frac{1}{1-x_{\mathrm{end}}}\right.\right. \\
& \left.\left.-\ln \left(1-x_{\mathrm{end}}\right)\right]\right\} \tag{4.275}
\end{align*}
$$

Making use of Eq. (4.273), and keeping only the dominant terms in $\phi_{0} / M_{\mathrm{Pl}}$, one obtains
$x_{*} \simeq 1-\frac{1}{12}\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2} \frac{1}{\Delta N_{*}}$.
This expression can be useful to determine typical values for the slow-roll parameters evaluated at Hubble crossing. One obtains
$\epsilon_{1 *} \simeq \frac{1}{288} \frac{1}{\Delta N_{*}^{4}} \frac{\phi_{0}^{6}}{M_{\mathrm{Pl}}^{6}}, \quad \epsilon_{2 *} \simeq \frac{4}{\Delta N_{*}}, \quad \epsilon_{3 *} \simeq \frac{1}{\Delta N_{*}}$,
hence
$r \simeq \frac{1}{18} \frac{1}{\Delta N_{*}^{4}} \frac{\phi_{0}^{6}}{M_{\mathrm{Pl}}^{6}}, \quad n_{\mathrm{S}}-1 \simeq-\frac{4}{\Delta N_{*}}$,
$\alpha_{\mathrm{S}} \simeq-\frac{4}{\Delta N_{*}^{2}}$.
One can see that these models typically predict a tiny amount of gravitational waves, but a substantial deviation from scale invariance $n_{S}-1 \simeq-4 / \Delta N_{*} \simeq 0.1$. The similarity with Eq. (4.260) is obvious.

Finally, the parameter $M$ can be determined from the amplitude of the CMB anisotropies and the observable field value $x_{*}=x\left(N_{*}\right)$ by

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=622080 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}} \pi^{2} \frac{\left(x_{*}-1\right)^{4}}{x_{*}^{4}\left(3 x_{*}^{2}-8 x_{*}+3\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{4.279}
\end{equation*}
$$

For $\phi_{0} / M_{\mathrm{Pl}} \ll 1$, one can make use of Eq. (4.276) to get the approximate expression

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4} \simeq 30 \frac{\pi^{2}}{\Delta N_{*}^{4}}\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{6} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{4.280}
\end{equation*}
$$

Using the typical value $\phi_{0} \simeq 10^{14} \mathrm{GeV}$, one gets $M / M_{\mathrm{PI}} \simeq 5 \times$ $10^{-11}$.

The reheating consistent slow-roll predictions of the renormalizable inflection point models are displayed in Fig. 105. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 since the potential is quadratic close to its minimum. One can check that in the limit $\phi_{0} / M_{\mathrm{PI}} \ll 1$, the first slow-roll parameter is indeed extremely small, while the second slow-roll parameter does not depend much on $\phi_{0}$. Remembering that $\phi_{0} / M_{\mathrm{PI}} \simeq 10^{-4}$, one can see that these models are disfavored by the CMB data since they predict a too large deviation from scale invariance. In order to remain inside the two-sigma confidence intervals, these models should be such that $\phi_{0} / M_{\mathrm{PI}} \gg 1$, for which they are close to the large field models (LFI, see Section 4.2). However, such values for $\phi_{0}$ are, a priori, outside the range of validity of the RIPI scenario. Finally, comparing Fig. 104 with Fig. 105, one can see that the general features of RIPI are very close to the MSSMI ones, and that the conclusions drawn before are therefore robust against the precise value of the power index $n$ in Eq. (4.244).

### 4.19. Arctan inflation (AI)

This scenario was originally introduced in Ref. [440] as a toy model where the equation of state changes rapidly around $\phi=0$. The potential reads
$V(\phi)=M^{4}\left[1-\frac{2}{\pi} \arctan \left(\frac{\phi}{\mu}\right)\right]$,
and depends on one free parameter, $\mu$. This model was considered in order to test the reliability of different computational methods and schemes of approximation used in the calculations of the inflationary cosmological perturbations power spectrum, see Ref. [440]. More precisely, in Ref. [189], it was also used to study with which accuracy the first and second slow-roll order power spectra can approximate the actual power spectrum of the fluctuations in the case where the underlying model has both quite large tilt and running. This potential was considered again in Refs. [441,442] in order to study whether it can lead to the formation of long-lived primordial black holes. In the following slow-roll analysis, $\mu$ will be viewed as a free parameter with no restricted range of variation. Let us notice, however, that since it characterizes the typical vev at which inflation takes place, it could also be limited to the sub-Planckian regime if one wants inflaton to proceed in a small field regime. As a matter of fact, it will be shown below that this needs to be the case if one wants inflation to end by slow-roll violation.

The potential (4.281), as well as its logarithm, are displayed in Fig. 34. They are decreasing functions of the field and, hence, inflation proceed from the left to the right, in the direction specified by the arrow in Fig. 34.

Let us now compute the three first slow-roll parameters. If one defines $x \equiv \phi / \mu$, their expressions are given by

$$
\begin{align*}
& \epsilon_{1}=\frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \frac{2}{\left(1+x^{2}\right)^{2}(\pi-2 \arctan x)^{2}},  \tag{4.282}\\
& \epsilon_{2}=8 \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \frac{1-\pi x+2 x \arctan x}{\left(1+x^{2}\right)^{2}(\pi-2 \arctan x)^{2}},
\end{align*}
$$

and

$$
\begin{align*}
\epsilon_{3} & =2 \frac{M_{\mathrm{PI}}^{2}}{\mu^{2}}\left[-4+6 \pi x+\pi^{2}\left(1-3 x^{2}\right)\right. \\
& \left.+4\left(3 \pi x^{2}-3 x-\pi\right) \arctan x+4\left(1-3 x^{2}\right) \arctan ^{2} x\right] \\
& \times\left[\left(1+x^{2}\right)^{2}(\pi-2 \arctan x)^{2}(-1+\pi x-2 x \arctan x)\right]^{-1} . \tag{4.283}
\end{align*}
$$



Fig. 76. Generalized Mixed Inflation (GMLFI) for $p=3, q=2$ and $\alpha=0.1$. Upper panels: the potential and its logarithm with respect the field value. Bottom left panel: slow-roll parameter $\epsilon_{1}$, the shaded region is where inflation stops. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line).

They are displayed in Fig. 34. The first slow-roll parameter $\epsilon_{1}$ increases during inflation, reaches a maximum at $x_{\epsilon_{1}}^{\max }$ and then decreases. Whether inflation can stop by violation of slow-roll or not depends on the value of $\epsilon_{1}$ at its maximum: $\epsilon_{1}^{\max }$. This value is a solution of the following equation
$2 x_{\epsilon_{1}^{\max }} \arctan \left(x_{\epsilon_{1}^{\max }}\right)+1=\pi x_{\epsilon_{1}^{\max }}$,
which can only be solved numerically. One gets $\chi_{\epsilon_{1}}^{\max } \simeq 0.428978$, from which one deduces that
$\epsilon_{1}^{\max } \simeq 0.262531 \frac{M_{P l}^{2}}{\mu^{2}}$.
Therefore, in order for inflation to end by slow-roll violation, one needs to work under the assumption that $\mu / M_{\mathrm{PI}}<0.512378$. In that case, inflation proceeds along the plateau located at values of $x$ such that $x<x_{\epsilon}^{\max }$, in the direction specified by the arrow in Fig. 34 (i.e. from the left to the right). Otherwise, if one wants inflation to occur in other parts of the potential and/or for values of $\mu$ such that $\mu / M_{\mathrm{PI}}>0.512378$, another mechanism needs to be consider in order to stop it (typically, we imagine a tachyonic instability in another direction in field space). This means that we also need to introduce an extra parameter $x_{\text {end }}$ which gives the location of the $v e v$ at which the tachyonic instability is triggered. Let us remark that we could also consider a model where the inflaton starts at $x<x_{\epsilon_{1}}^{\max }$, then crosses the region where $\epsilon_{1}$ has its maximum and then causes the end of inflation by tachyonic instability. This case would give a bump in the power spectrum and, clearly, cannot be properly described in the slow-roll framework. In this article, we restrict ourselves to the first version of the scenario mentioned above. In this situation $x_{\text {end }}$ is given by the smallest solution of the equation $\epsilon_{1}=1$ and needs to be computed numerically. Before inflation stops, one can see in Fig. 34 that the second slowroll parameter $\epsilon_{2}$ reaches a maximum, the location of which can
be numerically computed to be $x_{\epsilon_{2}^{\max }} \simeq-0.28539<x_{\epsilon_{1}^{\max }}$. At this point, one has $\epsilon_{2}^{\max } \simeq 1.02827 M_{\mathrm{Pl}}^{2} / \mu^{2}>\epsilon_{1}^{\max }$. As a consequence, the slow-roll approximation breaks down before the end of inflation. This conclusion is reinforced by the fact that $\epsilon_{3}$ diverges at $x_{\epsilon_{1}}^{\max }$. This means that the last $e$-folds of inflation cannot be properly described in the slow-roll framework.

Let us now turn to the slow-roll trajectory. It can be integrated exactly and yields the following expression

$$
\begin{align*}
N_{\mathrm{end}}-N= & \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\left[\frac{\pi x_{\mathrm{end}}}{2}+\frac{x_{\mathrm{end}}^{2}}{6}+\frac{\pi x_{\mathrm{end}}^{3}}{6}\right. \\
& -\left(1+\frac{x_{\mathrm{end}}^{2}}{3}\right) x_{\mathrm{end}} \arctan x_{\mathrm{end}}+\frac{1}{3} \ln \left(1+x_{\mathrm{end}}^{2}\right) \\
& -\frac{\pi x}{2}-\frac{x^{2}}{6}-\frac{\pi x^{3}}{6} \\
& \left.+\left(1+\frac{x^{2}}{3}\right) x \arctan x+\frac{1}{3} \ln \left(1+x^{2}\right)\right] \tag{4.286}
\end{align*}
$$

where $N_{\text {end }}$ is the number of $e$-folds at the end of inflation. In the vacuum dominated approximation where the potential is just given by $V(\phi) \simeq M^{4}$, this trajectory can be approximated by $N_{\text {end }}-N=\mu^{2} / M_{\mathrm{Pl}}^{2}\left(\pi x_{\text {end }}+x_{\text {end }}^{2} / 6+\pi x^{3} / 3-\pi x-x^{2} / 6-\pi x^{3} / 3\right)$, which can be inverted exactly if needed. This formula is valid if $\mu / M_{\mathrm{Pl}} \ll 1$, since in that case, $x_{\text {end }} \simeq-\sqrt{M_{\mathrm{PI}} /(\mu \pi \sqrt{2})} \ll-1$. Under this assumption, one has $x_{*}^{3} \simeq-3 M_{\mathrm{Pl}}^{2} /\left(\pi \mu^{2}\right) \Delta N_{*}$, from which one can approximate the values of the three first Hubble flow parameters at Hubble radius crossing
$\epsilon_{1 *}=\frac{\left(\mu / M_{\mathrm{PI}}\right)^{2 / 3}}{2\left(\pi \Delta N_{*}^{2}\right)^{2 / 3}}, \quad \epsilon_{2 *}=\frac{4}{3 \Delta N_{*}}$,
$\epsilon_{3 *}=\frac{1}{\Delta N_{*}}$,


Fig. 77. Logarithmic Potential Inflation (LPI) for $p=4, q=2$. Upper panels: the potential and its logarithm. Bottom left panel: slow-roll parameter $\epsilon_{1}$. Bottom right panel: slow-roll parameters $\epsilon_{2}$ (solid line) and $\epsilon_{3}$ (dotted line).

Then, one can calculate the tensor-to-scalar ratio, the spectral index and the running. One obtains the following expressions
$r=\frac{8\left(\mu / M_{\mathrm{PI}}\right)^{2 / 3}}{\left(\pi \Delta N_{*}^{2}\right)^{2 / 3}}, \quad n_{\mathrm{S}}-1=-\frac{4}{3 \Delta N_{*}} \simeq-0.03$,
$\alpha_{S}=-\frac{4}{3 \Delta N_{*}^{2}} \simeq-5 \times 10^{-4}$.
These formulas are in agreement with the consistency relation $\alpha_{S}=-3 / 4\left(n_{S}-1\right)^{2}$ obtained in Ref. [441].

Finally, it is interesting to estimate the energy scale $M$ from the CMB normalization. This leads to

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=\frac{2880 \pi^{3} M_{\mathrm{Pl}}^{2} / \mu^{2}}{\left(1+x_{*}^{2}\right)^{2}\left(\pi-2 \arctan x_{*}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{4.289}
\end{equation*}
$$

Under the vacuum dominated approximation ( $\mu / M_{\mathrm{PI}} \ll 1$ ), the above equation can be re-expressed as

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4} \simeq \frac{40 \times 3^{2 / 3} \pi^{4 / 3}}{\Delta N_{*}}\left(\frac{\mu}{M_{\mathrm{Pl}}}\right)^{2 / 3} \frac{\mathrm{Q}_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{4.290}
\end{equation*}
$$

The requirement $M<M_{\text {PI }}$ is always satisfied form sub-Planckian values of $\mu$. The typical value $M / M_{\mathrm{Pl}} \simeq 10^{-3}$ corresponds to $\mu / M_{\mathrm{PI}} \simeq 10^{-2}$.

The slow-roll predictions of the AI models are displayed in Fig. 106, in the range $\mu / M_{\mathrm{Pl}}<0.512378$ (so that inflation can end by slow-roll violation). The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to be 0 but since there is no potential minimum around which the inflaton field can oscillate at the end of inflation, this parameter is a priori unspecified. One can see that this model typically predicts a small amount of gravitational waves, and a deviation from scale invariance which is in accordance with the observations. The predictions in the planes ( $n_{\mathrm{s}}, r$ ) are qualitatively well described by the vacuum dominated analysis (4.288) presented before.

### 4.20. Constant $n_{\mathrm{S}}$ A inflation (CNAI)

This class of models is designed in order to produce power spectra with constant spectral index. It was studied for the first time in Ref. [443]. The rational behind this approach is that, so far, no evidence for a significant running has been found in the cosmological data. Since, from a Bayesian point of view, one should avoid introducing parameters that are unnecessary in order to reproduce the observations, it makes sense to consider models which lead to exact power-law power spectra. This is of course the case for power-law inflation as discussed in Section 4.8 and we will see other examples in Sections 4.21, 5.15 and 6.6. In fact, in Ref. [443], a systematic analysis of potentials that yield constant spectral index was carried out. It was found that the following potential belongs to this category of models
$V(\phi)=M^{4}\left[3-\left(3+\alpha^{2}\right) \tanh ^{2}\left(\frac{\alpha}{\sqrt{2}} \frac{\phi}{M_{\mathrm{Pl}}}\right)\right]$,
where $\alpha$ is a positive massless parameter (denoted $n_{0}^{2}$ in Ref. [443]) and, in this section, we study this case. This potential is represented in Fig. 35 and, since it is symmetrical under the transformation $\phi \rightarrow-\phi$, only the $\phi>0$ part is displayed. The potential is a decreasing function of the field vev and, therefore, inflation proceeds from the left to the right. It is positive provided $\phi<\phi_{0}$, where
$\frac{\phi_{0}}{M_{\mathrm{Pl}}}=\frac{\sqrt{2}}{\alpha} \operatorname{arctanh}\left(\sqrt{\frac{3}{3+\alpha^{2}}}\right)$.
There is no value of $\alpha$ for which the potential is always positive. Defining $x=\phi / M_{\mathrm{Pl}}$, the slow-roll parameters are given by
$\epsilon_{1}=\frac{4 \alpha^{2}\left(3+\alpha^{2}\right)^{2} \tanh ^{2}\left(\frac{\alpha x}{\sqrt{2}}\right)}{\left[6+\alpha^{2}-\alpha^{2} \cosh (\sqrt{2} \alpha x)\right]^{2}}$,


Fig. 78. Top left panel: constant $n_{\mathrm{S}} \mathrm{D}$ inflaton potential for $\alpha=1$ and two values of $\beta$, namely $\beta=0.7$ (solid blue line) and $\beta=1.3$ (solid pink line). Top right panel: logarithm of the potential for the same values of $\alpha$ and $\beta$ and with the same color code. Bottom left panel: first slow-roll parameter $\epsilon_{1}$ for a potential with $\alpha=1$ and $\beta=0.7$ (solid blue line), $\beta=1.8$ (solid pink line). The shaded area indicates the breakdown of slow-roll inflation (strictly speaking where acceleration cannot occur). Bottom right panel: second and third slow-roll parameters $\epsilon_{2}$ and $\epsilon_{3}$ for $\alpha=0.25$ and the same values of $\beta$ as in the other plots. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$
\begin{align*}
\epsilon_{2}= & \frac{2 \alpha^{2}\left(3+\alpha^{2}\right)\left[12+\alpha^{2}-2 \alpha^{2} \cosh (\sqrt{2} \alpha x)+\alpha^{2} \cosh (2 \sqrt{2} \alpha x)\right]}{\left[6+\alpha^{2}-\alpha^{2} \cosh (\sqrt{2} \alpha x)\right]^{2} \cosh ^{2}\left(\frac{\alpha x}{\sqrt{2}}\right)},  \tag{4.294}\\
\epsilon_{3}= & 2 \alpha^{2}\left(3+\alpha^{2}\right) \tanh ^{2}\left(\frac{\alpha}{\sqrt{2}} x\right)\left[6\left(-24+2 \alpha^{2}-\alpha^{4}\right)\right. \\
& +\left(120 \alpha^{2}+7 \alpha^{4}\right) \cosh (\sqrt{2} \alpha x) \\
& \left.-2 \alpha^{2}\left(\alpha^{2}-6\right) \cosh (2 \sqrt{2} \alpha x)+\alpha^{4} \cosh (3 \sqrt{2} \alpha x)\right] \\
& \times\left[6+\alpha^{2}-\alpha^{2} \cosh (\sqrt{2} \alpha x)\right]^{-2} \\
& \times\left[12+\alpha^{2}-2 \alpha^{2} \cosh (\sqrt{2} \alpha x)+\alpha^{2} \cosh (2 \sqrt{2} \alpha x)\right]^{-1} . \tag{4.295}
\end{align*}
$$

These slow-roll parameters are displayed in Fig. 35. They all increase as inflation proceeds and diverge when the field approaches $\phi_{0}$. Hence inflation ends by slow-roll violation. Notice that the equation $\epsilon_{1}=1$ can be solved analytically. If we define $y \equiv \sinh ^{2}(\alpha x / \sqrt{2})$, then one has to solve the following cubic equation $\alpha^{4} y^{3}+\left(\alpha^{4}-6 \alpha^{2}\right) y^{2}+\left[9-6 \alpha^{2}-\alpha^{2}\left(3+\alpha^{2}\right)\right] y+9=0$. The relevant solution reads

$$
\begin{align*}
y_{\mathrm{end}}= & \frac{6-\alpha^{2}}{3 \alpha^{2}}-\frac{1-i \sqrt{3}}{3 \times 2^{1 / 3}}\left(3+\alpha^{2}\right)^{2}\left(1+3 \alpha^{2}\right) P^{-1 / 3} \\
& -\frac{1+i \sqrt{3}}{6 \times 2^{1 / 3} \alpha^{4}} P^{1 / 3} \tag{4.296}
\end{align*}
$$

where we have defined $P$ by

$$
\begin{align*}
P & \equiv-\alpha^{6}\left(3+\alpha^{2}\right)^{2}\left(6-52 \alpha^{2}+9 \alpha^{4}\right) \\
& +\sqrt{-27 \alpha^{14}\left(3+\alpha^{2}\right)^{4}\left(36-60 \alpha^{2}+96 \alpha^{4}+25 \alpha^{6}+4 \alpha^{8}\right)} \tag{4.297}
\end{align*}
$$

The slow-roll parameters $\epsilon_{1}$ and $\epsilon_{3}$ both vanish when the field vev goes to 0 , whereas $\epsilon_{2}$ has a non-vanishing minimum value, given
by $\epsilon_{2} \rightarrow 2 \alpha^{2}\left(3+\alpha^{2}\right) / 3$ when $x=0$. Therefore, if $\alpha$ is larger than some maximum value
$\alpha_{\max }=\sqrt{\frac{1}{2}(\sqrt{15}-3)} \simeq 0.66$,
then $\epsilon_{2}$ is larger than 1 in the whole inflationary regime and the slow-roll approximation does not hold. It is therefore necessary to work under the assumption $\alpha<\alpha_{\max }$ which we assume in the following.

Let now us check that the spectral index $n_{S}-1=-2 \epsilon_{1}-\epsilon_{2}$ (at first order in slow-roll), can be made constant, as announced previously. Expanding the slow-roll parameters $\epsilon_{1}$ and $\epsilon_{2}$ in small values of $\alpha$, and crucially assuming that $\alpha x_{*}$ remains small, one obtains $\epsilon_{1}=\mathcal{O}\left(\alpha^{4}\right)$ and $\epsilon_{2}=2 \alpha^{2}+\mathcal{O}\left(\alpha^{4}\right)$, so that $n_{\mathrm{S}}-1=$ $-2 \alpha^{2}+\mathcal{O}\left(\alpha^{4}\right)$. Therefore, the corresponding expression is indeed a constant (i.e. does no depend on $\phi_{*}$ ). Since we have $\left|n_{S}-1\right| \ll 1$, this implies that $\alpha$ should be small which is consistent with the condition $\alpha<\alpha_{\text {max }}$ derived above.

Let us now study the slow-roll trajectory of the system. This one can be integrated exactly leading to the following formula

$$
\begin{align*}
& N-N_{\text {end }}=\frac{1}{\alpha^{2}\left(3+\alpha^{2}\right)}\left\{3 \ln \left[\sinh \left(\frac{\alpha}{\sqrt{2}} x\right)\right]\right. \\
& -\frac{\alpha^{2}}{2} \sinh ^{2}\left(\frac{\alpha}{\sqrt{2}} x\right) \\
& \left.\quad-3 \ln \left[\sinh \left(\frac{\alpha}{\sqrt{2}} x_{\text {end }}\right)\right]+\frac{\alpha^{2}}{2} \sinh ^{2}\left(\frac{\alpha}{\sqrt{2}} x_{\text {end }}\right)\right\} \tag{4.299}
\end{align*}
$$

Moreover, this trajectory can be inverted which allows us to explicitly express the vev of the inflaton field in terms of the $e$-folds


Fig. 79. Upper panel: various ASPIC scenarios in the $\left(n_{S}, r\right)$ plane using the Schwarz-Terrero-Escalante classification [444] and compared to the Planck data [65-70] (blue contours) and the WMAP9 data [72,73] (light gray shading). Bottom panel: same plot in logarithmic scale for another sample of models. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
number. One obtains

$$
\begin{align*}
& x=\frac{\sqrt{2}}{\alpha} \operatorname{arcsinh}\left[-\frac{3}{\alpha^{2}} \mathrm{~W}_{0}\left(-\frac{\alpha^{2}}{3} \exp \left\{\frac{2}{3} \alpha^{2}\left(3+\alpha^{2}\right)\left(N-N_{\mathrm{end}}\right)\right.\right.\right. \\
& \left.\left.\left.+2 \ln \left[\sinh \left(\frac{\alpha}{\sqrt{2}} x_{\text {end }}\right)\right]-\frac{\alpha^{2}}{3} \sinh ^{2}\left(\frac{\alpha}{\sqrt{2}} x_{\text {end }}\right)\right\}\right)\right]^{1 / 2}, \tag{4.300}
\end{align*}
$$

where $\mathrm{W}_{0}$ is the 0 branch of the Lambert function as required since $x(N)$ is an increasing function of $N$. It is displayed in Fig. 36 where the CNAI trajectory takes place between $\phi / M_{\mathrm{PI}}=0$ at the origin of the plot, and $x=\phi_{0} / M_{\text {PI }}$ at the junction between the -1 branch and the 0 branch.

The slow-roll predictions of the CNAI models are displayed in Fig. 107. When $\alpha$ is small (but not too small), the value of $n_{\mathrm{S}}$ is indeed constant (and compatible with the considerations presented above) but, unfortunately, too far from scale invariance to be compatible with CMB data. When $\alpha \ll 10^{-1}$, the predictions become roughly compatible with the data but, clearly, $n_{\mathrm{S}}$ is no longer constant and no longer given by $-2 \alpha^{2}$. At first sight, this is surprising since we expect the spectral index to tend towards $-2 \alpha^{2}$ when $\alpha$ goes to zero (see above). In order to understand this point, let us remark that, in the limit where $\alpha$ vanishes, one


Fig. 80. Observable predictions in the ( $n_{\mathrm{S}}, r$ ) plane for various models belonging to region 1 of the Schwarz-Terrero-Escalante classification (see Fig. 79). Despite the fact that they are in the same broad class, the accuracy of the CMB data allows us to discriminate among them thereby justifying a detailed navigation within the inflationary landscape.
can expand Eq. (4.296) to find $y_{\text {end }} \simeq 3 / \alpha^{2}-3 / \alpha+\mathcal{O}(\alpha)$ (the term at order $\alpha^{0}$ is absent and this plays an important role in what follows). This leads to $x_{\text {end }} \simeq(\sqrt{2} / \alpha) \ln (2 \sqrt{3} / \alpha)-1 / \sqrt{2}+\mathcal{O}(\alpha)$. Notice that this last equation is compatible with the behavior of the first Hubble-flow parameter (4.293) in the vicinity of $\phi_{0}: \epsilon_{1} \simeq$ $M_{\mathrm{Pl}}^{2} /\left[2\left(\phi-\phi_{0}\right)^{2}\right]$. Therefore, the expression of $x_{\text {end }}$ found before corresponds in fact to writing $\epsilon_{1}=1$ with this approximated $\epsilon_{1}$. Then, using the slow-roll trajectory (4.300), one gets
$\sinh ^{2}\left(\frac{\alpha x_{*}}{\sqrt{2}}\right)=-\frac{3}{\alpha^{2}} \mathrm{~W}_{0}\left(-\frac{\alpha^{2}}{3} e^{-2 A / 3}\right)$,
where $A$ is given by the following expression

$$
\begin{align*}
A \equiv & \alpha^{2}\left(3+\alpha^{2}\right) \Delta N_{*}-3 \ln \left[\sinh \left(\frac{\alpha x_{\text {end }}}{\sqrt{2}}\right)\right] \\
& +\frac{\alpha^{2}}{2} \sinh ^{2}\left(\frac{\alpha x_{\text {end }}}{\sqrt{2}}\right) . \tag{4.302}
\end{align*}
$$

This quantity can be expanded in $\alpha$ using the equation for $y_{\text {end }}$ derived above and, at leading order, one obtains
$-\frac{2}{3} A \simeq-\frac{2}{3} \alpha^{2} \Delta N_{*}+\ln \left(\frac{3}{\alpha^{2}}\right)-1-\frac{\alpha^{2}}{2}$.
For simplicity, the last term in the previous expression can be ignored since $2 \Delta N_{*} \gg 1 / 2$. It follows that, introducing the formula for $-2 A / 3$ into Eq. (4.301), one arrives at
$\sinh ^{2}\left(\frac{\alpha X_{*}}{\sqrt{2}}\right)=-\frac{3}{\alpha^{2}} \mathrm{~W}_{0}\left(-\frac{1}{e} e^{-2 \alpha^{2} \Delta N_{*}}\right)$.
If we ignore the exponential in the argument of the Lambert function (since $\alpha \ll 1$ ) and use the identity $\operatorname{arcsinh}(x)=\ln (x+$ $\sqrt{x^{2}+1}$ ), one finally arrives at
$\alpha x_{*} \underset{\alpha \rightarrow 0}{\sim} \sqrt{2} \ln \left(\frac{2 \sqrt{3}}{\alpha}\right)$.
We now understand why, in the limit $\alpha \rightarrow 0$, the spectral index is no longer constant. The naive expression $n_{\mathrm{S}} \simeq-2 \alpha^{2}$ is obtained by expanding the expressions of $\epsilon_{1}$ and $\epsilon_{2}$ in $\alpha$, including the hyperbolic function of argument $\alpha x_{*}$. But we have just shown that, when $\alpha \ll 1, \alpha x_{*}$ is not small and, therefore, the Taylor expansion of those terms is no longer justified. This is why, in Fig. 107, we see
a deviation from $n_{S}$ constant at very small values of $\alpha$. In fact, this questions the interest of this model since the condition of constant spectral index is obtained only for values of $n_{\mathrm{S}}$ that are already ruled out by the CMB data. On the other hand, when $\alpha \ll 1$, the model seems compatible with the data and, therefore, represents a legitimate inflationary scenario even if the spectral index is not constant in this case.

Finally, it is also interesting to study the energy scale at which inflation takes place in this model. The CMB normalization gives

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=\frac{11520 \pi^{2} \alpha^{2}\left(\alpha^{2}+3\right)^{2} \sinh ^{2}\left(\frac{\alpha}{\sqrt{2}} x_{*}\right)}{\left[\alpha^{2}+6-\alpha^{2} \cosh \left(\sqrt{2} \alpha x_{*}\right)\right]^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{4.306}
\end{equation*}
$$

Since we have established the expression of $\chi_{*}$ above, it is sufficient to use it in the above formula. We have, however, to be careful about the calculation of the denominator. Indeed, if we neglect again the exponential in the argument of the Lambert function, Eq. (4.301), then $\sinh ^{2}\left(\alpha x_{*} / \sqrt{2}\right) \simeq 3 / \alpha^{2}$ and the denominator in Eq. (4.306) vanishes. Therefore, one needs to evaluate the Lambert function more precisely and to keep the corrections proportional to $\Delta N_{*}$. This can be done with the help of Eq. (33) of Ref. [445] which implies that $\sinh ^{2}\left(\alpha x_{*} / \sqrt{2}\right) \simeq 3 / \alpha^{2}-6 \sqrt{\Delta N_{*}} / \alpha$. Using this expression, one arrives at
$\frac{M}{M_{\mathrm{Pl}}} \simeq 0.016 \alpha^{-3 / 4}\left(\Delta N_{*}\right)^{-3 / 8}$.
For an order of magnitude estimate, one can use the fiducial value $\Delta N_{*} \simeq 55$. This leads to $M / M_{\mathrm{PI}} \simeq 0.0035 \alpha^{-3 / 4}$. Requiring $M<$ $M_{\mathrm{PI}}$ puts a lower bound on the parameter $\alpha$, namely $\alpha \gtrsim 5 \times 10^{-4}$. This roughly corresponds to the range studied in Fig. 107.

### 4.21. Constant $n_{S} B$ inflation (CNBI)

This model is another representative of the class of scenarios studied in Ref. [443]. As was already discussed in Section 4.20, it is designed such that the corresponding power spectrum has a constant spectral index. The potential is given by
$V(\phi)=M^{4}\left[\left(3-\alpha^{2}\right) \tan ^{2}\left(\frac{\alpha}{\sqrt{2}} \frac{\phi}{M_{\mathrm{Pl}}}\right)-3\right]$,
where $\alpha$ is a positive dimensionless parameter [443]. Since the potential is periodic with period $\pi \sqrt{2} / \alpha$ and, moreover, invariant under $\phi \rightarrow-\phi$, one can restrict ourselves to the range $0<$ $\phi / M_{\mathrm{Pl}}<\pi /(\sqrt{2} \alpha)$ without loss of generality. The potential is an increasing function of the field and, as a consequence, inflation proceeds from the right to the left. Finally, $V(\phi)$ is positive provided $\phi>\phi_{0}$, where
$\frac{\phi_{0}}{M_{\mathrm{Pl}}}=\frac{\sqrt{2}}{\alpha} \arctan \left(\sqrt{\frac{3}{3-\alpha^{2}}}\right)$.
Obviously, in order for the potential not to be negative everywhere, one needs to impose that $\alpha<\sqrt{3}$ and, as a result, the previous expression is well defined. The potential (and its logarithm) is displayed in Fig. 37, in the relevant range $\phi_{0} / M_{\mathrm{PI}}<\phi / M_{\mathrm{PI}}<$ $\pi /(\sqrt{2} \alpha)$.

Then, defining $x=\phi / M_{\mathrm{Pl}}$, the slow-roll parameters are given by
$\epsilon_{1}=\frac{4 \alpha^{2}\left(\alpha^{2}-3\right)^{2} \tan ^{2}\left(\frac{\alpha}{\sqrt{2}} x\right)}{\left[\alpha^{2}+\left(6-\alpha^{2}\right) \cos (\sqrt{2} \alpha x)\right]^{2}}$,


Fig. 81. Reheating consistent slow-roll predictions for the Higgs model in the plane $\left(n_{s}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which the large field reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$.
$\epsilon_{2}=\frac{\alpha^{2}\left(3-\alpha^{2}\right)\left[6+\alpha^{2}+2\left(6-\alpha^{2}\right) \cos (\sqrt{2} \alpha x)+\left(\alpha^{2}-6\right) \cos (2 \sqrt{2} \alpha x)\right]}{2 \cos ^{6}\left(\frac{\alpha}{\sqrt{2}} x\right)\left[3+\left(\alpha^{2}-3\right) \tan ^{2}\left(\frac{\alpha x}{\sqrt{2}}\right)\right]^{2}}$,
and

$$
\begin{align*}
\epsilon_{3}= & 2 \alpha^{2}\left(\alpha^{2}-3\right) \tan ^{2}\left(\frac{\alpha}{\sqrt{2}} x\right)\left[6\left(-72+14 \alpha^{2}-\alpha^{4}\right)\right. \\
& +\left(\alpha^{2}-6\right)\left(7 \alpha^{2}+78\right) \cos (\sqrt{2} \alpha x) \\
& -2\left(\alpha^{4}-18 \alpha^{2}+72\right) \cos (2 \sqrt{2} \alpha x) \\
& \left.+\left(\alpha^{2}-6\right)^{2} \cos (3 \sqrt{2} \alpha x)\right] \\
& \times\left[\alpha^{2}+\left(6-\alpha^{2}\right) \cos (\sqrt{2} \alpha x)\right]^{-2} \\
& \times\left[6+\alpha^{2}+2\left(6-\alpha^{2}\right) \cos (\sqrt{2} \alpha x)\right. \\
& \left.+\left(\alpha^{2}-6\right) \cos (2 \sqrt{2} \alpha x)\right]^{-1} . \tag{4.312}
\end{align*}
$$

These slow-roll parameters are displayed in Fig. 37 (bottom panels). The first slow-roll parameter $\epsilon_{1}$ first decreases as the field $v e v$ increases and reaches a minimum value at $x_{\epsilon_{2}}=0$ where $\epsilon_{2}$


Fig. 82. Reheating consistent slow-roll predictions for the radiatively corrected Higgs model in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which the large field reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
vanishes and then increases. The value of $x_{\epsilon_{2}=0}$ is given by
$x_{\epsilon_{2}=0}=\frac{1}{\alpha \sqrt{2}} \arccos \left[\frac{\alpha^{2}-6+\sqrt{\alpha^{4}-36 \alpha^{2}+180}}{2\left(\alpha^{2}-6\right)}\right]$.
The second slow-roll parameter, $\epsilon_{2}$, always decreases as inflation proceeds, crossing $\epsilon_{2}=0$ at $x_{\epsilon_{2}=0}$. The third slow-roll parameter, $\epsilon_{3}$, is positive for $x<x_{\epsilon_{2}=0}$. In this domain, it decreases to reach a minimum and then increases and diverges when $x$ approaches $x_{\epsilon_{2}=0}$. On the contrary, for $x>x_{\epsilon_{2}=0}, \epsilon_{3}$ becomes negative. It first increases and reaches a local maximum, then decreases and goes to $-\infty$ at $x=\pi /(\sqrt{2} \alpha)$. The three slow roll parameters diverge when $\phi$ goes to $\phi_{0}$ and to $M_{\mathrm{PI}} \pi /(\sqrt{2} \alpha)$.

The minimum value of $\epsilon_{1}$ at $\chi_{\epsilon_{2}=0}$ turns out to be smaller than 1 only if $\alpha<\alpha_{\max } \simeq 0.2975$. A (rather long) analytic expression for $\alpha_{\text {max }}$ can be derived, but it does not provide much information to the present discussion. Therefore, one must require $\alpha<0.2975$ in order to realize slow-roll inflation in this model. Then, assuming this is the case, it is clear from Fig. 37 and from the previous considerations that inflation ends by slow-roll violation. If we define $y \equiv \sin ^{2}(\alpha x / \sqrt{2})$, then the condition $\epsilon_{1}=1$ is equivalent to $4\left(6-\alpha^{2}\right)^{2} y^{3}-4\left(12-\alpha^{2}\right)\left(6-\alpha^{2}\right) y^{2}+4\left(45+3 \alpha^{2}-6 \alpha^{4}+\alpha^{6}\right) y-36=$ 0 . The relevant solution is given by
$y_{\text {end }}=\frac{12-\alpha^{2}}{3\left(6-\alpha^{2}\right)}+\frac{4}{3} 2^{-2 / 3}(1-i \sqrt{3})$


Fig. 83. Reheating consistent slow-roll predictions for the large field models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The black solid lines represent the locus of different LFI-p models [for which $(1+2 / p) r=8\left(1-n_{\mathrm{S}}\right)$, i.e. $\epsilon_{1}=(p / 4) \epsilon_{2}$ ]. The annotations trace the energy scale at which the large field reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. Large reheating temperatures are preferred and models with $p>2$ are disfavored at two sigma confidence level. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$
\begin{align*}
& \times \frac{\left(3 \alpha^{2}-1\right)\left(18-9 \alpha^{2}+\alpha^{4}\right)^{2}}{\left(6-\alpha^{2}\right)^{2}} P^{-1 / 3} \\
& -(1+i \sqrt{3}) \frac{2^{-1 / 3}}{24\left(6-\alpha^{2}\right)^{2}} P^{1 / 3}, \tag{4.314}
\end{align*}
$$

where we have defined the quantity $P$ by

$$
\begin{align*}
P \equiv & 64\left(-6+\alpha^{2}\right)^{3}\left(-3+\alpha^{2}\right)^{2}\left(-6+110 \alpha^{2}-9 \alpha^{4}+3 \alpha \sqrt{3}\right. \\
& \left.\times \sqrt{-36+408 \alpha^{2}-12 \alpha^{4}-25 \alpha^{6}+4 \alpha^{8}}\right) \tag{4.315}
\end{align*}
$$

If $\alpha \ll 1$, then $y_{\text {end }} \simeq 1 / 2$ and $x_{\text {end }} \simeq \sqrt{2} / \alpha \arcsin (1 / \sqrt{2})=$ $\pi /(2 \sqrt{2} \alpha)$.

As for the CNAI model, the spectral index $n_{S}-1=-2 \epsilon_{1}-\epsilon_{2}$, at first order in slow-roll, can be made constant in some limit. Expanding the slow-roll parameters in $\alpha$, while assuming $\alpha x$ to be small, gives $\epsilon_{1}=x^{2} \alpha^{4} / 2+\mathcal{O}\left(\alpha^{6}\right)$ and $\epsilon_{2}=2 \alpha^{2}+\mathcal{O}\left(\alpha^{4}\right)$, so that $n_{S}-1=-2 \alpha^{2}+\mathcal{O}\left(\alpha^{4}\right)$. Therefore, approximate scale-invariance, $\left|n_{\mathrm{S}}-1\right| \ll 1$, implies $\alpha$ small.

Let us now turn to the slow-roll trajectory. This one can be integrated exactly, leading to the following formula

$$
\begin{align*}
N-N_{\text {end }}= & \frac{1}{\alpha^{2}\left(3-\alpha^{2}\right)}\left\{3 \ln \left[\sin \left(\frac{\alpha}{\sqrt{2}} x\right)\right]\right. \\
& -\frac{6-\alpha^{2}}{2} \sin ^{2}\left(\frac{\alpha}{\sqrt{2}} x\right)-3 \ln \left[\sin \left(\frac{\alpha}{\sqrt{2}} x_{\text {end }}\right)\right] \\
& \left.+\frac{6-\alpha^{2}}{2} \sin ^{2}\left(\frac{\alpha}{\sqrt{2}} x_{\text {end }}\right)\right\} . \tag{4.316}
\end{align*}
$$

This formula can be inverted and $x$ can be expressed explicitly in terms of the $e$-folds number. One obtains

$$
\begin{align*}
x & =\frac{\sqrt{2}}{\alpha} \arcsin \left[-\frac{3}{6-\alpha^{2}} \mathrm{~W}_{-1}\left(-\frac{6-\alpha^{2}}{3}\right.\right. \\
& \times \exp \left\{\frac{2}{3} \alpha^{2}\left(3-\alpha^{2}\right)\left(N-N_{\text {end }}\right)+2 \ln \left[\sin \left(\frac{\alpha}{\sqrt{2}} x_{\text {end }}\right)\right]\right. \\
& \left.\left.\left.-\frac{6-\alpha^{2}}{3} \sin ^{2}\left(\frac{\alpha}{\sqrt{2}} x_{\text {end }}\right)\right\}\right)\right]^{1 / 2}, \tag{4.317}
\end{align*}
$$

where $\mathrm{W}_{-1}$ is the -1 branch of the Lambert function. It is displayed in Fig. 38. When $x=\pi /(\sqrt{2} \alpha)$, the argument of the Lambert function is $\left(\alpha^{2}-6\right) \exp \left(\alpha^{2} / 3-2\right) / 3$ which is always larger than $-1 / e$ for any value of $\alpha$ (this expression decreases with $\alpha$ when $\alpha<\sqrt{3}$ ), whereas when $x=\phi_{0} / M_{P 1}$, the argument of the Lambert function is just given by $-1 / e$. For $x>\phi_{0} / M_{\mathrm{P}}$, the value taken by the Lambert function must be less than -1 which indicates that the -1 branch is the relevant one. Therefore, inflation proceeds in the domain displayed in Fig. 38 in which one easily checks that the above trajectory is always well defined.

The slow-roll predictions of the CNBI models are displayed in Fig. 108 for the range $10^{-5} \lesssim \alpha \lesssim 10^{-1.3}$. For very small values of $\alpha$, the predictions are in agreement with the data with a value of $n_{\mathrm{S}}$ centered around the constant value $n_{\mathrm{S}} \simeq 0.97$ and an amount of gravitational waves such that $r \gtrsim 0.07$. But one also notices that the spectral index is not really constant. In fact, it does not come as a surprise that the same phenomenon highlighted in Section 4.20 is at work here. Indeed, using the slow-roll trajectory (4.316), one has
$\sin ^{2}\left(\frac{\alpha x_{*}}{\sqrt{2}}\right)=-\frac{3}{6-\alpha^{2}} \mathrm{~W}_{-1}\left(-\frac{6-\alpha^{2}}{3} e^{-2 A / 3}\right)$,
where $A$ is given by the following expression

$$
\begin{align*}
A \equiv & \alpha^{2}\left(3-\alpha^{2}\right) \Delta N_{*}-3 \ln \left[\sin \left(\frac{\alpha x_{\text {end }}}{\sqrt{2}}\right)\right] \\
& +\frac{6-\alpha^{2}}{2} \sin ^{2}\left(\frac{\alpha x_{\text {end }}}{\sqrt{2}}\right) . \tag{4.319}
\end{align*}
$$

Using the formula for $x_{\text {end }}$ derived above, one obtains, in the limit $\alpha \ll 1$ and at this order of approximation that $x_{*} \simeq x_{\text {end }}$. Therefore, as in Section 4.20, $\alpha x_{*}$ is not a small quantity and one cannot always Taylor expand the trigonometric functions that appear in the expressions of the slow-roll parameters. This explains why, in the limit $\alpha \ll 1$, the spectral index is in fact not constant (see Section 4.20).

Finally, the CMB normalization gives

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=\frac{11520 \pi^{2} \alpha^{2}\left(3-\alpha^{2}\right)^{2} \sin ^{2}\left(\frac{\alpha}{\sqrt{2}} x_{*}\right)}{\left[\left(\alpha^{2}-6\right) \cos \left(\sqrt{2} \alpha x_{*}\right)-\alpha^{2}\right]^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{4.320}
\end{equation*}
$$

In the limit $\alpha \ll 1$ we are interested in (since we have seen that, if $\alpha$ is not small, then the model is ruled out), the above expression


Fig. 84. Reheating consistent slow-roll predictions for the mixed large field models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 since the potential is quadratic close to its minimum. The black solid lines represent the locus of the quadratic model (namely LFI with $p=2$ ) and of the quartic model (namely LFI with $p=4$ ) [for which $(1+2 / p) r=8\left(1-n_{\mathrm{S}}\right)$, i.e. $\left.\epsilon_{1}=(p / 4) \epsilon_{2}\right]$. The annotations trace the energy scale at which the mixed large field reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. Clearly, these values are limited from below to stay inside the two-sigma contours and models with $\alpha>10^{-3}$ are excluded at two-sigma confidence level. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
takes the form $M / M_{\mathrm{PI}} \simeq 0.02 \alpha^{-1 / 4}\left(\Delta N_{*}\right)^{-3 / 8}$. We obtain almost exactly the same result as for CNAI, see Eq. (4.306), except that the power of $\alpha$ is different. Taking the value $\Delta N_{*}=55$, it follows that $M / M_{\mathrm{PI}} \simeq 0.0044 \alpha^{-1 / 4}$ and requiring $M<M_{\mathrm{Pl}}$, one obtains the following lower bound, $\alpha \gtrsim 3.8 \times 10^{-10}$.

### 4.22. Open string tachyonic inflation (OSTI)

### 4.22.1. Theoretical justifications

In this section, we consider tachyon inflation. It was shown in Refs. [446-449] that, in bosonic string theory, the four-dimensional action for a tachyon field $T$ on a D3-brane can be approximated as [448,449]

$$
\begin{align*}
S_{T}= & T_{3} \int \mathrm{~d}^{4} \boldsymbol{x} \sqrt{-g}\left[\alpha^{\prime} e^{-T / T_{0}} \partial_{\mu}\left(\frac{T}{T_{0}}\right) \partial^{\mu}\left(\frac{T}{T_{0}}\right)\right. \\
& \left.+\left(1+\frac{T}{T_{0}}\right) e^{-T / T_{0}}\right], \tag{4.321}
\end{align*}
$$

where higher derivative terms have been ignored. In this stringy setting, $T_{0}$ is of the order of the string scale $T_{0} \simeq M_{\mathrm{s}}=\ell_{\mathrm{s}}^{-1}=$ $1 / \sqrt{\alpha^{\prime}}$, where $\ell_{s}$ is the string length. The constant $T_{3}$ is the brane tension which can be expressed as $T_{3} \propto M_{\mathrm{s}}^{4} / g_{s}, g_{s}$ being the string coupling. The tachyon is assumed to be minimally coupled
to Einstein gravity and the Planck mass in four dimensions can be written as $M_{\mathrm{Pl}}^{2}=M_{\mathrm{s}}^{2} v / g_{\mathrm{s}}^{2}$, where $v=\left(M_{\mathrm{s}} r\right)^{d} / \pi, r$ being a radius of compactification and $d$ the number of compactified dimensions. This four dimensional approximation is valid provided $r \gg \ell_{\mathrm{s}}$ or $v \gg 1$. The action (4.321) can be viewed as a truncated version of the action
$S_{\bar{T}}=\int \mathrm{d}^{4} \boldsymbol{x} \sqrt{-g} V(\bar{T}) \sqrt{1+\alpha^{\prime} \partial_{\mu}\left(\frac{\bar{T}}{T_{0}}\right) \partial^{\mu}\left(\frac{\bar{T}}{T_{0}}\right)}$.
Indeed, following Refs. [301,450,451], redefining the field $\bar{T}$ by $\bar{T} / T_{0} \equiv \sqrt{8\left(1+T / T_{0}\right)}$ with $V[\bar{T}(T)] \equiv T_{3}\left(1+T / T_{0}\right) \exp \left(-T / T_{0}\right)$, it is straightforward to show that the leading terms of Eq. (4.322) give back Eq. (4.321). Conversely, the full action of tachyonic inflation, under the assumptions discussed previously, can thus be described in terms of $\bar{T}$ by Eq. (4.322) with [450]
$V(\bar{T})=\frac{T_{3} e}{8} \frac{\bar{T}^{2}}{T_{0}^{2}} e^{-\bar{T}^{2} /\left(8 T_{0}^{2}\right)}$.
Because the action (4.322) is a particular case of k -inflation for which $S=\int \mathrm{d}^{4} \boldsymbol{x} \sqrt{-g} P(T, X)$ with $X \equiv-g^{\mu \nu} \partial_{\mu} T \partial_{\nu} T / 2$ and, here, $P(T, X)=\sqrt{1-2 X}$, tachyonic inflation could produce observable non-Gaussianities. Therefore, one may wonder how accurate is the truncated action to describe the observable features of the model. On the theoretical point of view, knowing whether the truncated action is a faithful representation of the actual action is a complicated question since even an exact derivation of the complete action is still an open problem. On a more phenomenological point of view, non-Gaussianities are not observed by Planck [67]. More precisely, the parameter $f_{\mathrm{NL}}$ (equilateral configuration) characterizing the amplitude of the bispectrum in Fourier space can be written as $[126,452]$
$f_{\mathrm{NL}}=\frac{35}{108}\left(\frac{1}{c_{\mathrm{S}}^{2}}-1\right)-\frac{5}{81}\left(\frac{1}{c_{\mathrm{S}}^{2}}-1-2 \Lambda\right)$,
where, in our case, $c_{\mathrm{S}}^{2}=1-2 X$ and $1 / c_{\mathrm{S}}^{2}-1=2 \Lambda$ so that the last term in the above equation cancels out [452]. This leads to $f_{\mathrm{NL}}=35 X /[54(1-2 X)]$. In the range of interest $X \in[0,1 / 2]$, the Planck constraint [67], $f_{\mathrm{NL}}=-42 \pm 75$, yields $X \lesssim 0.495$. As a result, departures from the leading order (4.321) are, a priori, still allowed by the CMB data. We will see at the end of this section that tachyonic inflation has however other problems. For the moment, given that Eq. (4.321) can always be seen as a phenomenological model, we can continue to work with this action in order to see if, at least, this can lead to an inflationary scenario compatible with the CMB data.

### 4.22.2. Slow-roll analysis

The inflationary dynamics can be studied directly from Eq. (4.321) but since it is linear in $X$, the field can be canonically normalized. Performing the change of variable $e^{-T / T_{0}} \equiv\left(\phi / T_{0}\right)^{2} / 8$, the Lagrangian can be re-written with an ordinary kinetic term, as a function of the field $\phi$ and with a potential given by
$V(\phi)=-M^{4}\left(\frac{\phi}{\phi_{0}}\right)^{2} \ln \left[\left(\frac{\phi}{\phi_{0}}\right)^{2}\right]$,
where $M^{4} \equiv e T_{3}$ and $\phi_{0}^{2} \equiv 8 e T_{0}^{2}$. We notice that it corresponds to a particular case of LPI discussed in Section 6.5, with $q=1$ and $p=2$. Such a potential was also introduced in Ref. [453] as a toy model of tachyon condensation. Let us also comment on the parameter $\phi_{0}$. In the original model $\phi_{0} \simeq M_{\mathrm{s}}$ and, as such, it is a zero-parameter scenario. Here, given the issues discussed before (see also the end of this section) we consider $\phi_{0}$ as a free parameter.


Fig. 85. Reheating consistent slow-roll predictions for the radiatively corrected massive models in the plane ( $n_{\mathrm{s}}, r$ ). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The black solid line represent the locus of the quadratic model [i.e. LFI with $p=2$, for which $r=4\left(1-n_{\mathrm{S}}\right)$, i.e. $\epsilon_{1}=\epsilon_{2} / 2$ ]. The annotations trace the energy scale at which the radiatively corrected massive reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. Clearly, these values are limited from below to stay inside the two-sigma contours and models with $\alpha>10^{-3.5}$ are disfavored at two sigma confidence level. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

If necessary, one can always recover the situation where $\phi_{0}$ is fixed to the string scale by assuming the corresponding prior $\phi_{0}=M_{\mathrm{s}}$.

The potential (4.325) in represented in Fig. 39, together with its logarithm (top panels), as a function of $x \equiv \phi / \phi_{0}$. Since it is invariant under $x \rightarrow-x$, and since it is positive definite only if $x^{2}<1$, it is only displayed in the range $0<x<1$. The potential vanishes at $x=0$, increases with $x$, reaches a maximum at $x_{V^{\prime}=0}=$ $e^{-1 / 2}$, then decreases with $x$ and vanishes at $x_{V=0}=1$. Inflation is supposed to take place between $x_{V^{\prime}=0}$, where the effective mass of the inflaton is negative $m_{\phi}^{2}=-4 \phi_{0}^{2}$, and $x=0$, where the effective mass is positive and infinite $m_{\phi}^{2} \rightarrow+\infty$. Hence it proceeds from the right to the left, at decreasing field values (see Fig. 39).

Let us now calculate the three first slow-roll parameters. They are given by
$\epsilon_{1}=2\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2}\left[\frac{1+\ln \left(x^{2}\right)}{x \ln \left(x^{2}\right)}\right]^{2}$,
$\epsilon_{2}=4\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} \frac{2+\ln \left(x^{2}\right)+\ln ^{2}\left(x^{2}\right)}{x^{2} \ln ^{2}\left(x^{2}\right)}$,
and
$\epsilon_{3}=4\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} \frac{1+\ln \left(x^{2}\right)}{x^{2} \ln ^{2}\left(x^{2}\right)} \frac{4+3 \ln \left(x^{2}\right)+\ln ^{2}\left(x^{2}\right)+\ln ^{3}\left(x^{2}\right)}{2+\ln \left(x^{2}\right)+\ln ^{2}\left(x^{2}\right)}$.


Fig. 86. Reheating consistent slow-roll predictions for the radiatively corrected quartic models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel), with $\bar{w}_{\text {reh }}=0$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The black solid line represent the locus of the quartic model [i.e. LFI with $p=4$, for which $r=$ $(16 / 3)\left(1-n_{S}\right)$, i.e. $\epsilon_{1}=\epsilon_{2}$ ]. The annotations trace the energy scale at which the radiatively corrected quartic reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. Clearly, these values are limited from below, and regardless of them, these models seem to be disfavored at two sigma confidence level. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

They are displayed in the bottom panels of Fig. 39. The first slowroll parameter $\epsilon_{1}$ diverges when $x \rightarrow 0$, decreases with $x$, vanishes at $x_{V^{\prime}=0}$ and then increases with $x$ and diverges when $x \rightarrow x_{V=0}$. As a consequence, inflation stops by slow-roll violation at a point $x_{\text {end }}$ where $\epsilon_{1}=1$ that needs to be determined numerically. The second slow-roll parameter $\epsilon_{2}$ has the same kind of behavior, except that it has a non-vanishing minimum located at a point $x_{\epsilon_{2} \text { min }}$, which is such that $0<x_{\epsilon_{2}^{\min }}<x_{V=0}$. An analytic expression for $x_{\epsilon_{2}^{\min }}$ can be derived but it does not add much to the discussion. It yields $\epsilon_{2}^{\min } \simeq$ $20.65 M_{\mathrm{Pl}}^{2} / \phi_{0}^{2}$. This means that in order for a slow-roll inflationary regime to take place, $\epsilon_{2}^{\min } \ll 1$ requires that the parameter $\phi_{0}$ be sufficiently super-Planckian. Finally, the third slow-roll parameter has the same behavior as the two previous ones, except that it has a negative minimum $\epsilon_{3}^{\min } \simeq-0.2733 M_{\mathrm{Pl}}^{2} / \phi_{0}^{2}$, located between $x_{\epsilon_{2}^{\min }}$ and $x_{V^{\prime}=0}$ where it vanishes.

Let us now turn to the slow-roll trajectory. It can be integrated, and gives rise to

$$
\begin{align*}
N_{\mathrm{end}}-N= & \frac{1}{4}\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2}\left[x^{2}-\frac{1}{e} \operatorname{Ei}\left(1+\ln x^{2}\right)-x_{\mathrm{end}}^{2}\right. \\
& \left.+\frac{1}{e} \operatorname{Ei}\left(1+\ln x_{\mathrm{end}}^{2}\right)\right], \tag{4.329}
\end{align*}
$$

where Ei is the exponential integral function [216,217] and $N_{\text {end }}$ is the number of $e$-folds at the end of inflation. This trajectory can only be inverted numerically to obtain $\phi(N)$.

Finally, it is interesting to constrain the value of the scale $M$ with the CMB normalization. It follows that

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=2880 \pi^{2}\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} \frac{\left[1+\ln \left(x_{*}^{2}\right)\right]^{2}}{x_{*}^{4}\left|\ln \left(x_{*}^{2}\right)\right|^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{4.330}
\end{equation*}
$$

The reheating consistent slow-roll predictions of the open string tachyonic inflation models are displayed in Fig. 109. It is interesting to notice that, as expected, these models are compatible with the CMB data only for super-Planckian values of $\phi_{0}, \phi_{0} / M_{\mathrm{PI}} \gg 1$. In this limit, one has $x_{\text {end }} \simeq \sqrt{2} M_{\mathrm{Pl}} / \phi_{0}$, the quadratic terms in the slow roll trajectory Eq. (4.329) dominate over the exponential integral ones, such that one has $x_{*} \simeq 2 M_{\mathrm{PI}} / \phi_{0} \sqrt{\Delta N_{*}+\frac{1}{2}}$. It follows that
$\epsilon_{1 *} \simeq \frac{1}{2 \Delta N_{*}+1}, \quad \epsilon_{2 *} \simeq \epsilon_{3 *} \simeq 2 \epsilon_{1 *}$,
hence

$$
\begin{align*}
& r \simeq \frac{16}{2 \Delta N_{*}+1}, \quad 1-n_{\mathrm{S}} \simeq \frac{4}{2 \Delta N_{*}+1}, \quad \text { and } \\
& \alpha_{\mathrm{S}} \simeq-\frac{8}{\left(2 \Delta N_{*}+1\right)^{2}} \tag{4.332}
\end{align*}
$$

One can check that indeed, in the $\phi_{0} / M_{\mathrm{PI}} \gg 1$ limit, the prediction points lie in the line $\epsilon_{2}=2 \epsilon_{1}$, or equivalently, $1-n_{\mathrm{S}}=r / 4$.

Finally, let us close this section by some additional considerations on the difficulties that tachyonic inflation faces [450]. Using the above equations, it is easy to show that

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4} \simeq \frac{2880 \pi^{2}}{16 \Delta N_{*}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}} \frac{\left[5-2 \ln \left(\phi_{0} / M_{\mathrm{Pl}}\right)\right]^{2}}{\left[4-2 \ln \left(\phi_{0} / M_{\mathrm{Pl}}\right)\right]^{3}} \ll 1 . \tag{4.333}
\end{equation*}
$$

Given that $T_{3} \simeq M^{4}$, this implies that $g_{\mathrm{s}}^{3} \ll v^{2}$. On the other hand, we have seen that the model is compatible with the CMB data only if $\phi_{0} / M_{\mathrm{PI}}=(\mathrm{g} / v)^{1 / 2} \gg 1$. This last inequality is consistent with $g_{s}^{3} \ll v^{2}$ only if $v \ll 1$. But $v \ll 1$ is in contradiction with the assumption that $r \gg \ell_{s}$, which implies that $v \gg 1$. Therefore, it seems that the constraints obtained from the CMB data invalidates the use of an effective four-dimensional approach to describe tachyonic inflation [450]. On the other hand, this can also justify our approach which just considers this scenario as a phenomenological model.

### 4.23. Witten-O'Raifeartaigh inflation (WRI)

### 4.23.1. Theoretical justifications

This model arises in different contexts and we now briefly review one of its theoretical motivation. The first situation originates from supersymmetric theories aimed at explaining the gauge hierarchy problem (that is to say why the GUT scale differs so much from the weak scale). In the supersymmetric scenario of Ref. [454], three chiral superfields $A, X$ and $Y$ are considered in a superpotential of the O'Raifeartaigh type [455],
$W=\lambda X\left(A^{2}-m^{2}\right)+g Y A$,
where $m$ and $g$ are constant of mass dimension. The corresponding (global) supersymmetric potential can be expressed as
$V=\lambda^{2}\left|A^{2}-m^{2}\right|^{2}+g^{2}|A|^{2}+|2 \lambda X A+g Y|^{2}$.
The minimum of this potential is given by $\langle Y\rangle=-2 \lambda\langle X\rangle\langle A\rangle / g$ and $\langle A\rangle=0$ [there is also another minimum at $\langle A\rangle=$


Fig. 87. Reheating consistent slow-roll predictions for the radiatively corrected quartic models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel), with $\bar{w}_{\text {reh }}=\frac{1}{3}$. This value of $\bar{w}_{\text {reh }}$ may be more physically justified if the reheating phase takes place at the bottom of the potential, which is quartic in a good approximation, and for which one has $\bar{w}_{\text {reh }}=1 / 3$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The black solid line represent the locus of the quartic model [i.e. LFI with $p=4$, for which $r=(16 / 3)\left(1-n_{\mathrm{S}}\right)$, i.e. $\epsilon_{1}=\epsilon_{2}$ ]. Clearly, these models are disfavored at two sigma confidence level. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
$\left.\sqrt{m^{2}-g^{2} /\left(2 \lambda^{2}\right)}\right]$. Clearly, the potential is minimized regardless of $\langle X\rangle$, that is to say we have a flat direction along $X$. Along that direction, $V=\lambda^{2} m^{4}$ and supersymmetry is broken since $F_{X} \equiv$ $\partial W / \partial X \neq 0$. As a consequence, the mass of the real part and imaginary parts of $A$ are split and are given by $4 \lambda^{2}|X|^{2}+g^{2} \pm 2 m^{2} \lambda^{2}$. The mass of the fermion field $\psi_{A}$ is $4 \lambda^{2}|X|^{2}+g^{2}$. The fact that supersymmetry is broken implies that the potential will receive corrections: as is well-known, if supersymmetry is preserved, the corrections originating from bosons and fermions exactly cancel out. Here, this is not the case and the amplitude of the corrections will be determined by the split between the bosonic and fermionic masses that we have just evaluated before. A simple calculation leads to
$V=\lambda^{2} m^{4}\left[1+\frac{\lambda^{2}}{8 \pi^{2}} \ln \left(\frac{|X|^{2}}{\mu^{2}}\right)\right]$,
where $\mu$ is the renormalization scale. Therefore, one obtains an increasing function of the field vev and this implies that $X$ cannot become large because it cannot climb its potential. As a consequence, one cannot generate a large hierarchy in this scenario. In fact, as explained in Ref. [454], this is due to the fact that the one loop correction is positive, as appropriate in a theory with scalars and fermions. This can also be understood from the renormalization group perspective where the appearance of the logarithm in the above expression of $V(X)$ can be viewed as the renormalization of the coupling constant such that $\lambda^{2} \rightarrow$
$\lambda^{2}\left[1+\lambda^{2} /\left(8 \pi^{2}\right) \ln \left(|X|^{2} / \mu^{2}\right)\right]$. The conclusion of Ref. [454] is that if $m$ is the small scale (the weak scale) and $\langle X\rangle$ the large one (the GUT scale), a large hierarchy cannot be achieved in this approach.

However, it is well-known that asymptotic freedom is possible in non-Abelian gauge theories. This means that the renormalization group equations have to produce negative one loop corrections. In such a situation, the field could run to infinity, in the nonperturbative regime. For this reason, it is interesting to re-consider the previous model in the framework of a non-Abelian gauge group such as in Grand Unified SU(5) theories. Refs. [456,457] consider two matter fields $A_{a}^{b}$ and $Z_{a}^{b}$ in the adjoint representation of $\operatorname{SU}(5)$ and one singlet $X$ in a superpotential given by
$W=\lambda_{1} \operatorname{Tr}\left(Z A^{2}\right)+\lambda_{2} X\left[\operatorname{Tr}\left(A^{2}\right)-m^{2}\right]$,
which is the non-Abelian generalization of Eq. (4.334). One can show that supersymmetry is again necessarily broken ${ }^{6}$ and that the potential exhibits a flat direction with the value $V=$ $\lambda_{1}^{2} \lambda_{2}^{2} m^{4} /\left(30 \lambda_{2}^{2}+\lambda_{1}^{2}\right)$. As it was the case in the first simple example presented above, and since supersymmetry is broken, quantum corrections modify the potential. At the one loop order, one obtains the following expression [456]
$V(X)=\frac{\lambda_{1}^{2} \lambda_{2}^{2} m^{4}}{30 \lambda_{2}^{2}+\lambda_{1}^{2}}\left(1+\frac{\lambda_{2}^{2}}{\lambda_{2}^{2}+\lambda_{1}^{2} / 30} \frac{29 \lambda_{1}^{2}-50 g^{2}}{80 \pi^{2}} \ln |X|^{2}\right)$,
where $g$ is the $\operatorname{SU}(5)$ gauge coupling constant. If $29 \lambda_{1}^{2}<50 g^{2}$, the correction is negative contrary to the case studied before. Again, this is precisely because we deal with non-Abelian gauge interaction. The field $X$ will grow and can reach a point where the perturbative approach is no longer valid. However, asymptotic freedom tells us that the potential could develop a minimum in this regime in which $X$ could be stabilized, hence the original motivation for this scenario: the scale $m$ can be taken to be relatively small while $\langle X\rangle$ can now be very large thereby addressing the gauge hierarchy problem.

This class of model was considered in Ref. [458] in order to build a new inflationary scenario. The idea is to start from a potential of the form derived above, namely $V(\phi)=M^{4}(1+\tilde{b} \ln \phi)$ with a negative coefficient $\tilde{b}$. Therefore, the field is driven towards a regime where higher corrections must become important. Typically, one expects $\tilde{b}$ to acquire a logarithmic dependence in $\phi$ and the potential to develop a minimum at, say $\phi=m_{\text {GUM }}$. Therefore, this leads to $V(\phi)=M^{4}\left[1+b \ln ^{2}\left(\phi / m_{\text {GUM }}\right)\right]$ where $b$ is a constant. Moreover, if one requires the potential to vanish at the minimum, we are led to $V(\phi) \propto \ln ^{2}\left(\phi / m_{\text {GUM }}\right)$ and this is the potential studied in this section. In Ref. [458], it is argued that $m_{\mathrm{GUM}} \simeq M_{\mathrm{PI}}$ and that, initially, $\phi \simeq \mu \simeq\left(m_{\text {weak }} m_{\mathrm{GUM}}\right)^{1 / 2} \simeq$ $10^{12} \mathrm{GeV}$. We will come back to these conditions in what follows.

Another way to obtain the same potential is based on Refs. [459, 460] in which one consider the following action
$S=-\int \mathrm{d}^{4} \boldsymbol{x} \sqrt{-g}\left[\hat{\mathrm{~g}}_{A \bar{B}}\left(z^{\mathrm{c}}, \bar{z}^{\bar{c}}\right) g^{\mu \nu} \partial_{\mu} z^{A} \partial_{\nu} \bar{z}^{\bar{B}}-V\left(z^{\mathrm{C}}, \bar{z}^{\bar{c}}\right)\right]$.

[^5]

Fig. 88. Reheating consistent slow-roll predictions for the natural inflation models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 since the potential is quadratic close to its minimum. The black solid line represent the locus of the quadratic model points [i.e. LFI with $p=2$, for which $r=4\left(1-n_{\mathrm{S}}\right)$, i.e. $\left.\epsilon_{1}=\epsilon_{2} / 2\right]$. The annotations trace the energy scale at which the natural reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. Clearly, high values of $f / M_{\text {PI }}$ seem to be favored by the data, as well as high reheating temperatures. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The $z^{A}$ 's are complex scalar fields and $\hat{g}_{A \bar{B}}$ is the Kähler metric. The corresponding equations of motion can be expressed as
$g^{\mu \nu} \nabla_{\mu} \nabla_{\nu} \bar{z}^{\bar{D}}+\Gamma_{\bar{A} \bar{D}}^{\bar{D}} g^{\mu \nu} \partial_{\mu} \bar{z}^{\bar{A}} \partial_{\nu} \bar{z}^{\bar{B}}-\hat{g}^{C \bar{D}} \frac{\partial V}{\partial z^{C}}=0$,
where $\Gamma_{\bar{A} \bar{B}}^{\bar{D}} \equiv \hat{g}^{C \bar{D}} \partial_{\bar{A}} \hat{g}_{C \bar{B}}$. If we restrict ourselves to cosmological spacetimes, the above equation becomes $\ddot{\ddot{z}}^{\bar{D}}+3 H \dot{z}^{\bar{D}}+\Gamma_{\bar{A} \bar{B}}^{\bar{D}} \dot{\bar{A}}_{\bar{z}}^{\bar{z}} \overline{\bar{B}}+$ $\hat{g}^{C \bar{D}} \partial V / \partial z^{C}=0$, where $H$ is the Hubble parameter. Then, for simplicity, we assume that there is only one field $Z$ and we denote its real part as $u$ and its imaginary part as $v$. We also assume that the potential is flat in the $v$-direction and take $V=V(z+\bar{z})$, $\hat{g}_{Z \bar{z}} \equiv \hat{g}(Z+\bar{Z})$. It follows that
$\ddot{u}+3 H \dot{u}+\Gamma(u)\left(\dot{u}^{2}-\dot{v}^{2}\right)+\partial_{u} V /(2 \hat{g})=0$,
$\ddot{v}+3 H \dot{v}+2 \Gamma(u) \dot{u} \dot{v}=0$,
with $\Gamma=\partial_{u} \hat{g} /(2 \hat{g})$. The second differential equation can be integrated and one obtains $\dot{v}=Q a^{-3} / \hat{g}$, where $Q$ is a constant. The next step consists in defining the field $\phi$ by $\dot{\phi} \equiv \sqrt{\hat{g}} \dot{u}$. As a consequence, the first differential equation can be re-written as $\ddot{\phi}+3 H \dot{\phi}+\partial_{\phi}\left[V+Q^{2} /\left(\hat{g} a^{6}\right)\right]=0$, that is to say $\phi$ is now canonically normalized and its evolution is controlled by the effective potential $V(\phi)+Q^{2} /\left(\hat{g} a^{6}\right)$. One can show that the presence of the additional term proportional to $Q^{2}$ is not


Fig. 89. Reheating consistent slow-roll predictions for the exponential Susy models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel), with $\bar{w}_{\text {reh }}=0$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The black solid line represent the locus obtained from the linear large field model [with $p=1$, for which $r=$ $(8 / 3)\left(1-n_{\text {S }}\right)$, i.e. $\left.\epsilon_{1}=\epsilon_{2} / 4\right]$. The annotations trace the energy scale at which the exponential Susy reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. Clearly, all these models seem to be consistent with observations. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
crucial $[459,460]$. Initially, it dominates because $a$ is small but, quickly, since it is proportional to $a^{-6}$, it goes to zero as the universe expands. As a consequence, one is left with $V(\phi)$ only. A specific version of this scenario has been studied in details in Ref. [459]. In that article, it is assumed that $\hat{g}=e^{-2 u} / 2$ and $V=0$. This corresponds to the bosonic action of a model which is superconformal invariant [461]. Then, this invariance is softly broken by adding a term $m^{2} u^{2} / 2$ and, through the redefinition of the field, one can check that this leads to a potential proportional to $m^{2}(\ln \phi)^{2}$, that is to say of the type studied in this section. Moreover, one can also verifies that, in the regime discussed above where the term $Q^{2} /\left(\hat{g} a^{6}\right)$ dominates, an exact solution can be found and reads: $a=a_{0} t^{1 / 3}$ and $\phi^{2}(t)=E^{2}(\ln t+C)^{2}+$ $4 Q^{2} /\left(a_{0}^{6} E^{2}\right)$, where $E$ and $C$ are two integration constants. As a consequence, when the universe expands, $Q^{2} /\left(\hat{g} a^{6}\right)$ goes to zero and one is left with the logarithmic potential only.

### 4.23.2. Slow-roll analysis

Based on the previous considerations, we study the WRI potential
$V(\phi)=M^{4} \ln ^{2}\left(\frac{\phi}{\phi_{0}}\right)$,
where $\phi_{0}$ is viewed as a free parameter but we also keep in mind that a natural prior is $\phi_{0}=M_{\mathrm{PI}}$. The potential Eq. (4.343) is displayed in Fig. 40, together with its logarithm (top panels). The
arrow indicates that inflation proceeds from the right to the left. Let us now calculate the Hubble flow parameters. If one defines $x \equiv \phi / \phi_{0}$, they are given by
$\epsilon_{1}=2 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}} \frac{1}{x^{2} \ln ^{2} x}$,
$\epsilon_{2}=4 \frac{M_{\mathrm{PI}}^{2}}{\phi_{0}^{2}} \frac{1+\ln x}{x^{2} \ln ^{2} x}$,
and
$\epsilon_{3}=2 \frac{M_{\mathrm{PL}}^{2}}{\phi_{0}^{2}} \frac{2+3 \ln x+2 \ln ^{2} x}{x^{2} \ln ^{2} x(1+\ln x)}$.
They are displayed in the bottom panels of Fig. 40. One can see that they all vanish when $x \rightarrow \infty$, that they increase as inflation proceed, diverging when $x \rightarrow 1$. At this stage, a remark is in order about Ref. [458]. As already mentioned above, a natural prior is $\phi_{0}=M_{\mathrm{Pl}}$. This means that if, initially, one has $\phi \simeq \mu$, one is in fact in the decreasing branch of the potential and, as a matter of fact, one cannot have inflation since $\epsilon_{1}>1$ always. Clearly, the only way to have inflation in this branch is to assume that $\phi_{0} \gg M_{\mathrm{Pl}}$, a case which appears to be difficult to justify in this context. Here, we do not consider this case. In the increasing branch of the potential, inflation stops by slow-roll violation when $\epsilon_{1}=1$, at a vev $x_{\text {end }}$ given by
$x_{\text {end }}=\exp \left[\mathrm{W}_{0}\left(\sqrt{2} \frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)\right]$,
where $\mathrm{W}_{0}$ is the 0 -branch of the Lambert function, which must be chosen in order to have $x>1$.

Let us now turn to the slow-roll trajectory. It can be integrated exactly and this leads to the following expression
$N_{\text {end }}-N=\frac{1}{4} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}}\left(x^{2} \ln x-\frac{x^{2}}{2}-x_{\text {end }}^{2} \ln x_{\text {end }}+\frac{x_{\text {end }}^{2}}{2}\right)$,
where $N_{\text {end }}$ is the number of $e$-folds at the end of inflation. Interestingly enough, this trajectory can be inverted, and one obtains

$$
\begin{align*}
x= & \exp \left\{\frac{1}{2} \mathrm{~W}_{0}\left[\frac{8}{e} \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}}\left(N_{\mathrm{end}}-N\right)+\frac{2}{e} x_{\mathrm{end}}^{2} \ln x_{\mathrm{end}}-\frac{x_{\mathrm{end}}^{2}}{e}\right]\right. \\
& \left.+\frac{1}{2}\right\} \tag{4.349}
\end{align*}
$$

where $W_{0}$ is still the 0 -branch of the Lambert function. It is displayed in Fig. 41, together with the region where inflation proceeds.

Finally, it is interesting to constrain the value of the scale $M$ with the CMB normalization. It follows that

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=2880 \pi^{2}\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} \frac{1}{x_{*}^{2} \ln ^{4} x_{*}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{4.350}
\end{equation*}
$$

The reheating consistent slow-roll predictions of the WittenO'Raifeartaigh inflation models are displayed in Fig. 110. One should remember that in principle, $\phi_{0} \simeq M_{\mathrm{Pl}}$, even if a wider range of values for $\phi_{0}$ is displayed in order to understand how the predictions depend on this parameter. In particular, when $\phi_{0} \gg$ $M_{\mathrm{PI}}$, the predictions lie along the line $\epsilon_{2}=2 \epsilon_{1}$. Indeed, in this limit, Eq. (4.347) shows that $x_{\text {end }} \rightarrow 1$ while Eq. (4.349) indicates that $x_{*} \rightarrow 1$. As a consequence, one obtains $\epsilon_{2 *} \simeq \epsilon_{1 *}$ from Eqs. (4.344) and (4.345).


Fig. 90. Reheating consistent slow-roll predictions for the exponential Susy models in the plane $\left(n_{\mathrm{S}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel), with $\bar{w}_{\text {reh }}=$ $-1 / 3$. This value of $\bar{w}_{\text {reh }}$ may be more physically justified (although rather extreme) if a parametric reheating feels the bottom of the potential, which is linear in a good approximation. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The black solid line represent the locus of the linear large field model [with $p=1$, for which $r=(8 / 3)\left(1-n_{\text {s }}\right)$, i.e. $\left.\epsilon_{1}=\epsilon_{2} / 4\right]$. The annotations trace the energy scale at which the exponential Susy reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. Clearly in that case, these values are limited from below. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

## 5. Two parameters models

### 5.1. Small field inflation (SFI)

This model is proto-typical of inflation occurring at the top of a flat-enough potential. As such it appears in very different contexts. It has been introduced in Refs. [2,397] and derived in Ref. [3] in the context of radiatively induced symmetry breaking. It appears within superstring models [462], low scale symmetry breaking [267,463], supersymmetry [352,464] and supergravity [246,247,251,266,465-469]. It is also obtained in non-linear sigma models [277] or using moduli as inflatons [470]. It has been discussed in braneworld cosmology in Refs. [471-473] and is more recently referred to as "hilltop inflation" from Refs. [411,412]. The potential is given by
$V(\phi)=M^{4}\left[1-\left(\frac{\phi}{\mu}\right)^{p}\right]$,
and has two parameters in addition to the overall normalization $M$ : a typical vev $\mu$ and the power index $p$. As this potential can be associated with very different physical frameworks, $\mu$ can take any values while $p>0$ for being at the top of a potential (in the small field limit, namely $\phi \ll \mu$ ). In particular, we will allow superPlanckian values for $\mu$ even though, in the supergravity context,


Fig. 91. Reheating consistent slow-roll predictions for the power law models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The black solid line represents the locus of the points such that $r=-8\left(n_{S}-1\right)$, i.e. $\epsilon_{2}=0$. The annotations of the energy scale at which reheating ends are not displayed since the predictions of these models do not depend on this parameter. Clearly, these models are excluded at more than two sigma confidence level. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
one would require $\mu<M_{\mathrm{PI}}$. Let us stress that Eq. (5.1) is defined only in the domain $\phi<\mu$ as one assumes that the small field potential describes only the field dynamics during inflation. The equation of state during reheating is thus not specified by Eq. (5.1). Defining
$x \equiv \frac{\phi}{\mu}$,
the first three Hubble flow functions read
$\epsilon_{1}=\frac{p^{2}}{2}\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{2} \frac{x^{2 p-2}}{\left(1-x^{p}\right)^{2}}$,
$\epsilon_{2}=2 p\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{2} x^{p-2} \frac{p-1+x^{p}}{\left(1-x^{p}\right)^{2}}$,
and
$\epsilon_{3}=p\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{2} \frac{x^{p-2}\left[2 x^{2 p}+(p-1)(p+4) x^{p}+(p-1)(p-2)\right]}{\left(1-x^{p}\right)^{2}\left(p-1+x^{p}\right)}$.

They are monotonic functions of the field value but also decreasing functions of the vev $\mu$. The potential, its logarithm and the Hubble flow functions are represented in Fig. 42.

The slow-roll trajectory is obtained by integrating Eq. (2.11) to get
$N-N_{\mathrm{end}}=\frac{1}{2 p} \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\left[-x^{2}+x_{\mathrm{end}}^{2}+\frac{2}{2-p}\left(x^{2-p}-x_{\mathrm{end}}^{2-p}\right)\right]$.
This equation seems to be well-defined only for $p \neq 2$. However, the particular case $p=2$ can be directly obtained from Eqs. (2.11) and (5.1) to get
$N-N_{\text {end }}=\frac{1}{4} \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\left[-x^{2}+x_{\text {end }}^{2}+2 \ln \left(\frac{x}{x_{\text {end }}}\right)\right]$.
This expression can also be viewed as the limit of Eq. (5.5) for $p \rightarrow 2$. In general, the trajectory cannot be analytically inverted to give the field value $x(N)$ but one can find some analytic form for almost all integer values of $p$ (e.g. for $p=1, p=2, p=3, p=4$, $p=6$ ) that we do not write down for the sake of clarity.

From the potential Eq. (5.1), inflation can stop naturally at $\epsilon_{1}\left(x_{\text {end }}\right)=1$ with $x_{\text {end }}<1$. This condition gives the algebraic equation
$x_{\text {end }}^{p}+\frac{p}{\sqrt{2}} \frac{M_{\mathrm{Pl}}}{\mu} x_{\text {end }}^{p-1}=1$,
which cannot be solved analytically in full generality. As for the trajectory, there are however explicit solutions for almost all integer values of $p$, the first two being
$x_{\mathrm{end}}^{(p=1)}=1-\frac{M_{\mathrm{Pl}}}{\sqrt{2} \mu}$,
$x_{\mathrm{end}}^{(p=2)}=\frac{M_{\mathrm{PI}}}{\sqrt{2} \mu}\left(-1+\sqrt{1+2 \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}}\right)$.
Together with Eq. (2.47), these equations are enough to allow the determination of the field value $x_{*}$ at which the observable modes crossed the Hubble radius during inflation. This fixes the value of the parameter $M$ to match the observed amplitude of the CMB anisotropies at
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=720 \pi^{2} p^{2} \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \frac{x_{*}^{2 p-2}}{\left(1-x_{*}^{p}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
The reheating consistent slow-roll predictions for the small field models are represented in Figs. 111-113 for $p=1, p=2$ and $p=4$. The $p=1$ case is trivial since one then has $\epsilon_{2 *}=4 \epsilon_{1 *}$. For $p=2$ or $p=4$, one sees that the reheating temperature is limited from below to fit in the observable range. For instance, with $p=2$, values of $\mu$ such that $\mu / M_{\mathrm{PI}}<10$ are clearly disfavored. Let us notice that the relation $\epsilon_{2 *}=4 \epsilon_{1 *}$ is recovered in the limit $\mu / M_{\mathrm{PI}} \gg 1$ whereas one clearly observes a systematic shift in $n_{\mathrm{S}}$ (or $\epsilon_{2}$ ) when $\mu \ll M_{\mathrm{PI}}$. These behaviors can in fact be understood analytically.

Small field models in the supergravity context are commonly studied in the limit $\mu \ll M_{\text {Pl }}$. In this situation it is possible to find some approximate solution to both the trajectory and $x_{\text {end }}$. Keeping only the dominant term in Eq. (5.7), one gets
$x_{\mathrm{end}}^{(p \neq 1)} \simeq\left(\frac{\sqrt{2}}{p} \frac{\mu}{M_{\mathrm{Pl}}}\right)^{1 /(p-1)}$,
the case $p \leq 1$ being incompatible with the limit $\mu \ll M_{\mathrm{PI}}$ and the consistency requirement that $x_{\text {end }}<1$. The small vev limit can also be used to invert Eq. (5.5). Assuming $\mu \ll M_{\text {PI }}$ and $x_{\text {end }} \ll$ 1 , neglecting the quadratic terms for $p>1$, the approximate trajectory reads
$N-N_{\mathrm{end}} \simeq \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}} \frac{x^{2-p}-x_{\mathrm{eld}}^{2-p}}{p(2-p)}$,


Fig. 92. Reheating consistent slow-roll predictions for the Kähler Moduli I models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The reheating equation of state parameter $\bar{w}_{\text {reh }}=0$ since the potential is quadratic close to its minimum. The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
which can be inverted to
$x \simeq\left[x_{\mathrm{end}}^{2-p}-\frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} p(2-p)\left(N_{\mathrm{end}}-N\right)\right]^{1 /(2-p)}$.
Notice that far from the end of inflation, i.e. $N \ll N_{\text {end }}$, the first term can be neglected (for $p>2$ ) since $x_{\text {end }}<1$ and $M_{\mathrm{PI}} / \mu \gg 1$. Defining $\Delta N_{*}=N_{\text {end }}-N_{*}$, one can now plug this expression for $x_{*}$ into the Hubble flow functions of Eqs. (5.3) and (5.4) to get their observable values:
$\epsilon_{1 *} \simeq \frac{p^{2}}{2}\left(\frac{M_{\mathrm{P}}}{\mu}\right)^{2}\left[\Delta N_{*} p(p-2)\left(\frac{M_{\mathrm{P}}}{\mu}\right)^{2}\right]^{-\frac{2(p-1)}{p-2}}$,
$\epsilon_{2 *} \simeq \frac{2}{\Delta N_{*}} \frac{p-1}{p-2}, \quad \epsilon_{3 *} \simeq \frac{1}{\Delta N_{*}}$.
It is crucial to keep in mind that the above formulas are valid only in the limit $\mu \ll M_{P I}$ and $p>2$. As before, the limiting case $p \rightarrow 2$ has to be taken with care and, starting with Eq. (5.6), one obtains
$\epsilon_{1 *}^{(p=2)}=\exp \left(-4 \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \Delta N_{*}\right), \quad \epsilon_{2 *}^{(p=2)}=4 \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}}$,
$\epsilon_{3 *}^{(p=2)}=6 \epsilon_{1 *}^{(p=2)}$.
Both Eqs. (5.13) and (5.14) describes the observed behavior in Figs. 111-113 when $\mu / M_{\mathrm{Pl}} \rightarrow 0$ but they do fail in the intermediate region as we have discussed in the introduction (see Fig. 3).

If the theoretical motivations underlying the potential Eq. (5.1) do not require the vev to be small, one can similarly derive approximate expressions for the observables in the limit $\mu / M_{\mathrm{PI}} \gg$ 1 (but still with $x<1$ ). Defining $\varepsilon \equiv M_{\mathrm{PI}} / \mu$, one has $x_{\text {end }}(\varepsilon)$ and we can search for a Taylor expanded solution of Eq. (5.7) to get
$x_{\text {end }}=1-\frac{\varepsilon}{\sqrt{2}}+\frac{p-1}{4} \varepsilon^{2}+\mathcal{O}\left(\varepsilon^{3}\right)$.
Similarly one can search for a Taylor expanded solution for the trajectory Eq. (5.5), plugging in the previous expression for $x_{\text {end }}$. Doing so yields
$x_{*}=1-\varepsilon \sqrt{\frac{1}{2}+2 \Delta N_{*}}+\mathcal{O}\left(\varepsilon^{2}\right)$.
From this, one gets the corresponding Hubble flow functions
$\epsilon_{1 *} \simeq \frac{1}{4 \Delta N_{*}+1} \quad \epsilon_{2 *} \simeq 4 \epsilon_{1 *}, \quad \epsilon_{3 *} \simeq \epsilon_{1}$.
This result is quite remarkable since the observable slow-roll parameters become $\mu$ and $p$ independent. Performing the same calculation in the singular case $p \rightarrow 2$ yields exactly the same result. The spectral index, tensor-to-scalar ratio and running are immediately obtained from Eq. (5.17) with $r=16 \epsilon_{1_{*}}, n_{\mathrm{S}}-$ $1 \simeq-3 r / 8$ and $\alpha \simeq-r$. Again, these expressions match with Figs. 111-113 when $\mu / M_{\mathrm{PI}} \rightarrow \infty$.

### 5.2. Intermediate inflation (II)

This model was introduced in Refs. [474-477] as an implementation of an equation of state of the form
$\rho+p=\gamma \rho^{\lambda}$,
where $\rho$ stands for the energy density and $p$ the pressure. Both $\gamma>0$ and $\lambda>1$ are dimensionless constants. As will be made explicit, this equation of state leads to a scale factor which is given by $a(t) \propto \exp \left(A t^{f}\right)$ where $0<f<1$. In some sense the expansion is thus faster than power law but slower than de Sitter, hence the name of the model. The pure de Sitter case corresponds to $f=1$. Inserting the Friedmann-Lemaître equation, $3 M_{P 1}^{2} H^{2}=\rho$ as well as the equation of state Eq. (5.18) into the equation of conservation $\dot{\rho}+3 H(\rho+p)=0$, one obtains a closed equation for $\rho$ which is solved by
$\rho=\rho_{0}\left[3 \gamma(\lambda-1) \ln \left(\frac{a}{a_{0}}\right)\right]^{1 /(1-\lambda)}$,
where $\rho_{0}$ and $a_{0}$ are positive constants. Making use of the Friedmann-Lemaître equation again, one deduces the behavior for $a$,

$$
\begin{align*}
& \ln \left(\frac{a}{a_{0}}\right) \\
& \quad=3^{\lambda /(1-2 \lambda)} \gamma^{1 /(1-2 \lambda)} \frac{\left(\lambda-\frac{1}{2}\right)^{(1-\lambda) /(1-2 \lambda)}}{\lambda-1}\left(\frac{t}{t_{0}}\right)^{(1-\lambda) /(1-2 \lambda)}, \tag{5.20}
\end{align*}
$$

i.e. the announced form $a(t) \propto \exp \left(A t^{f}\right)$, with $f=2(1-\lambda) /(1-$ $2 \lambda$ ). Since $\lambda>1$, this means that $0<f<1$. Then, one can notice that it is possible to reinterpret the matter source as that of a scalar field with the potential $V(\phi)$ given by

$$
\begin{align*}
V(\phi)= & 3 A^{2} f^{2} M_{\mathrm{Pl}}^{4}\left[\frac{\phi-\phi_{0}}{M_{\mathrm{Pl}} \sqrt{8 A\left(f^{-1}-1\right)}}\right]^{4(1-1 / f)} \\
& -M_{\mathrm{Pl}}^{4} A f(1-f)\left[\frac{\phi-\phi_{0}}{M_{\mathrm{Pl}} \sqrt{8 A\left(f^{-1}-1\right)}}\right]^{2-4 / f} . \tag{5.21}
\end{align*}
$$



Fig. 93. Reheating consistent (exact) predictions for the horizon flow inflation at first order models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours trace the two-sigma Planck confidence intervals (marginalized over second order slow-roll). The black solid line represent the locus of the quadratic large field model [with $p=2$, for which $r=4\left(1-n_{\mathrm{S}}\right)$, i.e. $\left.\epsilon_{1}=\epsilon_{2} / 2\right]$. The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. Clearly, a high energy scale reheating is preferred for these models to remain inside the two-sigma contours. Notice that, up to the amplitude of the CMB anisotropies, the predictions do not depend much on $A_{1}$ as they are all superimposed. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Indeed, starting from this potential, the Klein-Gordon equation with $H=A f t^{f-1}$, has an exact non-trivial solution given by
$\phi=\phi_{0}+M_{\mathrm{Pl}} \sqrt{8 A\left(f^{-1}-1\right)}\left(\frac{t}{t_{0}}\right)^{f / 2}$.
It is then straightforward to calculate $\rho=\dot{\phi}^{2} / 2+V$ and $p=$ $\dot{\phi}^{2} / 2-V$, and to show that they satisfy the equation of state Eq. (5.18). The potential can be recast in the form
$V(\phi)=M^{4}\left(\frac{\phi-\phi_{0}}{M_{\mathrm{Pl}}}\right)^{-\beta}-M^{4} \frac{\beta^{2}}{6}\left(\frac{\phi-\phi_{0}}{M_{\mathrm{Pl}}}\right)^{-\beta-2}$,
with $\beta=4(1 / f-1)$. The constraint $0<f<1$ means that $\beta>0$. Defining
$x \equiv \frac{\phi-\phi_{0}}{M_{\mathrm{Pl}}}$,
it is shown below that the model predictions do not depend on $\phi_{0}$. Therefore Intermediate Inflation is a priori a one parameter family of models, but as explained below, one needs an extra parameter $x_{\text {end }}$ specifying the field value at which an unspecified mechanism is triggered to end of inflation. It is thus a two parameters model.

This potential appears in the earlier work of Ref. [478] as a solution for a cosmological model containing a string creation term. It is also discussed in the context of tachyon
fields in Refs. [479,480]. Warm intermediate inflation was considered in Refs. [481,482], intermediate inflation within a Gauss-Bonnet braneworld was studied in Ref. [483], and with Jordan-Brans-Dicke theory in Refs. $[484,485]$.

The potential (5.23), as well as its logarithm, are displayed in Fig. 43. It is positive definite for $x>x_{V=0} \equiv \beta / \sqrt{6}$. Therefore, one must restrict the inflaton vev to lie beyond this value. The potential increases with $x$, reaches a maximum at $x_{V^{\prime}=0} \equiv \sqrt{\beta(\beta+2) / 6}$, then decreases with $x$ to asymptotically vanish when $x$ goes to infinity. Therefore, a priori, two regimes of inflation exist. Either inflation proceeds at $x<x_{V^{\prime}=0}$ from the right to the left, either it proceeds at $x>x_{V^{\prime}=0}$ from the left to the right. However, in Eq. (5.22), one can see that the inflaton vev has to increase with time. Therefore only the branch $x>x_{V^{\prime}=0}$ can produce an equation of state of the form of Eq. (5.18), which is where the model will be studied in the following.

Let us now turn to the slow-roll parameters. The first three Hubble flow functions in the slow-roll approximation are given by

$$
\begin{align*}
& \epsilon_{1}=\frac{1}{2}\left[\frac{\beta^{2}(\beta+2)-6 \beta x^{2}}{-\beta^{2} x+6 x^{3}}\right]^{2} \\
& \epsilon_{2}=\frac{-2 \beta x^{4}+\frac{\beta^{2}}{3}(2 \beta+6) x^{2}-\frac{\beta^{4}}{18}(\beta+2)}{\left(x^{3}-\frac{\beta^{2} x}{6}\right)^{2}} \tag{5.25}
\end{align*}
$$

and

$$
\begin{equation*}
\epsilon_{3}=\frac{\beta\left[6 x^{2}-\beta(2+\beta)\right]\left[\frac{\beta^{5}}{18}(2+\beta)-\beta^{3}(2+\beta) x^{2}+6 \beta(4+\beta) x^{4}-12 x^{6}\right]}{\left(x^{3}-\frac{\beta^{2}}{6} x\right)^{2}\left[\beta^{3}(\beta+2)-12 \beta(\beta+3) x^{2}+36 x^{4}\right]} . \tag{5.26}
\end{equation*}
$$

They are displayed in Fig. 43. The first slow-roll parameter diverges where the potential vanishes at $x_{V=0}$, decreases from here and vanishes at the maximum of the potential $x_{V^{\prime}=0}$. Then it increases again, reaches a local maximum at $x_{\epsilon_{1}}^{\max }$, and decreases to asymptotically vanish when $x$ goes to infinity. The location $x_{\epsilon_{1}}^{\max }$ is given by
$x_{\epsilon_{1}^{\max }}=\sqrt{\frac{\beta}{2}\left(1+\frac{\beta}{3}+\sqrt{1+\frac{4 \beta}{9}}\right)}$.
At this point, the maximum value of $\epsilon_{1}$ is
$\epsilon_{1}^{\max }=\frac{\beta}{9} \frac{(1+3 \sqrt{1+4 \beta / 9})^{2}}{(1+\sqrt{1+4 \beta / 9})^{2}(1+\beta / 3+\sqrt{1+4 \beta / 9})}$.
If $\beta<9 / 2(1+\sqrt{2}) \simeq 10.86$, this maximum value is smaller than one. In this case inflation cannot stop by slow-roll violation in the decreasing branch of the potential and an extra parameter $x_{\text {end }}$ must be added to the model to specify the location where another mechanism such as e.g. tachyonic instability could trigger the end of inflation. If $\beta>9 / 2(1+\sqrt{2}) \simeq 10.86$, the local maximum value of $\epsilon_{1}$ is higher than one and in the decreasing branch of the potential, either inflation takes place between $x_{V^{\prime}=0}$ and the first solution of $\epsilon_{1}=1$, either it takes place between the second solution of $\epsilon_{1}=1$ and $x=\infty$. As will be shown below, only the latter case is consistent with the exact trajectory Eq. (5.22) which allows for an equation of state of the form of Eq. (5.18).

The slow-roll trajectory of the model can be obtained from Eq. (2.11). However, as already mentioned, a non-trivial and exact field evolution is given by Eq. (5.22). Written in terms of the number of $e$-folds $N-N_{0}=\ln \left(a / a_{0}\right)=A\left(t^{f}-t_{0}^{f}\right)$, one obtains
$x=\sqrt{x_{\mathrm{end}}^{2}+2 \beta\left(N-N_{\mathrm{end}}\right)}$.


Fig. 94. Reheating consistent slow-roll predictions for the Colemann-Weinberg models in the plane ( $n_{S}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel), in the physical domain $Q / M_{\mathrm{PI}} \in\left[10^{-5}, 10^{-3}\right]$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The typical amount of gravitational waves is extremely small. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

This expression is exact and does not involve any approximations. It can be compared to slow-roll trajectory which reads

$$
\begin{align*}
N_{\mathrm{end}}-N= & \frac{1}{2 \beta}\left(x_{\mathrm{end}}^{2}-x^{2}\right)+\frac{1}{6} \ln \left[x_{\mathrm{end}}^{2}-\frac{\beta(\beta+2)}{6}\right] \\
& -\ln \left[x^{2}-\frac{\beta(\beta+2)}{6}\right] \tag{5.30}
\end{align*}
$$

where $N_{\text {end }}$ is the number of $e$-folds at the end of inflation and $N$ is the number of $e$-folds at some point when the scaled field vev is $x$. As mentioned above, the slow-roll trajectory should match the exact one in the decreasing branch of the potential. For $x \rightarrow \infty$, one can neglect the logarithmic terms in Eq. (5.30) and one indeed recovers Eq. (5.29). This is expected since in this limit, the slow-roll parameters all go to zero and the slow-roll approximation becomes increasingly accurate. As a result, the domain of validity lies at $x \gg$ $x_{V^{\prime}=0}$, i.e. between the second solution of $\epsilon_{1}=1$ and $x=\infty$ and inflation cannot stop by slow-roll violation. This justifies the need of the extra-parameter $x_{\text {end }}$. This parameter is thus constrained to $x_{\text {end }}>x_{V^{\prime}=0}$ and should be large enough to allow for a sufficient number of $e$-folding. In order to get $N_{\text {end }}-N_{\text {ini }} e$-folds, making use of Eq. (5.29), one gets
$x_{\text {end }}=\sqrt{x_{\text {ini }}^{2}+2 \beta\left(N_{\text {end }}-N_{\text {ini }}\right)}$.
If $\beta>9 / 2(1+\sqrt{2}) \simeq 10.86, x_{\text {ini }}$ is bounded from below by the highest solution of the equation $\epsilon_{1}=1$. This equation admits three
solutions which, from the smallest to the biggest, are given by

$$
\begin{align*}
x_{\epsilon_{1}=1}^{0}= & -\frac{\beta}{3 \sqrt{2}}+\frac{\sqrt{2}}{3} \frac{\beta^{4 / 3}}{\sqrt[3]{9+2 \beta+i \sqrt{-81-36 \beta+4 \beta^{2}}}} \\
& +\frac{\beta^{2 / 3}}{3 \sqrt{2}} \sqrt[3]{9+2 \beta+i \sqrt{-81-36 \beta+4 \beta^{2}}}  \tag{5.32}\\
x_{\epsilon_{1}=1}^{\mp}= & \frac{\beta}{3 \sqrt{2}}+\frac{1 \mp i \sqrt{3}}{3 \sqrt{2}} \frac{\beta^{4 / 3}}{\sqrt[3]{9+2 \beta+i \sqrt{-81-36 \beta+4 \beta^{2}}}} \\
+ & (1 \pm i \sqrt{3}) \frac{\beta^{2 / 3}}{6 \sqrt{2}} \sqrt[3]{9+2 \beta+i \sqrt{-81-36 \beta+4 \beta^{2}}} \tag{5.33}
\end{align*}
$$

The first solution is located below the maximum of the potential $x_{\epsilon_{1}=1}^{0}<x_{V^{\prime}=0}$, while the two others are located beyond it $x_{\epsilon_{1}=1}^{\mp}>$ $x_{V^{\prime}=0}$. Using the larger solution as a lower bound for $x_{\text {ini }}$, one gets
$x_{\text {end }}>\sqrt{\left(x_{\epsilon_{1}=1}^{+}\right)^{2}+2 \beta\left(N_{\text {end }}-N_{\text {ini }}\right)}$.
If $\beta<9 / 2(1+\sqrt{2})$, only one solution to $\epsilon_{1}=1$ exists,

$$
\begin{align*}
x_{\epsilon_{1}=1}= & -\frac{\beta}{3 \sqrt{2}}+\frac{\sqrt{2}}{3} \frac{\beta^{4 / 3}}{\sqrt[3]{9+2 \beta+\sqrt{81+36 \beta-4 \beta^{2}}}} \\
& +\frac{\beta^{2 / 3}}{3 \sqrt{2}} \sqrt[3]{9+2 \beta+\sqrt{81+36 \beta-4 \beta^{2}}} \tag{5.35}
\end{align*}
$$

which is located below the maximum of the potential $x_{\epsilon_{1}=1}^{0}<$ $x_{V^{\prime}=0}$. In principle $x_{\text {ini }}$ is now only bounded from below by $x_{V^{\prime}=0}$ and one can check from Eq. (5.30) that the total number of $e$-folds diverges close to $x_{V^{\prime}=0}$. As a result, provided $x_{\text {ini }}$ is fine-tuned to the top of the potential, there is no bound on $x_{\text {end }}$. The prior space described by these relations is displayed in Fig. 44.

According to the previous discussion, the observable field value, at which the pivot mode crossed the Hubble radius during inflation, is such that $x_{*} \gg 1$. In this limit, it is possible to approximate the slow-roll parameters at Hubble crossing with
$\epsilon_{1}^{*} \simeq \frac{\beta^{2}}{2 x_{*}^{2}}, \quad \epsilon_{2}^{*} \simeq \epsilon_{3}^{*} \simeq-\frac{2 \beta}{2 x_{*}^{2}}$,
hence
$r \simeq \frac{8 \beta^{2}}{x_{*}^{2}}, \quad n_{\mathrm{S}}-1 \simeq \frac{\beta(2-\beta)}{x_{*}^{2}}, \quad \alpha_{\mathrm{S}}=\frac{2 \beta^{2}(\beta-2)}{x_{*}^{4}}$.
These estimates match with those of Ref. [477]. Finally, the parameter $M$ is obtained from the amplitude of the CMB anisotropies

$$
\begin{align*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}= & 720 \pi^{2}\left[\frac{\beta^{2}(\beta+2)}{6}-\beta x_{*}^{2}\right]^{2}\left(x_{*}^{3}-\frac{\beta^{2} x_{*}}{6}\right)^{-2} \\
& \times\left(x_{*}^{-\beta}-\frac{\beta^{2}}{6} x_{*}^{-\beta-2}\right) \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{5.38}
\end{align*}
$$

In the $x_{*} \gg 1$ limit, this gives
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}} \simeq 720 \pi^{2} \beta^{2} x_{*}^{-2-\beta} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$,
which yields $M / M_{\mathrm{PI}} \lesssim 10^{-2}$.
The reheating consistent slow-roll predictions for the intermediate inflation models are displayed in Fig. 114, for different values of $\beta>0$, and for $x_{\text {end }}$ describing the prior space displayed in Fig. 44. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0



Fig. 95. Reheating consistent slow-roll predictions for the Colemann-Weinberg models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel), in the domain $Q / M_{\mathrm{PI}} \in[1,100]$. The two pink solid contours are the one and twosigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. When $Q / M_{\mathrm{Pl}} \gg 1$, the model is similar to a quadratic potential close to its minimum, and the predictions match the LFI $\epsilon_{1}=\epsilon_{2} / 2$ relation (see Section 4.2) represented by the black lines. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
but since there is no potential minimum around which the inflaton field can oscillate at the end of inflation, this parameter is a priori unspecified and can take different values. In any case the reheating temperature is fully degenerate with the parameter $x_{\text {end }}$, and therefore these two parameters cannot be constrained independently. However one can see that $x_{\text {end }}$ is clearly limited from below as expected. The black solid lines represent the locus of the points such that $\epsilon_{1}^{*}=-\beta / 4 \epsilon_{2}^{*}$, or equivalently, $n_{\mathrm{S}}-1=(1 / \beta-1 / 2) r / 4$, these consistency relations arising from Eq. (5.36). One can check that they provide a good qualitative description of the model predictions. In particular, they explain why, for $\beta<2$, one has a blue tilt $n_{S}>1$.

### 5.3. Kähler moduli inflation II (KMIII)

### 5.3.1. Theoretical justifications

These models are string motivated scenarios. They arise in the context of type IIB string theory via Calabi-Yau flux compactification. They have been derived and studied in Refs. [318-324], and a two-field generalization of this model has been investigated in Refs. [319-323]. They can be understood in the context of supergravity, viewed as an effective theory. In this framework, one starts with the following superpotential for the moduli $T_{i}$
$W=W_{0}+\sum_{i=2}^{n} A_{i} e^{-a_{i} T_{i}}$,
where $a_{i}=2 \pi /\left(g_{s} N\right), N$ being a positive integer (not to be confused with the $e$-fold number), $g_{s}$ the string coupling, and $W_{0}$ and $A_{i}$ are model dependent constants. The Kähler potential can be written as
$K=-2 M_{\mathrm{Pl}}^{2} \ln \left(\frac{\mathcal{V}}{2 \ell_{\mathrm{s}}^{6}}+\frac{\xi}{2}\right)$,
where the constant $\xi$ is given by $\xi=-\zeta(3) \chi(M) /\left[2(2 \pi)^{2}\right], \chi(M)$ being the Euler characteristic of the compactification manifold. The quantity $\mathcal{V}$ represents the overall volume of the Calabi-Yau manifold and can be taken to be
$\mathcal{V}=\frac{\gamma \ell_{s}^{6}}{2 \sqrt{2}}\left[\left(T_{1}+T_{1}^{\dagger}\right)^{3 / 2}-\sum_{i=2}^{n} \lambda_{i}\left(T_{i}+T_{i}^{\dagger}\right)^{3 / 2}\right]$,
where $\gamma$ and $\lambda_{i}$ are positive constants and depend on the details of the model. From the expression of the Kähler and superpotentials, it is then straightforward to calculate the corresponding F-term potential which is a relatively complex expression that can be found in Ref. [322]. If, however, one consider the limit $\mathcal{V} \gg 1$ (and $T_{1} \gg T_{i}$ ), then the F-term simplifies a lot and gives rise to the following equation

$$
\begin{align*}
V\left(\tau_{i}\right) \simeq & \frac{3 \xi W_{0}^{2}}{4 M_{\mathrm{Pl}}^{2} \mathcal{V}_{\mathrm{s}}^{3}}+\sum_{i=2}^{n}\left[\frac{4 W_{0} a_{i} A_{i}}{M_{\mathrm{Pl}}^{2} \mathcal{V}_{\mathrm{s}}^{2}} \tau_{i} e^{-a_{i} \tau_{i}} \cos \left(a_{i} \theta_{i}\right)\right. \\
& \left.+\frac{8\left(a_{i} A_{i}\right)^{2}}{3 M_{\mathrm{Pl}}^{2} \gamma \lambda_{i} \mathcal{V}_{\mathrm{s}}} \sqrt{\tau_{i}} e^{-2 a_{i} \tau_{i}}\right] \tag{5.43}
\end{align*}
$$

where we have written $T_{i}=\tau_{i}+i \theta_{i}$ and $\mathcal{V}_{\mathrm{s}} \equiv \mathcal{V} / \ell_{\mathrm{s}}^{6}$. We see that all the constants introduced before, namely $a_{i}, A_{i}, W_{0}, \xi, \gamma$ and $\lambda_{i}$ participate to the expression of the potential. From Eq. (5.43), solving $\partial V / \partial \tau_{i}=0$, one can estimate the value of each $\tau_{i}$ at the global minimum of the potential. In the following, we denote this quantity by $\tau_{i}^{\min }$. Then, one can also calculate the value of the potential at this minimum. One finds [where, as usual, we have taken $\left.\cos \left(a_{i} \theta_{i}\right)=-1\right]$
$V_{\min } \simeq \frac{3 \xi W_{0}^{2}}{4 M_{\mathrm{P}}^{2} \nu_{s}^{3}}-\frac{3 W_{0}^{2} \gamma}{2 M_{\mathrm{Pl}}^{2} \nu_{s}^{3}} \sum_{i=2}^{n} \frac{\lambda_{i}}{a_{i}^{3 / 2}}\left(a_{i} \tau_{i}^{\min }\right)^{3 / 2}$.
As a consequence, if for one of the fields, say $\tau_{n}$, one has $\left(\lambda_{n} / a_{n}^{3 / 2}\right) /\left[\sum_{i=2}^{n-1}\left(\lambda_{i} / a_{i}^{3 / 2}\right)\right] \ll 1$, then the value of $V_{\min }$ is not modified even if one displaces $\tau_{n}$ from $\tau_{n}^{\min }$. In other words, we have an inflationary valley along the $\tau_{n}$ direction and one can use it to produce inflation. In that case, the potential can be re-written as
$V\left(\tau_{n}\right) \simeq \frac{B W_{0}^{2}}{M_{\mathrm{Pl}}^{2} \mathcal{V}_{\mathrm{s}}^{3}}-\frac{4 W_{0} a_{n} A_{n}}{M_{\mathrm{PI}}^{2} \mathcal{V}_{\mathrm{s}}^{2}} \tau_{n} e^{-a_{n} \tau_{n}}$,
where the second exponential in Eq. (5.43) has been neglected, thanks to the condition $a_{n} \tau_{n} \gg 1$ and $B$ is a constant that includes the constant term in Eq. (5.43) as well as the contributions of the other fields at their minimum, i.e. $B=3 \xi / 4+\cdots$. It is important to notice that the assumption of large volume translates into a condition on the vev of $\tau_{n}$. The above potential is of the form of the toy model studied as "Kähler Moduli Inflation I (KMII)" in Section 4.9. The field is however not canonically normalized since it is a modulus. It is therefore necessary to first canonically normalize it and, then, re-derive the corresponding potential. Using the form of the Kähler potential given above, denoting by $\phi$ the canonical field, one arrives at
$\tau_{n}=\left(\frac{3 \mathcal{V}_{\mathrm{s}}}{4 \gamma \lambda_{n}}\right)^{2 / 3}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{4 / 3}$.


Fig. 96. Reheating consistent slow-roll predictions for the loop inflation models for $\alpha>0$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel), and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

As a consequence, the final form of the inflaton's potential is given by

$$
\begin{align*}
V(\phi)= & \frac{B W_{0}^{2}}{M_{\mathrm{Pl}}^{2} \mathcal{V}_{\mathrm{s}}^{3}}-\frac{4 W_{0} a_{n} A_{n}}{M_{\mathrm{Pl}}^{2} \mathcal{V}_{\mathrm{s}}^{2}}\left(\frac{3 \mathcal{V}_{\mathrm{s}}}{4 \gamma \lambda_{n}}\right)^{2 / 3}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{4 / 3} \\
& \times \exp \left[-a_{n}\left(\frac{3 \mathcal{V}_{\mathrm{s}}}{4 \gamma \lambda_{n}}\right)^{2 / 3}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{4 / 3}\right] \tag{5.47}
\end{align*}
$$

Let us now see what are the typical values that the parameters appearing in the above potential can take. As already mentioned, the quantity $\mathcal{V}_{s}$ represents the Calabi-Yau volume and is supposed to be such that $\mathcal{V}_{\mathrm{s}} \gg 1$ or $\mathcal{V} \gg \ell_{\mathrm{s}}^{6}$. In Ref. [324] the typical value $\mathcal{V}_{\mathrm{s}} \simeq 3 \times 10^{6}$ was chosen. The parameter $A_{n}$ depends on the complex structure moduli and is typically of order $\mathcal{O}\left(\ell_{\mathrm{s}}^{3}\right)$. This is also the case for $W_{0}$. One has $a_{n}=2 \pi / N$, where $N$ is a positive integer (for $D 3$-brane instantons, one has $N=1$ ). The dimensionless parameter $\lambda_{n}$ is model dependent but is considered to be of order $\mathcal{O}(1)$. The quantity $\xi=\zeta(3) \chi /\left[2(2 \pi)^{3}\right]$, where $\chi$ is the Euler number of the internal Calabi-Yau space, is also of order $\mathcal{O}(1)$ as well as the coefficient $\gamma$. This means that $B$ is of order $\mathcal{O}(1)$.

### 5.3.2. Slow-roll analysis

We now study the inflationary scenario based on the potential derived above. Re-writing $V(\phi)$ in a more convenient way, we have
$V(\phi)=M^{4}\left[1-\alpha\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{4 / 3} e^{-\beta\left(\phi / M_{\mathrm{Pl}}\right)^{4 / 3}}\right]$.
where we have defined the parameters $M, \alpha$ and $\beta$ by
$M^{4}=\frac{B W_{0}^{2}}{M_{\mathrm{Pl}}^{2} \mathcal{V}_{\mathrm{s}}^{3}}, \quad \alpha=\frac{16 \mathcal{V}_{\mathrm{s}} a_{n}}{3} \frac{A_{n}}{W_{0}}\left(\frac{3 \mathcal{V}_{\mathrm{s}}}{4 \gamma \lambda_{n}}\right)^{2 / 3}$,
$\beta=a_{n}\left(\frac{3 \mathcal{V}_{\mathrm{s}}}{4 \gamma \lambda_{n}}\right)^{2 / 3}$.
Making use of the typical orders of magnitude for the various quantities entering these expression, one sees that
$\alpha=\mathcal{O}\left(\mathcal{V}_{s}^{5 / 3}\right), \quad \beta=\mathcal{O}\left(\mathcal{V}_{s}^{2 / 3}\right)$,
with $\mathcal{V}_{\mathrm{s}} \gg 1$.
The potential (5.48) and its logarithm are displayed in Fig. 45. $V(\phi)$ decreases from $V / M^{4}=1$ at $\phi=0$, reaches a minimum at $\phi / M_{\mathrm{PI}}=\beta^{-3 / 4}$, and then increases to the asymptotic value $V / M^{4}=1$ when $\phi / M_{\mathrm{PI}} \rightarrow+\infty$. However, since the potential is derived under the large field assumption, only the increasing branch of the potential is relevant. Inflation proceeds from the right to the left along this branch. The minimum value of the potential at $\phi=M_{\mathrm{PI}} \beta^{-3 / 4}$ is given by $V_{\min }=M^{4}[1-\alpha /(\beta e)]$. Therefore, if one wants the potential to be definite positive everywhere, one must have $\alpha / \beta<e$. However, from Eq. (5.50), we see that this condition cannot be satisfied since $\alpha / \beta=\mathcal{O}\left(\mathcal{V}_{\mathrm{s}}\right) \gg 1$. This means that the potential necessarily vanishes at some point. In the increasing branch of the potential, this occurs for a vev given by
$x_{V=0} \equiv \frac{\phi_{V=0}}{M_{\mathrm{Pl}}}=\left[-\frac{1}{\beta} \mathrm{~W}_{-1}\left(-\frac{\beta}{\alpha}\right)\right]^{3 / 4}$.
Anyway, since the potential (5.48) is only valid in the large field region, this criterion does not play an important role in what follows.

Let us now calculate the three first Hubble flow parameters. Defining $x \equiv \phi / M_{\mathrm{P}}$, they are given by

$$
\begin{align*}
\epsilon_{1} & =\frac{8 \alpha^{2}}{9} x^{2 / 3} e^{-2 \beta x^{4 / 3}}\left(\frac{1-\beta x^{4 / 3}}{1-\alpha x^{4 / 3} e^{-\beta x^{4 / 3}}}\right)^{2},  \tag{5.52}\\
\epsilon_{2} & =\frac{8 \alpha}{9} x^{-2 / 3} e^{-2 \beta x^{4 / 3}} \\
& \times \frac{3 \alpha x^{4 / 3}+\alpha \beta x^{8 / 3}+e^{\beta x^{4 / 3}}\left(1-9 \beta x^{4 / 3}+4 \beta^{2} x^{8 / 3}\right)}{\left(1-\alpha x^{4 / 3} e^{-\beta x^{4 / 3}}\right)^{2}}, \tag{5.53}
\end{align*}
$$

and

$$
\begin{align*}
\epsilon_{3}= & \left\{8 \alpha ( 1 - \beta x ^ { 4 / 3 } ) \left[\alpha^{2} x^{8 / 3}\left(9+\beta x^{4 / 3}\right)\right.\right. \\
& -2 \alpha e^{\beta x^{4 / 3}} x^{4 / 3}\left(-4+19 \beta x^{4 / 3}-9 \beta^{2} x^{8 / 3}\right. \\
& \left.\left.\left.+4 \beta^{3} x^{4}\right)-e^{2 \beta x^{4 / 3}}\left(1+11 \beta x^{4 / 3}-30 \beta^{2} x^{8 / 3}+8 \beta^{3} x^{4}\right)\right]\right\} \\
& \times\left\{9 x ^ { 2 / 3 } ( e ^ { \beta x ^ { 4 / 3 } } - \alpha x ^ { 4 / 3 } ) ^ { 2 } \left[\alpha x^{4 / 3}\left(3+\beta x^{4 / 3}\right)\right.\right. \\
& \left.\left.+e^{\beta x^{4 / 3}}\left(1-9 \beta x^{4 / 3}+4 \beta^{2} x^{8 / 3}\right)\right]\right\}^{-1} \tag{5.54}
\end{align*}
$$

Inflation stops when $\epsilon_{1}\left(x_{\text {end }}\right)=1$. As can be seen in Fig. 45, for $\alpha / \beta \gg 1$, the first slow-roll parameter $\epsilon_{1}$ starts increasing from $\epsilon_{1}=0$ at $x=0$, diverges at a $v e v$ that we do not need to compute here, and then decreases to vanish at $x=\beta^{-3 / 4}$. Then, it increases again, blows up at $x_{V=0}$ and, finally, asymptotically


Fig. 97. Reheating consistent slow-roll predictions for the loop inflation models for $\alpha<0$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel), and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
vanishes when $x \rightarrow \infty$. Since inflation proceeds at $x>x_{V=0}$ it always stops by violation of the slow-roll conditions. Unfortunately is not possible to find an analytic expression for $x_{\text {end }}$ but one can provide the following approximated formula,
$x_{\mathrm{end}} \simeq\left[-\frac{5}{4 \beta} \mathrm{~W}_{-1}\left(-\frac{4 \times 9^{2 / 5}}{5 \times 8^{2 / 5}} \alpha^{-4 / 5} \beta^{1 / 5}\right)\right]^{3 / 4}$,
where $\mathrm{W}_{-1}$ is the Lambert function. It is compared to the numerical solution for $x_{\text {end }}$ implemented in the ASPIC code in Fig. 46. The agreement is excellent.

Let us now turn to the slow-roll trajectory. Unfortunately, KMIII is one of the rare cases for which it cannot be integrated by quadrature. As such, in the ASPIC library, the slow-roll trajectory is numerically integrated. However, in the large field limit $x \gg$ $\beta^{-3 / 4}$, one can obtain an approximate analytic formula given by
$N_{\text {end }}-N \simeq \frac{9}{16 \alpha \beta^{2}}\left(\frac{e^{\beta x^{4 / 3}}}{x^{2}}-\frac{e^{\beta x_{\text {end }}^{4 / 3}}}{x_{\text {end }}^{2}}\right)$,
from which one deduces that

$$
\begin{align*}
x \simeq & \left(-\frac{3}{2 \beta} \mathrm{~W}_{-1}\left\{-\frac{2}{3} \beta\left[\frac{e^{\beta x_{\mathrm{end}}^{4 / 3}}}{x_{\mathrm{end}}^{2}}\right.\right.\right. \\
& \left.\left.\left.+\frac{16 \alpha \beta^{2}}{9}\left(N_{\mathrm{end}}-N\right)\right]^{-2 / 3}\right\}\right)^{3 / 4} . \tag{5.57}
\end{align*}
$$

This approximation is in agreement with what was obtained in Ref. [324], up to an incorrect choice of the Lambert function branch. The Lambert function is displayed in Fig. 47 and the part of the curve where inflation proceeds is indicated by the arrow. The fact that the -1 branch of the Lambert function has to be chosen comes from the fact that, when $N_{\text {end }}-N \rightarrow \infty$, one must have $x \rightarrow \infty$. On the other hand, when $N_{\text {end }}-N \rightarrow 0, x \rightarrow x_{\text {end }}>\beta^{-3 / 4}$ and this is again consistent with the choice of the -1 branch.

Finally, one can use the CMB normalization to calculate the mass scale $M$. Without any approximation on top of slow-roll, this leads to the following expression

$$
\begin{align*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}= & 1280 \pi^{2} \alpha^{2} x_{*}^{2 / 3} e^{-2 \beta x_{*}^{4 / 3}}\left(1-\beta x_{*}^{4 / 3}\right)^{2} \\
& \times\left(1-\alpha x^{4 / 3} e^{-\beta x_{*}^{4 / 3}}\right)^{-2} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{5.58}
\end{align*}
$$

Making use of the approximated trajectory and of the expression for the scale $M$, one roughly obtains

$$
\begin{align*}
\mathcal{V}_{\mathrm{s}} \simeq & \frac{\Delta N_{*}}{\pi \sqrt{720}} \frac{1}{\left(M_{\mathrm{Pl}} \ell_{\mathrm{s}}\right)^{3}}\left[\frac{4 B a_{n}\left(W_{0} \ell_{\mathrm{s}}^{3}\right)^{2}}{3 \gamma \lambda_{n}}\right] \\
& \times \ln ^{-5 / 4}\left(\frac{16 \alpha \beta^{2}}{9} \Delta N_{*}\right) \frac{T}{Q_{\mathrm{rms}-\mathrm{PS}}} . \tag{5.59}
\end{align*}
$$

Given that $a_{n}, B, \gamma, \lambda_{n}, W_{0} \ell_{\mathrm{s}}^{3}$ are a priori coefficients of order one, we see that the above expression roughly implies that $\mathcal{V}$ is of the order $10^{6} \ell_{s}$.

The reheating consistent slow-roll predictions for the Kähler moduli inflation II models are displayed in Fig. 115, for $\mathcal{V} \in$ [ $10^{5}, 10^{7}$ ], and taking $\alpha=\mathcal{V}^{5 / 3}$ and $\beta=\mathcal{V}^{2 / 3}$. One can check that even if one adds $\mathcal{O}(1)$ factors in these relations, the slowroll predictions do not depend significantly on them. Also, we notice that $\epsilon_{1}$ is typically extremely small and that $\epsilon_{2}$ is almost independent of $\mathcal{V}$. These effects can be analytically understood. Working out Eq. (5.55) and Eqs. (5.52)-(5.54) in the large field limit, one obtains
$\epsilon_{1 *} \simeq \frac{1}{324 \beta^{3 / 2}\left(\Delta N_{*}\right)^{2}} \ln ^{5 / 2}\left(16 \sqrt{\frac{9}{8}} \alpha \beta^{1 / 2} \Delta N_{*}\right)$,
$\epsilon_{2 *} \simeq \frac{2}{\Delta N_{*}}, \quad \epsilon_{3 *} \simeq \frac{1}{\Delta N_{*}}$,
from which one deduces that
$n_{\mathrm{S}} \simeq 1-\frac{2}{\Delta N_{*}}$,
$r \simeq \frac{4}{81 \beta^{3 / 2}\left(\Delta N_{*}\right)^{2}} \ln ^{5 / 2}\left(16 \sqrt{\frac{9}{8}} \alpha \beta^{1 / 2} \Delta N_{*}\right)$,
$\alpha_{\mathrm{S}} \simeq-\frac{2}{\Delta N_{*}^{2}}$.
Firstly, we see that the slow-roll parameters at Hubble crossing depend on $\alpha$ logarithmically only. This explains the weak dependence in the $\mathcal{O}(1)$ factors mentioned above. Secondly, we also notice that $\epsilon_{2 *}$ and $\epsilon_{3 *}$ do not depend on $\beta$. In a third place, $\epsilon_{1}$ is a very small number since it is proportional to the inverse of $\beta^{3 / 2}$. This also means that, when $\mathcal{V}$ increases, $\epsilon_{1}$ decreases. All these considerations can be checked in Fig. 115 and the amount of gravitational waves predicted by this model is very small. This is in agreement with the rough estimates given in Refs. [318,321,322,324]. However, contrary to what is claimed in Ref. [324], the predicted value for the running of the spectral index is not excluded by observations since, according to the Planck results [70], $\alpha_{\mathrm{S}}=-0.013 \pm 0.009$ while, for the fiducial value $\Delta N_{*} \simeq 55$, one obtains $\alpha_{\mathrm{S}} \simeq-0.0006$.


Fig. 98. Reheating consistent slow-roll predictions for the $R+R^{2 p}$ inflation models in the Rpl1 regime, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel), and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

### 5.4. Logamediate inflation (LMI)

Logamediate inflation has been discussed in Refs. [486,487] and refers to inflationary scenarios in which the scale factor evolves according to
$a(t)=a_{0} \exp \left[A\left(\ln \frac{t}{t_{0}}\right)^{\lambda}\right]$,
where $A$ and $\lambda$ are two dimensionless parameters and where $t_{0}$ has the dimension of a cosmic time. This evolution form for the scale factor is required to occur "at late times", i.e. when $t \gg t_{0}$. If $\lambda=1$, one recovers the power law model (see Section 4.8), and in that case, $t_{0}$ can be absorbed in a rescaling of the scale factor. Otherwise, these three parameters are relevant and one therefore expects LMI to be a two parameters models according to our classification. Following Ref. [486], from Eq. (5.62), one has
$H \equiv \frac{\dot{a}}{a}=\frac{A \lambda}{t}\left(\ln \frac{t}{t_{0}}\right)^{\lambda-1}$,
from which one deduces that $A \lambda>0$ in order to have expansion ( $H>0$ ). From Eq. (5.62), one can also establish that

$$
\begin{align*}
\frac{\ddot{a}}{a}= & \frac{A \lambda}{t^{2}}\left(\ln \frac{t}{t_{0}}\right)^{\lambda-1}\left[(\lambda-1)\left(\ln \frac{t}{t_{0}}\right)^{-1}-1\right. \\
& \left.+A \lambda\left(\ln \frac{t}{t_{0}}\right)^{\lambda-1}\right] \tag{5.64}
\end{align*}
$$

from which one deduces that in order to have inflation at late times (when $t>t_{0}$ ), one must have $\lambda>1$, or if $\lambda=1, A>1$. If this inflationary scenario is implemented within a single minimally coupled scalar field $\phi$, one can derive the corresponding potential. From the Friedmann-Lemaître and Klein-Gordon equations one can show that [486]
$\frac{\dot{\phi}(t)}{M_{\mathrm{Pl}}}=\frac{\sqrt{2 A \lambda}}{t}\left(\ln \frac{t}{t_{0}}\right)^{\frac{\lambda-1}{2}}$.
This equation can easily be integrated into
$\frac{\phi(t)}{M_{\mathrm{Pl}}}=\frac{\phi_{0}}{M_{\mathrm{Pl}}}+2 \frac{\sqrt{2 A \lambda}}{\lambda+1}\left(\ln \frac{t}{t_{0}}\right)^{\frac{\lambda+1}{2}}$.
Combining the Friedmann-Lemaître equation $3 M_{\mathrm{Pl}}^{2} H^{2}=V(\phi)+$ $\dot{\phi}^{2} / 2$ and the relation $2 M_{\mathrm{Pl}}^{2} \dot{H}=-\dot{\phi}^{2}$, one obtains $V(\phi)=$ $3 M_{\mathrm{Pl}}^{2} H^{2}+M_{\mathrm{Pl}}^{2} \dot{H}$, namely

$$
\begin{align*}
V(\phi)= & \frac{3 M_{\mathrm{P}}^{2} A^{2} \lambda^{2}}{t^{2}}\left(\ln \frac{t}{t_{0}}\right)^{2(\lambda-1)}+\frac{M_{\mathrm{PI}}^{2} A \lambda}{t^{2}}(\lambda-1)\left(\ln \frac{t}{t_{0}}\right)^{\lambda-2} \\
& -\frac{M_{\mathrm{P}}^{2} A \lambda}{t^{2}}\left(\ln \frac{t}{t_{0}}\right)^{\lambda-1} \tag{5.67}
\end{align*}
$$

Together with Eq. (5.66), this gives a parametric representation of the field potential in terms of $t$. It can be further simplified since the Logamediate regime occurs in the limit $t \gg t_{0}$. If $\lambda>1$, the first term of this expression dominates at late times and one has $V(\phi)=3 M_{P 1}^{2} A^{2} \lambda^{2}\left(\ln t / t_{0}\right)^{2(\lambda-1)} / t^{2}$. Defining $x \equiv\left(\phi-\phi_{0}\right) / M_{P 1}$, one makes use of Eq. (5.66) to obtain
$V(\phi)=M^{4} x^{\alpha} \exp \left(-\beta x^{\gamma}\right)$,
where the new parameters are defined by
$\alpha=4 \frac{\lambda-1}{\lambda+1}, \quad \beta=2\left(\frac{\lambda+1}{2 \sqrt{2 A \lambda}}\right)^{2 /(\lambda+1)}$,
$\gamma=\frac{2}{\lambda+1}$,
and
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=\frac{3 A^{2} \lambda^{2}}{M_{\mathrm{Pl}}^{2} t_{0}^{2}}\left(\frac{\lambda+1}{2 \sqrt{2 A \lambda}}\right)^{4 \frac{\lambda-1}{\lambda+1}}$.
The same potential has been studied for $\alpha=2, \beta=1 / 8$ and $\gamma=2$ within tachyon inflation models in Ref. [450]. The case $\lambda=1$ is particular. At late times, the first term and the last term must be kept in Eq. (5.67), such that $V(\phi)=(3 A-1) A M_{\mathrm{Pl}}^{2} / t^{2}$. In that situation, one has $x=\sqrt{2 A} \ln t / t_{0}$, and the derived potential shares the same expressions for $\alpha, \beta$ and $\gamma$ as in Eq. (5.69) but evaluated at $\lambda=1$. There is a difference however because $M^{4}$ now reads $M^{4}=(3 A-1) A M_{\mathrm{Pl}}^{2} / t_{0}^{2}$. We recover explicitly that $\lambda=1$ corresponds to power law inflation and has already been treated in Section 4.8.

In the following, we will work only with the derived parameters $\beta, \gamma$ and $M^{4}$, noticing that
$\alpha=4(1-\gamma)$.
The restrictions $A \lambda>0$ and $\lambda \geq 1$ translates into the conditions $0<\gamma \leq 1$ and $\beta>0$. Following Ref. [487], since there is no fundamental reasons preventing it, we will generalize this model to any possible values of these parameters supporting inflation.


Fig. 99. Reheating consistent slow-roll predictions for the $R+R^{2 p}$ inflation models in the Rpl2 regime, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel), and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The color of the data points encodes the value of $p$, while different data blocks correspond to different values of $y_{\text {end }}$. Inside a given bock, the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. When $y_{\text {end }} \gg 1$, one has $\epsilon_{2} \rightarrow 0$ which is denoted by the black line. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The three first Hubble flow functions in the slow-roll approximation read
$\epsilon_{1}=\frac{\left(\alpha-\beta \gamma \chi^{\gamma}\right)^{2}}{2 x^{2}}, \quad \epsilon_{2}=\frac{2}{x^{2}}\left[\alpha+\beta(\gamma-1) \gamma \chi^{\gamma}\right]$,
$\epsilon_{3}=\frac{\alpha-\beta \gamma \chi^{\gamma}}{x^{2}} \frac{2 \alpha-\beta(\gamma-2)(\gamma-1) \gamma x^{\gamma}}{\alpha+\beta(\gamma-1) \gamma \chi^{\gamma}}$.
The potential and the Hubble flow functions in the slow-roll approximation have been represented in Fig. 48.

Inflation can proceed in two regimes: either at decreasing field values, left to the top of the potential (LMI1), or at increasing field values, right to the top of the potential (LMI2). Notice that from Eq. (5.66), $\phi$ has to increase with time to reproduce the scale factor expansion Eq. (5.62) and this happens only in the regime LMI2 for large values of $x$. As can be seen in Fig. 48, the slow-roll parameter $\epsilon_{1}$ diverges when $x$ approaches zero, it vanishes at the top of the potential for $x=x_{V \text { max }}$ and it is maximal at $x=x_{\epsilon_{1}} \max$ with
$x_{V^{\max }} \equiv\left(\frac{\alpha}{\beta \gamma}\right)^{1 / \gamma}, \quad x_{\epsilon_{1}^{\max }}=\left[\frac{\alpha}{\beta \gamma(1-\gamma)}\right]^{1 / \gamma}$.
Finally it asymptotes to zero for large values of the field. The value of the local maximum of $\epsilon_{1}$ reads
$\epsilon_{1}^{\max }=\frac{\alpha^{2}}{2}\left[\frac{\beta \gamma(1-\gamma)}{\alpha}\right]^{\frac{2}{\gamma}}\left(\frac{\gamma}{1-\gamma}\right)^{2}$.


Fig. 100. Reheating consistent slow-roll predictions for the $R+R^{2 p}$ inflation models in the RpI3 regime, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel), and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$.

Thus in the regime LMI1, inflation always stops naturally as $\epsilon_{1}$ becomes larger than unity whereas in the regime LMI2, this may occur only if $\epsilon_{1}^{\max }>1$ and if inflation has started from $x_{\text {ini }}<x_{\epsilon_{1}^{\max }}$. Otherwise, if inflation starts with $x_{\text {ini }}>x_{\epsilon_{1}^{\max }}$, or if $\epsilon_{1}^{\max }<1$, one needs to add an extra-parameter $x_{\text {end }}$ encoding an unspecified mechanism to end inflation. In that situation, the model becomes a three parameters one. If one makes use of $\alpha=4(1-\gamma)$, one obtains $\epsilon_{1}^{\max }=8 \gamma^{2}(\beta \gamma / 4)^{2 / \gamma}$. Solving $\epsilon_{1}^{\max } \geq 1$ for $\beta$ gives
$\beta \geq \frac{4}{\gamma\left(8 \gamma^{2}\right)^{\gamma / 2}}$.
This condition is therefore required for the model LMI2, if one wants inflation to end naturally. As we will see below, LMI2 inflating in the domain $x_{V \max }<x<x_{\epsilon_{1}}^{\max }$ is a very fine-tuned situation which is strongly disfavored by the observations. Notice that if one assumes $0<\gamma \leq 1$, this conditions translates into $\beta>\sqrt{2}$.

Finally, let us notice that for the value of $\epsilon_{2}$ at the top of the potential to be smaller than some maximal value $\epsilon_{2, \text { top }}^{\max }$, one needs to impose the condition
$\beta<\beta^{\max }\left(\gamma, \epsilon_{2, \text { top }}^{\max }\right)=2^{2-3 \gamma / 2}\left(\epsilon_{2, \text { top }}^{\max }\right)^{\gamma / 2} \frac{(1-\gamma)^{1-\gamma / 2}}{\gamma^{1+\gamma / 2}}$.
In the LMI1 model, a slow roll regime of inflation can proceed only if such a condition is verified (with typically $\epsilon_{2, \text { top }}^{\max } \simeq 10^{-1}$ ).

The slow-roll trajectory can be integrated thanks to the hypergeometric function $[216,217]{ }_{2} F_{1}$, leading to
$N-N_{\text {end }}=\frac{x_{\text {end }}^{2}}{2 \alpha}{ }_{2} F_{1}\left[1, \frac{2}{\gamma}, \frac{2}{\gamma}+1,\left(\frac{x_{\text {end }}}{x_{V \max }}\right)^{\gamma}\right]$

$$
\begin{equation*}
-\frac{x^{2}}{2 \alpha}{ }_{2} F_{1}\left[1, \frac{2}{\gamma}, \frac{2}{\gamma}+1,\left(\frac{x}{x_{V \max }}\right)^{\gamma}\right] . \tag{5.78}
\end{equation*}
$$

One can notice that inserting $\alpha=4(1-\gamma)$, as a function of $x / x_{V \max }$, this trajectory only involves $\gamma$. Plugging $x=x_{V \text { max }}$ into Eq. (5.78) one gets an infinite number of $e$-folds. This means that the required number of $e$-folds to solve the problems of the standard Big-Bang scenario can always be realized, both in the decreasing branch of the potential and the increasing one, provided that inflation starts close enough to $x_{V \max }$. However, it can numerically be checked that in the case of LMI2 with $\epsilon_{1}^{\max }>1$ and inside the $x_{V} \max <x<x_{\epsilon_{1}}^{\max }$ region, one has to fine-tune $x_{\text {ini }}$ and $x_{*}$ extremely close to $x_{V \text { max }}$. In that situation $n_{S}=1$, with vanishing $r$ and vanishing running of the spectral index, can be considered as generic predictions of the model. For this reason, it is more natural to consider LMI2 in the large field regime, namely $x>\max \left(x_{V \max }, x_{\epsilon_{1}^{\max }}\right)$, together with the extra-parameter $x_{\text {end }}$.

The trajectory in Eq. (5.78) cannot be inverted analytically. However, one can perform some consistency checks in the limit $x / x_{V^{\max }} \gg 1$ in which
$N-N_{\mathrm{end}} \simeq \frac{1}{\beta \gamma(2-\gamma)}\left(x^{2-\gamma}-x_{\mathrm{end}}^{2-\gamma}\right)$,
and
$x \simeq\left[x_{\text {end }}^{2-\gamma}+\beta \gamma(2-\gamma)\left(N-N_{\mathrm{end}}\right)\right]^{\frac{1}{2-\gamma}}$.
These expressions can be compared to Eq. (5.66)
$x=2 \frac{\sqrt{2 A \lambda}}{\lambda+1}\left(\ln \frac{t}{t_{0}}\right)^{\frac{\lambda+1}{2}}$,
where $t$ in terms of the number of $e$-folds $N$ can be obtained from Eq. (5.62). With $N-N_{0}=A\left(\ln t / t_{0}\right)^{\lambda}$, one gets
$x=2 \frac{\sqrt{2 A \lambda}}{\lambda+1}\left(\frac{N-N_{0}}{A}\right)^{\frac{\lambda+1}{2 \lambda}}$.
The previous calculations are consistent since, making use of Eqs. (5.69), (5.80) and (5.82) are the same when setting the constants $N_{0}=N_{\text {ini }}$ and $x_{0}=x_{\text {ini }}=0$. This means that in the late times limit $x / x_{V^{\max }} \gg 1$, the slow-roll trajectory coincides with the exact one, as expected.

The amplitude of the CMB anisotropies fixes the value of the parameter $M$ according to
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=720 \pi^{2}\left(\alpha-\beta \gamma \chi_{*}^{\gamma}\right)^{2} e^{\beta \chi_{*}^{\gamma}} \chi_{*}^{-\alpha-2} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$,
where $x_{*}$ is the observable field value obtained by solving Eq. (2.47) given some assumptions on the reheating. The reheating consistent slow-roll predictions for the models LMI1 and LMI2 (at $x>x_{\epsilon_{1}^{\max }}$ ) are displayed in Figs. 116-118 for LMI1, and in Figs. 119-121 for LMI2. In the case of LMI2, the turning points in the plots precisely correspond to the case where inflation occurs in the fine-tuned domain $x_{V \max }<x_{*}<x_{\epsilon_{1}^{\max }}$ and in which the model behaves like a pure de Sitter era.

### 5.5. Twisted inflation (TWI)

### 5.5.1. Theoretical justifications

This model was introduced in Ref. [488] and is based on higher dimensional supersymmetric gauge theories. The idea is to assume that, in higher dimensions, we have a flat direction $\phi$ in the potential. Since the theory is supersymmetric, this flat direction will not receive corrections because the bosonic and fermionic


Fig. 101. Reheating consistent slow-roll predictions for the double well models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The shape of the zone covered by the models predictions is similar to the one for Small Field Inflation (SFI, see Fig. 112), except in the domain $\phi_{0} \gg M_{\mathrm{Pl}}$, which is the one favored by the observations. The black solid line represents the locus of the points such that $r=4\left(1-n_{\text {s }}\right)$, i.e. $\epsilon_{2}=2 \epsilon_{1}$, on which this model lies for $\phi_{0} / M_{P 1} \gg 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
contributions exactly cancel out. Then, we compactify the theory down to $3+1$ dimensions but with boundary conditions that break supersymmetry. The typical example given in Ref. [488] is "twisted" circle compactification, hence the name of the model. Since supersymmetry is broken, the "Kaluza-Klein" masses of bosons and fermions will differ. Typically, they can be written as
$m_{\mathrm{b}}=\sqrt{\phi^{2}+\frac{n^{2}}{R^{2}}}, \quad m_{\mathrm{f}}=\sqrt{\phi^{2}+\frac{(n+1 / 2)^{2}}{R^{2}}}$,
where $R$ is the radius of compactification and $n$ an integer. Since $m_{\mathrm{b}} \neq m_{\mathrm{f}}$, this time, the potential will receive one loop corrections which lift the potential. However, it is clear that, when $\phi R \gg n$, one has approximately $m_{\mathrm{b}} \simeq m_{\mathrm{f}}$. Therefore, in this regime, we expect the corrections to vanish and the flat direction to remain flat. This is thus particularly well-suited for inflation. In practice, the higher dimensional model considered to implement the above discussed mechanism is a maximally supersymmetric $4+1 \mathrm{U}(\mathcal{N})$ Yang-Mills theory compactified on a circle of radius $R$. A priori, we have therefore two parameters: $\mathcal{N}$ and the compactification scale $R$.

### 5.5.2. Slow-roll analysis

As shown in Ref. [488], the above considerations leads to the following expression for the inflaton potential
$V(\phi)=M^{4}\left[1-A\left(\frac{\phi}{\phi_{0}}\right)^{2} e^{-\phi / \phi_{0}}\right]$,
where $A$ is a constant parameter given by
$A \equiv \frac{32}{93 \zeta(5)} \simeq 0.33$,
and where $\phi_{0}$ is related to the compactification scale $R$ through the following equation
$\frac{\phi_{0}}{M_{\mathrm{PI}}}=\frac{1}{2 \pi R M_{\mathrm{PI}}}$.
Since the radius $R$ must be larger than the Planck length, i.e. $R M_{\mathrm{PI}} \gg$ 1 , this implies that $\phi_{0} / M_{\mathrm{PI}} \ll 1$. On the other hand, the overall normalization can be expressed as
$M^{4}=\frac{8 \mathcal{N}}{A \pi^{2}(2 \pi R)^{4}}$.
We see that the scale $M$ depends on the compactification radius $R$ but also on the number $\mathcal{N}$. In addition, one must have $\phi<$ $\sqrt{3 / \mathcal{N}} M_{\mathrm{Pl}}$ or $\phi \ll M_{\mathrm{Pl}}$ which guarantees that the higher order Planck suppressed operators do not alter the potential. The potential (5.85) is the small coupling limit of the model, while the strong coupling limit corresponds to a BI model with $p=3$, see Section 5.19.

The potential Eq. (5.85), as well as its logarithm, is displayed in Fig. 49. Inflation is supposed to take place for vev's larger than the scale $\phi_{0}$, i.e. for $\phi>\phi_{0}$, in the increasing branch of the potential. This means that it proceeds from the right to the left in the direction indicated by the arrow. The minimum of the potential is located at $\phi / \phi_{0}=2$.

Let us now turn to the calculation of the Hubble flow parameters. If one defines $x$ by $x \equiv \phi / \phi_{0}$, then they are given by
$\epsilon_{1}=\frac{A^{2}}{2}\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} e^{-2 x}\left[\frac{x(x-2)}{1-A x^{2} e^{-x}}\right]^{2}$,
$\epsilon_{2}=2 A\left(\frac{M_{\mathrm{PI}}}{\phi_{0}}\right)^{2} e^{-2 x} \frac{2 A x^{2}+e^{x}\left(x^{2}-4 x+2\right)}{\left(1-A x^{2} e^{-x}\right)^{2}}$,
and

$$
\begin{align*}
\epsilon_{3} & =A\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} x(2-x) e^{-2 x} \\
& \times \frac{4 A^{2} x^{3}-e^{2 x}\left(x^{2}-6 x+6\right)-A x e^{x}\left(x^{3}-6 x^{2}+18 x-12\right)}{\left(1-A x^{2} e^{-x}\right)^{2}\left[2 A x^{2}+e^{x}\left(x^{2}-4 x+2\right)\right]} \tag{5.90}
\end{align*}
$$

They are displayed in Fig. 49. The first slow-roll parameter $\epsilon_{1}$ vanishes at the minimum of the potential when $x=2$, then increases with $x$ and reaches a maximum at $x_{\epsilon_{1}}^{\max }$, and finally decreases to zero when $x$ goes to infinity. The value of $\epsilon_{1}$ at this local maximum is larger than one if $\phi_{0}$ is smaller than some value that can only be determined numerically. We find
$\phi_{0}<0.04228 M_{\mathrm{PI}}$.
Therefore, a priori, inflation could stop by slow-roll violation. However, by numerically integrating the exact trajectory (i.e. if one does not make use of the slow-roll approximation), one realizes that, in fact, the first Hubble flow function, which is defined by $\epsilon_{1}^{H}=-\dot{H} / H^{2}$, remains smaller than one for all field values,


Fig. 102. Reheating consistent slow-roll predictions for the mutated hilltop models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. For small values of $\mu / M_{\mathrm{Pl}}$, this model predicts a very small amount of gravitational waves. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
see Fig. 50. This is due to the fact that while the inflaton rolls down its potential and approaches its minimum, the slow-roll parameters continuously increase and the slow-roll approximation is broken before $\epsilon_{1}$ becomes $\mathcal{O}(1)$. Usually, this leads only to small corrections at the end of inflation. However, in the case of twisted inflation, this leads to a radically different picture because the potential does not vanish at its minimum and, therefore, acts as a cosmological constant. In practice, the numerical calculations indicate that the field oscillates around its minimum but always such that $\epsilon_{1}^{H}<1$ and independently on the value of $\phi_{0}$, see Fig. 50. In principle, inflation can never stops in this model since the final stage of the evolution corresponds to an inflaton field sitting for ever at the bottom of the potential and, as already mentioned, it acts as a cosmological constant. However, as explained in Ref. [488], the interactions of the inflaton field with the other degrees of freedom of the standard model starts to play a role in this regime. As a consequence, the energy contained in the inflaton field should quickly be transferred to other fields and a phase of reheating starts. The details of this process are complicated and are discussed in Ref. [488]. In order to model the end of inflation, we therefore introduce the extra parameter $x_{\text {end }}$ giving the vev at which inflation stops. As a consequence, TWI is in fact a two parameter model, $\phi_{0}$ and $\phi_{\text {end }}$.

Let us now turn to the slow-roll trajectory. It can be integrated exactly and leads to the following expression
$N_{\text {end }}-N=\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2}\left\{\frac{1}{2 A}\left[\operatorname{Ei}\left(x_{\text {end }}\right)-\operatorname{Ei}(x)\right]\right.$

$$
\begin{align*}
& -\frac{e^{2}}{2 A}\left[\operatorname{Ei}\left(x_{\mathrm{end}}-2\right)-\operatorname{Ei}(x-2)\right] \\
& \left.+x_{\mathrm{end}}-x+2 \ln \left(\frac{x_{\mathrm{end}}-2}{x-2}\right)\right\} \tag{5.92}
\end{align*}
$$

where $N_{\text {end }}$ is the number of $e$-folds at the end of inflation and Ei is the exponential integral function [216,217]. This expression is the one used in the ASPIC library. However, if one makes the vacuum dominated approximation, $x \gg 1$, then a simpler formula can be derived for the trajectory, namely
$N_{\text {end }}-N \simeq \frac{1}{A}\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2}\left(\frac{e^{x}}{x^{2}}-\frac{e^{x_{\text {end }}}}{x_{\text {end }}^{2}}\right)$.
This allows us to obtain an approximated expression for the vev of the field at Hubble radius crossing which reads
$x_{*} \simeq \ln \left[4 A \Delta N_{*}\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2}\right]$.
It is valid provided $\phi_{0} / M_{\mathrm{Pl}} \ll 1$, i.e. precisely in the regime for which the TWI potential was derived. Using this formula, one can estimate the value of the three first Hubble flow parameters at Hubble crossing. One arrives at
$\epsilon_{1 *} \simeq \frac{A^{2}}{2}\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} e^{-2 x_{*} x_{*}^{4}} \simeq \frac{1}{32 \Delta N_{*}^{2}}\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2}$,
$\epsilon_{2 *} \simeq \frac{\epsilon_{3 *}}{2} \simeq 2 A\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} e^{-x_{*}} x_{*}^{2} \simeq \frac{1}{2 \Delta N_{*}}$.
Finally, we can derive an expression for the tensor-to-scalar ratio, the spectral index
$r \simeq 8 A^{2}\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} e^{-2 x_{*}} x_{*}^{4} \sim \frac{1}{2 \Delta N_{*}^{2}}\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2}$,
$n_{\mathrm{S}}-1 \simeq-2 A\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} x_{*}^{2} e^{-x_{*}} \sim \frac{1}{2 \Delta N_{*}}$,
and the running
$\alpha_{\mathrm{S}} \simeq-2 A^{2}\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{4} x_{*}^{4} e^{-2 x_{*}} \simeq-\frac{1}{8 \Delta N_{*}^{2}}$.
These estimates are in agreement with the ones of Ref. [488], up to a missing factor 4 in Eq. (5.94). However, we have checked that this does not affect the predictions in a significant way.

It is also interesting to discuss the value of the scale $M$ since this is important from the model building point of view. The CMB normalization gives
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=720 \pi^{2} A^{2}\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} \frac{\left[e^{-x_{*}} x_{*}\left(x_{*}-2\right)\right]^{2}}{\left(1-A x_{*}^{2} e^{-x_{*}}\right)^{3}} \frac{Q_{\mathrm{mms}-\mathrm{PS}}^{2}}{T^{2}}$.
In the vacuum dominated approximation, the above expression simplifies and gives $M^{4} / M_{\mathrm{Pl}}^{4} \simeq 45 \pi^{2} / \Delta N_{*}^{2} \phi_{0}^{2} / M_{\mathrm{Pl}}^{2} Q_{\mathrm{rms}-\mathrm{PS}}^{2} / T^{2}$. This leads to
$M_{\mathrm{Pl}} R=\sqrt{\frac{2 \mathcal{N}}{45 A}} \frac{\Delta N_{*}}{\pi^{3}} \frac{T}{Q_{\mathrm{rms}-\mathrm{PS}}} \simeq 1.2 \times 10^{5} \sqrt{\mathcal{N}}$,
where we have taken $\Delta N_{*} \simeq 60$. This also implies that
$\frac{\phi_{0}}{M_{\mathrm{Pl}}} \simeq \frac{1.35}{\sqrt{\mathcal{N}}} \times 10^{-5}$.
Therefore, we have a rough determination of the compactification radius. The model seems consistent since we obtain that $M_{\mathrm{P} 1} R \gg 1$,


Fig. 103. Reheating consistent slow-roll predictions for the radion gauge models in the plane ( $\left.n_{\mathrm{S}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. At large values of $\alpha$, the predictions are the same as the large field model with $p=2$ (see Fig. 83) for which $\epsilon_{2}=2 \epsilon_{1}$ (black solid line).
in agreement with the assumptions made at the beginning of this section.

The predictions for TWI are presented in Fig. 122. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to be 0 since the potential is quadratic close to its minimum. However, since the details of reheating depend on the details of the interactions between the inflaton field and the others degrees of freedom in the theory, this parameter is a priori unspecified and could very well take different values. In the ASPIC code, $\bar{w}_{\text {reh }}$ can be freely chosen. Anyway, since the reheating temperature is in fact fully degenerate with the parameter $x_{\text {end }}$, these two parameters can not be constrained independently. One can check that the rough description provided by Eq. (5.96) is correct: the model typically predicts a small amount of gravitational waves which increases with $\phi_{0}$, and a deviation from scale invariance which does not significantly depends on $\phi_{0}$. When $\phi_{0} / M_{\mathrm{PI}}=\mathcal{O}(1)$, however, one notices a turning point (at fixed values of $\phi_{0}$ ). This corresponds to the separation between two regimes, one where $x_{*}<x_{\epsilon_{1}}^{\max }$ and $\epsilon_{1}$ is an increasing function of $x$ (hence $\epsilon_{1 *}$ increases with $x_{\text {end }}$ ) and another where $x_{*}>x_{\epsilon_{1}}^{\max }$ and $\epsilon_{1}$ is a decreasing function of $x$ (hence $\epsilon_{1 *}$ decreases with $x_{\text {end }}$ ). If a sufficient number of $e$-folds can be realized in the $2<x<x_{\epsilon_{1}^{\max }}$ part of the potential, then $\epsilon_{2 *}$ can become negative. However, this mostly happens for fine-tuned values of $x_{\text {end }} \simeq 2$.

### 5.6. Generalized MSSM inflation (GMSSMI)

As for the MSSMI models, see Section 4.17, GMSSMI scenarios are based on the Minimal Supersymmetric Model (MSSM) in which


Fig. 104. Reheating consistent slow-roll predictions for the MSSMI models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid line represent the locus of the points such that $r=4\left(1-n_{\mathrm{S}}\right)$, i.e. $\epsilon_{2}=2 \epsilon_{1}$, on which this model lies for for $\phi_{0} / M_{\mathrm{Pl}} \gg 1$. However, the physical relevant value is closer to $\phi_{0} / M_{\mathrm{Pl}} \simeq 10^{-4}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
a flat direction direction is lifted by soft supersymmetry breaking terms and by superpotential corrections. The potential is of the form
$V(\phi)=\frac{1}{2} m_{\phi}^{2} \phi^{2}-A \frac{\lambda_{n}}{n} \frac{\phi^{n}}{M_{\mathrm{Pl}}^{n-3}}+\lambda_{n}^{2} \frac{\phi^{2(n-1)}}{M_{\mathrm{Pl}}^{2(n-3)}}$.
The MSSMI model corresponds to $n=6$ and $A^{2}=8(n-1) m_{\phi}^{2}$. This last relation is of crucial importance since it implies an exact flat inflection point. Following Refs. [423,424,427,489-492], one may wonder whether the model is robust when this relation is not exactly satisfied. In order to investigate this question, we therefore relax the condition $A^{2}=8(n-1) m_{\phi}^{2}$. In this more general case, the potential can be reparametrized in the form
$V(\phi)=M^{4}\left[\left(\frac{\phi}{\phi_{0}}\right)^{2}-\frac{2}{3} \alpha\left(\frac{\phi}{\phi_{0}}\right)^{6}+\frac{\alpha}{5}\left(\frac{\phi}{\phi_{0}}\right)^{10}\right]$,
where $\phi_{0} \simeq 10^{14} \mathrm{GeV}$, this value being the same as the one found in Section 4.17. The positive dimensionless parameter $\alpha$ encodes any deviations from the MSSM case for which it equals unity, $\alpha_{\text {MSSM }}=$ 1.

The potential is displayed in Fig. 51, where four cases can be distinguished. In the following, we define the quantity $x$ by the expression
$x \equiv \frac{\phi}{\phi_{0}}$.

If $\alpha<9 / 25$, the second derivative of the potential does not vanish and the potential is convex everywhere. This corresponds to the case $\alpha=0.1$ case in Fig. 51. If $9 / 25<\alpha<1$, the potential has two inflection points $x_{V^{\prime \prime}=0}^{ \pm}$and is concave in between. It remains an increasing function of the field since its first derivative never vanishes. This is illustrated with the case $\alpha=0.7$ in Fig. 51. If $\alpha=$ 1 , this is the MSSM inflation models (see Section 4.17) where the potential has a flat inflection point. If $1<\alpha<9 / 5$, the potential decreases in between $x_{V^{\prime}=0}^{ \pm}$but remains positive everywhere. This is exemplified by the case $\alpha=1.5$ in Fig. 51. Finally, if $\alpha>9 / 5$, the potential becomes negative (hence is not properly defined) between the two points $x_{V=0}^{ \pm}$(see $\alpha=2.5$ in Fig. 51). The values of the field vev's appearing in this discussion are given by the following formulas:
$x_{V^{\prime \prime}=0}^{ \pm}=\left[\frac{5}{9}\left(1 \pm \sqrt{1-\frac{9}{25 \alpha}}\right)\right]^{1 / 4}$,
$x_{V^{\prime}=0}^{ \pm}=\left(1 \pm \sqrt{1-\frac{1}{\alpha}}\right)^{1 / 4}$,
and
$x_{V=0}^{ \pm}=\left[\frac{5}{3}\left(1 \pm \sqrt{1-\frac{9}{5 \alpha}}\right)\right]^{1 / 4}$.
Let us now calculate the first three Hubble flow functions in the slow-roll approximation. They are given by
$\epsilon_{1}=450\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} \frac{\left(1-2 \alpha x^{4}+\alpha x^{8}\right)^{2}}{x^{2}\left(15-10 \alpha x^{4}+3 \alpha x^{8}\right)^{2}}$,
$\epsilon_{2}=60\left(\frac{M_{\mathrm{PI}}}{\phi_{0}}\right)^{2} \frac{15+40 \alpha x^{4}+\alpha(20 \alpha-78) x^{8}+3 \alpha^{2} x^{16}}{x^{2}\left(15-10 \alpha x^{4}+3 \alpha x^{8}\right)^{2}}$,
and

$$
\begin{align*}
\epsilon_{3} & =60\left(\frac{M_{\mathrm{PI}}}{\phi_{0}}\right)^{2}\left[225-1800 \alpha x^{4}+60 \alpha(69+10 \alpha) x^{8}\right. \\
& -40(189-100 \alpha) \alpha^{2} x^{12}+10 \alpha^{2}\left(243-504 \alpha+402 \alpha^{2}\right) x^{16} \\
& +40 \alpha^{3}(117-20 \alpha) x^{20}+12 \alpha^{3}(10 \alpha-123) x^{24} \\
& \left.+72 \alpha^{4} x^{28}+9 \alpha^{4} x^{3} 2\right] \\
& \times\left[3375 x^{2}+4500 \alpha x^{6}-600 \alpha(27+10 \alpha) x^{10}\right. \\
& +100 \alpha^{2}(261-20 \alpha) x^{14}+10 \alpha^{2}\left(200 \alpha^{2}-840 \alpha-621\right) x^{18} \\
& +60 \alpha^{3}(69-20 \alpha) x^{22} \\
& \left.+48 \alpha^{3}(10 \alpha-9) x^{26}-180 \alpha^{4} x^{30}+27 \alpha^{4} x^{34}\right]^{-1} . \tag{5.107}
\end{align*}
$$

The first two slow-roll parameters diverge when $x \rightarrow 0$ and vanish asymptotically. In between, their shape depends on $\alpha$ as it is represented in Fig. 51. If $\alpha<1, \epsilon_{1}$ first decreases, reaches a local non-zero minimum where $\epsilon_{2}$ vanishes, then increases to reach a local maximum where $\epsilon_{2}$ vanishes again, and eventually decreases again. Let $x_{\epsilon_{2}=0}^{ \pm}$be the position of these two local extrema. From Ferrari's solutions for depressed quartic equations one gets

$$
\begin{align*}
& x_{\epsilon_{2}=0}^{ \pm} \\
& =\left[\frac{1}{2 \alpha} \sqrt{\frac{5}{3}}\left(\sqrt{\Sigma} \pm 2 \sqrt{\frac{39}{5} \alpha-2 \alpha^{2}-\frac{\Sigma}{4}-\frac{12}{\sqrt{15 \Sigma}} \alpha^{2}}\right)\right]^{1 / 4}, \tag{5.108}
\end{align*}
$$

where
$\delta=\frac{736 \alpha^{2}}{25}-\frac{208 \alpha^{3}}{15}+\frac{16 \alpha^{4}}{9}$,
$\Delta=-\frac{430336 \alpha^{4}}{625}+\frac{612352 \alpha^{5}}{1125}-\frac{20992 \alpha^{6}}{225}+\frac{256 \alpha^{8}}{243}$,
$\sigma=-\frac{12896 \alpha^{3}}{125}+\frac{2944 \alpha^{4}}{25}-\frac{416 \alpha^{5}}{15}+\frac{64 \alpha^{6}}{27}+\frac{6}{5} \sqrt{15 \Delta} \alpha$,
$\Sigma=\frac{52 \alpha}{5}-\frac{8 \alpha^{2}}{3}+\frac{\delta}{\sigma^{1 / 3}}+\sigma^{1 / 3}$,
are intermediate quantities introduced solely to reduce the size of Eq. (5.108). If $\alpha>1, \epsilon_{1}$ has two local minimums located at $x_{V^{\prime}=0}^{ \pm}$ where it vanishes. In between it reaches a local maximum or may even diverges for $\alpha>9 / 5$ (see Fig. 51). The slow-roll parameter $\epsilon_{2}$ vanishes when $\epsilon_{1}$ reaches these local maxima, or diverge when $\epsilon_{1}$ does (for $\alpha>9 / 5$ ). As explained in Section 4.17, inflation is meant to proceed at $\phi \lesssim \phi_{0}$. Let us assume that inflation can end for $\epsilon_{1}>1$ between $x=0$ and the position of the first minimum $x_{\epsilon_{1}}$ min. Following the previous considerations, this latter location is defined as
$x_{\epsilon_{1} \min }= \begin{cases}x_{\epsilon_{2}=0}^{-} & \text {if } \alpha<1 \\ x_{V^{\prime}=0}^{2} & \text { if } \alpha>1,\end{cases}$
and provides an upper bound to $x_{\text {end }}$ the solution of $\epsilon_{1}\left(x_{\text {end }}\right)=1$. This one can only be determined numerically. The values of $x_{\epsilon_{2}=0}^{ \pm}$ and $x_{V^{\prime}=0}^{ \pm}$in terms of $\alpha$ are displayed in the left panel in Fig. 52 together with $x_{\epsilon_{1}^{\min }}$. The right panel of Fig. 52 represents the value of the first slow-roll parameter at this minimum, $\epsilon_{1}^{\min }=\epsilon_{1}\left(x_{\epsilon_{1} \text { min }}\right)$. For $\alpha<1$, one can see that $\epsilon_{1}^{\min }<1$ only if the parameter $\alpha \lesssim 1$. This defines a minimum value for $\alpha$, which depends on $\phi_{0}$, such that inflation can take place within this domain. When $\alpha \simeq 1$, one can derive an approximated version of Eq. (5.108), namely, $x_{\epsilon_{2}=0}^{-} \simeq 1-(1-\alpha) / 32$. Plugging it into the expression for $\epsilon_{1}$ one obtains
$\epsilon_{1}^{\min } \simeq \frac{225}{32}(\alpha-1)^{2} \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}}$,
from which one gets
$\alpha>1-\frac{4 \sqrt{2}}{15} \frac{\phi_{0}}{M_{\mathrm{PI}}}$.
For the value suggested in Ref. [420], $\phi_{0} / M_{\mathrm{PI}} \simeq 10^{-4}$, one obtains $\alpha>1-10^{-5}$, which is in agreement with Ref. [489], and shows that the model needs to be sufficiently fine-tuned (i.e. sufficiently close to regular MSSM inflation) in order to be a viable inflationary model.

On top of that, as shall be seen now, the constraints on $\alpha$ are even tighter if one wants a sufficient number of $e$-folds to be produced. Let us thus turn to the slow-roll trajectory. It can be integrated, and leads to

$$
\begin{align*}
& N_{\text {end }}-N=\frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}}\left\{-\frac{x_{\mathrm{end}}^{2}-x^{2}}{20}\right. \\
& \quad-\frac{b_{+}}{10 \sqrt{a_{+}}}\left[\arctan \left(\sqrt{a_{+}} x_{\mathrm{end}}^{2}\right)-\arctan \left(\sqrt{a_{+}} x^{2}\right)\right] \\
& \left.\quad-\frac{b_{-}}{10 \sqrt{a_{-}}}\left[\arctan \left(\sqrt{a_{-}} x_{\mathrm{end}}^{2}\right)-\arctan \left(\sqrt{a_{-}} x^{2}\right)\right]\right\}, \tag{5.113}
\end{align*}
$$

where
$a_{ \pm}=-\alpha \pm \sqrt{\alpha^{2}-\alpha}, \quad b_{ \pm}=2 \frac{a_{ \pm}+\alpha / 3}{a_{ \pm}-a_{\mp}}$.


Fig. 105. Reheating consistent slow-roll predictions for the renormalizable inflection point models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

A few remarks are in order. Firstly, even if the terms appearing in the previous expression are complex, their imaginary contributions cancel out and the resulting expression is truly a real quantity. Then, one can check that formally, when $\alpha \rightarrow 0$, one has $a_{ \pm} \rightarrow 0$ and $b_{ \pm} \rightarrow 1$, hence $N \simeq-\left(x^{2}-x_{\mathrm{ini}}^{2}\right) / 4$, which is precisely the LFI slow-roll trajectory for $p=2$, see Section 4.2. This is just a formal check since $\alpha$ is meant to be tuned close to 1 in the GMSSMI scenario. Finally, let us notice that, in the case $\alpha<1$, and contrary to the MSSM models ( $\alpha=1$ ), the number of $e$-folds never diverges at a given point $x$. Therefore, the total number of $e$-folds is bounded from above for the field vev's considered here. Working out the limit of Eq. (5.113) when $\alpha \rightarrow$ 1, one has
$N_{\text {end }}-N_{\text {ini }} \leq\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2} \frac{\pi}{30} \frac{1}{\sqrt{1-\alpha}}$.
If one require at least $\Delta N=N_{\text {end }}-N_{\text {ini }} e$-folds during inflation, then $\alpha$ has to be fine-tuned to
$\alpha>1-\left(\frac{\phi_{0}}{M_{\mathrm{PI}}}\right)^{4} \frac{\pi^{2}}{900 \Delta N^{2}}$.
Remembering that the small parameter here is $\phi_{0} / M_{\mathrm{Pl}}$, one can see that it is a much tighter constraint than the one of Eq. (5.112). Taking $\phi_{0} / M_{\mathrm{Pl}} \simeq 10^{-4}$ and $\Delta N \simeq 50$, one obtains $\alpha>1-10^{-22}$. This is clearly an extreme fine-tuning which can even make the numerical investigation of the model challenging. ${ }^{7}$ As explained

[^6]below, the same condition $|\alpha-1|<\phi_{0}^{4} / M_{\mathrm{Pl}}^{4} / \Delta N^{2}$ also applies to the case $\alpha>1$ in order to maintain an acceptable deviation from scale invariance. This makes GMSSM inflation a severely finetuned scenario. Let us also notice that our parameter $\alpha$ is related to the parameter $\delta$ of Ref. [490] by $\delta=\sqrt{\alpha^{-2}-1}$. Ref. [490] finds that, in order for the model to be compatible with the data, $\delta \simeq 10^{-20}$. Therefore, although our method slightly differs from that of Ref. [490], our results are in broad agreement.

Finally, the amplitude of the CMB anisotropies fixes the parameter $M$ to

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=2880 \pi^{2} \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}} \frac{\left(1-2 \alpha x_{*}^{4}+\alpha x_{*}^{8}\right)^{2}}{x_{*}^{4}\left(1-\frac{2}{3} \alpha x_{*}^{4}+\frac{\alpha}{5} x_{*}^{8}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{5.117}
\end{equation*}
$$

As explained in Section 4.17, this leads to $M / M_{\mathrm{PI}} \simeq 10^{8} \mathrm{GeV}$ for $\phi_{0} / M_{\mathrm{PI}} \simeq 10^{-4}$.

The reheating consistent slow-roll predictions of the GMSSMI models are displayed in Figs. 123, 124, for $\alpha>1$ and $\alpha<$ 1 , respectively. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 since the potential is quadratic close to its minimum. In both cases, one can see that in the limit $\alpha \rightarrow 1$, the standard MSSM predictions are recovered, see Fig. 104. The amount of gravitational waves $r$ seems to be quite independent on $\alpha$ and, therefore, is similar to its regular MSSM counterpart. On the other hand, the spectral index $n_{S}$ strongly depends on $\alpha$. In the case $\alpha>1$, larger values of $\alpha-1$ worsens the spectral index problem, already present in standard MSSMI. These models are therefore strongly disfavored by the data. In the case $\alpha<1$ however, there is a very narrow range of acceptable values for $\alpha$. They are well inside the $|\alpha-1|<\phi_{0}^{4} / M_{\mathrm{Pl}}^{4} / \Delta N^{2}$ condition and the spectral index is inside the two-sigma confidence intervals. But, as can be seen in Fig. 124, the spectral index varies so quickly with $\alpha$ that one has to fine-tune the power of the fine-tuning to remain inside the twosigma contours. In Refs. [424,489-492], it is argued that, since the flat saddle point condition is robust against radiative corrections, such a fine-tuning may not be a problem. However, as explained here and in Section 4.17, if the flat saddle point condition is exactly satisfied, the model is disfavored by the observations because the spectral index is too red. The only way out is therefore to detune the condition $\alpha=1$ at an extremely fine-tuned level.

### 5.7. Generalized renormalizable point inflation (GRIPI)

As for the MSSMI models (see Section 4.17) and for the RIPI models (see Section 4.18), the GRIPI models have a potential of the form
$V(\phi)=\frac{1}{2} m_{\phi}^{2} \phi^{2}-A \frac{\lambda_{n}}{n} \frac{\phi^{n}}{M_{\mathrm{Pl}}^{n-3}}+\lambda_{n}^{2} \frac{\phi^{2(n-1)}}{M_{\mathrm{Pl}}^{2(n-3)}}$.
In Section 4.18, the particular example $n=3$ is discussed in the case where the potential has a flat inflection point, i.e. when $A^{2}=16 m_{\phi}^{2}$. Then, as studied in Section 5.6 for MSSMI, comes the question of what happens when we relax this condition. To address this issue, it is convenient to reparametrize the potential as
$V(\phi)=M^{4}\left[\left(\frac{\phi}{\phi_{0}}\right)^{2}-\frac{4}{3} \alpha\left(\frac{\phi}{\phi_{0}}\right)^{3}+\frac{\alpha}{2}\left(\frac{\phi}{\phi_{0}}\right)^{4}\right]$,
where the positive dimensionless parameter $\alpha$ encodes the deviation from the RIPI case (that is to say $\alpha_{\text {RIPI }}=1$ ). This model was studied in Ref. [493] and in Refs. [494,495]. In the first reference, the mass $m_{\phi}$ is fixed by the soft supersymmetry breaking terms and, in Section 4.18, it was shown that this leads to $\phi_{0} \simeq 10^{14} \mathrm{GeV}$. However, in Refs. [494,495], the scale $m_{\phi}$ is no longer controlled by the soft supersymmetry breaking terms but by


Fig. 106. Reheating consistent slow-roll predictions for the ArcTan models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel), when the reheating equation of state is $\bar{w}_{\text {reh }}=0$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
the right-handed neutrino mass in Type I supersymmetric seesaw and this leads to a different value for $\phi_{0}$, namely $\phi_{0} \simeq 10^{17} \mathrm{GeV}$. Therefore, in what follows, we will use both values.

The potential is displayed in Fig. 53, where four cases can be distinguished. In the following, for convenience, we use the quantity $x$ defined by
$x \equiv \frac{\phi}{\phi_{0}}$.
If $\alpha<3 / 4$, the second derivative of the potential does not vanish and the potential is convex everywhere. This corresponds to the case $\alpha=0.7$ case in Fig. 53. If $3 / 4<\alpha<1$, the potential has two inflection points $x_{V^{\prime \prime}=0}^{ \pm}$and is concave in between. It remains an increasing function of the field since its first derivative never vanishes. This is illustrated by the case $\alpha=0.85$ in Fig. 53. If $\alpha=1$, then this is the RIPI model (see Section 4.18) where the potential has a flat inflection point. If $1<\alpha<9 / 8$, then the potential decreases between the two values of $x, x_{V^{\prime}=0}^{ \pm}$, for which the derivative is zero, but remains positive everywhere. Typically, this corresponds to the case $\alpha=1.094$ in Fig. 53. Finally, if $\alpha>$ $9 / 8$, then the potential becomes negative (hence is not properly defined everywhere) between $x_{V=0}^{ \pm}$(see the case $\alpha=1.188$ in Fig. 53). The values of the field vev in this discussion are given by the following formulas:
$x_{V^{\prime \prime}=0}^{ \pm}=\frac{2}{3}\left(1 \pm \sqrt{1-\frac{3}{4 \alpha}}\right), \quad x_{V^{\prime}=0}^{ \pm}=1 \pm \sqrt{\frac{\alpha-1}{\alpha}}$,
and
$x_{V=0}^{ \pm}=\frac{4}{3}\left(1 \pm \sqrt{1-\frac{9}{8 \alpha}}\right)$.
Let us now calculate the first Hubble flow functions in the slowroll approximation. They are given by

$$
\begin{align*}
\epsilon_{1}= & 72\left(\frac{M_{\mathrm{PI}}}{\phi_{0}}\right)^{2} \frac{\left(1-2 \alpha x+\alpha x^{2}\right)^{2}}{x^{2}\left(6-8 \alpha x+3 \alpha x^{2}\right)^{2}} \\
\epsilon_{2}= & 24\left(\frac{M_{\mathrm{PI}}}{\phi_{0}}\right)^{2}  \tag{5.123}\\
& \times \frac{6-16 \alpha x+(3+16 \alpha) \alpha x^{2}-12 \alpha^{2} x^{3}+3 \alpha^{2} x^{4}}{x^{2}\left(6-8 \alpha x+3 \alpha x^{2}\right)^{2}}
\end{align*}
$$

and

$$
\begin{align*}
& \epsilon_{3}=24\left(\frac{M_{P 1}}{\phi_{0}}\right)^{2}\left[36-216 \alpha x+30 \alpha(3+16 \alpha) x^{2}\right. \\
& -8(45+64 \alpha) \alpha^{2} x^{3}+2\left(27+276 \alpha+128 \alpha^{2}\right) \alpha^{2} x^{4} \\
& -2(208 \alpha+81) \alpha^{3} x^{5}+9(1+28 \alpha) \alpha^{3} x^{6} \\
& \left.-72 \alpha^{4} x^{7}+9 \alpha^{4} x^{8}\right]\left[x^{2}\left(6-8 \alpha x+3 \alpha x^{2}\right)^{2}\right. \\
& \left.\times\left(6-16 \alpha x+3 \alpha x^{2}+16 \alpha^{2} x^{2}-12 \alpha^{2} x^{3}+3 \alpha^{2} x^{4}\right)\right]^{-1} \tag{5.124}
\end{align*}
$$

The first two slow-roll parameters diverge when $x \rightarrow 0$ and asymptotically goes to zero when $x \rightarrow \infty$. In between, their behavior depends on $\alpha$ as can be seen in Fig. 53. If $\alpha<\alpha_{0}$, where
$\alpha_{0}=\frac{3}{16}\left[5-3^{2 / 3}(6-2 \sqrt{3})^{-1 / 3}-2^{-2 / 3}(9-3 \sqrt{3})^{1 / 3}\right]$

$$
\begin{equation*}
\simeq 0.4671 \tag{5.125}
\end{equation*}
$$

$\epsilon_{1}$ monotonously decreases with $x$. If $\alpha_{0}<\alpha<1, \epsilon_{1}$ first decreases, reaches a local non-vanishing minimum at a value of $x$ for which $\epsilon_{2}$ vanishes, then increases to reach a local maximum where $\epsilon_{2}$ vanishes again, and eventually decreases for $x \rightarrow \infty$, as already mentioned. Let $x_{\epsilon_{2}=0}^{ \pm}$be the position of these two local extrema. Similarly to Eq. (5.108) for the generalized MSSM inflation models, analytic expressions can be obtained for these two quantities using Ferrari's solutions for depressed quartic equations. They are implemented in ASPIC but are not displayed here since this does not add much to the discussion. If $\alpha>1, \epsilon_{1}$ has two local minima located at $\chi_{V^{\prime}=0}^{ \pm}$where it vanishes. In between it reaches a local maximum or may even diverge for $\alpha>9 / 8$ (see Fig. 53). The slow-roll parameter $\epsilon_{2}$ vanishes when $\epsilon_{1}$ reaches these local maxima, or diverge when $\epsilon_{1}$ itself diverges (for $\alpha>9 / 8$ ).

As explained in Section 4.18, inflation is supposed to proceed at $\phi \lesssim \phi_{0}$. Let us assume that inflation ends by violation of slow-roll between $x=0$ and the position of the first minimum $x_{\epsilon_{1}^{\min }}$. Following the previous considerations, this latter value of $x$ is defined by
$x_{\epsilon_{1}^{\text {min }}}= \begin{cases}x_{\epsilon_{2}=0}^{-} & \text {if } \alpha_{0}<\alpha<1 \\ x_{V^{\prime}=0}^{-} & \text {if } \alpha>1,\end{cases}$
and, moreover, provides an upper bound to determine $x_{\text {end }}$ [i.e. the solution of the equation $\left.\epsilon_{1}\left(x_{\text {end }}\right)=1\right]$. Let us emphasize that this one can only be determined numerically. The values of $x_{\epsilon_{2}=0}^{ \pm}$ and $x_{V^{\prime}=0}^{ \pm}$in terms of $\alpha$ are displayed in the left panel of Fig. 54 together with $x_{\epsilon_{1}^{\min }}$. The right panel of Fig. 54 represents the value of the first slow-roll parameter at this minimum, $\epsilon_{1}^{\min }=\epsilon_{1}\left(x_{\epsilon_{1}^{\min }}\right)$. For $\alpha<\alpha_{0}$, one has $\epsilon_{1}(x=1)>1.5 M_{\mathrm{Pl}}^{2} / \phi_{0}^{2}$ and, recalling that


Fig. 107. Reheating consistent slow-roll predictions for the constant $n_{S} A$ models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
typically $\phi_{0} \simeq 10^{14} \mathrm{GeV}$ or $\phi_{0} \simeq 10^{17} \mathrm{GeV}$, one sees that inflation cannot proceed in this case. For $\alpha_{0}<\alpha<1$, one has $\epsilon_{1}^{\min }<1$ only if the parameter $\alpha \lesssim 1$. This defines a minimum value for $\alpha$, which depends on $\phi_{0}$, allowing for inflation to take place. When $\alpha \simeq 1$, one can derive an approximated formula for $x_{\epsilon_{2}=0}^{-}$, namely, $x_{\epsilon_{2}=0}^{-} \simeq 1-(1-\alpha) / 2$. Plugging it into the expression for $\epsilon_{1}$ one obtains
$\epsilon_{1}^{\min } \simeq 72(\alpha-1)^{2} \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}}$,
from which it follows that
$\alpha>1-\frac{\sqrt{2}}{12} \frac{\phi_{0}}{M_{\mathrm{Pl}}}$.
With $\phi_{0} / M_{\mathrm{Pl}} \simeq 10^{-1}$, one obtains $\alpha>0.99$, which shows that the model needs to be sufficiently fine-tuned such that it becomes very similar to the regular RIPI scenario. If, on the other hand, $\phi_{0} / M_{\mathrm{Pl}} \simeq 10^{-4}$, the constraint is much tighter. As discussed in Refs. [494,495], one of the main advantage of the model studied in those references is that a value $\phi_{0} \simeq 10^{17} \mathrm{GeV}$ leads to a less severe fine tuning problem than $\phi_{0} \simeq 10^{14} \mathrm{GeV}$.

However, the constraints on $\alpha$ are tighter to get a sufficient number of $e$-folds. Let us therefore now turn to the determination of the slow-roll trajectory. It can be integrated exactly to give
$N_{\text {end }}-N=\frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}}\left\{\frac{5-4 \alpha}{12 \sqrt{\alpha(1-\alpha)}} \arctan \left(\frac{x-1}{\sqrt{1 / \alpha-1}}\right)\right.$

$$
\begin{align*}
& +\frac{x}{2}\left(\frac{x}{4}-\frac{1}{3}\right)+\left(\frac{1}{8 \alpha}-\frac{1}{6}\right) \ln [1+\alpha x(x-2)] \\
& -\frac{5-4 \alpha}{12 \sqrt{\alpha(1-\alpha)}} \arctan \left(\frac{x_{\text {end }}-1}{\sqrt{1 / \alpha-1}}\right)-\frac{x_{\text {end }}}{2}\left(\frac{x_{\text {end }}}{4}-\frac{1}{3}\right) \\
& \left.-\left(\frac{1}{8 \alpha}-\frac{1}{6}\right) \ln \left[1+\alpha x_{\text {end }}\left(x_{\text {end }}-2\right)\right]\right\} \tag{5.129}
\end{align*}
$$

Exactly the same remarks we have made for the GMSSMI model also applies here (see Section 5.6). In particular, for $\alpha<1$, and contrary to the RIPI models ( $\alpha=1$ ), the number of $e$-folds never diverges at a given point $x$. Therefore, the total number of $e$-folds is bounded by some maximal finite value. From Eq. (5.129) when $\alpha \rightarrow 1$, one has
$N_{\text {end }}-N_{\text {ini }} \leq\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2} \frac{\pi}{24} \frac{1}{\sqrt{1-\alpha}}$.
Therefore, if one require at least $\Delta N=N_{\text {end }}-N_{\text {ini }} e$-folds, one has to fine-tune $\alpha$ to
$\alpha>1-\left(\frac{\phi_{0}}{M_{\mathrm{PI}}}\right)^{4} \frac{\pi^{2}}{576 \Delta N^{2}}$.
Remembering that the small parameter here is $\phi_{0} / M_{\mathrm{Pl}}$, one can see that it is a much tighter constraint than the one of Eq. (5.128). Taking $\phi_{0} / M_{\mathrm{Pl}} \simeq 10^{-1}$ and $\Delta N \simeq 50$, one obtains $\alpha>1-10^{-10}$. This makes the fine-tuning quite important and, as explained below, the same condition $|\alpha-1|<\phi_{0}^{4} / M_{\mathrm{Pl}}^{4} / \Delta N^{2}$ also applies to the case $\alpha>1$ to maintain an acceptable deviation from scale invariance, making the whole class of models fine-tuned. However, as already mentioned above, the value $\phi_{0} \simeq 10^{17} \mathrm{GeV}$ makes the fine-tuning issue easier to accept than the value $\phi_{0} \simeq 10^{14} \mathrm{GeV}$.

Finally, the amplitude of the CMB anisotropies fixes the parameter $M$ to

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=622080 \pi^{2} \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}} \frac{\left(1-2 \alpha x_{*}+\alpha x_{*}^{2}\right)^{2}}{x_{*}^{4}\left(6-8 \alpha x_{*}+3 \alpha x_{*}^{2}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{5.132}
\end{equation*}
$$

As explained in Section 4.17, this leads to $M / M_{\mathrm{PI}} \simeq 10^{13} \mathrm{GeV}$ for $\phi_{0} / M_{\mathrm{PI}} \simeq 10^{-4}$.

The reheating consistent slow-roll predictions of the GRIPI models are displayed in Figs. 125 and 126, for $\alpha>1$ and $\alpha<$ 1 respectively, and for values of $\phi_{0}$ such that $\phi_{0} \simeq 10^{17} \mathrm{GeV}$ : $\phi_{0} / M_{\mathrm{Pl}}=10^{-2}, 10^{-1.5}, 10^{-1}, 10^{-0.5}, 1$. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 since the potential is quadratic close to its minimum. In both cases, one can see that in the limit $\alpha \rightarrow 1$, the standard RIPI predictions are recovered, see Fig. 105. The amount of gravitational waves $r$ seems to be quite independent on $\alpha$ while the spectral index $n_{\mathrm{S}}$ strongly depends on it. In the case $\alpha>1$, the fine-tuning is as important as in the case $\alpha<1$ as mentioned above. Considering values of $\alpha$ very different from 1 worsens the spectral index problem, already present in standard RIPI. These models are therefore strongly disfavored by the data. In the case $\alpha<1$ however, there is a very narrow range of acceptable values for $\alpha$. They are well inside the $|\alpha-1|<$ $\phi_{0}^{4} / M_{\mathrm{Pl}}^{4} / \Delta N^{2}$ condition and the spectral index is inside the twosigma confidence intervals. But as can be seen in Fig. 126, the spectral index varies so quickly with $\alpha$ that, even if the fine-tuning is less problematic than in the GMSSMI case (due to the different value of $\phi_{0}$ ), it is still very important.

### 5.8. Brane SUSY breaking inflation (BSUSYBI)

This model has been studied in Ref. [496] in the context of superstrings models ${ }^{8}$. The potential is a sum of two exponential

[^7]

Fig. 108. Reheating consistent slow-roll predictions for the constant $n_{S} B$ models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
terms
$V(\phi)=M^{4}\left(e^{\sqrt{6} \frac{\phi}{M_{P l}}}+e^{\sqrt{6} \gamma \frac{\phi}{M_{P l}}}\right)$,
one is a "hard" exponential brought about by a SUSY breaking mechanism and the other is a "slow-roll term" having $0<\gamma<$ $1 / \sqrt{3}$ and that dominates the eventual inflationary dynamics. It was shown in Ref. [496] that the inflationary dynamics can also generate superimposed oscillations in the primordial power spectrum but we will not focus on this case since, obviously, slowroll is not satisfied in this situation [497-499]. Let us also notice that if the term in $\sqrt{6}$ in the first exponential function is relaxed to be a free parameter, the potential becomes as in Ref. [500], i.e. a general exponential brane potential. Defining
$x \equiv \frac{\phi}{M_{\mathrm{Pl}}}$,
the first three Hubble flow functions in the slow-roll approximation read
$\epsilon_{1}=3\left(\frac{e^{\sqrt{6} x}+\gamma e^{\sqrt{6} \gamma x}}{e^{\sqrt{6} x}+e^{\sqrt{6} \gamma x}}\right)^{2}$,
$\epsilon_{2}=-12(\gamma-1)^{2} \frac{e^{\sqrt{6}(\gamma+1) x}}{\left(e^{\sqrt{6} x}+e^{\sqrt{6} \gamma x}\right)^{2}}$,
and
$\epsilon_{3}=6(1-\gamma) \frac{\left(e^{\sqrt{6} x}-e^{\sqrt{6} \gamma x}\right)\left(e^{\sqrt{6} x}+\gamma e^{\sqrt{6} \gamma x}\right)}{\left(e^{\sqrt{6} x}+e^{\sqrt{6} \gamma x}\right)^{2}}$.
These functions together with the potential are displayed in Fig. 55. The two exponential components are clearly visible on the plot of the logarithm of the potential. The required flatness of the potential is realized only along the $\gamma$ branch and for negative values of $x$. The first Hubble flow function $\epsilon_{1}$ is an increasing function of $x$ which varies between its asymptotic values:
$\lim _{x \rightarrow-\infty} \epsilon_{1}=3 \gamma^{2}, \quad \lim _{x \rightarrow+\infty}=3$.
For $\gamma$ small enough $(\gamma<1 / \sqrt{3})$, there is a regime where it is less than unity. This regime is given by the condition $x<x_{\epsilon_{1}=1}$ with
$x_{\epsilon_{1}=1}=\frac{1}{\sqrt{6}(\gamma-1)} \ln \left(\frac{\sqrt{3}-1}{1-\gamma \sqrt{3}}\right)$.
As a result, inflation can only proceed in the domain $x<x_{\epsilon_{1}=1}$ and it never stops. Hence the need for an extra-parameter $x_{\text {end }}$ encoding the field value at which some unspecified mechanism (such as a tachyonic instability) is triggered and stops inflation. Let us notice that the slow-roll parameter $\epsilon_{2}$ is always negative and goes to zero at large $|x|$ with a local minimum in $x=0$ equals to $\epsilon_{2}^{\min }=-3(\gamma-1)^{2}$. Finally, the slow-roll parameter $\epsilon_{3}$ vanishes when $x=0$ and shares the same sign as $x$. Its asymptotic values are
$\lim _{x \rightarrow-\infty} \epsilon_{3}=6 \gamma(\gamma-1), \quad \lim _{x \rightarrow+\infty} \epsilon_{3}=6(1-\gamma)$.

The slow-roll trajectory can be integrated and gives

$$
\begin{align*}
N-N_{\mathrm{end}}= & -\frac{1}{\sqrt{6}}\left(x-x_{\mathrm{end}}\right) \\
& +\frac{1}{6 \gamma} \ln \left[\frac{1+\gamma e^{\sqrt{6}(\gamma-1) x}}{1+\gamma e^{\sqrt{6}(\gamma-1) x_{\mathrm{end}}}}\right] . \tag{5.140}
\end{align*}
$$

This equation cannot be analytically inverted but since inflation requires $x<x_{\epsilon_{1}=1}$, it shows that $x_{\text {end }}$ should not be too close to $x_{\epsilon_{1}=1}$ in order to realize enough $e$-folds of inflation. This puts some upper bound on $x_{\text {end }}$, that can be computed numerically and that is displayed in Fig. 56. This value $x_{\text {end }}^{\max }$ defines a prior for the model parameter $x_{\text {end }}$, which is the region lying under the curves on the figure.

Integrating Eq. (2.47) finally gives the field value $x_{*}$ at which the pivot mode crossed the Hubble radius during inflation. The parameter $M$ being fixed by the amplitude of the CMB anisotropies

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=4320 \pi^{2} \frac{\left(e^{\sqrt{6} x_{*}}+\gamma e^{\sqrt{6} \gamma x_{*}}\right)^{2}}{\left(e^{\sqrt{6} x_{*}}+e^{\sqrt{6} \gamma x_{*}}\right)^{3}} \frac{Q_{\mathrm{mss}-\mathrm{PS}}^{2}}{T^{2}} \tag{5.141}
\end{equation*}
$$

The reheating consistent slow-roll predictions of the BSUSYBI models have been plotted in Fig. 127. The parameter $x_{\text {end }}$ varies between $2 x_{\text {end }}^{\max }<x_{\text {end }}<x_{\text {end }}^{\max }$ with $x_{\text {end }}^{\max }<0$, under which the predictions of the model coincide with those of PLI (see Section 4.8). Large values for the parameter $\gamma$ are disfavored and it has to be smaller than $\lesssim 5 \times 10^{-2}$ to generate a reasonable amount of gravitational waves.



Fig. 109. Reheating consistent slow-roll predictions for the open string tachyonic models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid line represents the locus of the points such that $r=4\left(1-n_{\mathrm{s}}\right)$, i.e. $\epsilon_{2}=2 \epsilon_{1}$, on which this model lies for $\phi_{0} / M_{\mathrm{PI}} \gg 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

### 5.9. Tip inflation (TI)

### 5.9.1. Theoretical justifications

This model is a scenario based on string theory in which the motion of branes in extra-dimensions causes the four-dimensional spacetime to inflate, see for instance Refs. [158,225,501-506]. Let us assume string theory with flux compactification. In this situation, the six-dimensional Calabi-Yau space has generically the shape of a bulk with warped throat(s) attached to it. The metric in the bulk is usually not known but, along the throat, explicit examples are available. A representative case is the Klebanov-Strassler throat [507] for which one can write the metric as
$\mathrm{ds} s^{2}=h^{-1 / 2}(r) \eta_{\mu \nu} \mathrm{d} x^{\mu} \mathrm{d} x^{\nu}+h^{1 / 2}(r)\left(\mathrm{d} r^{2}+r^{2} \mathrm{~d} s_{5}^{2}\right)$.
The function $h(r)$ describes the warping along the radial coordinate $r$ of the throat. We see that the throat is in fact a cone with five-dimensional sections given by the metric $\mathrm{d} s_{5}^{2}$. For a conifold, these sections are two spheres $S_{2} \times S_{3}$ which shrink to zero at the tip of the cone [508]. Let us recall that a conifold can also be defined by the equation $\sum_{A=1}^{4}\left(Z_{A}\right)^{2}=0$, i.e. a six-dimensional (or three complex dimension) surface in $\mathbb{C}^{4}$. However, if one has a deformed conifold, then, at the tip the $S_{2}$ sphere shrinks to zero but the $S_{3}$ remains finite [508]. A deformed conifold can similarly be defined by the equation $\sum_{A=1}^{4}\left(Z_{A}\right)^{2}=\varepsilon^{2}$ and, at the tip, one has $\sum_{A=1}^{4}\left|Z_{A}\right|^{2}=\varepsilon^{2}$. Usually brane inflation takes place when a brane is moving along the radial direction of the throat, see Section 5.19.


Fig. 110. Reheating consistent slow-roll predictions for the Witten-O'Raifeartaigh models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid line represents the locus of the points such that $r=4\left(1-n_{\mathrm{S}}\right)$, i.e. $\epsilon_{2}=2 \epsilon_{1}$, on which this model lies for $\phi_{0} / M_{\mathrm{PI}} \gg 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Here, following Ref. [505], we will consider a different situation, namely the case of a brane moving at the tip of the deformed conifold. In addition, we will not only consider radial motion only but also angular motion.

Technically, the above model can be described in the framework of supergravity (viewed, in this context, as a low energy effective field theory). Let us assume that there is a D3-brane moving at the tip and that complex structure moduli and the dilaton are stabilized, thanks to the presence of fluxes. Furthermore, following Ref. [505], we suppose that there is only one volume modulus, $\rho$, plus three fields $z_{i}, i=1, \ldots, 3$ describing the $D 3$-brane position. It follows that the corresponding Kähler potential is given by
$K\left(\rho, z_{i}, z_{i}^{\dagger}\right)=-3 M_{\mathrm{PI}}^{2} \ln \left[\rho+\rho^{\dagger}-\gamma k\left(z_{i}, z_{i}^{\dagger}\right)\right]$,
where $k$ is a function of the brane coordinates and $\gamma$ is a constant (of mass dimension -2 ) related to the brane tension $T_{3}$, an approximate expression of which will be given below. In the vicinity of the deformed conifold tip, the function $k$ takes the form
$k\left(z_{i}, z_{i}^{\dagger}\right)=k_{0}+c \varepsilon^{-2 / 3}\left(\sum_{A=1}^{4}\left|Z_{A}\right|^{2}-\varepsilon^{2}\right)$.
Here $c$ is a numerical constant $c=2^{1 / 6} / 3^{1 / 3} \simeq 0.77$ and $k_{0}$ stands for the value of the function $k$ at the tip. The quantity $\varepsilon^{2 / 3}=r_{\text {tip }}$ can be viewed as the radius of the tip as illustrated in Figs. 1 and 2 of Ref. [505].


Fig. 111. Reheating consistent slow-roll predictions for the small field models with $p=1$ in the plane $\left(n_{\mathrm{s}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid line represent the locus of the points such that $r=(8 / 3)\left(1-n_{\mathrm{S}}\right)$, i.e. $\epsilon_{2}=4 \epsilon_{1}$, on which this model must lie. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The last ingredient of the model is a stack of $n$ D7-branes placed far from the tip. Then, the superpotential (Kuperstein embedding [509]) can be written as
$W=W_{0}+A\left(z_{1}\right) e^{-a \rho}=W_{0}+A_{0}\left(1-\frac{z_{1}}{\mu}\right)^{1 / n} e^{-a \rho}$.
In this expression, $\mu^{2 / 3}$ represents the distance between the stack of D7-branes and the tip (see Fig. 2 of Ref. [505] for an illustration). We always assume that this distance is much larger than the size of the tip, i.e. $\epsilon / \mu \ll 1$. The quantities $W_{0}, A_{0}$ and $a$ are constants. It is interesting to remark that the above superpotential only depends on $z_{1}$ and therefore breaks the symmetry of the tip.

We are now in a position where the potential and the kinetic term can be calculated for the fields $z_{i}$ and $\rho$. The $F$-term potential reads

$$
\begin{align*}
& V\left(\sigma, x_{1}\right)=\frac{2 a e^{-a \sigma}}{M_{\mathrm{Pl}}^{2} U^{2}}\left(\frac{a U}{6}|A|^{2} e^{-a \sigma}+|A|^{2} e^{-a \sigma}-\left|W_{0} A\right|\right) \\
& \quad+\frac{e^{-2 a \sigma}}{3 M_{\mathrm{Pl}}^{2} \gamma U^{2}} \frac{|A|^{2}}{n^{2} \mu^{2}} \frac{\varepsilon^{2 / 3}}{c}\left(1-\frac{x_{1}^{2}}{\varepsilon^{2}}\right)\left(1-\frac{x_{1}}{\mu}\right)^{-2}+\frac{D}{U^{b}} \tag{5.146}
\end{align*}
$$

where we have taken, from the definition $z_{i}=x_{i}+i y_{i}, z_{1}=x_{1}$ at the tip. Because of our choice of the superpotential, $V$ no longer depends on $x_{2}, x_{3}$. In the above expression, we have defined $\rho=$ $\sigma+i \tau$ and $\tau$ is chosen such that $V$ is minimal. The quantity $U$ is defined by $U=\rho+\rho^{\dagger}-k=2 \sigma-k_{0}$ at the tip. Finally, the last term $D / U^{b}$, with $D$ and $b$ constant, is an uplifting term which
is added in order to avoid having an anti-de Sitter minimum. In practice, uplifting potentials generically have $b=3$ [510].

The calculation of the kinetic term is difficult since the Kähler matrix mixes all the fields $z_{i}$. For this reason, it is easier to use another parametrization such where $z_{1}=\varepsilon \cos \varphi, z_{2}=$ $\varepsilon \sin \varphi \cos \theta, z_{3}=\varepsilon \sin \varphi \sin \theta \cos \psi$ and $z_{4}=\varepsilon \sin \varphi \sin \theta \sin \psi$, as appropriate since the tip of the deformed conifold is $S_{3}$. In this case, the Kähler matrix becomes diagonal and expanding everything in the small parameter $\epsilon / \mu \ll 1$, one obtains
$V(\sigma, \varphi)=\Lambda(\sigma)+B(\sigma) \cos \varphi+C(\sigma) \sin ^{2} \varphi+\cdots$,
where

$$
\begin{align*}
\Lambda(\sigma)= & \frac{2 a\left|A_{0}\right| e^{-a \sigma}}{M_{\mathrm{Pl}}^{2} U^{2}}\left(\frac{a U}{6}\left|A_{0}\right| e^{-a \sigma}+\left|A_{0}\right| e^{-a \sigma}-\left|W_{0}\right|\right) \\
& +\frac{D}{U^{b}} \tag{5.148}
\end{align*}
$$

$B(\sigma)=\frac{2 a\left|A_{0}\right| e^{-a \sigma}}{M_{\mathrm{Pl}}^{2} U^{2} n} \frac{\varepsilon}{\mu}\left(-\frac{a U\left|A_{0}\right|}{3} e^{-a \sigma}-2\left|A_{0}\right| e^{-a \sigma}+\left|W_{0}\right|\right)$,
$C(\sigma)=\frac{\left|A_{0}\right|^{2} e^{-2 a \sigma}}{3 M_{\mathrm{P} 1}^{2} U^{2} \gamma \mu^{2} n^{2}} \frac{\varepsilon^{2 / 3}}{c}$.
Let us now discuss this result. If one ignores, for the moment, all terms depending on the brane position, it remains only the term $\Lambda(\sigma)$ which is nothing but the Kachru-Kallosh-Linde-Trivedi (KKLT) potential for the volume modulus [510]. We see that in absence of the uplifting term $D / U^{b}$, its minimum given by $\partial \Lambda / \partial \sigma=0$ would be located at $\sigma=\sigma_{0}$, solution of the implicit equation
$W_{0}=-A_{0}\left[1+\frac{a}{3}\left(2 \sigma_{0}-k_{0}\right)\right] e^{-a \sigma_{0}}$.
The corresponding value of the potential would actually be negative (anti-de Sitter) and given by
$\Lambda\left(\sigma_{0}\right)=-\frac{a^{2}\left|A_{0}\right|^{2}}{3 M_{\mathrm{Pl}}^{2} U} e^{-2 a \sigma_{0}}<0$.
Hence the required uplifting term from which one can find a new minimum at which $V$ is positive. This is precisely how KKLT managed to find a de Sitter minimum instead of an anti de Sitter one for the first time in string theory [510].

If the position of the minimum were not changed by adding the uplifting term, one would obtain a vanishing value of $V$ for
$D_{0}=\frac{a^{2}\left|A_{0}\right|^{2} U^{b-1}\left(\sigma_{0}\right)}{3 M_{\mathrm{Pl}}^{2}} e^{-2 a \sigma_{0}}$.
This suggests to introduce a new parameter $\beta$, defined by
$\beta \equiv D \frac{3 M_{\mathrm{Pl}}^{2}}{a^{2}\left|A_{0}\right|^{2} U^{b-1}\left(\sigma_{0}\right)} e^{2 a \sigma_{0}}$,
such that one can trade $D$ for $\beta$ in all the uplifting terms. Therefore, $\beta=1$ represents a situation in which the potential is uplifted while the position of its minimum is unchanged. In general, as expected in presence of the brane, the KKLT minimum $\sigma_{0}$ of $\Lambda(\sigma)$ will be shifted. The correction due to the uplifting terms can be evaluated perturbatively and one obtains the following expression
$\sigma_{\text {min }}=\sigma_{0}+\frac{b \beta}{2 a^{2} \sigma_{0}}+\cdots$,
valid provided $b \beta /\left(2 a^{2} \sigma_{0}\right) \ll 1$. For $\beta=0$, one recovers that $\sigma_{\min }=\sigma_{0}$ as expected without uplifting terms (and with a negative minimum for $V$ ). There are other corrections to the position of


Fig. 112. Reheating consistent slow-roll predictions for the small field models with $p=2$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. Clearly, if $\mu / M_{\mathrm{PI}}$ is not too high these values are limited from below to stay inside the two-sigma contours, and $\mu / M_{\mathrm{Pl}}<10$ seems to be disfavored by the data. The black solid line represent the locus of the points such that $r=(8 / 3)\left(1-n_{\mathrm{s}}\right)$, i.e. $\epsilon_{2}=4 \epsilon_{1}$, on which this model lies for $\mu / M_{\mathrm{PI}} \gg 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
the minimum due to the presence of the brane but one can show that they do not play an important role (they are calculated in Ref. [505]). The final argument consists in considering that the modulus is stabilized at this minimum. Then, one obtains a single field model $V(\varphi)=V\left(\sigma_{\min }, \varphi\right)$ where the coefficients in Eq. (5.147) are now given by

$$
\begin{align*}
\Lambda\left(\sigma_{\min }\right) \equiv & \Lambda \simeq \frac{a^{2}\left|A_{0}\right|^{2} e^{-2 a \sigma_{0}}}{6 M_{\mathrm{Pl}}^{2} \sigma_{0}}[(\beta-1)+\cdots],  \tag{5.156}\\
B\left(\sigma_{\min }\right) \equiv & B \simeq \frac{a\left|A_{0}\right|^{2} \varepsilon e^{-2 a \sigma_{0}}}{6 M_{\mathrm{Pl}}^{2} n \mu \sigma_{0}^{2}} \\
& \times\left[(b \beta-3)+\frac{b \beta}{4 a \sigma_{0}}(14-3 b \beta)+\cdots\right],  \tag{5.157}\\
C\left(\sigma_{\min }\right) \equiv & C \simeq \frac{\left|A_{0}\right|^{2} \varepsilon^{2 / 3} e^{-2 a \sigma_{0}}}{12 M_{\mathrm{Pl}}^{2} n^{2} \mu^{2} \sigma_{0}^{2} \gamma c}+\cdots . \tag{5.158}
\end{align*}
$$

The above relations express the parameters of the potential in terms of the stringy parameters. We see that, if $\beta>1$, we have that the KKLT potential is positive at the minimum that could account for a cosmological constant today for $\beta-1=\mathcal{O}\left(\sigma_{0}^{-2}\right)$ [505].

Finally, the kinetic term for $\varphi$ remains to be calculated. Using the explicit form of the Kähler metric, one obtains
$K_{I J} \partial_{\mu} z^{I} \partial^{\mu} z^{\bar{J}} \simeq \frac{3 M_{\mathrm{PI}}^{2}}{U} \gamma c \varepsilon^{4 / 3} \partial_{\mu} \varphi \partial^{\mu} \varphi+\cdots$,
where, at the minimum, one has
$\gamma \simeq \frac{\sigma_{0} T_{3}}{3 M_{\mathrm{Pl}}^{2}}$,
$T_{3}$ being the brane tension. Therefore, in the large volume limit, the canonical field $\phi$ is $\phi=\sqrt{T_{3} c} \varepsilon^{2 / 3} \varphi$. As a consequence, the final form of the potential reads
$V(\phi)=\Lambda+B \cos \left(\frac{\phi}{\sqrt{T_{3} c} \varepsilon^{2 / 3}}\right)+C \sin ^{2}\left(\frac{\phi}{\sqrt{T_{3} c} \varepsilon^{2 / 3}}\right)$.
To end this section, it is interesting to discuss the orders of magnitude of the parameters appearing in the above potential. For this purpose, it is useful to recall that $\sigma_{0}$, being a volume modulus, is related to the size (or volume) of the extra-dimensions, $V_{6} \simeq$ $\sigma_{0}^{3 / 2} \alpha^{\prime 3}$. The brane tension can be written as $T_{3}=(2 \pi)^{-3} g_{\mathrm{s}}^{-1} \alpha^{\prime-2}$ while the Planck mass takes the form $M_{\mathrm{Pl}}^{2}=2(2 \pi)^{-7} V_{6} \mathrm{~g}_{\mathrm{s}}^{-2} \alpha^{\prime-4}$ ( $g_{s}$ is the string coupling). As already mentioned, the distance $\mu^{2 / 3}$ can be viewed as the distance between the stack of $D 7$-branes and the tip. It is therefore of the order of the size of the throat which allows us to write that $\mu \simeq\left(27 \pi g_{s} \mathcal{N} \alpha^{\prime 2} / 4\right)^{3 / 8}$ where the positive integer $\mathcal{N}$ is the total background Ramond-Ramond charge.

In order to have a successful slow-roll scenario, we must assume that the potential vanishes at its minimum. This amounts to take $\Lambda=B$ which can always be achieved by choosing $\beta=\beta_{\text {sr }}$ such that (with $b=3$, see before)
$\beta_{\mathrm{sr}}=1+\frac{45 \varepsilon}{4 n \mu a^{2} \sigma_{0}^{2}}+\cdots$,
where we have performed a large volume expansion. Then, at the top of the potential, one has $\partial^{2} V / \partial \phi^{2} \simeq 2 C-\Lambda$ and if one wants a flat potential $2 C-\Lambda=2 C-B$ must be a very small quantity, i.e. $C / B \simeq 1 / 2$. Using the equations established above, one can write
$\frac{C}{B}=\Upsilon \frac{\sigma_{0}^{3 / 2}}{g_{s}\left(g_{s} \pi \mathcal{N}\right)^{3 / 8}}\left(\frac{r_{\text {tip }}}{\ell_{s}}\right)^{-1 / 2}$,
where the numerical factor $\Upsilon=(12 / 15) \times(4 / 27)^{3 / 8} /\left[(2 \pi)^{4} n c\right] \simeq$ $5 \times 10^{-5}$ and $r_{\text {tip }} \equiv \varepsilon^{2 / 3}$. The string length is given by $\ell_{\mathrm{s}}=\sqrt{\alpha^{\prime}}$. Let us also recall that we have taken $b=3$. We see in the above expressions, especially Eq. (5.157), that this case is special because $\beta_{\text {sr }} \simeq 1$ and we have an additional suppression. It is also interesting to discuss the mass scale which appears in the arguments of the trigonometric functions. Straightforward calculations lead to
$\frac{\sqrt{T_{3}} c \varepsilon^{2 / 3}}{M_{\mathrm{PI}}}=(2 \pi)^{2} \sqrt{\frac{c}{2}} g_{\mathrm{s}}^{1 / 2} \sigma_{0}^{-3 / 4}\left(\frac{r_{\text {tip }}}{\ell_{\mathrm{s}}}\right)$.
For fixed $g_{s}$ and $\mathcal{N}$, the two inflationary parameters $C / B$ and $\sqrt{T_{3} c} \varepsilon^{2 / 3} / M_{\text {Pl }}$ are in fact controlled by the radius of the tip and the volume of the extra-dimensions.

Finally, if one requires $C / B=1 / 2$, as appropriate in a slow-roll analysis, then the above equations imply that
$\frac{\sqrt{T_{3}} \varepsilon^{2 / 3}}{M_{\mathrm{Pl}}} \simeq 2 \times 10^{8} \sigma_{0}^{9 / 4}$.
This equation is relevant for the question of the priors that should be put on the model parameters.

### 5.9.2. Slow-roll analysis

We now turn to the slow-roll analysis of the model. For the canonically normalized inflaton field, we have just seen that the potential is given by
$V=M^{4}\left(1+\cos \frac{\phi}{\mu}+\alpha \sin ^{2} \frac{\phi}{\mu}\right)$,


Fig. 113. Reheating consistent slow-roll predictions for the small field models with $p=4$ in the plane $\left(n_{\mathrm{s}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. Clearly, if $\mu / M_{\mathrm{PI}}$ is not too high these values are limited from below to stay inside the two-sigma contours. The black solid line represent the locus of the points such that $r=$ $(8 / 3)\left(1-n_{\mathrm{S}}\right)$, i.e. $\epsilon_{2}=4 \epsilon_{1}$, on which this model lies for $\mu / M_{\mathrm{PI}} \gg 1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
where inflation proceeds in the region $0<\phi / \mu<\pi$. Here, we have written $\Lambda=M^{4}, C / B=\alpha$ and $\mu=\sqrt{T_{3} c} \varepsilon^{2 / 3}$ (not to be confused with the scale $\mu$ introduced above and related to the distance between the stack of branes and the tip). When $\alpha \ll 1$, the potential reduces to the natural inflation (NI) one. Yet, it was shown in Section 4.6 that only super-Planckian decay constants $\mu / M_{\mathrm{PI}}>\mathcal{O}(1)$ could make the natural inflation models compatible with observations (see e.g. Fig. 88). As noticed in Ref. [505], this means that tip inflation models with $\alpha \ll 1$ are not viable. On the other hand, as was discussed in detail in the previous subsection, if $\alpha$ is fine-tuned to $\alpha \simeq 1 / 2$, then the potential of Eq. (5.166) becomes very flat at the top and a phenomenologically successful slow-roll inflationary stage could occur. This is why, in the following, these models are studied with $\alpha \simeq 1 / 2$.

Defining
$x \equiv \frac{\phi}{\mu}$,
the potential of Eq. (5.166) and its logarithm with respect to $x$ are displayed in Fig. 57. Its general shape depends on the value of $\alpha$. If $\alpha<1 / 2$, it is a decreasing function of the field $v e v$, hence inflation proceeds from the left to the right, and it has a vanishing minimum at $x=\pi$. Its first derivative vanishes at the top of the potential for $x=0$ while its second derivative $V^{\prime \prime}(x=0) \propto 2 \alpha-1$. It vanishes there when $\alpha=1 / 2$ and the potential becomes flat enough to
support inflation. If $\alpha>1 / 2$, the potential maximum is not located at $x=0$ anymore but at $x=\arccos [1 /(2 \alpha)]$. Let us thus define
$x_{V^{\prime}=0}= \begin{cases}0 & \text { if } \alpha<1 / 2, \\ \arccos \left(\frac{1}{2 \alpha}\right) & \text { if } \alpha>1 / 2 .\end{cases}$
If $\alpha>1 / 2$, the potential decreases with the field $v e v$ in the range $x_{V^{\prime}=0}<x<\pi$, where inflation proceeds from the left to the right. Again, the first derivative of the potential vanishes at the top of the potential while its second derivative $V^{\prime \prime}\left(x=x_{V^{\prime}=0}\right) \propto 1 /(2 \alpha)-2 \alpha$ again vanishes when $\alpha=1 / 2$. This is why $\alpha$ must be close enough to $1 / 2$ in order for a viable slow-roll inflationary regime to take place.

Let us calculate the Hubble flow functions within the slow-roll approximation. They read

$$
\begin{align*}
\epsilon_{1} & =\frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \frac{(1-2 \alpha \cos x)^{2} \sin ^{2} x}{2\left(1+\cos x+\alpha \sin ^{2} x\right)^{2}}  \tag{5.169}\\
\epsilon_{2} & =\frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \frac{2 \cos ^{2} \frac{x}{2}}{\left(1+\cos x+\alpha \sin ^{2} x\right)^{2}} \\
& \times[2+\alpha(3+4 \alpha)-2 \alpha(3+2 \alpha) \cos x-\alpha \cos (2 x)] \tag{5.170}
\end{align*}
$$

and

$$
\begin{align*}
\epsilon_{3} & =\frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}}\left\{-2-\frac{2+4 \alpha}{(1+\alpha-\alpha \cos x)^{2}}+\frac{5+3 \alpha}{1+\alpha-\alpha \cos x}\right. \\
& \left.+\frac{1}{\cos ^{2}\left(\frac{x}{2}\right)}+\frac{4\left(1+\alpha+3 \alpha^{2}\right)-2 \alpha(7+4 \alpha) \cos x}{\alpha[\cos (2 x)+(6+4 \alpha) \cos x-3-4 \alpha]-2}\right\} . \tag{5.171}
\end{align*}
$$

They are displayed in Fig. 57 and are increasing functions of the field $v e v$ in the inflationary domain $x_{V^{\prime}=0}<x<\pi$. Notice that they diverge when $x \rightarrow \pi$. The first and third slow-roll parameters $\epsilon_{1}$ and $\epsilon_{3}$ vanish at the potential maximum. However, the second slow-roll parameter $\epsilon_{2}$ takes a non-vanishing positive value given by
$\epsilon_{2}\left(x=x_{V^{\prime}=0}\right)= \begin{cases}\frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}}(1-2 \alpha) & \text { if } \alpha<1 / 2, \\ 4 \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \frac{2 \alpha-1}{2 \alpha+1} & \text { if } \alpha>1 / 2 .\end{cases}$
Requiring $\left|\epsilon_{2}\right|<1$ implies again to adjust $\alpha$ close to $1 / 2$ such that $|\alpha-1 / 2| \ll \mu^{2} / M_{\mathrm{Pl}}^{2} \ll 1$.

Inflation stops when $\epsilon_{1}=1$ at the position $x_{\text {end }}$ given by
$x_{\text {end }}=\arccos \left[\Sigma+\frac{(1+i \sqrt{3}) \sigma}{3 \times 2^{2 / 3}(\delta+\sqrt{\Delta})^{1 / 3}}\right.$

$$
\begin{equation*}
\left.-\frac{(1-i \sqrt{3}) \sigma^{\prime}}{6 \times 2^{1 / 3}}(\delta+\sqrt{\Delta})^{1 / 3}\right] \tag{5.173}
\end{equation*}
$$

In this formula, we have defined

$$
\begin{align*}
\Delta= & -864 \alpha^{6}(2 \alpha+1)^{3} \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\left(\frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}+2\right)^{2} \\
& \times\left\{(2 \alpha-1)^{3}+2(2 \alpha+1)[(\alpha-10) \alpha-2] \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\right. \\
& \left.-4(2 \alpha+1)^{2} \frac{\mu^{4}}{M_{\mathrm{Pl}}^{4}}\right\} \tag{5.174}
\end{align*}
$$



Fig. 114. Reheating consistent slow-roll predictions for the intermediate inflation models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). Four different values of $\beta$ are displayed (namely $\beta=1,4.1,17,70$ ), and for each of them the black solid lines correspond to the points such that $\epsilon_{1}=-(\beta / 4) \epsilon_{2}$, on which the predictions should lie for $x_{\text {end }} \gg 1$, which is very well verified. The annotations of the energy scale at which reheating ends are not displayed since this parameter is degenerated with $x_{\text {end }}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
and

$$
\begin{align*}
\delta= & 8 \alpha^{3}\left[2(2 \alpha-1)^{3}-3(1+2 \alpha)(5+2 \alpha)(1+4 \alpha) \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\right. \\
& \left.-15(1+\alpha)(1+2 \alpha)^{2} \frac{\mu^{4}}{M_{\mathrm{Pl}}^{4}}-2(1+2 \alpha)^{3} \frac{\mu^{6}}{M_{\mathrm{Pl}}^{6}}\right] \tag{5.175}
\end{align*}
$$

together with
$\sigma=3+4 \alpha(1-\alpha)-2 \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}(1+2 \alpha)^{2}-\frac{8}{2+\frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}}$,
$\sigma^{\prime}=\frac{1}{2 \alpha^{2}\left(2+\frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\right)}$.
Let us now turn to the slow-roll trajectory. It can be integrated explicitly, leading to

$$
\begin{align*}
N_{\mathrm{end}}-N= & \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}} \frac{1}{2 \alpha-1} \ln \left(\frac{1-\cos x}{1-\cos x_{\mathrm{end}}}\right) \\
& -\frac{\mu^{2}}{2 M_{\mathrm{Pl}}^{2}} \frac{2 \alpha+1}{2 \alpha-1} \ln \left(\frac{1-2 \alpha \cos x}{1-2 \alpha \cos x_{\mathrm{end}}}\right) . \tag{5.177}
\end{align*}
$$

For $\alpha=1 / 2$, this expression is singular, and one has
$N_{\mathrm{end}}-N=\frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\left[\frac{1}{1-\cos x}-\frac{1}{1-\cos x_{\mathrm{end}}}\right.$


Fig. 115. Reheating consistent slow-roll predictions for the Kähler moduli III models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel), for $10^{5}<\mathcal{V}<10^{7}, \alpha=\mathcal{V}^{5 / 3}$ and $\beta=\mathcal{V}^{2 / 3}$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slowroll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$
\begin{equation*}
\left.-\frac{1}{2} \ln \left(\frac{1-\cos x}{1-\cos x_{\mathrm{end}}}\right)\right] . \tag{5.178}
\end{equation*}
$$

Finally, the parameter $M$ can be determined from the amplitude of the CMB anisotropies and the observable field value $x_{*}$ [see Eq. (2.47)], and one gets
$\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 \pi^{2} \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \frac{\left(1-2 \alpha \cos x_{*}\right)^{2} \sin ^{2} x_{*}}{\left(1+\cos x_{*}+\alpha \sin ^{2} x_{*}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
The reheating consistent slow-roll predictions of the TI models are displayed in Fig. 128 for $\alpha<1 / 2$ and in Fig. 129 for $\alpha>$ $1 / 2$, with $\mu / M_{\mathrm{PI}}=10^{-6}, 10^{-4}$ and $10^{-2}$. In both cases, one can see that $\alpha$ needs to be sufficiently adjusted to $1 / 2$, namely $|2 \alpha-1| \ll \mu^{2} / M_{\mathrm{Pl}}^{2}$, otherwise the deviation from scale invariance is too important. The typical amount of gravitational waves is very small. To see how $\mu / M_{\mathrm{PI}}$ is constrained, the slow-roll predictions are displayed for $\alpha=1 / 2$ in Fig. 130, and with $\mu$ varying. One can see that even if one allows values of $\mu$ larger than the typical ones ( $\mu / M_{\mathrm{PI}} \simeq 10^{-4}$ ) these models are disfavored by the observations since they deviate too much from scale invariance.

### 5.10. $\beta$ exponential inflation (BEI)

This model was introduced and studied in Ref. [511] as a phenomenological generalization of the PLI exponential potential (see Section 4.8). The potential is given by
$V(\phi)=M^{4} \exp _{1-\beta}\left(-\lambda \frac{\phi}{M_{\mathrm{Pl}}}\right)$,
where the generalized exponential function $\exp _{1-\beta}$ is defined by

$$
\exp _{1-\beta}(f)= \begin{cases}(1+\beta f)^{1 / \beta} & \text { for } 1+\beta f>0  \tag{5.181}\\ 0 & \text { otherwise }\end{cases}
$$

As discussed in Ref. [511], for $f>0$ and $g>0$, this function satisfies the following identities:

$$
\begin{align*}
& \exp _{1-\beta}\left[\ln _{1-\beta}(f)\right]=f \\
& \ln _{1-\beta}(f)+\ln _{1-\beta}(g)  \tag{5.182}\\
& \quad=\ln _{1-\beta}(f g)-\beta\left[\ln _{1-\beta}(f) \ln _{1-\beta}(g)\right]
\end{align*}
$$

where $\ln _{1-\beta}(f)=\left(f^{\beta}-1\right) / \beta$ is the generalized logarithmic function. In the limit $\beta \rightarrow 0$, all the above expressions reproduce the usual exponential and logarithm properties. Therefore, the limit $\beta \rightarrow 0$ reproduces the PLI potential (see Section 4.8). However, as discussed below, this is not the case for the observable predictions which remain different. Defining the quantity $x$ by
$x \equiv \frac{\phi}{M_{\mathrm{Pl}}}$,
the range of field vev for which inflation occurs depends on the sign of $\beta$. For $\beta>0$, the field values are such that $x<1 /(\beta \lambda)$, whereas if $\beta<0$, the potential is defined for $x>1 /(\beta \lambda)$. In both cases, inflation proceeds from the left to the right. The first three Hubble flow functions in the slow-roll approximation are given by
$\epsilon_{1}=\frac{\lambda^{2}}{2(1-\beta \lambda x)^{2}}, \quad \epsilon_{2}=\frac{2 \beta \lambda^{2}}{(1-\beta \lambda x)^{2}}=4 \beta \epsilon_{1}$,
$\epsilon_{3}=\epsilon_{2}$.
Together with the potential, they are represented in Fig. 58.
One immediately sees that $\epsilon_{1}$ is an increasing function of $x$ only for the case where $\beta>0$. Therefore inflation can naturally stop at $x_{\text {end }}$ such that $\epsilon_{1}\left(x_{\text {end }}\right)=1$. In the opposite situation, namely $\beta<0$, inflation has to be ended by some additional mechanism and $x_{\text {end }}$ would become an extra-parameter. Since this model is purely phenomenological, in the following, we restrict ourselves to the case $\beta>0$ for which
$x_{\mathrm{end}}=\frac{1}{\beta}\left(\frac{1}{\lambda}-\frac{1}{\sqrt{2}}\right)$.
The next step consists in determining the slow-roll trajectory. It can be integrated explicitly and the result reads
$N-N_{\text {end }}=\frac{1}{\lambda}\left(x-x_{\text {end }}\right)-\frac{\beta}{2}\left(x^{2}-x_{\text {end }}^{2}\right)$.
It can also be inverted and one obtains the following expression for $x$ as a function of the $e$-folds number
$x=\frac{1}{\lambda \beta}-\sqrt{\left(x_{\mathrm{end}}-\frac{1}{\lambda \beta}\right)^{2}-\frac{2}{\beta}\left(N-N_{\mathrm{end}}\right)}$.
Using these expressions, the observable field value $\chi_{*}$ can be related to the number of $e$-folds $\Delta N_{*}=N_{\text {end }}-N_{*}$ at which the pivot scale crossed out the Hubble radius during inflation. Making use of Eq. (5.185), one gets
$x_{*}=\frac{1}{\lambda \beta}-\sqrt{\frac{1}{2 \beta^{2}}+\frac{2}{\beta} \Delta N_{*}}$.
Inserting this expression into the slow-roll parameters formulas yields
$\epsilon_{1 *}=\frac{1}{1+4 \beta \Delta N_{*}}, \quad \epsilon_{2 *}=\epsilon_{3 *}=4 \beta \epsilon_{1 *}$.


Fig. 116. Reheating consistent slow-roll predictions for the Logamediate Inflation 1 models with $\beta=10^{-3}$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). Inflation proceeds at decreasing field values $x<x_{V}$ max . The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. For $\beta \ll 1$, the exponential term in the potential Eq. (5.68) is almost constant so that the model is close to large field inflation (LFI, see Section 4.2). In that limit, one has $\epsilon_{1}=\alpha \epsilon_{2} / 4=(1-\gamma) \epsilon_{2}$, which corresponds to the black solid lines. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Therefore, the slow-roll predictions of these models do not depend on the parameter $\lambda$. Moreover, the limit $\beta \rightarrow 0$ does not give the same observable predictions as for the PLI models due to the singular behavior of $x_{\text {end }}$. These models can therefore be viewed as a completely different class.

Finally, the amplitude of the CMB anisotropies fixes the parameter $M$ with

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 \pi^{2} \lambda^{2}\left(1-\beta \lambda x_{*}\right)^{-2-\frac{1}{\beta}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{5.190}
\end{equation*}
$$

Notice that, from Eq. (5.188), the above expression can be written in terms of $\Delta N_{*}$ and that it does not depend on $\lambda$ anymore. The reheating consistent slow-roll predictions for the BEI models are displayed in Fig. 131. The parameter $\beta$ must be such that $\beta \gtrsim 0.6$ in order for the predictions of the model to remain inside the twosigma confidence intervals, while the parameter $\lambda$ remains totally unconstrained.

### 5.11. Pseudo natural inflation (PSNI)

### 5.11.1. Theoretical justifications

Pseudo Natural Inflation (PSNI) was introduced and studied in Ref. [272]. This model has common points with NI, see Section 4.6. Indeed, in PSNI, the inflaton field is also a pseudoNambu Goldstone boson which appears after symmetry breaking.

The corresponding potential is nearly flat which is well-suited for inflation. The main ideas behind this construction are reviewed in Section 4.6. The main difference with respect to natural inflation, for which the broken symmetry is a shift symmetry, is that in pseudo natural inflation the broken symmetry is now a $U(1)$ one. A concrete implementation of this idea has been proposed in Ref. [272] and starts with the following supersymmetric hybrid superpotential

$$
\begin{align*}
W\left(S, X, \varphi, \psi_{1}, \psi_{2}\right)= & \lambda_{0} S\left(\psi_{1}^{2}+\psi_{2}^{2}-f^{2}\right) \\
& +\frac{\lambda_{1}}{2} \psi_{1} \varphi^{2}+\lambda_{2} X\left(\varphi^{2}-v^{2}\right) \tag{5.191}
\end{align*}
$$

with $\lambda_{1}^{2} f^{2}>2 \lambda_{2}^{2} v^{2}$, where $S, X, \psi_{1}, \psi_{2}$ and $\varphi$ are scalar fields and $\lambda_{0}, \lambda_{1}$ and $\lambda_{2}$ are coupling constants. We see that the $U(1)$ symmetry is explicitly broken by the term proportional to $\lambda_{1}$. The corresponding potential can be written as

$$
\begin{align*}
V & =\lambda_{0}^{2}\left|\psi_{1}^{2}+\psi_{2}^{2}-f^{2}\right|^{2}+\left|2 \lambda_{0} S \psi_{1}+\frac{\lambda_{1}}{2} \varphi^{2}\right|^{2} \\
& +4 \lambda_{0}^{2}\left|S \psi_{2}\right|^{2}+|\varphi|^{2}\left|\lambda_{1} \psi_{1}+2 \lambda_{2} X\right|^{2}+\lambda_{2}^{2}\left|\varphi^{2}-v^{2}\right|^{2} \tag{5.192}
\end{align*}
$$

The flat directions of this superpotential can be reparametrized as

$$
\begin{equation*}
\psi_{1}+i \psi_{2} \equiv(f+\sigma) e^{i \phi / f}, \quad \psi_{1}-i \psi_{2} \equiv(f-\sigma) e^{-i \phi / f} \tag{5.193}
\end{equation*}
$$

where $\phi$ is the Nambu-Goldstone boson associated to the broken $U(1)$ symmetry and $\sigma$ is a modulus. One can assume that $\sigma$ is stabilized and sits at $\sigma=0$, the minimum of a potential originating from supersymmetry breaking. The field $\phi$ plays the role of the inflaton. Using the above expressions and the condition $\sigma=0$, one obtains that $\psi_{1}=f \cos (\phi / f)$ and $\psi_{2}=f \sin (\phi / f)$. In that case, a flat direction for $\phi$ is obtained for $\varphi=0$ and $S=0$ since then we have
$V=\lambda_{2}^{2} v^{4}$.
Notice that SUSY is broken because $F_{X} \equiv\langle\partial W / \partial X\rangle=\lambda_{2} v^{2} \neq 0$. As a consequence, the corresponding vacuum energy density is indeed given by $V_{0} \simeq\left|F_{X}\right|^{2}=\lambda_{2}^{2} v^{4}$.

This tree level potential is corrected by two kind of contributions. First, supergravity induces a soft SUSY breaking mass of order $H$ for every scalar, but since $\phi$ is a pseudo Nambu-Goldstone boson, it only receives a potential due to the explicit breaking term proportional to $\lambda_{1}$. The corresponding contribution is loop suppressed, $m_{\phi}^{2} \simeq 3 \lambda_{1}^{2} H^{2} /\left(16 \pi^{2}\right)$, as soon as $\lambda_{1} \lesssim 1$ which will be assumed. Second, the potential receives a direct Yukawa mediated contribution through a $\varphi$ loop and Ref. [272] has shown that it takes the form

$$
\begin{align*}
V(\phi) & \simeq V_{0}\left(1+\frac{\lambda_{2}^{2}}{4 \pi^{2}} \ln \frac{\lambda_{1} \psi_{1}}{\mu}\right) \\
& =V_{0}\left[1+\frac{\lambda_{2}^{2}}{4 \pi^{2}} \ln \frac{\cos (\phi / f)}{\mu / f}\right] \tag{5.195}
\end{align*}
$$

where $\mu$ is some renormalization scale. The above formula gives rise to a new type of potential that we study in the next sub-section.

### 5.11.2. Slow-roll analysis

We now turn to the slow-roll analysis of the PSNI model. Using more friendly notations, the potential (5.195) can be re-expressed as
$V=M^{4}\left[1+\alpha \ln \left(\cos \frac{\phi}{f}\right)\right]$,
with the following definitions


Fig. 117. Reheating consistent slow-roll predictions for the Logamediate Inflation 1 models with $\beta=1$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). Inflation proceeds as in Fig. 116, at decreasing field values and with $x<x_{V^{\max }}$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
$M^{4}=\lambda_{2}^{2} v^{4}\left[1+\frac{\lambda_{2}^{2}}{4 \pi^{2}} \ln \left(\frac{\lambda_{1} f}{\mu}\right)\right]$,
$\alpha=\frac{\lambda_{2}^{2} /\left(4 \pi^{2}\right)}{1+\lambda_{2}^{2} /\left(4 \pi^{2}\right) \ln \left(\frac{\lambda_{1} f}{\mu}\right)}$.
Therefore, one typically has $\alpha \ll 1$, and the scale $f$ should a priori be such that $f \lesssim M_{\text {Pl }}$ in order to avoid the usual problems of natural inflation.

The potential (5.196) as well as its logarithm are displayed in Fig. 59. Since $\phi$ is assumed to be such that $\phi \simeq 0$ initially, the potential must be studied in the range $\phi / f \in[0, \pi / 2]$. It is positive definite in the range $\phi / f \in\left[0, \arccos \left(e^{-1 / \alpha}\right)\right]$. We see that it is a decreasing function of the inflaton vev, which means that inflation proceeds from the left to the right in the direction specified by the arrow in Fig. 59.

Let us now turn to the slow-roll parameters. If one defines $x \equiv$ $\phi / f$, then the three first Hubble flow parameters are given by

$$
\begin{align*}
\epsilon_{1} & =\frac{M_{\mathrm{PI}}^{2}}{2 f^{2}} \frac{\alpha^{2} \tan ^{2} x}{(1+\alpha \ln \cos x)^{2}},  \tag{5.198}\\
\epsilon_{2} & =2 \alpha \frac{M_{\mathrm{Pl}}^{2}}{f^{2}} \frac{1+\alpha+\alpha \ln \cos x-\alpha \cos ^{2} x}{\cos ^{2} x(1+\alpha \ln \cos x)^{2}}, \\
\epsilon_{3} & =\alpha \frac{M_{\mathrm{Pl}}^{2}}{f^{2}}(\tan x)^{2} \\
& \times \frac{2+3 \alpha+\alpha^{2}-\alpha^{2} \cos (2 x)+(4+3 \alpha) \alpha \ln \cos x+2 \alpha^{2} \ln ^{2} \cos x}{(1+\alpha \ln \cos x)^{2}\left(1+\alpha \ln \cos x+\alpha \sin ^{2} x\right)} . \tag{5.199}
\end{align*}
$$



Fig. 118. Reheating consistent slow-roll predictions for the Logamediate Inflation 1 models ( $x<x_{V^{\max }}$ ) with $\beta=50$, in the plane ( $n_{S}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and twosigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. For such high values of $\beta$, only small values of $\gamma$ are in agreement with observations. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

They are displayed in Fig. 59. We see on this plot that the slow-roll parameters $\epsilon_{1}$ and $\epsilon_{3}$ vanish when $x$ goes to 0 and diverge when $x$ goes to $\pi / 2$. On the other hand, the slow-roll parameter $\epsilon_{2}$ has a non-zero limit when $x$ goes to 0 , namely
$\lim _{x \rightarrow 0} \epsilon_{2}=2 \frac{M_{\mathrm{Pl}}^{2}}{f^{2}} \alpha$.
This quantity should be small in order for slow-roll to be valid. This means that, at a fixed scale $f$, the parameter $\alpha$ needs to be smaller than $f^{2} / M_{\mathrm{Pl}}^{2}$. From the monotonous behavior of $\epsilon_{1}$, one also notices that inflation naturally stops at $\epsilon_{1}=1$. Unfortunately, this equation cannot be solved exactly and the solution needs to be determined numerically. However, since we are in a regime where $f / M_{\mathrm{Pl}} \ll 1$ and $\alpha M_{\mathrm{Pl}}^{2} / f^{2} \ll 1, x_{\text {end }}$ must be close to $\pi / 2$. One can derive a better approximation by solving the equation $\epsilon_{1}=1$ using an expansion in the small quantities of the problem. One arrives at
$x_{\mathrm{end}} \simeq \frac{\pi}{2}-\frac{\alpha}{\sqrt{2}} \frac{M_{\mathrm{PI}}}{f}$,
that is to say the first correction to $\pi / 2$ is linear in $\alpha M_{\mathrm{PI}} / f$ and, as expected, negative. As usual, the ASPIC code makes use of the complete slow-roll solution.

Let us now turn to the slow-roll trajectory. It can be integrated exactly in terms of the dilogarithm function $\mathrm{Li}_{2}$ (also referred to as Spence's function, or Joncquière function). This function was already used in this paper, for instance in Section 4.1. The explicit
expression of the trajectory reads

$$
\begin{align*}
& N_{\mathrm{end}}-N=\frac{f^{2}}{\alpha M_{\mathrm{Pl}}^{2}}\left[\left(1+\alpha \ln \cos x_{\mathrm{end}}\right) \ln \sin x_{\mathrm{end}}\right. \\
& \left.\quad+\frac{\alpha}{4} \mathrm{Li}_{2}\left(\cos ^{2} x_{\mathrm{end}}\right)\right] \\
& \quad-\frac{f^{2}}{\alpha M_{\mathrm{Pl}}^{2}}\left[(1+\alpha \ln \cos x) \ln \sin x+\frac{\alpha}{4} \mathrm{Li}_{2}\left(\cos ^{2} x\right)\right], \tag{5.202}
\end{align*}
$$

where $N_{\text {end }}$ is the number of $e$-folds at the end of inflation. Unfortunately, this trajectory cannot be inverted analytically. However, if one uses the two conditions $f / M_{\mathrm{Pl}} \ll 1$ and $\alpha M_{\mathrm{Pl}}^{2} / f^{2} \ll 1$, one can simplify a lot its expression. In particular, at Hubble crossing, one can write
$\Delta N_{*} \simeq \frac{f^{2}}{2 \alpha M_{\mathrm{Pl}}^{2}}\left[\left(x_{*}-\frac{\pi}{2}\right)^{2}-\left(x_{\text {end }}-\frac{\pi}{2}\right)^{2}\right]$,
from which one can obtain an explicit formula for $X_{*}$
$x_{*} \simeq \frac{\pi}{2}-\sqrt{2 \alpha \Delta N_{*}} \frac{M_{\mathrm{PI}}}{f}$.
Then, this also allows us to derive useful approximated equations for the first three Hubble flow parameters, namely
$\epsilon_{1 *} \simeq \frac{\alpha}{4 \Delta N_{*}}, \quad \epsilon_{2 *} \simeq \epsilon_{3 *} \simeq \frac{1}{\Delta N_{*}}$.
The expressions of the tensor-to-scalar ratio, spectral index and running are
$r \simeq \frac{4 \alpha}{\Delta N_{*}}, \quad n_{\mathrm{S}}-1 \simeq \alpha_{\mathrm{S}} \simeq-\frac{1}{\Delta N_{*}}$.
These formulas are in agreement with the estimates given in Ref. [272]. Interestingly enough, we see that these predictions are independent of the scale $f$ and that the spectral index (and the running) is even independent of $\alpha$.

The last step consists in using the CMB normalization in order to extract the mass scale $M$. Straightforward manipulations lead to

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 \pi^{2} \alpha^{2} \frac{M_{\mathrm{Pl}}^{2}}{f^{2}} \frac{\tan ^{2} x_{*}}{\left(1+\alpha \ln \cos x_{*}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{5.207}
\end{equation*}
$$

Under the two conditions $f / M_{\mathrm{Pl}} \ll 1$ and $\alpha M_{\mathrm{Pl}}^{2} / f^{2} \ll 1$ and using the same method as before, this leads to

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4} \simeq \frac{360 \pi^{2} \alpha}{\Delta N_{*}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{5.208}
\end{equation*}
$$

Requiring $M<M_{\mathrm{PI}}$ is easily achieved since, for the fiducial value $\Delta N_{*} \simeq 55$, this is equivalent to $\alpha \lesssim 2580$ whereas we have $\alpha \ll 1$. Taking the more realistic value $\alpha \simeq 10^{-6}$ and $\Delta N_{*} \simeq 55$, one typically obtains that $M / M_{P 1} \simeq 10^{-3}$.

The predictions of the PSNI models are displayed in Fig. 132 for $f / M_{\mathrm{PI}}=10^{-3}, 10^{-1}$, 10 respectively (although this last value is considered just for the purpose of illustration since superPlanckian values of $f$ are not very physical). The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 but since there is no potential minimum around which the inflaton field can oscillate at the end of inflation, this parameter is a priori unspecified and can take different values (in the ASPIC code, this parameter can be freely chosen). One can see that the rough description provided by Eq. (5.205) is correct: when $\alpha M_{\mathrm{Pl}}^{2} / f^{2} \ll 1$, the deviation from scale invariance does not depend on the model parameters and is of the order of $n_{S} \simeq 1-1 / \Delta N_{*} \simeq 0.975$, while $r \simeq 4 \alpha / \Delta N_{*}$ is typically very small.


Fig. 119. Reheating consistent slow-roll predictions for the Logamediate Inflation 2 models with $\beta=0.1$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right.$ ) (bottom panel). Inflation proceeds at increasing field values and with $x>x_{V^{\max }}$. The color of the data points encodes the value of $\gamma$, while different data blocks correspond to different values of $x_{\text {end }}$. Inside a given bock, the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

### 5.12. Non canonical Kähler inflation (NCKI)

### 5.12.1. Theoretical justifications

This model was introduced and studied in Ref. [411] as a way to model hilltop inflation. The idea is to consider $F$ or $D$ term inflation in which we have a flat direction lifted by one loop corrections. This gives rise to loop inflation as discussed in Section 4.12. The LI potential has been obtained, however, under the assumption of a minimal Kähler potential. Now, corrections originating from higher order operators, always present in the Kähler potential, should typically produce a mass term and, therefore, the scalar potential gets modified and takes the form
$V(\phi) \simeq V_{0}+\alpha \ln \left(\frac{\phi}{Q}\right)+b \phi^{2}$,
where $Q$ is a renormalization scale. This is the model we study in this section. Let us notice that the coefficient $b$ can be positive or negative. The case $b>0$ has been investigated in Refs. [512,513] as "hybrid inflation with quasi-canonical supergravity" and the case $b<0$ was studied in Ref. [411]. For $b>0$, the potential (5.209) can be viewed as a valley hybrid potential [VHI, see Section 6.2 and Eq. (6.29)] plus logarithmic radiative corrections. Therefore, a consistency check of our calculations will be that, when $\alpha \rightarrow$ 0 , all the formulas derived below must reproduce those derived in Section 6.2. Finally, let us mention that the potential (5.209) has also been studied in Ref. [514] for $b<0$ under the name


Fig. 120. Reheating consistent slow-roll predictions for the Logamediate Inflation 2 models ( $x>x_{V \max }$ ) with $\beta=1$, in the plane $\left(n_{\mathrm{s}}, r\right)$ (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The color of the data points encodes the value of $\gamma$, while different data blocks correspond to different values of $x_{\text {end }}$. Inside a given bock, the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). For fixed $\gamma$, the turning point in the predictions line occurs when $x_{\text {end }}$ lies in the fine-tuned region of LMI2, i.e. $x_{V^{\max }}<x<x_{\epsilon_{1}}^{\max }$. One sees that the predictions become infinitely close to pure de-Sitter. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
"SUSY breaking potential" and in Ref. [515] in the context of supersymmetric hybrid inflation.

### 5.12.2. Slow-roll analysis

In this sub-section, we now turn to the slow-roll analysis of the NCKI scenario. For this purpose, it is convenient to re-write the potential (5.209) under the following form
$V=M^{4}\left[1+\alpha \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right)+\beta\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{2}\right]$,
where $\alpha$ is a small positive dimensionless parameter and $\beta$ a dimensionless parameter of order $\mathcal{O}(1)$ which can be either positive or negative. Notice that the coefficient $\alpha$ has be redefined and that $\beta$ is directly related to $b$.

The potential (5.210), as well as its logarithm, are displayed in Fig. 60. We now describe its shape. For this purpose, let us first define the quantity $x \equiv \phi / M_{\mathrm{Pl}}$. If $\beta>0$, the potential is definite positive provided $x>x_{V=0}^{-}$, where
$x_{V=0}^{-}=\left[\frac{\alpha}{2 \beta} \mathrm{~W}_{0}\left(\frac{2 \beta}{\alpha} e^{-2 / \alpha}\right)\right]^{1 / 2}$,
and where $W_{0}$ is the " 0 "-branch of the Lambert function. In this case, the potential is an increasing function of the field vev and,
therefore, inflation proceeds from the right to the left in the direction indicated by the arrow in Fig. 60. Let us also notice that, in this case, the potential has an inflection point located at $x_{V^{\prime \prime}=0}=$ $\sqrt{\alpha /(2 \beta)}$. If $\beta<0$, we must have $2 \beta / \alpha \exp (1-2 / \alpha)>-1$ in order to avoid the situation where the potential is everywhere negative. This implies that either $\beta>-1$ or $\beta<-1$ and, in this last case, $\alpha<-2 / \mathrm{W}_{-1}[1 /(e \beta)]$ or $\alpha>-2 / \mathrm{W}_{0}[1 /(e \beta)]$. If one of these conditions is satisfied (which is generically the case when $\alpha \ll 1$ ), the potential is positive provided $x_{V=0}^{-}<x<x_{V=0}^{+}$, where $x_{V=0}^{-}$is defined in Eq. (5.211) and where
$x_{V=0}^{+}=\left[\frac{\alpha}{2 \beta} \mathrm{~W}_{-1}\left(\frac{2 \beta}{\alpha} e^{-2 / \alpha}\right)\right]^{1 / 2}$,
$\mathrm{W}_{-1}$ being the -1 branch of the Lambert function. In this case, the potential is a concave function of the field $v e v$, with a maximum located at $x_{V^{\prime}=0}=\sqrt{-\alpha /(2 \beta)}$. Typically, inflation proceeds from the right to the left at small values of the field vev compared to the Planck mass.

The Hubble flow functions in the slow-roll approximation are given by

$$
\begin{align*}
& \epsilon_{1}=\frac{\left(\alpha+2 \beta x^{2}\right)^{2}}{2 x^{2}\left(1+\alpha \ln x+\beta x^{2}\right)^{2}}  \tag{5.213}\\
& \epsilon_{2}=2 \frac{\alpha(\alpha+1)+(5 \alpha-2) \beta x^{2}+2 \beta^{2} x^{4}+\alpha\left(\alpha-2 \beta x^{2}\right) \ln x}{x^{2}\left(1+\alpha \ln x+\beta x^{2}\right)^{2}} \tag{5.214}
\end{align*}
$$

and

$$
\begin{align*}
\epsilon_{3} & =\frac{1}{x^{2}}\left[\frac{2\left(\alpha+2 \beta x^{2}\right)^{2}}{\left(1+\alpha \ln x+\beta x^{2}\right)^{2}}+\frac{\alpha-2 \beta x^{2}}{1+\alpha \ln x+\beta x^{2}}\right. \\
& \left.+\frac{\alpha^{2}+8 \alpha \beta x^{2}-4 \beta^{2} x^{4}}{\alpha(\alpha+1)+(5 \alpha-2) \beta x^{2}+2 \beta^{2} x^{4}+\alpha\left(\alpha-2 \beta x^{2}\right) \ln x}\right] \tag{5.215}
\end{align*}
$$

The are displayed in the bottom panels in Fig. 60. If $\beta>0$, the first slow-roll parameter $\epsilon_{1}$ diverges when $x \rightarrow x_{V=0}^{-}$. For $x>x_{V=0}^{-}$, it first decreases, then reaches a minimum, then increases and reaches a local maximum. Finally, from this maximum, it decreases again and vanishes at infinity. Therefore, inflation stops at a vev $x_{\text {end }}$ solution of $\epsilon_{1}\left(x_{\text {end }}\right)=1$, which cannot be solved analytically. It can be noticed that the value of $\epsilon_{1}$ as its local maximum increases when $\alpha$ decreases. In the limit $\alpha \ll 1$, one has
$\epsilon_{1}^{\max } \simeq \frac{\beta}{2}$,
which is reached at $x_{\epsilon_{1}^{\max }} \simeq 1 / \sqrt{\beta}$ (still in the limit of very small $\beta$ ). This sets an upper bound on $\beta$ in order for this local maximum to satisfy $\epsilon_{1} \ll 1$. If not, inflation would proceed in the part of the potential beyond its inflection point, corresponding to "large values" of the field vev and the model would formally be equivalent to a quadratic model $\left(\mathrm{LFI}_{2}\right.$, see Section 4.2).

If $\beta<0$, the first slow-roll parameter diverges when $x \rightarrow$ $x_{V=0}^{-}$. For $x>x_{V=0}^{-}, \epsilon_{1}$ decreases, vanishes at the potential local maximum $x_{V^{\prime}=0}$, and then increases to blow up when $x \rightarrow x_{V=0}^{+}$. At the same time, the second slow-roll parameter $\epsilon_{2}$ decreases in the inflationary range $x_{V=0}^{-}<x<x_{V^{\prime}=0}$. Let us also notice that, since $\epsilon_{2}\left(x_{V^{\prime}=0}\right) \propto 2 \alpha-\alpha^{2}+\alpha^{2} \ln [-\alpha /(2 \beta)]$, one has $\epsilon_{2}>0$, thanks to the condition $2 \beta / \alpha \exp (1-2 / \alpha)>-1$. Therefore the minimum value of $\epsilon_{2}$ in the increasing branch of the potential is reached at the potential maximum and is given by

$$
\begin{equation*}
\epsilon_{2}^{\min }=\frac{-16 \beta}{2-\alpha\left[1+\ln \left(-2 \frac{\beta}{\alpha}\right)\right]} \tag{5.217}
\end{equation*}
$$




Fig. 121. Reheating consistent slow-roll predictions for the Logamediate Inflation 2 models ( $x>x_{V} \max$ ) with $\beta=10$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The color of the data points encodes the value of $\gamma$, while different data blocks correspond to different values of $x_{\text {end }}$. Inside a given bock, the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). For fixed $\gamma$, the turning point in the predictions line occurs when $x_{V \max }<x<\chi_{\epsilon_{1}^{\max }}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

For $\alpha<-2 \beta / e$ (which is generically the case since $\alpha \ll 1$ ), this number is such that $\epsilon_{2}^{\min }>-8 \beta$, which puts a lower bound on $\beta$ in order for $\epsilon_{2}$ to remain small and slow-roll to be satisfied. As it was the case for $\beta>0$, inflation also ends when $\epsilon_{1}=1$. Notice that the exact calculations are implemented in the ASPIC routines.

Let us now turn to the slow-roll trajectory. It can be analytically integrated using the dilogarithm function $\mathrm{Li}_{2}$ and the corresponding expression reads

$$
\begin{align*}
N_{\mathrm{end}}-N= & \left(1-\frac{\alpha}{2}+\alpha \ln x\right) \frac{\ln \left(\alpha+2 \beta x^{2}\right)}{4 \beta}+\frac{x^{2}}{4} \\
& -\frac{\alpha}{4 \beta} \ln \alpha \ln x+\frac{\alpha}{8 \beta} \mathrm{Li}_{2}\left(-2 \frac{\beta}{\alpha} x^{2}\right) \\
& -\left(1-\frac{\alpha}{2}+\alpha \ln x_{\text {end }}\right) \frac{\ln \left(\alpha+2 \beta x_{\mathrm{end}}^{2}\right)}{4 \beta}-\frac{x_{\mathrm{end}}^{2}}{4} \\
& +\frac{\alpha}{4 \beta} \ln \alpha \ln x_{\mathrm{end}}-\frac{\alpha}{8 \beta} \operatorname{Li}_{2}\left(-2 \frac{\beta}{\alpha} x_{\mathrm{end}}^{2}\right) \tag{5.218}
\end{align*}
$$

where $N_{\text {end }}$ is the number of $e$-folds at the end of inflation. An approximate and simpler expression can be derived in the limit $\alpha \ll 1$. In that limit, one obtains $N_{\text {end }}-N=x^{2} / 4+\ln (x) /(2 \beta)-$ $x_{\text {end }}^{2} / 4-\ln \left(x_{\text {end }}\right) /(2 \beta)$, which is precisely the slow-roll trajectory for the VHI models with $\mu=M_{\mathrm{PI}} / \sqrt{\beta}$ and $p=2$, see Eq. (6.35). For $\alpha \neq 0$, the exact trajectory cannot be inverted analytically.



Fig. 122. Reheating consistent slow-roll predictions for the twisted models in the plane $\left(n_{\mathrm{S}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The color of the data points encodes the value of $\phi_{0}$, while different data blocks correspond to different values of $x_{\text {end }}$. Inside a given bock, the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Finally, the parameter $M$ can be determined from the CMB normalization. One obtains the following expression

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 \pi^{2} \frac{\left(\alpha+2 \beta x_{*}^{2}\right)^{2}}{x_{*}^{2}\left(1+\alpha \ln x_{*}+\beta x_{*}^{2}\right)^{3}} \frac{Q_{\mathrm{rmS}-\mathrm{PS}}^{2}}{T^{2}} . \tag{5.219}
\end{equation*}
$$

The slow-roll predictions of the NCKI models are displayed in Figs. 133 and 134 for $\beta>0$ and $\beta<0$, respectively. The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to be 0 but, since there is no potential minimum around which the inflaton field can oscillate at the end of inflation, this parameter is in fact unspecified. Some remarks are in order at this point. Firstly, when $\beta>0$, we notice that $\epsilon_{2}$ at Hubble crossing is either positive or negative while, when $\beta<0$, it is always positive. This is in agreement with what we have discussed before. Secondly, when $\beta>0$ and $\alpha \ll 1$, one can check that the predictions of the models are similar to the VHI ones with $p=2$ (compare with Fig. 174). Again, this is consistent with the previous considerations. Thirdly, when $|\beta| \gtrsim \mathcal{O}(1)$, the predictions of the models do not depend much on $\beta$. Finally, as expected, when $\beta \rightarrow 0$, one recovers the predictions of the LI models, see Section 4.12 and Fig. 96. Now, in the regime $|\beta|=\mathcal{O}(1)$ and $\alpha \ll 1$, Figs. 133 and 134 indicate that the case $\beta>0$ is disfavored by the observations. The situation is even worst for $\beta<0$, the deviation from scale invariance being clearly too important to satisfy the observational constraints.

### 5.13. Constant spectrum inflation (CSI)

This potential belongs to the class of models discussed in Ref. [516] and is constructed in order to produce a power spectrum $P(k) \propto k^{0}$ for the primordial density fluctuations, i.e. a power spectrum with constant spectral index such that $n_{S}=1$ (exact scale invariance). It reads
$V(\phi)=\frac{M^{4}}{\left(1-\alpha \frac{\phi}{M_{\mathrm{Pl}}}\right)^{2}}$.
There is a symmetry for $\phi / M_{\mathrm{PI}} \rightarrow 2 / \alpha-\phi / M_{\mathrm{PI}}$ and inflation can proceed indifferently in the branch $\phi / M_{\mathrm{Pl}}<1 / \alpha$ or in the branch $\phi / M_{\mathrm{Pl}}>1 / \alpha$, leading to the same physical predictions. For this reason, in the following, we will be interested in the branch $\phi / M_{\mathrm{Pl}}<1 / \alpha$. Defining the quantity $x$ by
$x \equiv \frac{\phi}{M_{\mathrm{Pl}}}$,
the first three Hubble flow functions in the slow-roll approximation are given by
$\epsilon_{1}=\frac{2 \alpha^{2}}{(\alpha x-1)^{2}}, \quad \epsilon_{2}=\epsilon_{3}=-2 \epsilon_{1}$.
The previous relation $\epsilon_{2}=-2 \epsilon_{1}$ means that, at first order in slowroll, the spectral index is indeed equals to unity, $n_{S}-1=0$. Recall that the potential of this model is precisely constructed in order for this relation to be true. Let us notice, however, that, at second order in slow-roll, $\epsilon_{2}=\epsilon_{3}=-2 \epsilon_{1}$ yields $n_{S}-1=4 \epsilon_{1}^{2}>0$. One should note that another way to realize $n_{S}-1=0$ at first order in slow-roll is to take the large field inflation potential LFI (see Section 4.2) with a negative power index $p=-2$. In that case one also has $\epsilon_{2}=\epsilon_{3}=-2 \epsilon_{1}$ and, at second order, $n_{S}-1=4 \epsilon_{1}^{2}$ is also verified. However, since the explicit expressions of $\epsilon_{1}$ for CSI and LFI $(p=-2)$ are different, the actual value of the spectral index at second order is also different. The potential and the Hubble flow functions have been represented in Fig. 61.

As can be checked in this figure, $\epsilon_{1}$ is a monotonous function of $x$ in both branches of the potential. It diverges at $x=1 / \alpha$ and vanishes for $x \rightarrow \pm \infty$. Inflation can therefore take place in the region $x<x_{\epsilon_{1}=1}^{-}$for the branch $x<1 / \alpha$ (or $x>x_{\epsilon_{1}=1}^{+}$for the branch $x>1 / \alpha$ ), where $x_{\epsilon_{1}=1}^{ \pm}$are the field values at which $\epsilon_{1}=1$ :
$x_{\epsilon_{1}=1}^{ \pm}=\frac{1 \pm \sqrt{2} \alpha}{\alpha}$.
Since the field evolution proceeds from the right to the left from $x_{\epsilon_{1}=1}^{ \pm}$, inflation does not stop by slow-roll violation and an extra mechanism parametrized by $x_{\text {end }}$ should be considered in order to end it. For this reason, CSI is in fact a two parameters model. Let us also notice that the slow-roll parameters $\epsilon_{2}=\epsilon_{3}$ are negative monotonous functions of $x$ in both branches of the potential and cross the line $\epsilon_{2}=\epsilon_{3}=-1$ at
$x_{\epsilon_{2}=-1}^{ \pm}=x_{\epsilon_{3}=-1}^{ \pm}=\frac{1 \pm 2 \alpha}{\alpha}$.
As a result, there is a small domain $x_{\epsilon_{2}=-1}^{-}<x<x_{\epsilon_{1}=1}^{-}$where we have inflation but where the slow-roll approximation is violated (this is also true for the other branch). This is not problematic since the system is driven away from this regime towards a situation in which all the Hubble flow functions become small (see Fig. 61).

The slow-roll trajectory can be integrated explicitly and reads
$N-N_{\text {end }}=\frac{x^{2}}{4}-\frac{x}{2 \alpha}+\frac{x_{\text {end }}^{2}}{4}-\frac{x_{\text {end }}}{2 \alpha}$.


Fig. 123. Reheating consistent slow-roll predictions for the GMSSMI models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel), for $1<\alpha<$ $1+\phi_{0}^{4} / M_{\mathrm{Pl}}^{4} \pi^{2} / 900 /\left(N_{\text {end }}-N_{\text {ini }}\right)^{2}$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. When $\alpha \rightarrow 1$, one recovers the standard MSSM predictions, see Fig. 104. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

It can also be inverted analytically and it follows that
$x=\frac{1 \pm \sqrt{1-2 \alpha x_{\text {end }}+\alpha^{2} x_{\text {end }}^{2}+4 \alpha^{2}\left(N-N_{\text {end }}\right)}}{\alpha}$.
The sign $\mp$ depends on whether one works in the $x<1 / \alpha$ branch or in the $x>1 / \alpha$ branch, respectively. A consequence of this formula is the fact that, if one requires $N_{\text {end }}-N_{\text {ini }} e$-folds during inflation, then $x_{\text {end }}$ should be smaller than some value $x_{\text {end }}^{\max }$ given by
$x_{\text {end }}^{\max }=\frac{1}{\alpha}-\sqrt{2+4\left(N_{\text {end }}-N_{\text {ini }}\right)}$,
in the $x<1 / \alpha$ branch. Equivalently, taking the minus sign in this expression would lead to $x_{\text {end }}^{\min }$ for the branch $x>1 / \alpha$.

Finally, the observable field value $x_{*}$ is obtained by solving Eq. (2.47) while the amplitude of the CMB anisotropies fixes the parameter $M$ to

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=2880 \pi^{2} \alpha^{2} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{5.228}
\end{equation*}
$$

Interestingly enough, it only depends on $\alpha$, and not on $x_{*}$ (i.e. it has no explicit dependence on the reheating). The reheating consistent slow-roll predictions for the CSI models are represented in Figs. 135 and 136 for $\alpha=10^{-3}$ and $\alpha=1$, respectively.

### 5.14. Orientifold inflation (OI)

### 5.14.1. Theoretical justifications

The model is based on the following considerations. Let us start with a $N=1$ supersymmetric Yang-Mills gauge theory the Lagrangian of which can be written as
$\mathcal{L}=-\frac{1}{4} F_{\mu \nu}^{a} F^{a \mu \nu}+\frac{i}{2} \bar{\lambda}^{a} \not D_{a b} \lambda^{b}$,
with $a=1, \ldots, N_{c}^{2}, N_{c}$ being the number characterizing the group $\operatorname{SU}\left(N_{\mathrm{c}}\right) . F_{\mu \nu}^{a}$ is the field strength, $\lambda^{a}$ a spinor field and $\not D$ a covariant derivative. A is a composite scalar field, i.e. a bound state denoted by $\varphi \simeq \lambda \bar{\lambda}$, can actually appear in the theory if a strongly interacting regime takes place. The effective Lagrangian aimed at describing its dynamics has been derived in Ref. [517] and reads

$$
\begin{align*}
\mathscr{L}_{\mathrm{YV}}= & -\frac{N_{\mathrm{c}}^{2}}{\alpha_{\mathrm{OI}}}\left(\varphi \varphi^{\dagger}\right)^{-2 / 3} \partial_{\mu} \varphi \partial^{\mu} \varphi^{\dagger} \\
& -\frac{4 \alpha_{\mathrm{OI}} N_{\mathrm{c}}^{2}}{9}\left(\varphi \varphi^{\dagger}\right)^{2 / 3} \ln \left(\frac{\varphi}{\Lambda^{3}}\right) \ln \left(\frac{\varphi^{\dagger}}{\Lambda^{3}}\right), \tag{5.230}
\end{align*}
$$

where $\alpha_{\text {OI }}$ is a constant and $\Lambda$ a mass scale. This class of theories are discussed in more detail in Section 6.5. However, in Ref. [518], it was argued that in "orientifold theories", the above Lagrangian can be slightly deformed and now takes the form

$$
\begin{align*}
& \mathcal{L}_{\mathrm{OI}}=-\frac{N_{\mathrm{c}}^{2}}{\alpha_{\mathrm{OI}}}\left(\varphi \varphi^{\dagger}\right)^{-2 / 3} \partial_{\mu} \varphi \partial^{\mu} \varphi^{\dagger} \\
& \quad-\frac{4 \alpha_{\mathrm{OI}} N_{\mathrm{c}}^{2}}{9}\left(\varphi \varphi^{\dagger}\right)^{2 / 3}\left[\ln \left(\frac{\varphi}{\Lambda^{3}}\right) \ln \left(\frac{\varphi^{\dagger}}{\Lambda^{3}}\right)-\beta\right], \tag{5.231}
\end{align*}
$$

where $\beta=\mathcal{O}\left(1 / N_{c}\right)$. Ref. [518] raised the possibility that $\varphi$ (or, rather, its canonically conjugated version) could be the inflaton. In fact, in order to study this question, one must also specify the gravitational coupling. In Ref. [518], the scalar field $\varphi$ is nonminimally coupled to gravity such that, in the Jordan frame,
$S=\int \mathrm{d}^{4} \boldsymbol{x} \sqrt{-g}\left[-\frac{M^{2}+N_{\mathrm{c}}^{2} \xi\left(\varphi \varphi^{\dagger}\right)^{1 / 3}}{2} R+\mathscr{L}_{\mathrm{OI}}\right]$,
where $M$ is a mass scale. There is a new parameter in the problem, $\xi$, which describes the strength of the non-minimal coupling to gravity (as it was the case for Higgs inflation, see Section 3.1). Then, in the Einstein frame, one can write the above model as Ref. [518]

$$
\begin{align*}
S= & \int \mathrm{d}^{4} \boldsymbol{x} \sqrt{-g}\left\{-\frac{1}{2} M_{\mathrm{Pl}}^{2} R-\frac{N_{\mathrm{c}}^{2}}{\alpha_{\mathrm{OI}}} \Omega^{-2}\right. \\
& \times\left[1+\frac{\alpha_{\mathrm{OI}} N_{\mathrm{c}}^{2} \xi^{2}}{3 M_{\mathrm{Pl}}^{2}} \Omega^{-2}\left(\varphi \varphi^{\dagger}\right)^{1 / 3}\right]\left(\varphi \varphi^{\dagger}\right)^{-2 / 3} \partial_{\mu} \varphi \partial^{\mu} \varphi^{\dagger} \\
& \left.-\Omega^{-4} V_{\mathrm{OI}}\right\} . \tag{5.233}
\end{align*}
$$

In this expression, $V_{\text {OI }}$ refers to the second term in Eq. (5.231) and
$\Omega^{2} \equiv \frac{M^{2}+N_{\mathrm{c}}^{2} \xi\left(\varphi \varphi^{\dagger}\right)^{1 / 3}}{M_{\mathrm{Pl}}^{2}}$.
In the following, we consider two situations: the case where $\xi \neq 0$ such that $\Omega^{2} \simeq N_{\mathrm{c}}^{2} \xi \varphi^{2 / 3} / M_{\mathrm{pl}}^{2}$, i.e. the second term in the definition of $\Omega^{2}$ dominates (the large field limit) and the case $\xi=0$. In the first case, taking $\varphi=\varphi^{\dagger}$ and canonically normalizing the field one finds
$V(\varphi)=\frac{4 \alpha_{\mathrm{OI}} M_{\mathrm{PI}}^{4}}{9 N_{\mathrm{c}}^{2} \xi^{2}}\left[\left(\ln \frac{\varphi}{\Lambda^{3}}\right)^{2}-\beta\right]$.


Fig. 124. Reheating consistent slow-roll predictions for the GMSSMI models in the plane $\left(n_{\mathrm{S}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel), for $1-$ $\phi_{0}^{4} / M_{\mathrm{Pl}}^{4} \pi^{2} / 900 /\left(N_{\text {end }}-N_{\mathrm{ini}}\right)^{2}<\alpha<1$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\ln \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. When $\alpha \rightarrow 1$, one recovers the standard MSSM predictions, see Fig. 104. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The canonically normalized field is $\phi / M_{\mathrm{Pl}} \propto \ln \varphi$. Since $\beta$ is a small number, it can be neglected and this model is in fact a LFI model with $V(\phi) \propto \phi^{2}$ which was already studied in Section 4.2. For the second case, it is sufficient to restart from Eq. (5.231). Then, the canonically normalized field reads
$\frac{\varphi}{\Lambda^{3}}=\left(\frac{\phi}{\phi_{0}}\right)^{3}$,
with
$\phi_{0}=3 N_{c}\left(\frac{2}{\alpha_{\mathrm{OI}}}\right)^{1 / 3} \Lambda$.
It follows that the potential can be written as
$V=\alpha_{\mathrm{OI}} N_{\mathrm{c}}^{2} \Lambda^{4}\left(\frac{\phi}{\phi_{0}}\right)^{4}\left[\ln ^{2}\left(\frac{\phi}{\phi_{0}}\right)-\frac{\beta}{9}\right]$.
This model is studied in detail in the next subsection. The case $\beta=0$ will also be investigated in Section 6.5.

### 5.14.2. Slow-roll analysis

We now turn to the slow-roll study of the potential derived previously in Eq. (5.238). This one can be re-written as
$V(\phi)=M^{4}\left(\frac{\phi}{\phi_{0}}\right)^{4}\left[\left(\ln \frac{\phi}{\phi_{0}}\right)^{2}-\alpha\right]$,
where we have defined
$M^{4}=\alpha_{\mathrm{OI}} N_{\mathrm{c}}^{2} \Lambda^{4}, \quad \alpha \equiv \frac{\beta}{9}$.
One should be careful that $\alpha_{\text {OI }}$ appearing in the first of the two above equations stems from the Lagrangian used in the previous subsection while the observable constant $\alpha$ only refers to the quantity $\beta / 9=\mathcal{O}\left(1 / N_{c}\right) \ll 1$. The scale $\phi_{0}$ is defined in Eq.(5.237) and will be chosen such that $\phi_{0} \simeq 10^{16} \mathrm{GeV}$. The potential as well as its logarithm are displayed in Fig. 62.

Defining the quantity $x$ by the following expression
$x \equiv \frac{\phi}{\phi_{0}}$,
the potential remains positive provided $x<x_{V=0}^{-}$or $x>x_{V=0}^{+}$, where
$x_{V=0}^{ \pm}=e^{ \pm \sqrt{\alpha}}$.
It vanishes at $x=0$, then increases to reach a local maximum at $x_{V^{\prime}=0}^{-}$, decreases again to become negative at $x_{V=0}^{-}$, reaches a local minimum at $x_{V^{\prime}=0}^{+}$, then increases again to become positive at $x_{V=0}^{+}$ and diverges asymptotically. The values of $x_{V^{\prime}=0}^{-}$and $x_{V^{\prime}=0}^{+}$are given by
$x_{V^{\prime}=0}^{ \pm}=e^{-\frac{1}{4} \pm \sqrt{\frac{1}{16}+\alpha}}$.
A priori three regimes of inflation may exist: $x<x_{V^{\prime}=0}^{-}$and inflation proceeds from the right to the left, $x_{V^{\prime}=0}^{-}<x<x_{V=0}^{-}$and inflation proceeds from the left to the right, $x_{V=0}^{+}<x$ and inflation proceeds from the right to the left in the direction specified by the arrow in Fig. 62. As explained below, only the third possibility allows us to have a slow-roll inflationary regime.

Let us now calculate the quantities $\epsilon_{n}$. The first three Hubble flow functions in the slow-roll approximation are given by
$\epsilon_{1}=2 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}}\left(\frac{2 \ln ^{2} x+\ln x-2 \alpha}{x \ln ^{2} x-\alpha x}\right)^{2}$,
$\epsilon_{2}=4 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}} \frac{2 \ln ^{4} x+\ln ^{3} x+(1-4 \alpha) \ln ^{2} x-\alpha \ln x+\alpha+2 \alpha^{2}}{\left(x \ln ^{2} x-\alpha x\right)^{2}}$,
and

$$
\begin{align*}
\epsilon_{3} & =2 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}}\left[8 \alpha^{4}+6 \alpha^{3}-\alpha^{2}(8 \alpha+15) \ln x\right. \\
& +2 \alpha\left(3-16 \alpha^{2}-2 \alpha\right) \ln ^{2} x \\
& +8 \alpha(3 \alpha+1) \ln ^{3} x+2\left(24 \alpha^{2}-5 \alpha+1\right) \ln ^{4} x \\
& +(7-24 \alpha) \ln ^{5} x+8(1-4 \alpha) \ln ^{6} x \\
& \left.+8 \ln ^{7} x+8 \ln ^{8} x\right]\left(x \ln ^{2} x-\alpha x\right)^{-2} \\
& \times\left[2 \alpha^{2}+\alpha-\alpha \ln x+(1-4 \alpha) \ln ^{2} x+\ln ^{3} x+2 \ln ^{4} x\right]^{-1} . \tag{5.246}
\end{align*}
$$

They have been represented in Fig. 62. One can see that the slowroll regime can only take place in the $x>x_{V=0}^{+}$region, where $\epsilon_{1}$ continuously increase as inflation proceeds from the right to the left, and diverges at $x_{V=0}^{+}$. In the other domains, $\epsilon_{2}$ remains too large to support slow-roll inflation. Within the $x>x_{V=0}^{+}$ domain, inflation naturally ends by slow-roll violation, but the field value $x_{\text {end }}$ at which this occurs has to be determined numerically.

However, since $\phi_{0} \simeq 10^{16} \mathrm{GeV}$, one can derive an approximated formula for $x_{\text {end }}$ in the $\phi_{0} \ll M_{\text {PI }}$ limit, namely
$x_{\text {end }} \simeq 2 \sqrt{2} \frac{M_{\mathrm{PI}}}{\phi_{0}}$.
The next step is to derive the slow-roll trajectory. It can be obtained from Eq. (2.11) and reads

$$
\begin{align*}
& N_{\text {end }}-N=-\frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}} \frac{x_{\text {end }}^{2}-x^{2}}{8}+\frac{\ln ^{2}\left(x_{V^{\prime}=0}^{+}\right)-\alpha}{2 \sqrt{1+16 \alpha}}\left(x_{V^{\prime}=0}^{+}\right)^{2} \\
& \quad \times\left[\operatorname{Ei}\left(2 \ln \frac{x_{\text {end }}}{x_{V^{\prime}=0}^{+}}\right)-\operatorname{Ei}\left(2 \ln \frac{x}{x_{V^{\prime}=0}^{+}}\right)\right] \\
& \quad-\frac{\ln ^{2}\left(x_{V^{\prime}=0}^{-}\right)-\alpha}{2 \sqrt{1+16 \alpha}}\left(x_{V^{\prime}=0}^{-}\right)^{2} \\
& \left.\quad \times\left[\operatorname{Ei}\left(2 \ln \frac{x_{\text {end }}^{-}}{x_{V^{\prime}=0}^{-}}\right)-\operatorname{Ei}\left(2 \ln \frac{x}{x_{V^{\prime}=0}^{-}}\right)\right]\right\}, \tag{5.248}
\end{align*}
$$

where Ei is the exponential integral function, and where $x_{V^{\prime}=0}^{ \pm}$have been defined in Eq. (5.243). In the $\phi_{0} \ll M_{P 1}$ limit, this trajectory reduces to $\Delta N_{*} \simeq \phi_{0}^{2} /\left(8 M_{\mathrm{Pl}}^{2}\right)\left(x_{*}^{2}-x_{\text {end }}^{2}\right)$, where we have introduced the observable field value $x_{*}$ at which the pivot scale crossed the Hubble radius during inflation. It can be inverted to give $x_{*}$ in terms of $\Delta N_{*}=N_{\text {end }}-N_{*}$ and one gets
$x_{*} \simeq 2 \sqrt{2} \frac{M_{\mathrm{Pl}}}{\phi_{0}} \sqrt{\Delta N_{*}+1}$.
Plugging this into Eqs. (5.244)-(5.246) gives the approximated expressions
$\epsilon_{1 *} \simeq \epsilon_{2 *} \simeq \epsilon_{3 *} \simeq \frac{1}{\Delta N_{*}+1}$,
hence
$r \simeq \frac{16}{\Delta N_{*}+1}, \quad n_{\mathrm{S}}-1 \simeq-\frac{3}{\Delta N_{*}+1}$,
$\alpha_{\mathrm{S}} \simeq-\frac{3}{\left(\Delta N_{*}+1\right)^{2}}$.
From $x_{*}$, the parameter $M$ is fixed by the amplitude of the CMB anisotropies and one obtains

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=\frac{2880 \pi^{2}\left(2 \ln ^{2} x_{*}+\ln x_{*}-2 \alpha\right)^{2}}{x_{*}^{6}\left(\ln ^{2} x_{*}-\alpha\right)^{3}} \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{5.252}
\end{equation*}
$$

In the $\phi_{0} \ll M_{\mathrm{Pl}}$ limit, the previous expression reduces to the following formula

$$
\begin{align*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4} \simeq & \frac{45 \pi^{2}}{2\left(\Delta N_{*}+1\right)^{3}}\left(\frac{\phi_{0}}{M_{\mathrm{PI}}}\right)^{4} \frac{1}{\ln ^{2}\left(2 \sqrt{2} \frac{M_{\mathrm{Pl}}}{\phi_{0}} \sqrt{\Delta N_{*}+1}\right)} \\
& \times \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{5.253}
\end{align*}
$$

With $\phi_{0} \simeq 10^{16} \mathrm{GeV}$, this typically gives $M / M_{\mathrm{PI}} \simeq 5 \times 10^{-4}$.
The reheating consistent slow-roll predictions for the orientifold inflation models are displayed in Fig. 137, for $\phi_{0} / M_{\mathrm{Pl}}=$ $10^{-4}, 10^{-2}$, and 1 . Let us recall that natural values are around $\phi_{0} \simeq$ $10^{16} \mathrm{GeV}$ and $\alpha \in\left[10^{-3}, 1\right]$. The reheating equation of state parameter has been fixed to $\bar{w}_{\text {reh }}=0$ since the potential is quadratic in the vicinity of its minimum. According to the rough picture provided by Eq. (5.250), the predictions of these models almost do not depend on its parameters $\phi_{0}$ and $\alpha$, which is why all the points in Fig. 137 are superimposed. In particular, one can see that these models generically predict an important amount of gravitational waves which is disfavored by the observations.


Fig. 125. Reheating consistent slow-roll predictions for the generalized renormalizable inflection point models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel), for $1<\alpha<1+\phi_{0}^{4} / M_{\mathrm{Pl}}^{4} \pi^{2} / 576 /\left(N_{\text {end }}-N_{\mathrm{ini}}=60\right)^{2}$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\mathrm{reh}} / \mathrm{GeV}\right)$. When $\alpha \rightarrow 1$, one recovers the standard RIPI predictions, see Fig. 105. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

### 5.15. Constant $n_{S} C$ inflation (CNCI)

This model has been obtained in Ref. [443] and is the third example of a class of scenarios already studied in Sections 4.20 and 4.21. As explained in those sections, the corresponding potential is designed in order to produce a power spectrum with constant spectral index. The potential studied in this section reads
$V(\phi)=M^{4}\left[\left(3+\alpha^{2}\right) \operatorname{coth}^{2}\left(\frac{\alpha}{\sqrt{2}} \frac{\phi}{M_{\mathrm{Pl}}}\right)-3\right]$,
where $\alpha$ is a positive dimensionless parameter (denoted $n_{0}$ in Ref. [443]). The potential being symmetrical in $\phi \rightarrow-\phi$, only the $\phi>0$ part is displayed in Fig. 63. It is a decreasing function of the field vev, and its asymptotic value when $\phi / M_{\mathrm{PI}}$ goes to infinity is given by $\alpha^{2} M^{4}$, hence the potential is always positive.

Defining $x=\phi / M_{\mathrm{P}}$, the three first slow-roll parameters are given by

$$
\begin{align*}
\epsilon_{1} & =\frac{4 \alpha^{2}\left(3+\alpha^{2}\right)^{2} \operatorname{coth}^{2}\left(\frac{\alpha x}{\sqrt{2}}\right)}{\left[6+\alpha^{2}+\alpha^{2} \cosh (\sqrt{2} \alpha x)\right]^{2}}  \tag{5.255}\\
\epsilon_{2} & =-\frac{2 \alpha^{2}\left(3+\alpha^{2}\right)\left[12+\alpha^{2}+2 \alpha^{2} \cosh (\sqrt{2} \alpha x)+\alpha^{2} \cosh (2 \sqrt{2} \alpha x)\right]}{\left[6+\alpha^{2}+\alpha^{2} \cosh (\sqrt{2} \alpha x)\right]^{2} \sinh ^{2}\left(\frac{\alpha x}{\sqrt{2}}\right)}
\end{align*}
$$



Fig. 126. Reheating consistent slow-roll predictions for the generalized renormalizable inflection point models in the plane ( $\left.n_{s}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel), for $1-\phi_{0}^{4} / M_{\mathrm{Pl}}^{4} \pi^{2} / 576 /\left(N_{\text {end }}-N_{\text {ini }}=60\right)^{2}<\alpha<1$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\ln \left(g_{*}^{1 / 4} T_{\mathrm{reh}} / \mathrm{GeV}\right)$. When $\alpha \rightarrow 1$, one recovers the standard RIPI predictions, see Fig. 105. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
and

$$
\begin{align*}
\epsilon_{3}= & -2 \alpha^{2}\left(3+\alpha^{2}\right)\left[6\left(24-2 \alpha^{2}+\alpha^{4}\right)\right. \\
& +\left(120 \alpha^{2}+7 \alpha^{4}\right) \cosh (\sqrt{2} \alpha x) \\
& +2 \alpha^{2}\left(\alpha^{2}-6\right) \cosh (2 \sqrt{2} \alpha x) \\
& \left.+\alpha^{4} \cosh (3 \sqrt{2} \alpha x)\right] \operatorname{coth}^{2}\left(\frac{\alpha}{\sqrt{2}} x\right) \\
& \times\left[6+\alpha^{2}+\alpha^{2} \cosh (\sqrt{2} \alpha x)\right]^{-2} \\
& \times\left[12+\alpha^{2}+2 \alpha^{2} \cosh (\sqrt{2} \alpha x)\right. \\
& \left.+\alpha^{2} \cosh (2 \sqrt{2} \alpha x)\right]^{-1} . \tag{5.257}
\end{align*}
$$

These slow-roll parameters are displayed in Fig. 63 (bottom panels). We see that the first slow-roll parameters monotonously decreases during inflation. It blows up as the field vev approaches zero and tends to zero when the field vev goes to infinity. On the contrary, the second and third slow-roll parameters monotonously increase from $-\infty$ to zero as inflation proceeds.

Given the above described behavior of $\epsilon_{1}$, it is clear that inflation cannot stop by slow-roll violation. Therefore, it should be stopped by instability which means that an extra parameter $x_{\text {end }}$ should be added to the model.

As for CNAI and CNBI, the spectral index $n_{\mathrm{S}}-1=-2 \epsilon_{1}-\epsilon_{2}$ at first order in slow-roll, can be made constant in some limit. Expanding the slow-roll parameters $\epsilon_{1}$ and $\epsilon_{2}$ in $\alpha$, assuming that $x \alpha$ remains small, one obtains $\epsilon_{1}=2 / x^{2}+2 \alpha^{2} / 3+\mathcal{O}\left(\alpha^{4}\right)$ and $\epsilon_{2}=-4 / x^{2}+2 \alpha^{2} / 3+\mathcal{O}\left(\alpha^{4}\right)$, so that $n_{S}-1=-2 \alpha^{2}+\mathcal{O}\left(\alpha^{4}\right)$. As for the similar calculations performed in Sections 4.20 and 4.21, one should remark that, if $x_{\text {end }}$ is such that $\alpha x_{*} \gtrsim 1$, the previous expansion can be inaccurate and some deviations from constant $n_{S}$ may appear.

Let us now consider the slow-roll trajectory. It can be integrated analytically and is given by the following formula

$$
\begin{align*}
N-N_{\mathrm{end}}= & \frac{1}{\alpha^{2}\left(3+\alpha^{2}\right)}\left\{3 \ln \left[\cosh \left(\frac{\alpha}{\sqrt{2}} x\right)\right]\right. \\
& +\frac{\alpha^{2}}{2} \cosh ^{2}\left(\frac{\alpha}{\sqrt{2}} x\right)-3 \ln \left[\cosh \left(\frac{\alpha}{\sqrt{2}} x_{\mathrm{end}}\right)\right] \\
& \left.-\frac{\alpha^{2}}{2} \cosh ^{2}\left(\frac{\alpha}{\sqrt{2}} x_{\text {end }}\right)\right\} . \tag{5.258}
\end{align*}
$$

Moreover, this expression can be explicitly inverted. As a consequence, the function $x(N)$ can be written as

$$
\begin{align*}
x & =\frac{\sqrt{2}}{\alpha} \operatorname{arccosh}\left[\frac { 3 } { \alpha ^ { 2 } } \mathrm { W } _ { 0 } \left(\frac { \alpha ^ { 2 } } { 3 } \operatorname { e x p } \left\{\frac{2}{3} \alpha^{2}\left(3+\alpha^{2}\right)\left(N-N_{\mathrm{end}}\right)\right.\right.\right. \\
& \left.\left.\left.+2 \ln \left[\cosh \left(\frac{\alpha}{\sqrt{2}} x_{\text {end }}\right)\right]+\frac{\alpha^{2}}{3} \cosh ^{2}\left(\frac{\alpha}{\sqrt{2}} x_{\text {end }}\right)\right\}\right)\right]^{1 / 2}, \tag{5.259}
\end{align*}
$$

where $\mathrm{W}_{0}$ is the Lambert function. The fact that we deal with the 0 -branch is obvious since the argument of this function is positive definite.

The predictions of the CNCI models are displayed in Fig. 138, for $\alpha=10^{-3}, 0.1$ and 0.2 . The thin black solid lines are the lines such that $n_{\mathrm{S}}-1=-2 \alpha^{2}$. We see that, for very small values of $\alpha$, the predictions are indeed such that the spectral index is constant. For $\alpha$ not too small, however, we also notice deviations from this law and the larger $\alpha$ the stronger these deviations. This is reminiscent with the phenomenon observed in Sections 4.20 and 4.21 but now $x_{\text {end }}$ is a free parameter and, for a given value of $\alpha$, the deviations from $n_{S}-1=-2 \alpha^{2}$ become larger when $x_{\text {end }}$ increase (i.e. when the line becomes redder in Fig. 138). In this case, the Taylor expansion of the trigonometric functions which appear in the expressions of the slow-roll parameters is no longer valid because a larger $x_{\text {end }}$ implies a larger $x_{*}$. This has for consequence that CNCI inflation is only marginally consistent with the data. Indeed, it is precisely in the region where $n_{S}-1=-2 \alpha^{2}$ would be compatible with the observations that the deviations play an important role and push the predictions away from the allowed contours. In fact, these properties can be better illustrated by deriving explicitly $X_{*}$. Using Eq. (5.258), one gets
$\cosh ^{2}\left(\frac{\alpha X_{*}}{\sqrt{2}}\right)=\frac{3}{\alpha^{2}} \mathrm{~W}_{0}\left(\frac{\alpha^{2}}{3} e^{2 A / 3}\right)$,
where we have defined the quantity $A$ by

$$
\begin{align*}
A \equiv & -\alpha^{2}\left(3+\alpha^{2}\right) \Delta N_{*}+3 \ln \left[\cosh \left(\frac{\alpha x_{\text {end }}}{\sqrt{2}}\right)\right] \\
& +\frac{\alpha^{2}}{2} \cosh ^{2}\left(\frac{\alpha x_{\text {end }}}{\sqrt{2}}\right) \tag{5.261}
\end{align*}
$$

In the regime where both $\alpha \ll 1$ and $\alpha x_{\text {end }} \ll 1$, the previous expression reduces to $x_{*}^{2} \simeq x_{\text {end }}^{2}-4 \Delta N_{*}$. This last formula is identical to the slow-roll trajectory for LFI provided $p=-2$, see Eq. (4.36). At the beginning of this section, we have show that, at


Fig. 127. Reheating consistent slow-roll predictions for the BSUSYBI models in the plane $\left(n_{\mathrm{S}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The parameter $x_{\text {end }}$ varies between $2 x_{\text {end }}^{\max }<x_{\text {end }}<x_{\text {end }}^{\max }$ ( $x_{\text {end }}^{\max }<0$ ), under which the predictions of the model coincide with the line $\epsilon_{2}=0$ (black solid), i.e. PLI (see Section 4.8). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The parameter $\gamma$ should be $\lesssim 5 \times 10^{-2}$ to predict a reasonable amount of gravitational waves. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
leading order $\epsilon_{1} \simeq 2 / x^{2}$ and $\epsilon_{2} \simeq-4 / x^{2}$ and, comparing with Eq. (4.35), we notice that these are also the slow-roll parameters for LFI with $p=-2$. In fact, expanding Eq. (5.254), one sees that $V(\phi) \propto \phi^{-2}$ which confirms the previous considerations. In the regime where $\alpha \ll 1$ and $\alpha x_{\text {end }} \ll 1$, the model is very close to LFI with $p=-2$. On the contrary, if $\alpha x_{\text {end }}$ is not small, then the above relation does not hold anymore and one does not recover a constant spectral index.

Finally, we conclude this section by discussing how the mass scale $M$ can be chosen. The CMB normalization gives

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=\frac{11520 \pi^{2} \alpha^{2}\left(3+\alpha^{2}\right)^{2} \cosh ^{2}\left(\frac{\alpha}{\sqrt{2}} x_{*}\right)}{\left[6+\alpha^{2}+\alpha^{2} \cosh \left(\sqrt{2} \alpha x_{*}\right)\right]^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} . \tag{5.262}
\end{equation*}
$$

From Eq. (5.260), one deduces that $\cosh ^{2}\left(\alpha x_{*} / \sqrt{2}\right) \simeq 1-$ $2 \alpha^{2} \Delta N_{*}+\alpha^{2} x_{\text {end }}^{2} / 2 \simeq 1$. Inserting this formula into Eq. (5.262), and taking the leading order in $\alpha$, one obtains $M / M_{\mathrm{PI}} \simeq 0.02 \sqrt{\alpha}$. This implies that $M<M_{\mathrm{PI}}$ if $\alpha \lesssim 2420$, which is largely the case for the predictions displayed in Fig. 138.



Fig. 128. Reheating consistent slow-roll predictions for the tip inflation models with $\alpha<1 / 2$, and for $\mu / M_{\mathrm{Pl}}=10^{-6}, 10^{-4}, 10^{-2}$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

### 5.16. Supergravity brane inflation (SBI)

### 5.16.1. Theoretical justifications

This model can emerge in different contexts. Following Ref. [245], let us consider a model with a scalar field and a massive fermion interacting through a Yukawa type term (with a coupling constant $g$ ). The corresponding Lagrangian can be written as

$$
\begin{align*}
-\mathcal{L}= & \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi+\frac{i}{2} \bar{\psi} \gamma^{\mu} \partial_{\mu} \psi+\frac{1}{2} m^{2} \phi^{2} \\
& +\frac{\lambda}{4!} \phi^{4}+m_{\mathrm{f}} \bar{\psi} \psi+\frac{1}{2} g \phi \bar{\psi} \psi, \tag{5.263}
\end{align*}
$$

where we have assumed the most general renormalizable scalar potential. At one loop level, the potential takes the form

$$
\begin{align*}
V(\phi)= & V_{0}+\frac{1}{2} m^{2} \phi^{2}+\frac{\lambda}{4!} \phi^{4} \\
& +\frac{1}{64 \pi^{2}}\left(m^{2}+\frac{\lambda}{2} \phi^{2}\right)^{2} \ln \left(\frac{m^{2}+\lambda \phi^{2} / 2}{\mu^{2}}\right) \\
& -\frac{2}{64 \pi^{2}}\left(g \phi+m_{\mathrm{f}}\right)^{4} \ln \left[\frac{\left(g \phi+m_{\mathrm{f}}\right)^{2}}{\mu^{2}}\right] \tag{5.264}
\end{align*}
$$

where $\mu$ is a renormalization scale. Then, assuming that, for some reason, the bosonic and fermionic massive terms are negligible, the potential can be expressed as
$V(\phi) \simeq V_{0}+\left[\frac{\lambda}{4!}+\frac{\lambda^{2}}{256 \pi^{2}} \ln \left(\frac{\lambda}{2}\right)-\frac{g^{4}}{16 \pi^{2}} \ln g\right] \phi^{4}$


Fig. 129. Reheating consistent slow-roll predictions for the tip inflation models with $\alpha>1 / 2$, and for $\mu / M_{\mathrm{Pl}}=10^{-6}, 10^{-4}, 10^{-2}$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$
\begin{equation*}
+\frac{1}{64 \pi^{2}}\left(\frac{\lambda^{2}}{2}-\frac{g^{4}}{4}\right) \phi^{4} \ln \left(\frac{\phi}{\mu}\right) \tag{5.265}
\end{equation*}
$$

This is the type of potential that we study in this section. Notice that a change in the renormalization scale $\mu$ is in fact equivalent to a change in the coefficient of the terms $\propto \phi^{4}$ and $\propto \phi \ln (\phi / \mu)$. This potential was also studied in Ref. [519] but the coefficient of the $\phi^{4}$ term was chosen such that, at its minimum, the potential exactly vanishes. This particular case will also be treated in what follows. Finally, it is interesting to remark that this model was also proposed in Refs. [520,521] in the context of brane cosmology within a supergravity bulk spacetime.

### 5.16.2. Slow-roll analysis

Let us now turn to the slow-roll analysis of the potential given by Eq. (5.265). It is more convenient to write it under the following form
$V(\phi)=M^{4}\left\{1+\left[-\alpha+\beta \ln \left(\frac{\phi}{M_{\mathrm{Pl}}}\right)\right]\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{4}\right\}$,
where $\alpha$ and $\beta$ are dimensionless quantities that must be considered as small quantities since they are typically proportional to coupling constants, see Eq. (5.265). It is worth noticing that setting $\alpha=0$ in the above expression allows us to recover the Coleman-Weinberg CWI models already studied in Section 4.11. Defining the quantity $x$ by the following expression
$x \equiv \frac{\phi}{M_{\mathrm{PI}}}$,

$$
\begin{align*}
& \epsilon_{1}=\frac{x^{6}(-4 \alpha+\beta+4 \beta \ln x)^{2}}{2\left(1-\alpha x^{4}+\beta x^{4} \ln x\right)^{2}},  \tag{5.272}\\
& \epsilon_{2}=2 \frac{(12 \alpha-7 \beta-12 \beta \ln x) x^{2}+\left(4 \alpha^{2}-\alpha \beta+\beta^{2}+\beta^{2} \ln x-8 \alpha \beta \ln x+4 \beta^{2} \ln ^{2} x\right) x^{6}}{\left[1+x^{4}(-\alpha+\beta \ln x)\right]^{2}},  \tag{5.273}\\
& \epsilon_{3}=\frac{8}{x^{2}}+2 \frac{\left(-4+\beta x^{4}\right)^{2}}{x^{2}\left(1-\alpha x^{4}+\beta x^{4} \ln x\right)^{2}}+\frac{1}{x^{2}} \frac{-52+9 \beta x^{4}}{1-\alpha x^{4}+\beta x^{4} \ln x} \\
& +\frac{144 \alpha-84 \beta+(28 \alpha-11 \beta) \beta x^{4}-4 \beta\left(36+7 \beta x^{4}\right) \ln x}{(12 \alpha-7 \beta-12 \beta \ln x) x^{2}+\left(4 \alpha^{2}-\alpha \beta+\beta^{2}-8 \alpha \beta \ln x+\beta^{2} \ln x+4 \beta^{2} \ln ^{2} x\right) x^{6}} .
\end{align*}
$$

Box III.
one sees that the potential decreases from $x=0$ to reach a minimum located at $x=x_{V^{\prime}=0}$, then increases and diverges when $x$ goes to infinity. The value of $x_{V^{\prime}=0}$ is given by
$x_{V^{\prime}=0}=\exp \left(\frac{\alpha}{\beta}-\frac{1}{4}\right)$.
Since the logarithm terms in Eq. (5.266) are one loop corrections, they should not dominate the leading order terms. As a result, inflation can take place only in the domain $x<x_{V^{\prime}=0}$ if one wants the model to be such that additional corrections to $V(\phi)$ are negligible. The value of the potential at the minimum reads
$V_{\text {min }}=V\left(x_{V^{\prime}=0}\right)=M^{4}\left(1-\frac{\beta}{4} e^{4 \alpha / \beta-1}\right)$,
which is negative or vanishing if the following condition is satisfied
$\alpha \geq \alpha_{\text {min }}(\beta)=\frac{\beta}{4}\left[1-\ln \left(\frac{\beta}{4}\right)\right]$.
Inflation proceeds from the left to the right in the range $0<$ $x<x_{V=0}<x_{V^{\prime}=0}$ where $x_{V=0}$ is the value at which the potential vanishes. It is given by
$x_{V=0}=\left[\frac{-4 / \beta}{\mathrm{W}_{-1}\left(-4 / \beta e^{-4 \alpha / \beta}\right)}\right]^{1 / 4}$,
where $\mathrm{W}_{-1}$ is the -1 branch of the Lambert function. In this situation, inflation stops by slow-roll violation at $x=x_{V=0}$. As noticed above, the case $\alpha=\alpha_{\min }(\beta)$ is also interesting. It corresponds to tuning the parameters $\alpha$ and $\beta$ such that the minimum of the potential exactly vanishes. When this condition is satisfied the previous formula reduces to $x_{V=0}=x_{V^{\prime}=0}=$ $(\beta / 4)^{-1 / 4}$. Then, the first slow roll parameter $\epsilon_{1}$ diverges at this point (see below) and, as a consequence, inflation also ends by slow roll violation.

The first three Hubble flow functions in the slow-roll approximation are given by the equation in Box III. Together with the potential, they are represented in Fig. 64 for the physical branch $0<x<x_{V=0}$.

As already mentioned, inflation stops by violation of the slowroll conditions. This happens when $x=x_{\text {end }}$ where $x_{\text {end }}$ is the solution of $\epsilon_{1}\left(x_{\text {end }}\right)=1$. We see in Eq. (5.272) that there is no simple analytic solution for $x_{\text {end }}$ and this equation must in fact be solved numerically. We have, however, already stressed that, when $\alpha \leq \alpha_{\text {min }}(\beta), \epsilon_{1}$ diverges for $x \rightarrow \chi_{V=0}$, and therefore one already knows that $x_{\text {end }}<x_{V=0}$.


Fig. 130. Reheating consistent slow-roll predictions for the tip inflation models with $\alpha=1 / 2$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Let us now consider the slow-roll trajectory. It can be integrated analytically and one obtains the following expression

$$
\begin{aligned}
N-N_{\text {end }}= & \frac{e^{2 \frac{\alpha}{\beta}-\frac{1}{2}}}{16}\left[\operatorname{Ei}\left(\frac{1}{2}-2 \frac{\alpha}{\beta}+2 \ln x\right)\right. \\
& \left.-\operatorname{Ei}\left(\frac{1}{2}-2 \frac{\alpha}{\beta}+2 \ln x_{\text {end }}\right)\right] \\
& -\frac{e^{\frac{1}{2}-2 \frac{\alpha}{\beta}}}{4 \beta}\left[\operatorname{Ei}\left(-\frac{1}{2}+2 \frac{\alpha}{\beta}-2 \ln x\right)\right.
\end{aligned}
$$



Fig. 131. Reheating consistent slow-roll predictions for the $\beta$ exponential inflation models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The parameter $\lambda$ varies in the range $10^{-6}<\lambda<10^{3}$ but the predictions almost do not depend on it (and cannot be distinguished in the figure). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The black solid lines represent the locus of the points such that $\epsilon_{2}=4 \beta \epsilon_{1}$. The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$
\begin{equation*}
\left.-\operatorname{Ei}\left(-\frac{1}{2}+2 \frac{\alpha}{\beta}-2 \ln x_{\mathrm{end}}\right)\right]-\frac{x^{2}-x_{\mathrm{end}}^{2}}{8} \tag{5.275}
\end{equation*}
$$

The field value $x_{*}$ at which the pivot scale crossed the Hubble radius during inflation is obtained by solving Eq. (2.47). Clearly, it must also been done numerically and those calculations are implemented in the corresponding ASPIC routines.

Finally, the parameter $M$ is fixed by the amplitude of the CMB anisotropies and one obtains

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=\frac{720 \pi^{2}\left(4 \alpha-\beta-4 \beta \ln x_{*}\right)^{2}}{\left(1-\alpha x_{*}^{4}+\beta x_{*}^{4} \ln x_{*}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{5.276}
\end{equation*}
$$

The reheating consistent slow-roll predictions for the SBI models are displayed in Figs. 139 and 140, for $\beta=5 \times 10^{-5}$ and $\beta=10^{-3}$, respectively, and with $\alpha \leq \alpha_{\min }(\beta)$. These plots show that the larger values of $\beta$, the more negligible the amount of gravitational waves. The predictions for the special case $\alpha=\alpha_{\min }(\beta)$ are also displayed in Fig. 141, where it is clear that smaller values of $\beta$ are preferred.

### 5.17. Spontaneous symmetry breaking inflation (SSBI)

### 5.17.1. Theoretical justifications

The potential that we study in this section is given by the following expression

$$
\begin{equation*}
V(\phi)=V_{0}+a \phi^{2}+b \phi^{4} \tag{5.277}
\end{equation*}
$$

where $a$ and $b$ are constant coefficients the sign of which is not a priori determined. Before turning to the slow-roll analysis, it is interesting to study in which context such a potential can arise.

First of all, it is clear that this potential is very general since it is just made of the three first terms of a general Taylor expansion. Therefore, it can just be considered as a phenomenological description of a generic inflaton potential. This view was for instance adopted in Ref. [342], where this potential was used as a toy model to implement "new inflation". In the same fashion, it was also considered in Ref. [522] (with the assumptions $a<0$ and $b>0$ ) in the framework of models with spontaneous symmetry breaking where $\phi$ represents one of the components of a Higgs field. In Ref. [523], it was also studied in the context of "mixmaster inflation".

However, there are also models where this specific shape explicitly arises and, here, when necessary, we also briefly review them.

The first example is given by Refs. [524,525]. In these articles, inflation was investigated in the context of gauge mediated SUSY breaking scenarios. One of the basic idea of this approach is that the inflaton field should not be an extra field added to the theory on purpose but rather a field which is already present in known high energy theories. In the MSSM, see also Section 4.17, we know that the Higgs sector superpotential contains the term $\mu H_{u} \cdot H_{d}$ where $\mu$ should be of the order of the electroweak scale, that is to say far from the Planck scale. This is the so-called $\mu$-problem. One possible solution is to consider that this term dynamically arises due to the presence of another superfield (usually a singlet), $S$, in the theory. Refs. $[524,525]$ take advantage of this fact and build a model where $S$ can also play the role of the inflaton. Since the model is also formulated in the framework of gauge-mediated supersymmetry breaking scenarios, there is an additional superfield $X$ such that its scalar component (also denoted $X$ ) and auxiliary component $F_{X}$ acquire non-vanishing vev. Let us now consider the following super-potential
$W=-\beta \frac{X S^{4}}{M_{\mathrm{Pl}}^{2}}+\frac{S^{5}}{M_{\mathrm{Pl}}^{2}}+\lambda \frac{S^{2}}{M_{\mathrm{Pl}}} H_{\mathrm{u}} \cdot H_{\mathrm{d}}+\bar{W}$,
where the function $\bar{W}$ describes all the other extra terms in $W$ and, crucially, is assumed to be independent of $S$. The quantities $\lambda$ and $\beta$ are constant coefficients. As argued in Refs. [524,525], this form of $W$ can be enforced by discrete symmetries. In particular, we notice the absence of a term $S H_{u} \cdot H_{\mathrm{d}}$. Another important ingredient of the model is the assumption that the vev $F_{X}$ comes from the extraterms in the above superpotential, i.e. $F_{X} \simeq \partial \bar{W} / \partial X$. Then, the scalar potential reads
$V=\left(F_{X}-\beta \frac{S^{4}}{M_{\mathrm{Pl}}^{2}}\right)^{2}+\left(5 \frac{S^{4}}{M_{\mathrm{Pl}}^{2}}-4 \beta \frac{X}{M_{\mathrm{Pl}}^{2}} S^{3}\right)^{2}$.
Taking into account supergravity corrections, which are typically of the form $(\partial W / \partial X) / M_{\mathrm{Pl}}^{2}$, i.e. $m^{2}=a F_{X}^{2} / M_{\mathrm{Pl}}^{2}$, where $a$ is a coefficient of order one we are led to

$$
\begin{align*}
V \simeq & F_{X}^{2}-a \frac{F_{X}^{2}}{M_{\mathrm{Pl}}^{2}} S^{2}-2 \beta F_{X} \frac{S^{4}}{M_{\mathrm{Pl}}^{2}}+16 \beta^{2} \frac{X^{2}}{M_{\mathrm{Pl}}^{4}} S^{6}-40 \beta \frac{X}{M_{\mathrm{Pl}}^{4}} S^{7} \\
& +\left(25+\beta^{2}\right) \frac{S^{8}}{M_{\mathrm{Pl}}^{4}} \tag{5.280}
\end{align*}
$$

In addition, making the reasonable assumption that the field $X$ is stabilized at a vev such that $X / M_{\mathrm{Pl}} \ll 1$, one can neglect higher order terms in this expression. Then, we see that $S$ can play the role of the inflaton with a potential of the form given by Eq. (5.277), namely
$V \simeq F_{X}^{2}\left(1-a \frac{S^{2}}{M_{\mathrm{Pl}}^{2}}-\frac{2 \beta M_{\mathrm{Pl}}^{2}}{F_{X}} \frac{S^{4}}{M_{\mathrm{Pl}}^{4}}\right)$.


Fig. 132. Reheating consistent slow-roll predictions for the pseudo natural inflation models with $\mu / M_{\mathrm{PI}}=10,10^{-1}, 10^{-3}$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

At the minimum of the potential, $S^{4} \simeq M_{\mathrm{PI}}^{2} F_{X}$ and this implies a $\mu$ term for the MSSM of the form $\mu \simeq \lambda \sqrt{F_{X}}$. As explained before, this model dynamically produces the $\mu$ term while obtaining a candidate for the inflaton field. Finally, let us remark that the CMB normalization will determine the scale $F_{X}$ and that the spectrum of the superparticles depends on the ratio $F_{X} / X$. Therefore, given a value of $F_{X}$, one can always choose $X$ in order to obtain reasonable values for the superparticle masses.

The SSBI potential was also used, as a toy model, in Refs. [526,527] to study a model of "Spinodal Inflation". After the 90 's, it was considered again several times: in the context of the Randall-Sundrum model in Ref. [528] (but within the framework of Brans-Dicke theories), in the context of the little Higgs model in Ref. [272] and in the context of induced gravity inflation in Ref. [529]. In this last reference, a potential of the form (5.277) was considered but in the Jordan frame. Since the potential is different in the Einstein frame, in fact, this model does not belong to the class of scenarios studied here. Finally, it was also considered in the context of electroweak inflation in Ref. [530].

In Ref. [531], an inflationary scenario was studied in which the superpartner of the right-handed neutrino plays the role of the inflaton field. Let us denote by $N$ the singlet neutrino superfield, $\phi$ the super waterfall field (that can be put to zero during inflation) and $S$ another singlet superfield (which can also be put to zero during inflation). Then, on very general grounds, the

Kähler potential can be written as

$$
\begin{align*}
K= & |S|^{2}+|\phi|^{2}+|N|^{2}+\kappa_{S} \frac{|S|^{4}}{4 M_{\mathrm{Pl}}^{2}}+\kappa_{N} \frac{|N|^{4}}{4 M_{\mathrm{Pl}}^{2}}+\kappa_{\phi} \frac{|\phi|^{4}}{4 M_{\mathrm{Pl}}^{2}} \\
& +\kappa_{S \phi} \frac{|S|^{2}|\phi|^{2}}{M_{\mathrm{Pl}}^{2}}+\kappa_{S N} \frac{|S|^{2}|N|^{2}}{M_{\mathrm{Pl}}^{2}}+\kappa_{N \phi} \frac{|N|^{2}|\phi|^{2}}{M_{\mathrm{Pl}}^{2}}+\cdots, \tag{5.282}
\end{align*}
$$

where the dimensionless coefficients $\kappa$ are a priori of order one. The superpotential can be expressed as
$W=\kappa S\left(\frac{\phi^{4}}{M^{\prime 2}}-M^{2}\right)+\frac{\lambda}{M_{*}} N^{2} \phi^{2}+\cdots$,
where $M, M^{\prime}$ and $M_{*}$ are three mass scales and $\kappa$ and $\lambda$ are coupling constants. Since the three fields introduced before are singlets the potential does not contain $D$-term contributions. As a consequence, for $S \simeq 0$ and $\phi \simeq 0$, we are left with the $F$-term potential only and this one can be written as

$$
\begin{align*}
V(N) \simeq & \kappa^{2} M^{4}\left[1+\left(1-\kappa_{S N}\right) \frac{N^{2}}{M_{\mathrm{Pl}}^{2}}\right. \\
& \left.+\left(\frac{1}{2}+\frac{\kappa_{N}}{4}-\kappa_{S N}+\kappa_{S N}^{2}\right) \frac{N^{4}}{M_{\mathrm{Pl}}^{4}}+\cdots\right] . \tag{5.284}
\end{align*}
$$

We see that it has the form of Eq. (5.277). Ref. [531] also discusses how to stop inflation by tachyonic instability. Since the field $\phi$ is viewed as the waterfall field, one has to calculate his mass to see when the instability is triggered. This can be done by evaluating the quadratic correction in $\phi$ to the potential calculated before. This leads to
$m_{\phi}^{2}=\left(1+\kappa_{N \phi} \frac{N^{2}}{M_{\mathrm{Pl}}^{2}}-\kappa_{S \phi}\right) \frac{\kappa^{2} M^{4}}{M_{\mathrm{Pl}}^{2}}+4 \frac{\lambda^{2}}{M_{*}^{2}} N^{4}$.
Neglecting the term $N^{2} / M_{\mathrm{Pl}}^{2} \ll 1$ in this expression, the effective mass vanishes for
$N_{\text {cri }} \simeq \frac{\kappa M^{2} M_{*}}{2 \lambda M_{\mathrm{Pl}}} \sqrt{-\left(1-\kappa_{S \phi}\right)}$.
We see that this requires $1-\kappa_{S \phi}<0$. On the other hand, this model also provides an expression for the coefficients $a$ and $b$ in terms of the fundamental coefficients of the Kähler potential. Except from the above mentioned condition, there is no other constraint on the coefficients $\kappa$ and, as a consequence, the sign of $a$ and $b$ is, a priori, not fixed in this scenario.

Another context in which Eq. (5.277) arises is "racetrack inflation" [532,533]. Racetrack inflation is a string inspired inflationary scenario where the inflaton is a volume modulus. Therefore, this model belongs to the same class as KMIII, see Section 5.3. The Kähler and super potentials are given by standard formulas, namely
$K=-\frac{3}{\kappa} \ln \left(T+T^{\dagger}\right), \quad W=W_{0}+A e^{-a T}+B e^{-b T}$.
Writing $T=X+i Y$, it follows that the scalar $F$-term potential reduces to

$$
\begin{align*}
& V(X, Y)=\frac{\kappa}{6 X^{2}}\left\{a A^{2}(3+a X) e^{-2 a X}+b B^{2}(3+b X) e^{-2 b X}\right. \\
& \quad+3 a A W_{0} e^{-a X} \cos (a Y)+3 b B W_{0} e^{-b X} \cos (b Y)+A B \\
& \left.\quad \times[2 a b X+3(a+b)] e^{-(a+b) X} \cos [(a-b) Y]\right\}+\frac{E}{X^{\alpha}} \tag{5.288}
\end{align*}
$$

where an uplifting term $\propto X^{-\alpha}$ has been added. Let us mention that $X$ and $Y$ are not canonically normalized and their kinetic term reads $3\left[\left(\partial_{\mu} X\right)^{2}+\left(\partial_{\mu} Y\right)^{2}\right] /\left(4 \kappa X^{2}\right)$. The above potential has a very


Fig. 133. Reheating consistent slow-roll predictions for the non canonical Kähler inflation models with $\beta>0$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. When $\beta \gtrsim 1$, the predictions are almost identical to those displayed here. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
rich structure and for $W_{0}=0$ and $a=b$, we have a flat direction in $Y$. Moreover, for $Y=0$, one can find a minimum in the $X$ direction. If we then combine the two above remarks, then it is clear that there exists a choice of parameters such that one has a saddle point around $Y=0$ (a specific example was exhibited in Ref. [532]). This point seems suitable for inflation. Around such a point, it is argued in Ref. [533] that one can write
$V(Y)=V_{0}\left(1+\frac{\eta_{0}}{2} y^{2}+\frac{C}{4} y^{4}+\cdots\right)$,
where $y$ is now the canonically normalized field when $X$ is stabilized. This is again a potential of the type given by Eq. (5.277). In order to phenomenologically reproduce racetrack inflation, one should have $\eta_{0}$ small and negative and $C$ large and positive.

The potential of Eq. (5.277) was also used, as a toy model, in the context of minimal left-right symmetric models with spontaneous D-parity breaking in Ref. [534] and in the context of hilltop supernatural inflation in Refs. [535-537]. A justification based on high energy physics was offered and the idea is to assume that the full potential has a SUSY flat direction. The approach is therefore similar to what was already investigated in Section 4.17. In that situation, one can write $V(\phi)$ as
$V=V_{0}+\frac{1}{2} m^{2} \phi^{2}-A \frac{\lambda_{p} \phi^{p}}{p M_{\mathrm{Pl}}^{p-3}}+\lambda_{p}^{2} \frac{\phi^{2 p-2}}{M_{\mathrm{Pl}}^{2 p-6}}$,
where the term $V_{0}$ is added by hand. If one chooses $p=4$ and neglects the last term (for instance if $\phi \ll M_{\mathrm{PI}}$ ), then one arrives at
$V(\phi) \simeq V_{0}+\frac{1}{2} m^{2} \phi^{2}-\frac{\lambda_{4} A}{4 M_{\mathrm{Pl}}} \phi^{4}$,
which is of the form of Eq. (5.277). In this framework, $m$ and $A$ are SUSY soft terms and, therefore, should be taken of $\mathcal{O}(\mathrm{TeV})$. The term $V_{0}=M_{\mathrm{s}}^{4}$ where $M_{\mathrm{s}}$ is the SUSY breaking scale, $M_{\mathrm{s}} \simeq 10^{11} \mathrm{GeV}$.

Finally, let us mention that SSBI was also considered in the context of a supersymmetric $B-L$ extension of the standard model in Refs. [538,539] and in the context of Kähler-driven "tribrid inflation" in Ref. [540]. In this last case, one obtains a situation very similar to the one discussed above for sneutrino inflation. In particular, the coefficients $a$ and $b$ can be expressed in terms of the coefficients appearing in the Kähler potential. To end this part, let us notice that the potential (5.277) also arises in the context of Higgs inflation in a false vacuum, as shown in Refs. [541-543].

As already mentioned above, these works differ on the signs of $\alpha$ and $\beta$. Summarizing, Refs. [523,531] require $\alpha>0, \beta>0$ while Refs. [272,342,522,526,527,529,530,533,534] assume $\alpha<0$, $\beta>0$. On the other hand, Refs. [535-537] consider that $\alpha>0$ and $\beta<0$ and Refs. [524,525,541-543] have $\alpha<0, \beta<0$. We see that the four possible combinations have all been studied. Also, in Refs. [538,539], one has $\alpha, \beta \lesssim \mathcal{O}(1)$ and inflation only takes place in the increasing branches of the potential (see below). Finally, in Refs. [528,540], $\beta$ is taken to be positive and the sign of $\alpha$ is left unspecified.

### 5.17.2. Slow-roll analysis

Let us now turn to the slow-roll analysis of SSBI. For this purpose, it is more convenient to rewrite the potential (5.277) as
$V(\phi)=M^{4}\left[1+\alpha\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{2}+\beta\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{4}\right]$,
where $\alpha$ and $\beta$ are two dimensionless parameters. Based on the previous brief review of the literature, we conclude that it is necessary to study the model in full generality and, therefore, in what follows, we investigate all possible situations. As mentioned above, four cases should be distinguished: $\alpha>0, \beta>0$; $\alpha<$ $0, \beta<0$; $\alpha>0, \beta<0$ and $\alpha<0, \beta>0$, with two possible domains of inflation in the two latter cases. Therefore we have six regimes of inflation that we label SSBI1, SSBI2, SSBI3, SSBI4, SSBI5 and SSBI6. The different potentials and inflationary regimes are displayed and defined in Figs. 65 and 66. Since the potential is symmetric under $\phi / M_{\mathrm{Pl}} \rightarrow-\phi / M_{\mathrm{PI}}$, it is only displayed and studied for $\phi>0$.

Let us now calculate the slow-roll parameters. If one defines $x$ by $x \equiv \phi / M_{\mathrm{P}}$, then the three first Hubble parameters are given by the following expressions
$\epsilon_{1}=\frac{2\left(\alpha x+2 \beta x^{3}\right)^{2}}{\left(1+\alpha x^{2}+\beta x^{4}\right)^{2}}$,
$\epsilon_{2}=\frac{4\left[-\alpha+\left(\alpha^{2}-6 \beta\right) x^{2}+\alpha \beta x^{4}+2 \beta^{2} x^{6}\right]}{\left(1+\alpha x^{2}+\beta x^{4}\right)^{2}}$,
and
$\epsilon_{3}$

$$
\begin{equation*}
=\frac{4 x^{2}\left(\alpha+2 \beta x^{2}\right)\left[-3 \alpha^{2}+6 \beta+\alpha\left(\alpha^{2}-12 \beta\right) x^{2}+3\left(\alpha^{2}-8 \beta\right) \beta x^{4}+2 \beta^{3} x^{8}\right]}{\left(1+\alpha x^{2}+\beta x^{4}\right)^{2}\left[-\alpha+\left(\alpha^{2}-6 \beta\right) x^{2}+\alpha \beta x^{4}+2 \beta^{2} x^{6}\right]} . \tag{5.294}
\end{equation*}
$$

The first slow-roll parameter $\epsilon_{1}$ is displayed in the right panels of Figs. 65 and 66 while the second and third slow-roll parameters $\epsilon_{2}$



Fig. 134. Reheating consistent slow-roll predictions for the non canonical Kähler inflation models with $\beta<0$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. When $\beta \lesssim-1$, the predictions remain almost unchanged. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
and $\epsilon_{3}$ are displayed in Fig. 67. Let us describe the behavior of these slow-roll parameters, for the six models under consideration. For SSBI1, $\epsilon_{1}$ vanishes at $x=0$, reaches a maximum at $x_{\epsilon_{2}=0}^{\text {SSI1 }}$ (where $\epsilon_{2}$ vanishes and $\epsilon_{3}$ diverges) and then decreases to asymptotically vanish when $x$ goes to infinity. The value of $x_{\epsilon_{2}=0}^{\text {SSI1 }}$ is given by

$$
\begin{align*}
& x_{\epsilon_{2}=0}^{\mathrm{SB} 118386}=\left\{-\frac{\alpha}{6 \beta}+\frac{1}{6 \beta}\left[8 \alpha^{3}+\sqrt{64 \alpha^{6}+\left(5 \alpha^{2}-36 \beta\right)^{3}}\right]^{1 / 3}\right. \\
& \left.\quad+\frac{36 \beta-5 \alpha^{2}}{6 \beta}\left[8 \alpha^{3}+\sqrt{64 \alpha^{6}+\left(5 \alpha^{2}-36 \beta\right)^{3}}\right]^{-1 / 3}\right\}^{1 / 2} . \tag{5.295}
\end{align*}
$$

Whether the maximum of $\epsilon_{1}$ at this point is larger or smaller than 1 depends on $\alpha$ and $\beta$. In the following, we restrict ourselves to the physical regime where $\alpha, \beta \lesssim \mathcal{O}(1)$. For each value of $\beta$, there is a minimum value of $\alpha$, denoted $\alpha_{\min }$, above which the maximum is larger than 1. The line $\alpha_{\min }(\beta)$ is displayed in Fig. 68 and the shaded area in this plot represents the region in the parameter space where inflation stops by slow-roll violation. When $\beta \ll 1$, $\alpha_{\min }(\beta)$ approaches 2 as can be noticed in the figure. In addition, for $\beta \gtrsim 0.25$, the maximum value for $\epsilon_{1}$ becomes larger than 1 for any value of $\alpha$.

For SSBI2, the three first slow-roll parameters are monotonic increasing functions of the field vev and diverge when the potential
vanishes at
$x_{V=0}^{\text {SSBI28485 }}=\sqrt{-\frac{\alpha+\sqrt{\alpha^{2}-4 \beta}}{2 \beta}}$.
Hence inflation ends by slow-roll violation at $x_{\text {end }}$. Unfortunately, the corresponding vev cannot be found exactly and one has to rely on numerical calculations. Let us also notice that, while the first and third slow-roll parameters $\epsilon_{1}$ and $\epsilon_{3}$ vanish at $x=0, \epsilon_{2}$ is equal to $\epsilon_{2}^{\min }=-4 \alpha$ at this point. Therefore, in order for the slow-roll approximation to be valid, one needs to work with $|\alpha| \ll 1$.

For SSBI3, the first slow-roll parameter $\epsilon_{1}$ vanishes at $x=0$ and at $x=\sqrt{-\alpha /(2 \beta)}$. In between, it reaches a maximum located at
$x_{\epsilon_{2}=0}^{\text {SSBI3 }}=x_{\epsilon_{2}=0}^{\text {SSI1 }}$,
a point where $\epsilon_{2}$ vanishes and $\epsilon_{3}$ diverges. Whether the maximum of $\epsilon_{1}$ at this point is larger or smaller than 1 depends again on $\alpha$ and $\beta$. For each value of $\beta$, there is a minimum value for $\alpha$ above which inflation stops by slow-roll violation, similarly to the SSBI1 case. This corresponds to the green dotted line in Fig. 68 (top right panel). One way to estimate whether a slow roll regime of inflation can occur in the decreasing branch of $\epsilon_{1}$ is to look at the value of $\epsilon_{2}$ at the top of the potential. It is given by
$\epsilon_{2}^{\mathrm{top}}=\frac{-32 \alpha \beta}{\alpha^{2}-4 \beta}$.
This number is smaller than one when $\beta<-1 / 64$, or when $\alpha$ lies outside the range with limits given by $-16 \beta \pm \sqrt{\beta(1+64 \beta)}$, displayed in Fig. 68 with the red and cyan dotted lines. Therefore, requiring that $\epsilon_{2}^{\text {top }}<1$ and that inflation stops by slow roll violation leads to the allowed space $\alpha>\alpha_{\min }$, represented by the shaded region in Fig. 68.

For SSBI4, the three first slow-roll parameters are monotonic increasing functions of the field vev and diverge when the potential vanishes at $X_{V=0}^{\mathrm{SSBI284}}$. The first and third slow-roll parameters $\epsilon_{1}$ and $\epsilon_{3}$ vanish when $x=\sqrt{-\alpha /(2 \beta)}$ while $\epsilon_{2}$ has a non-zero value $\epsilon_{2}^{\min }=8 \alpha \beta /\left(\beta^{2}-\alpha^{2} / 4\right)$ at this point. From the above discussion, it is clear that, in this version of the scenario, inflation also stops by violation of the slow-roll condition. As for SSBI2, however, the corresponding vev can not be determined exactly and a numerical calculation is needed.

For SSBI5, the behavior of the slow-roll parameters depend on $\alpha^{2} / \beta$. If $\alpha^{2} / \beta \geq 4$, the minimum of the potential at $x=$ $\sqrt{-\alpha /(2 \beta)}$ is negative. The potential vanishes at $\chi_{V=0}^{\text {SSBI28485 }}$ and the three first slow-roll parameters continuously increase between $x=0$ where they vanish (except $\epsilon_{2}$ for which $\epsilon_{2}^{\min }=-4 \alpha$ ) and $x_{V=0}^{\text {SSI2\&4\&5 }}$ where they diverge. Inflation ends by slow-roll violation at some point $x_{\text {end }}$ that needs to be determined numerically. On the other hand, if $\alpha^{2} / \beta \leq 4, \epsilon_{1}$ vanishes at $x=0$, reaches a maximum at $\chi_{\epsilon_{2}=0}^{\text {SSI5 }}$ (where $\epsilon_{2}$ vanishes and $\epsilon_{3}$ diverges), then decreases and finally vanishes at $x=\sqrt{-\alpha /(2 \beta)}$. The value of $x_{\epsilon_{2}=0}^{\text {SSI5 }}$ is given by

$$
\begin{align*}
x_{\epsilon_{2}=0}^{\text {SSBI5 }}= & \left\{-\frac{\alpha}{6 \beta}-\frac{1+i \sqrt{3}}{12 \beta}\left[8 \alpha^{3}+\sqrt{64 \alpha^{6}+\left(5 \alpha^{2}-36 \beta\right)^{3}}\right]^{1 / 3}\right. \\
& +\frac{5 \alpha^{2}-36 \beta}{12 \beta}(1-i \sqrt{3}) \\
& \left.\times\left[8 \alpha^{3}+\sqrt{64 \alpha^{6}+\left(5 \alpha^{2}-36 \beta\right)^{3}}\right]^{-1 / 3}\right\}^{1 / 2} . \tag{5.299}
\end{align*}
$$

Whether the maximum of $\epsilon_{1}$ at this point is larger or smaller than 1 depends on $\alpha$ and $\beta$ and is again similar to what has already been discussed before. The region in the parameter space where inflation ends by slow-roll violation is displayed in Fig. 68 and


Fig. 135. Reheating consistent slow-roll predictions for the Constant Spectrum models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel), for $\alpha=10^{-3}$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The black solid lines correspond to $n_{S}=1$, and the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
corresponds to the points such that $\alpha<-\left|\alpha_{\min }\right|$. In this plot, the dotted line represents the curve $\alpha^{2}=4 \beta$, above which one is sure that inflation ends by slow-roll violation since the minimum of the potential is negative in this case. For values of $\beta \ll 1$, one can see that $\left|\alpha_{\min }\right| \simeq 2 \sqrt{\beta}$ and the allowed region becomes negligible.

Finally the case SSBI6 remains to be treated. The behavior of the slow roll parameters depend on $\alpha^{2} / \beta$ in the same way as before. If $\alpha^{2} / \beta \geq 4$, the minimum of the potential at $x=\sqrt{-\alpha /(2 \beta)}$ is negative. The potential vanishes at $\chi_{V=0}^{\text {SSB6 }}$ and the slow-roll parameters continuously decrease from this value (where they blow up) and go to zero at infinity. The value of $x_{V=0}^{\text {SSI6 }}$ can be expressed as
$x_{V=0}^{\text {SBIG }}=\sqrt{\frac{-\alpha+\sqrt{\alpha^{2}-4 \beta}}{2 \beta}}$.
On the other hand, if $\alpha^{2} / \beta \leq 4, \epsilon_{1}$ vanishes at $x=\sqrt{-\alpha /(2 \beta)}$, reaches a maximum at $x_{\epsilon_{2}=0}^{\text {SSI6 }}$ and then decreases. At infinity, it goes to zero. The value of $x_{\epsilon_{2}=0}^{\text {SSBI6 }}$ is given by
$x_{\epsilon_{2}=0}^{\mathrm{SSBI} 6}=x_{\epsilon_{2}=0}^{\mathrm{SSBI} 3}=x_{\epsilon_{2}=0}^{\mathrm{SSBI} 1}$.
Whether the maximum of $\epsilon_{1}$ at this point is larger or smaller than 1 depends on $\alpha$ and $\beta$. The corresponding region in the parameter space is displayed in Fig. 68 and corresponds to the inequality $\alpha<-\left|\alpha_{\text {min }}\right|$. The dotted line represents the law $\alpha^{2}=4 \beta$. Above this line, one is sure that inflation can stop by slow-roll violation


Fig. 136. Reheating consistent slow-roll predictions for the Constant Spectrum models in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel), for $\alpha=1$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The black solid lines correspond to $n_{S}=1$, and the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\mathrm{reh}} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
since, in this case, the potential becomes negative at some point. It is also interesting to notice that, when $\beta \gtrsim 1.48$, the maximum value of $\epsilon_{1}$ is larger than 1 for any value of $\alpha$. On the other hand, if $\beta \ll 1$, the allowed region shrinks to zero.

Let us now turn to the slow-roll trajectory. This one can be integrated analytically to get

$$
\begin{align*}
N_{\mathrm{end}}-N= & -\frac{1}{2 \alpha} \ln \left(\frac{x_{\mathrm{end}}}{x}\right)-\frac{x_{\mathrm{end}}^{2}-x^{2}}{8} \\
& -\frac{\alpha^{2}-4 \beta}{16 \alpha \beta} \ln \left(\frac{1+\frac{2 \beta}{\alpha} x_{\mathrm{end}}^{2}}{1+\frac{2 \beta}{\alpha} x^{2}}\right), \tag{5.302}
\end{align*}
$$

where $N_{\text {end }}$ is the number of $e$-folds at the end of inflation. It is important to notice that the argument of the logarithm is always positive. This trajectory cannot be inverted analytically. But, numerically, it is easy to use this expression in order to determine $x_{*}$, the value of $x$ at Hubble radius crossing.

Finally, it is interesting to constrain the value of the scale $M$ with the CMB normalization. It follows that
$\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=\frac{2880\left(\alpha x_{*}+2 \beta x_{*}^{3}\right)^{2} \pi^{2}}{\left(1+\alpha x_{*}^{2}+\beta x_{*}^{4}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
We are now in a position where we can discuss the predictions of the six versions of this model. The reheating consistent slowroll predictions for the SSBI1 models are displayed in Figs. 142144 for $\beta=10^{-3}, \beta=10^{-1}$ and $\beta=10$, respectively. SSBI1


Fig. 137. Reheating consistent slow-roll predictions for the orientifold inflation models for $\phi_{0} / M_{\mathrm{PI}}=10^{-4}, 10^{-2}, 1$ and $\alpha \in\left[10^{-3}, 10^{-1}\right]$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slowroll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. Since the predictions of these models almost do not depend on its parameters, they are all superimposed and one cannot distinguish the different values of $\phi_{0}$ are $\alpha$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
seems to be disfavored by the observations. The predictions of SSBI2 models are displayed in Fig. 145 for different values of $\beta$ and $\alpha$. We notice that they depend on the parameter $\alpha$ quite strongly. The spectral index is clearly red and, for values of $\beta$ of order one, the contribution of gravity waves becomes very small. For SSBI3, the predictions are presented in Figs. 146-148 for $\beta=-10^{-3}$, $\beta=-5 \times 10^{-3}$ and $\beta=-10^{-2}$, respectively. As we increase $\beta$, the points start spreading in the plane ( $n_{\mathrm{S}}, r$ ). For this class of models, the spectrum is red and the level of gravity waves quite important. The predictions for the SSBI4 models are displayed in Figs. 149151 for $\beta=-10^{-5}, \beta=-10^{-4}, \beta=-10^{-3}$, respectively. One can notice that the typical predicted values for $\epsilon_{1}$ decrease with the absolute value of $\beta$. As before the spread of the points increases with $\beta$. The tilt is still red and the contribution of gravity waves is small for small values of $\alpha$. The predictions for the SSBI5 models are displayed in Figs. 152-154 for $\beta=10^{-6}, \beta=10^{-5}$ and $\beta=10^{-4}$, respectively. Once again, for $\mathcal{O}(1)$ values of $\beta$, one can see that the model predict a small amount of gravitational waves but has a deviation from scale invariance strongly disfavored by the observational constraints. Finally, the reheating consistent slowroll predictions for the SSBI6 models are displayed in Figs. 155-157 for $\beta=10^{-6}, \beta=10^{-1}$ and $\beta=1$, respectively. When $\beta \ll 1$ the predictions of the model do not depend on $\beta$. Moreover, for values of $\beta$ of order one, the predictions become almost independent of the two parameters of the model.

### 5.18. Inverse monomial inflation (IMI)

These models are characterized by the inverse monomial potential given by
$V(\phi)=M^{4}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{-p}$,
where $p$ is a positive number. This scenario has been studied in many different situations: in Refs. [294,544,545] it was considered in the context of quintessential inflation, in Refs. [546-549] in the context of tachyon inflation, in Refs. [475,477] in the context of intermediate inflation and in Ref. [307] in the context of Randall-Sundrum braneworld models. In all these articles, the potential was just postulated. An attempt to derive this potential from high energy considerations was made in Refs. [550,551] in the context of supersymmetric QCD. Let us, however, notice that this was done in order to build a model of quintessence and not of inflation. The model uses the group $\mathrm{SU}\left(N_{\mathrm{c}}\right)$ and has $N_{\mathrm{f}}$ flavors. The quarks $Q^{i}, i=1, \ldots, N_{\mathrm{f}}$ are placed in the fundamental representation of $\operatorname{SU}\left(N_{c}\right)$ and the anti-quarks $Q_{i}^{\dagger}$ in the conjugate representation [550]. At scales below the gauge breaking scale $\Lambda$, the relevant degrees of freedom are the pions $\pi_{j}^{i}=Q^{i} Q_{j}^{\dagger}$ and one can show that the corresponding superpotential is given by [552,553]
$W=\left(N_{\mathrm{c}}-N_{\mathrm{f}}\right) \frac{\Lambda^{3\left(N_{\mathrm{c}}-N_{\mathrm{f}}\right) /\left(N_{\mathrm{c}}-N_{\mathrm{f}}\right)}}{(\operatorname{det} \pi)^{1 /\left(N_{\mathrm{c}}-N_{\mathrm{f}}\right)}}$.
The potential (5.304) then follows from the F-term associated to the above superpotential.

The potential is represented in Fig. 69 for $p=2$. It is a decreasing function of the field vev and, hence, inflation proceeds from the left to the right, in the direction specified by the arrow in the figure.

The three Hubble flow functions are straightforwardly obtained from Eqs. (2.4)-(2.6). Defining $x \equiv \phi / M_{\mathrm{Pl}}$, one gets
$\epsilon_{1}=\frac{p^{2}}{2 x^{2}}, \quad \epsilon_{2}=-\frac{2 p}{x^{2}}, \quad \epsilon_{3}=\epsilon_{2}$.
These functions are represented in the two bottom panels in Fig. 69. The first slow-roll parameter is a monotonic decreasing function of $\phi$ while $\epsilon_{2}$ and $\epsilon_{3}$ are negative increasing functions. From these expressions, one can also immediately deduce that, for a given $p$, the model in the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ is represented by the line $\epsilon_{1}=-(p / 4) \epsilon_{2}$. Since inflation proceeds from the left to the right, it cannot stop by slow-roll violation. As a consequence, an extra-mechanism, such as e.g. tachyonic instability, must be implemented to end inflation. Let us denote $x_{\text {end }}$ the position at which such a process occurs. The model has therefore two free parameters: $p$ and $x_{\text {end }}$.

The slow-roll trajectory can be obtained by quadrature from Eq. (2.11), and one obtains
$N-N_{\text {end }}=\frac{1}{2 p}\left(x^{2}-x_{\text {end }}^{2}\right)$.
This expression can be inverted and reads
$x=\sqrt{x_{\mathrm{end}}^{2}+2 p\left(N-N_{\mathrm{end}}\right)}$.
Let us now derive some prior condition on $x_{\text {end }}$. One can notice that when $x<x_{\epsilon_{1}=1}=p / \sqrt{2}$, one has $\epsilon_{1}>1$ and inflation cannot take place. This means that inflation can only proceed between $x_{\epsilon_{1}=1}$ and $x_{\text {end }}$, where the maximum number of $e$-folds is, using Eq. (5.307), $\Delta N_{\max }\left(x_{\text {end }}\right)=\left(x_{\text {end }}^{2}-x_{\epsilon_{1}=1}^{2}\right) /(2 p)$. Put it differently, if


Fig. 138. Reheating consistent slow-roll predictions for the constant $n_{S} C$ inflation models for $\alpha=10^{-3}, 0.1,0.2$ in the plane ( $n_{\mathrm{s}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The black solid lines are the $n_{S}-1=-2 \alpha^{2}$ contours, for the displayed values of $\alpha$. The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The energy scale at which reheating ends is degenerated with the parameter $x_{\text {end }}$, which is why it is not labeled. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
one wants to realize at least $\Delta N e$-folds, then one has to work with $x_{\text {end }}>x_{\text {end }}^{\text {min }}$ where
$x_{\text {end }}^{\min }(\Delta N)=\sqrt{p^{2} / 2+2 p \Delta N}$.
This defines a prior condition on $x_{\text {end }}$.
Finally, the parameter $M$ can be determined from the amplitude of the CMB anisotropies, and it follows that

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 \pi^{2} p^{2} x_{*}^{p-2} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \tag{5.310}
\end{equation*}
$$

The reheating consistent slow-roll predictions for the IMI models are displayed in Fig. 158. For a given value of $p$, they lie along the line $(1-2 / p) r=8\left(1-n_{\text {S }}\right)$, i.e. $\epsilon_{1}=-(p / 4) \epsilon_{2}$. As expected, large values of $x_{\text {end }}$, or small values of the reheating temperature (these two parameters being degenerate), are preferred.

### 5.19. Brane inflation (BI)

### 5.19.1. Theoretical justifications

This section is devoted to brane inflation, a class of models widely discussed in the literature [ $158,351,367,401,554-567]$. The idea is that inflation is caused by branes moving in the extra dimensions as it was already the case in TI, see Section 5.9. For this reason, the setup is very similar to the one considered in that section. One starts from type IIB superstring theory
where six dimensions are compactified. The effective, low energy, description of the model contains various fields among which are the dilaton, the axion and the (tensorial) gravitational field. One also has anti-symmetric fields with their corresponding field strength. The compact dimensions form a Calabi-Yau space and, generically, this Calabi-Yau space is made of a bulk plus throats attached to it. Along a given throat, a solution for the tendimensional metric is given by the conifold already discussed in Section 5.9 whose metric is given in Eq. (5.142). In this equation, the metric $\mathrm{ds}_{5}^{2}$ lives on the five-dimensional section $\Sigma_{5}$ and $r$ is the "radial" coordinate. In the following, we will denote by $r_{\mathrm{Uv}}$ the radial coordinate at which the cone is glued to the bulk and $r_{0}$ the coordinate at the tip of the cone. The volume of the cone section is denoted by $\operatorname{Vol}\left(\Sigma_{5}\right)$ and will be measured in terms of the volume of the five-dimensional sphere, namely
$v \equiv \frac{\operatorname{Vol}\left(\Sigma_{5}\right)}{\operatorname{Vol}\left(S_{5}\right)}$.
The geometry of the section $\Sigma_{5}$ depends on the background fluxes, denoted by $\mathcal{M}$ and $\mathcal{K}$, that are quantities related to the values of the anti-symmetric fields. If these fluxes vanish then the fivedimensional sections are simply given by $S_{2} \times S_{3}$. In that case, the conifold can be written as $\sum_{i=1}^{4} w_{i}^{2}=0$ where $w_{i}$ are four complex coordinates, see also Section 5.9. Moreover, an exact expression for the warp function $h(r)$ can be found and reads
$h(r)=C_{2}+\frac{C_{1}}{r^{4}}$,
$C_{1}$ and $C_{2}$ being constants. On the other hand, if the fluxes are turned on, then the background geometry responses accordingly and, as a consequence, the geometry of the cone is modified. It is now given by a "deformed conifold", $\sum_{i=1}^{4} w_{i}^{2}=z$, where $z$ is a number which depends on $\mathcal{M}$ and $\mathcal{K}$. The warp function acquires a more complicated form and, obviously, becomes $z$-dependent, i.e. $h(r, z)$. The explicit form of this warp function is not needed here but it is interesting to notice that, far from the tip, one has $h(r, z) \simeq h(r)$. In other words, the modification of the extradimensional geometry due to the fluxes is significant only in the vicinity of the tip. Notice that, provided the depth of the throat is comparable to its width, the radial coordinate $r_{\mathrm{UV}}$ can be expressed in terms of the quantity $\mathcal{N} \equiv \mathcal{M} \mathcal{K}$. One obtains [568]
$r_{\mathrm{UV}}^{4}=4 \pi g_{\mathrm{s}} \alpha^{\prime 2} \frac{\mathcal{N}}{v}$,
where $g_{\mathrm{s}}$ is the string coupling and $\alpha^{\prime} \equiv \ell_{\mathrm{s}}^{2}, \ell_{\mathrm{s}}$ being the string length.

Finally, an anti-D3 brane is placed at the tip of the conifold, i.e. at the bottom of the throat. This brane is heavy and is supposed to slightly disturb the geometry of the throat in a way that has been calculated for instance in Refs. [158,566,569]. Then, in this geometry, one studies the motion of a light $D 3$ brane with tension
$T_{3}=\frac{1}{(2 \pi)^{3} g_{s} \alpha^{\prime 2}}$.
This brane is attracted by the anti- $D 3$ brane and as a consequence moves radially along the throat. In principle it possesses a DBI kinetic term but one can show that, in the regime considered here, it always reduces to an ordinary, minimal, kinetic term, see Ref. [158]. If $r$ represents the distance between the two branes, then the effective Lagrangian of the system can be expressed as
$\mathscr{L}=-\frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^{2}-\frac{2 T_{3} r_{0}^{4}}{r_{\mathrm{UV}}^{4}}\left(1-\frac{r_{0}^{4} T_{3}^{2}}{\mathcal{N}} \frac{1}{\phi^{4}}\right)$,


Fig. 139. Reheating consistent slow-roll predictions for the supergravity brane inflation models for $\beta=5 \times 10^{-5}$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and twosigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
where $\phi \equiv \sqrt{T_{3}} r$. The shape of the potential is now completely fixed and the behavior $\propto \phi^{-4}$ is of course due to the particular scaling $\propto r^{-4}$ of the warp function given by Eq. (5.312).

In order to be valid, the effective model described above must satisfy some conditions that we now discuss in more detail. Defining $\phi_{0} \equiv \sqrt{T_{3}} r_{0}$ and $\phi_{\mathrm{UV}} \equiv \sqrt{T_{3}} r_{\mathrm{UV}}$, it is clear that the presence of the brane in the throat implies that $\phi_{0}<\phi<\phi_{\mathrm{UV}}$. In addition, as discussed for instance in Ref. [158], from the trivial fact that the volume of the throat, $V_{6}^{\text {throat }}=2 \pi^{4} g_{s} \mathcal{N} \alpha^{\prime 2} r_{\mathrm{UV}}^{2}$, cannot be bigger than the volume of the total Calabi-Yau manifold $V_{6}^{\text {tot }}$, one can derive the bound
$\phi_{\mathrm{UV}}<\frac{m_{\mathrm{Pl}}}{\sqrt{2 \pi \mathcal{N}}}$,
where the Planck mass can be expressed as $m_{\mathrm{Pl}}^{2}=8 \pi V_{6}^{\text {tot }} / \kappa_{10}$ and $\kappa_{10}=(2 \pi)^{7} g_{s}^{2} \alpha^{\prime 4} / 2$. Another constraint comes from the fact that the effective model is valid only if the proper distance between the two branes is larger than the Planck length. One can show, see Ref. [158], that this means $r>r_{\text {stg }}$ where
$r_{\text {stg }} \equiv r_{0} e^{\sqrt{\alpha^{\prime}} / r_{\mathrm{UV}}}$.
In particular, as will be seen in the following, the value of $r_{\text {stg }}$ plays an important role regarding the mechanism ending inflation. In the next section, we carry out the slow-roll analysis of this model.

Let us also mention that the same potential arises in the context of tachyon inflation [570,571], in the context of SQCD inflation [572] and in the context of the strong coupling limit of twisted models of SQCD inflation, (see TWI, Section 5.5 and


Fig. 140. Reheating consistent slow-roll predictions for the supergravity brane inflation models for $\beta=10^{-3}$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Ref. [488]). It is also worth noticing that the same kind of inverse power law potential is sometimes used in quintessence models [294,544,545]. The brane inflation potential can also receive power law corrections [573] with either positive (UV models) or negative sign (IR models). The UV case is similar to RIPI models while the IR corresponds to SFI models.

### 5.19.2. Slow-roll analysis

We now turn to the slow-roll analysis of BI. For this purpose, it is more convenient to re-write the potential appearing in Eq. (5.315) in the following way
$V(\phi)=M^{4}\left[1-\left(\frac{\phi}{\mu}\right)^{-p}\right]$,
where $\mu$ and $p$ are free parameters. Compared to Eq. (5.315), we have generalized by hand the expression of $V(\phi)$ by considering an arbitrary $p$. In such a way, this potential can be viewed as a generalization of the small field models to negative values of $p$ (see Section 5.1). In the following, we will also consider the nonapproximated KKLT potential
$V(\phi)=\frac{M^{4}}{1+\left(\frac{\phi}{\mu}\right)^{-p}}$,
from which (5.318) is the $\mu \ll M_{\mathrm{Pl}}$ limit.

In the context of the brane inflationary scenario, the value $p=4$ is special in the sense that, as explained above, it corresponds to the motion of a test $D 3$ brane in a warped throat and is, therefore, a case of physical interest. Let us notice that the parameters of the potential are related to their stringy counterparts by
$M^{4}=\frac{2 T_{3} r_{0}^{4}}{r_{\mathrm{UV}}^{4}}=\frac{4 \pi^{2} v}{\mathcal{N}} \phi_{0}^{4}, \quad \mu^{4}=\frac{T_{3}^{2} r_{0}^{4}}{\mathcal{N}}=\frac{M^{4}}{4 \pi^{2} v}$.
Moreover, brane inflation proceeds under the condition $\mu / M_{\mathrm{Pl}} \ll$ 1. Indeed, using the formulas established in the previous subsection, it is easy to show that
$\frac{\mu^{4}}{M_{\mathrm{Pl}}^{4}}=\frac{1}{\mathcal{N}}\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{4}<\frac{1}{\mathcal{N}}\left(\frac{\phi_{\mathrm{UV}}}{M_{\mathrm{Pl}}}\right)^{4}<\frac{16}{\mathcal{N}^{3}} \ll 1$,
where we have used the condition $\phi_{0}<\phi_{\mathrm{UV}}$ and Eq. (5.316). Finally, let us stress that the brane motion in the throat ends by a tachyonic instabilities at $\phi=\phi_{\text {stg }}$. As we discuss below, the observable predictions of the model crucially depends on whether the universe is still inflating at $\phi \gtrsim \phi_{\mathrm{stg}}$, or not. Therefore, in the context of string theory, we necessarily have $\mu / M_{\mathrm{Pl}} \ll 1, p=4$ and an additional model parameter $\phi_{\text {stg }}$.

In the following, we will first consider arbitrary values for $\mu$ and $p$ viewing Eq. (5.318) as a phenomenological potential in which $\phi_{\text {stg }}$ has no meaning, and then, the discussion will be focused on the stringy scenario. BI is another proto-typical case exemplifying how two models having exactly the same potential can lead to different observable predictions. Here this will be due to the mechanism ending inflation.

The potential (5.318), as well as its logarithm, are displayed in Fig. 70. It is an increasing function of the field, hence inflation proceeds from the right to the left. It vanishes for $\phi / \mu=1$ and, hence, it should be studied in the $\phi / \mu>1$ region only. Let us calculate the slow-roll parameters. Defining the quantity $x$ by the following expression
$x \equiv \frac{\phi}{\mu}$,
one can express the first three Hubble flow functions in the slowroll approximation as
$\epsilon_{1}=\left(\frac{M_{\mathrm{PI}}}{\mu}\right)^{2} \frac{p^{2}}{2 x^{2}\left(1-x^{p}\right)^{2}}$,
$\epsilon_{2}=2 p\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{2} \frac{(1+p) x^{p}-1}{x^{2}\left(1-x^{p}\right)^{2}}$,
and
$\epsilon_{3}=p\left(\frac{M_{\mathrm{PI}}}{\mu}\right)^{2} \frac{2+(p-4)(p+1) x^{p}+(1+p)(2+p) x^{2 p}}{x^{2}\left(1-x^{p}\right)^{2}\left[(1+p) x^{p}-1\right]}$.

These functions are displayed in Fig. 70. They become very small at large fields $x \gg 1$, and diverge when the potential vanishes at $x \rightarrow 1$. Therefore inflation can naturally end with slow-roll violation at a field value $x_{\text {end }}$, solution of $\epsilon_{1}\left(x_{\text {end }}\right)=1$, i.e., verifying
$x_{\text {end }}^{p+1}-x_{\text {end }}=\frac{p}{\sqrt{2}} \frac{M_{\mathrm{PI}}}{\mu}$.
Unless $p$ takes integer values, this equation has to be solved numerically (see also Section 5.1).

However, in the limits $\mu / M_{\mathrm{PI}} \ll 1$ and $\mu / M_{\mathrm{PI}} \gg 1$ we can find an approximate expression for $x_{\text {end }}$. Solving perturbatively the



Fig. 141. Reheating consistent slow-roll predictions for the supergravity brane inflation models for $\alpha=\alpha_{\min }(\beta)$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and twosigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
equation $\epsilon_{1}=1$, one obtains
$\chi_{\mathrm{end}} \underset{\mu \lll M_{\mathrm{Pl}}}{\simeq}\left(\frac{p M_{\mathrm{Pl}}}{\sqrt{2} \mu}\right)^{\frac{1}{p+1}}+\frac{1}{p+1}\left(\frac{p M_{\mathrm{PI}}}{\sqrt{2} \mu}\right)^{\frac{1-p}{1+p}}$,
$x_{\text {end }} \underset{\mu \gg M_{\mathrm{Pl}}}{\simeq} 1+\frac{1}{\sqrt{2}} \frac{M_{\mathrm{PI}}}{\mu}-\frac{p+1}{4} \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}}$.
It is also interesting to find the solution of $\epsilon_{2}=1$. As before, this cannot be done exactly but, perturbatively, one obtains
$x_{\epsilon_{2}=1} \underset{\mu \lll M_{\mathrm{Pl}}}{\sim}\left[2 p(1+p)\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{2}\right]^{\frac{1}{p+2}}$,
$x_{\epsilon_{2}=1} \underset{\mu \gg M_{\mathrm{Pl}}}{\simeq} 1+\sqrt{2} \frac{M_{\mathrm{PI}}}{\mu}$.
From the above expressions, we deduce that slow-roll violation always occurs before the end of inflation, that is to say $\epsilon_{2}$ becomes unity before $\epsilon_{1}$. This has not effect on the observable predictions since only a few $e$-folds of inflation are spent in this regime (see Fig. 70).

The slow-roll trajectory can be integrated explicitly from Eq. (2.11) and one obtains
$N_{\text {end }}-N=\frac{\mu^{2}}{2 p M_{\mathrm{Pl}}^{2}}\left(x_{\text {end }}^{2}-\frac{2}{p+2} x_{\text {end }}^{p+2}-x^{2}+\frac{2}{p+2} x^{p+2}\right)$,
an expression which cannot be inverted in general. However, in the $\mu \ll M_{\mathrm{PI}}$ and $\mu>M_{\mathrm{Pl}}$ limits, one has $x \gg 1$ and $x \simeq$ 1 respectively and the previous equation can be approximately inverted leading to the following expressions
$x_{*} \underset{\mu \lll M_{\mathrm{Pl}}}{\simeq}\left[p(p+2) \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \Delta N_{*}+x_{\mathrm{end}}^{p+2}\right]^{\frac{1}{p+2}}$,
$x_{*} \underset{\mu \gg M_{\mathrm{PI}}}{\sim} 1+\frac{M_{\mathrm{Pl}}}{\mu} \sqrt{\frac{1}{2}+2 \Delta N_{*}}$,
where use has been made of Eq. (5.326). Also, making use of the full KKLT potential (5.319), the slow roll trajectory reads
$N_{\mathrm{end}}-N=\frac{\mu^{2}}{2 p M_{\mathrm{Pl}}^{2}}\left(-x_{\text {end }}^{2}-\frac{2}{p+2} x_{\mathrm{end}}^{p+2}+x^{2}+\frac{2}{p+2} x^{p+2}\right)$,
which coincides with (5.328) in the limit $\mu \ll M_{\mathrm{PI}}$.
The mass scale $M$ is given by the CMB normalization and verifies
$\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 \pi^{2} p^{2}\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{2} \frac{x_{*}^{p-2}}{\left(x_{*}^{p}-1\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$
which can be further simplified in the appropriate limits using Eqs. (5.326) and (5.329).

The reheating consistent slow-roll predictions for the phenomenological models are displayed in Figs. 159-161 for $p=2$, $p=3$ and $p=4$, respectively, and with $\mu / M_{\mathrm{PI}} \in\left[10^{-3}, 10^{3}\right]$. The reheating equation of state parameter $\bar{w}_{\text {reh }}=0$ but since the shape of the potential is unknown at $x<1$, this parameter is a priori unspecified and could take different values. For small values of $\mu$, we see that $n_{\mathrm{S}} \simeq 0.96$ and $r \ll 1$. In the opposite case, $\mu \gg M_{\mathrm{Pl}}$, the model predictions lie around $\epsilon_{2} \simeq 4 \epsilon_{1}$ with $n_{\mathrm{S}} \simeq 0.97$ and $r \simeq 0.08$. These behaviors can be recovered by plugging the approximated expressions given in Eqs. (5.326) and (5.329) into the Hubble flow functions. For $\mu \ll M_{\mathrm{Pl}}$, one obtains
$\epsilon_{1 *} \simeq \frac{p^{2}}{2}\left[p(p+2) \Delta N_{*}\right]^{-\frac{2 p+2}{p+2}}\left(\frac{\mu}{M_{\mathrm{Pl}}}\right)^{\frac{2 p}{p+2}}$,
$\epsilon_{2 *} \simeq \frac{2}{\Delta N_{*}} \frac{p+1}{p+2}, \quad \epsilon_{3 *} \simeq \frac{1}{\Delta N_{*}}$,
and the spectral index is of the order $n_{\mathrm{S}} \simeq 1-2 / \Delta N_{*}(p+1) /(p+$ 2) $\sim 0.96$ with $r \ll 1$. Similarly, for $\mu \gg M_{\text {PI }}$ limit, the Hubble flow parameters at Hubble crossing behave as
$\epsilon_{1 *} \simeq \frac{1}{4 \Delta N_{*}}, \quad \epsilon_{2 *} \simeq \frac{1}{\Delta N_{*}}, \quad \epsilon_{3 *} \simeq \frac{1}{\Delta N_{*}}$.
Therefore, the predicted level of gravity waves is now of the order $r \simeq 4 / \Delta N_{*} \simeq 0.08$ and the spectral index is $n_{\mathrm{S}} \simeq 1-3 /\left(2 \Delta N_{*}\right) \simeq$ 0.97 , which is again in agreement with the numerical results.

Finally, the predictions for the KKLTI models, i.e. using the full potential (5.319), are displayed in Figs. 163-165 for the same parameters. One can see that they deviate from the ones of brane inflation only when $\mu \gg M_{\mathrm{Pl}}$.

### 5.19.3. Slow-roll analysis of the stringy scenario

In the case where the model is interpreted as a stringy scenario, with $p=4$, we have seen before that the low energy description is valid provided $r>r_{\text {stg }}$, or $x>x_{\text {stg }}$ with
$x_{\mathrm{stg}} \equiv \frac{\sqrt{T_{3}} r_{\mathrm{stg}}}{\mu}=\mathcal{N}^{1 / 4} \exp \left[\left(4 \pi g_{\mathrm{s}} \frac{\mathcal{N}}{v}\right)^{-1 / 4}\right]$.
If slow-roll violation occurs before the system reaches $x_{\text {stg }}$, then the effective string description is always valid and the observable


Fig. 142. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 1 inflation $(\alpha>0, \beta>0)$ models with $\beta=10^{-3}$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The parameter $\alpha$ is varied between $\alpha_{\min }(\beta)<$ $\alpha<10^{6} \alpha_{\text {min }}(\beta)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
predictions will be exactly the same as those derived in the previous paragraph (for $p=4$ and $\mu \ll M_{\mathrm{PI}}$ ). However, if, on the contrary, slow-roll violation occurs after the field crosses the value $x_{\text {stg }}$, then inflation stops by instability at $x_{\text {stg }}$ instead of the naively expected $x_{\text {end }}$. Indeed, in this case, a tachyon appears and triggers the process of branes annihilation. Therefore, the mechanism ending inflation in this model depends on whether slow-roll violation occurs in a regime where the distance between the branes is larger or smaller than the string length. And this question depends on the value of the parameters characterizing BI. One can determine the two regimes by evaluating the ratio

$$
\begin{align*}
\frac{x_{\epsilon_{2}=1}}{x_{\text {stg }}}= & 40^{1 / 6}\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{-1 / 3} \mathcal{N}^{-1 / 4}\left(4 \pi^{2} v\right)^{1 / 12} \\
& \times \exp \left[-\left(4 \pi g_{\mathrm{s}} \frac{\mathcal{N}}{v}\right)^{-1 / 4}\right] \tag{5.335}
\end{align*}
$$

in which we have used Eqs.(5.320), (5.327) and (5.334) (with $p=4$ and $\mu \ll M_{\mathrm{PI}}$ ). If this ratio is larger than one, inflation stops by slow-roll violation and if it is smaller than one by instability. The complicated part of the analysis lies in the fact that the above equation depends on the mass scale $M$. In order to have an explicit expression of $M$ in terms of the parameters of the model, one must first CMB normalize the model which, in turn, requires the knowledge of the mechanism ending inflation. However, we are interested in calculating the frontier where $x_{\epsilon_{2}=1}=x_{\text {stg }}$ and, therefore, the two possible mechanisms for stopping inflation
coincide in that case. Replacing $x_{\text {end }}$ by $x_{\text {stg }}=x_{\epsilon_{2}=1}$ in Eq. (5.329) yields
$x_{*}^{\mathrm{f}} \simeq\left[24 \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}}\left(\Delta N_{*}+\frac{5}{3}\right)\right]^{1 / 6}$,
from which one can obtain an explicit formula for the first slow-roll coefficient (5.323) at Hubble radius crossing
$\epsilon_{1 *}^{\mathrm{f}} \simeq 8\left[24\left(\Delta N_{*}+\frac{5}{3}\right)\right]^{-5 / 3}\left(\frac{\mu}{M_{\mathrm{Pl}}}\right)^{4 / 3}$.
Comparing this expression to Eq. (5.332), we see that there is a very small shift by $5 / 3$ in $\Delta N_{*}$. It accounts for the difference of $e$ folds between the time at which slow-roll violations occur, i.e. for $x=x_{\epsilon_{2}=1}$, and the end of inflation at $x_{\text {end }}$. As argued before, we see that these effects are too small to be observable and completely degenerated with the reheating duration. Plugging this expression into the CMB normalization, and using the relation $M^{4}=4 \pi^{2} v \mu^{4}$, one arrives at the following expression for $M$
$\frac{M}{M_{\mathrm{Pl}}}=C\left(4 \pi^{2} v\right)^{-1 / 8}\left(\Delta N_{*}+\frac{5}{3}\right)^{-5 / 8}$,
where we have defined
$C \equiv 3^{-5 / 8}\left(8 \pi^{2} Q_{*}\right)^{3 / 8}, \quad Q_{*} \equiv 45 \frac{Q_{\text {rms-PS }}^{2}}{T^{2}}=2700 P_{*}$.
We can now insert this expression of $M$ in Eq. (5.335) to get the equation defining the frontier in the string parameter space, namely

$$
\begin{align*}
\left.\frac{x_{\epsilon_{2}=1}}{x_{\text {stg }}}\right|_{\mathrm{f}}= & 1=\left(\frac{40}{C^{2}}\right)^{1 / 6}\left(\Delta N_{*}+\frac{5}{3}\right)^{5 / 24}\left(4 \pi^{2} v\right)^{1 / 8} \mathcal{N}^{-1 / 4} \\
& \times \exp \left[-\left(4 \pi g_{\mathrm{s}} \frac{\mathcal{N}}{v}\right)^{-1 / 4}\right] \tag{5.340}
\end{align*}
$$

Following Ref. [158], if one defines the two following rescaled stringy parameters
$y \equiv 4 \pi g_{\mathrm{s}} \frac{\mathcal{N}}{v}, \quad \bar{v} \equiv \frac{v}{\left(4 \pi g_{\mathrm{s}}\right)^{2}}$,
then the frontier (5.340) is defined by the following "universal" form
$y^{1 / 4} e^{y^{-1 / 4}} \bar{v}^{1 / 8}-\left(\frac{40}{C^{2}}\right)^{1 / 6}\left(\Delta N_{*}+\frac{5}{3}\right)^{5 / 24}\left(4 \pi^{2}\right)^{1 / 8}=0$,
which is independent of the string coupling $g_{s}$. As represented in Fig. 71, in the plane $(y, \bar{v})$, this relation is a curve that separates the region where inflation stops by slow-roll violation (below the curve) and the region where inflation stops by instability due to brane annihilation (above the curve).

The requirement of having the throat contained within the Calabi-Yau manifold can equally be written in terms of the universal variables. From Eqs. (5.316) and (5.341), one gets
$y^{3 / 2} \bar{v}<8 \pi^{2} M_{\mathrm{Pl}}^{2} \ell_{\mathrm{s}}^{2}$,
which therefore depends on the string length $\ell_{\mathrm{s}}=\sqrt{\alpha^{\prime}}$ but not on the string coupling $g_{s}$.

Finally, the last theoretical prior comes from requiring that the brane motion remains located inside the throat, i.e. $x<x_{\mathrm{UV}}$ with
$x_{\mathrm{UV}} \equiv \frac{\sqrt{T_{3}} r_{\mathrm{UV}}}{\mu}=\frac{M_{\mathrm{Pl}}}{M}\left(\frac{\mathcal{N}}{4 \pi^{3} \alpha^{\prime 2} \mathrm{~g}_{\mathrm{s}}}\right)^{1 / 4}$.
Since during inflation $x$ decreases, this condition gives an upper limit on the admissible initial field values. However, the initial


Fig. 143. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 1 inflation $(\alpha>0, \beta>0)$ models with $\beta=10^{-1}$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The parameter $\alpha$ is varied between $\alpha_{\min }(\beta)<$ $\alpha<10^{6} \alpha_{\text {min }}$ ( $\beta$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
field values depends on the total number of $e$-folds of inflation, say $\Delta N_{\text {tot }}$, and on the field value at which inflation ends, i.e. either $x_{\text {stg }}$ or $x_{\epsilon_{2}=1}$ depending on if brane annihilation occurs before slow-roll violations.

Let us first assume that brane annihilation occurs well after the end of inflation, i.e. we are in lower part of the string parameter space ( $y, \bar{v}$ ) separated by Eq. (5.342). For the relevant limit, $\mu \ll$ $M_{\mathrm{PI}}$, the initial field value is given by
$x_{\mathrm{ini}}^{\epsilon_{2}} \simeq\left[24 \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}}\left(\Delta N_{\mathrm{tot}}+\frac{5}{3}\right)\right]^{1 / 6}$.
This expression involves $\mu$ and therefore $M$ through Eq. (5.320). Again, one has to determine $M$ using the CMB normalization and we are assuming that inflation ends at $x_{\epsilon_{2}}=1$, i.e. exactly Eq. (5.338). Plugging everything together and making use of the universal variables, one gets

$$
\begin{align*}
& y \bar{v} \underset{x_{\mathrm{stg}}<x_{\epsilon_{2}=1}}{>} C^{8 / 3} \pi^{2} M_{\mathrm{Pl}}^{2} \ell_{\mathrm{s}}^{4}\left[24\left(\Delta N_{\mathrm{tot}}+\frac{5}{3}\right)\right]^{2 / 3} \\
& \quad \times\left(\Delta N_{*}+\frac{5}{3}\right)^{-5 / 3} . \tag{5.346}
\end{align*}
$$

If inflation ends by brane annihilation at $x=x_{\text {stg }}$, i.e. the string parameters $(y, \bar{v})$ lie above the curve given by Eq. (5.338), then $x_{\text {ini }}$ and $x_{*}$ are accordingly modified. For $\mu \ll M_{\mathrm{PI}}$, their new expressions are however still given by Eq. (5.329), up to the
replacement $x_{\text {end }} \rightarrow x_{\text {stg }}$, i.e.
$x_{\mathrm{ini}}^{\mathrm{stg}} \simeq\left(24 \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \Delta N_{\mathrm{tot}}+x_{\mathrm{stg}}^{6}\right)^{1 / 6}$,
$x_{*}^{\mathrm{stg}} \simeq\left(24 \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \Delta N_{*}+x_{\mathrm{stg}}^{6}\right)^{1 / 6}$.
As before, $x_{\mathrm{ini}}^{\text {stg }}$ and $x_{*}^{\mathrm{stg}}$ depend on $\mu$ and therefore on $M$, which is determined by the CMB normalization. However, since inflation now ends by tachyonic instability this one has to be re-determined by plugging $x_{*}^{\text {stg }}$ into Eq. (5.331). Doing so gives an implicit expression for $M$

$$
\begin{align*}
\frac{M}{M_{\mathrm{Pl}}} & \simeq C\left(4 \pi^{2} v\right)^{-1 / 8}\left(\Delta N_{*}+\frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}} \frac{x_{\mathrm{stg}}^{6}}{24}\right)^{-5 / 8} \\
& =C\left(4 \pi^{2} v\right)^{-1 / 8}\left[\Delta N_{*}+\frac{5}{3}\left(\frac{x_{\mathrm{stg}}}{x_{\epsilon_{2}=1}}\right)^{6}\right]^{-5 / 8} \tag{5.348}
\end{align*}
$$

where use has been made of Eq. (5.327), for $\mu \ll M_{\mathrm{PI}}$. This equation cannot be analytically solved for $M$ because $\mu$, and $x_{\epsilon_{2}=1}$, depends on $M$. However, if brane annihilation occurs well before slow-roll violation, one has $x_{\text {stg }} \gg x_{\epsilon_{2}=1}$ such that the term in $\Delta N_{*}$ can be neglected. In that situation, from $\mu^{4}=M^{4} /\left(4 \pi^{2} v\right)$, one gets the approximate expression
$\frac{M}{M_{\mathrm{Pl}}} \underset{\mathrm{xtg} \gg x_{\epsilon_{2}}=1}{\sim} 24^{5 / 18} C^{4 / 9}\left(4 \pi^{2} v\right)^{1 / 12} x_{\mathrm{stg}}^{-5 / 3}$.
Requiring $x_{\text {ini }}^{\text {stg }}<x_{\text {UV }}$ finally yields

$$
\begin{align*}
& y^{19 / 6} \bar{v}^{7 / 3} \exp \left(\frac{20}{3} y^{-1 / 4}\right) \underset{x_{\mathrm{stg}} \gg x_{\epsilon_{2}}=1}{>}\left(8 \pi^{2} \ell_{\mathrm{s}}^{2}\right)^{3} Q_{*} \\
& \quad \times\left[y^{2 / 3} \bar{v}^{1 / 3} \exp \left(\frac{8}{3} y^{-1 / 4}\right)+\frac{6 \Delta N_{\text {tot }}}{Q_{*}^{1 / 3}}\right] \tag{5.350}
\end{align*}
$$

which completes the bounds coming from $x_{\mathrm{UV}}$.
Brane inflation within the string scenario has therefore a rather involved set of priors. In addition to have $p=4$ and $\mu \ll M_{\mathrm{PI}}$, the model parameters should simultaneously verify Eq. (5.343) and either Eq. (5.346), or Eq. (5.350), according to the sign of the left hand side of Eq. (5.342). All these equations involve the amplitude of the CMB anisotropies, which is well measured, the total number of e-folds $\Delta N_{\text {tot }}$, which is an unknown quantity, and the number of $e$-folds $\Delta N_{*}$ before the end of inflation at which the pivot mode crossed the Hubble radius. As discussed in Section $2.2, \Delta N_{*}$ can only be obtained by solving Eq. (2.44), i.e. after having specified the reheating parameter. As the result, the reheating slow-roll predictions for the string scenario can only be sorted out numerically, paying attention that for a given reheating history, all of the previous theoretical constraints are satisfied. As an illustration, we have plotted in Fig. 71 the bounds for the typical values $\Delta N_{*}=50$ and $\Delta N_{\text {tot }}=60$ with $\alpha^{\prime} M_{\mathrm{Pl}}^{2} \simeq 1 / 4[158,574]$.

The reheating consistent slow-roll predictions for the string models are displayed in Fig. 162 for a set of realistic fundamental parameters. Also, making use of the full potential (5.319), the predictions of the corresponding KKLT inflation models are displayed in Fig. 166. One can check that they match perfectly.

## 6. Three parameters models

### 6.1. Running-mass inflation (RMI)

### 6.1.1. Theoretical justifications

This model has been derived and studied in Refs. [352,575-583]. Following Ref. [578], let us briefly discuss its physical origin.


Fig. 144. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 1 inflation ( $\alpha>0, \beta>0$ ) models with $\beta=10$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The parameter $\alpha$ is varied between $\alpha_{\min }(\beta)<$ $\alpha<10^{6} \alpha_{\text {min }}$ ( $\beta$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

At tree level, a potential can always be expanded as $V(\phi) \simeq$ $M^{4}+m^{2} \phi^{2} / 2+\lambda \phi^{4} / 4+\cdots$. Since the potential must be flat to support inflation, quantum corrections may play an important role. Typically, they modify the potential with a term of the form $\left(c_{1}+c_{2} \phi^{2}+c_{4} \phi^{4}\right) \ln (\phi / \mu)$, where $\mu$ is the renormalization scale. In a non-supersymmetric framework, the quartic term dominates and one is led to models similar to RCMI, RCQI or CWI, see Sections 4.4, 4.5 and 4.11. On the other hand, in a supersymmetric context, at least if supersymmetry is spontaneously broken, the quadratic and the quartic terms cancel and one is left with a model similar to LI, see Section 4.12. If, however, supersymmetry is explicitly broken by the presence of soft terms, then the most important term will be the quadratic one.

Concretely, the above reasoning leads to a specific shape for the inflaton potential. We start from a flat direction in supersymmetry. Then, we assume that supersymmetry is explicitly broken and, as a consequence, that the potential receives corrections $\propto m^{2} \phi^{2}$, where $m$ is a soft mass. Higher order terms are supposed to be negligible since we assume $\phi / M_{\mathrm{PI}} \ll 1$. We thus have
$V=V_{0}+\frac{1}{2} m^{2} \phi^{2}+\cdots$.
The one loop corrections to this tree potential will typically induces a logarithmic dependence of the soft mass through the renormalization group equation
$\frac{\mathrm{d} m^{2}}{\mathrm{~d} \ln \phi}=\beta_{\mathrm{mat}}$,


Fig. 145. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 2 inflation ( $\alpha<0, \beta<0$ ) models, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
where $\beta_{\text {mat }}$ is proportional to the inflaton couplings with the other fields present in the theory. Therefore, by Taylor expanding the solution of the previous equation aroused $\phi=\bar{\phi}$, we can write
$m^{2}=m^{2}(\bar{\phi})+\beta_{\text {mat }} \ln \left(\frac{\phi}{\bar{\phi}}\right)+\cdots$.
As a consequence, the potential (6.1) can be re-expressed as
$V(\phi)=V_{0}+\frac{1}{2} m^{2}(\bar{\phi}) \phi^{2}+\frac{1}{2} \beta_{\mathrm{mat}} \phi^{2} \ln \left(\frac{\phi}{\bar{\phi}}\right)$.
As noticed in Refs. [578,581,583], the beta function can typically be expressed as
$\beta_{\text {mat }}=\frac{-2 C}{\pi} \alpha \tilde{m}^{2}+\frac{D}{16 \pi^{2}}|\lambda|^{2} m_{\text {loop }}^{2}$,
if we assume that the inflaton interacts with gauge bosons and fermions. The quantity $\alpha$ is the coupling constant between $\phi$ and the gauge boson, $\lambda$ is a Yukawa coefficient, $\tilde{m}$ is the gaugino mass, $m$ the fermionic mass and $C$ and $D$ are dimensionless numbers of order one.

In the next section, we explore the cosmological consequences of this type of potential. In particular, we will see that it can lead to four different kind of inflationary scenarios.

### 6.1.2. Slow-roll analysis

We now perform the slow-roll analysis of the potential previously derived. In order to carry out this task, it is more


Fig. 146. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 3 inflation $\left[\alpha>0, \beta<0, x^{2}<-\alpha /(2 \beta)\right]$ models for $\beta=-10^{-3}$, in the plane $\left(n_{\mathrm{S}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The parameter $\alpha$ is varied between $\alpha_{\min }(\beta) \simeq 2<\alpha<10^{3} \alpha_{\min }(\beta)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
convenient to re-write the potential as follows
$V(\phi)=M^{4}\left[1-\frac{c}{2}\left(-\frac{1}{2}+\ln \frac{\phi}{\phi_{0}}\right) \frac{\phi^{2}}{M_{\mathrm{Pl}}^{2}}\right]$,
where we have defined the two parameters $c$ and $\phi_{0}$ by
$c=-\frac{M_{\mathrm{P}}^{2} \beta_{\mathrm{mat}}}{2 V_{0}}, \quad m^{2}(\bar{\phi})=-\beta_{\mathrm{mat}}\left[\frac{1}{2}+\ln \left(\frac{\phi_{0}}{\bar{\phi}}\right)\right]$.
In this expression, $M, c$ and $\phi_{0}$ are free parameters. The dimensionless parameter $c$ can be positive or negative. With the form of the beta function given in Eq. (6.5), the coefficient $c$ is given by $\alpha m^{2} M_{\mathrm{Pl}}^{2} / V_{0}$. If one assumes that the soft masses are of order $m \simeq H \simeq V_{0}^{1 / 2} / M_{\mathrm{Pl}}^{2}$, then $c \simeq \alpha \simeq 10^{-2}$ to $10^{-1}$ or may be smaller depending on the assumption on the couplings. This also mean that, in order for the expansion (6.3) to be valid, one has $\left|\ln \left(\phi / \phi_{0}\right)\right| \ll 1$. Also, the model is commonly worked out in the vacuum dominated regime (otherwise it is equivalent to a large field model, LFI, see Section 4.2), which means that $c \phi_{0}^{2} / M_{\mathrm{Pl}}^{2} \ll 1$. The location $\phi=\phi_{0}$ is an extremum of $V(\phi)$, a maximum if $c>0$ and a minimum if $c<0$. The potential and its logarithm are represented in Fig. 72.

Running mass inflation can be realized in four different ways [578], denoted as RMI1, RMI2, RMI3 and RMI4 in what follows. RMI1 corresponds to the case where $c>0$ and $\phi<\phi_{0}$, see Fig. 72 (top panels). In this case, $\phi$ decreases during inflation which proceeds from the right to the left. RMI2 also corresponds


Fig. 147. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 3 inflation $\left[\alpha>0, \beta<0, x^{2}<-\alpha /(2 \beta)\right]$ models for $\beta=-5 \times 10^{-3}$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The parameter $\alpha$ is varied between $\alpha_{\min }(\beta) \simeq 2<\alpha<10^{3} \alpha_{\min }(\beta)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
to $c>0$ but with $\phi>\phi_{0}$ and $\phi$ increases during inflation which now proceeds from the left to the right. RMI3 refers to the situation where $c<0$ and $\phi<\phi_{0}$ all the time. In this case, $\phi$ increases during inflation which proceeds from the left to the right. Finally, RMI4 has $c<0$ and $\phi>\phi_{0}$ decreases as inflation proceeds from the right to the left.

Using the potential (6.6), one can calculate the three slow-roll parameters $\epsilon_{1}, \epsilon_{2}$ and $\epsilon_{3}$. Defining $x \equiv \phi / \phi_{0}$, one obtains the following expressions

$$
\begin{equation*}
\epsilon_{1}=\frac{c^{2}}{2}\left[\frac{\frac{\phi_{0}}{M_{\mathrm{Pl}}} x \ln x}{1-\frac{c}{2} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}}\left(-\frac{1}{2}+\ln x\right) x^{2}}\right]^{2}, \tag{6.8}
\end{equation*}
$$

and

$$
\begin{aligned}
\epsilon_{3}= & \frac{c \ln x}{\left[1-\frac{c}{2} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}}\left(-\frac{1}{2}+\ln x\right) x^{2}\right]^{2}}\left[1+\frac{c}{4} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}} x^{2}\right. \\
& \left.+\left(1-\frac{c}{4} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}} x^{2}\right) \ln x+\frac{c}{2} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}} x^{2} \ln ^{2} x\right]^{-1}
\end{aligned}
$$



Fig. 148. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 3 inflation $\left[\alpha>0, \beta<0, x^{2}<-\alpha /(2 \beta)\right]$ models for $\beta=-10^{-2}$, in the plane $\left(n_{S}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The parameter $\alpha$ is varied between $\alpha_{\min }(\beta) \simeq 2<\alpha<10^{3} \alpha_{\min }(\beta)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$
\begin{align*}
& \times\left[1+\frac{c}{2} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}} x^{2}+\frac{c^{2}}{16} \frac{\phi_{0}^{4}}{M_{\mathrm{Pl}}^{4}} x^{4}+c\left(2 \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}} x^{2}+\frac{c}{2} \frac{\phi_{0}^{4}}{M_{\mathrm{Pl}}^{4}} x^{4}\right) \ln x\right. \\
& \left.+c\left(3 \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}} x^{2}-\frac{c}{2} \frac{\phi_{0}^{4}}{M_{\mathrm{Pl}}^{4}} x^{4}\right) \ln ^{2} x+\frac{c^{2}}{2} \frac{\phi_{0}^{4}}{M_{\mathrm{Pl}}^{4}} x^{4} \ln ^{3} x\right] . \tag{6.10}
\end{align*}
$$

The slow-roll parameters are represented in the bottom panels in Fig. 72.

Let us now examine how inflation ends in this model. The slow-roll parameter $\epsilon_{1}$ has a maximum in the $x<1$ region and a maximum in the $x>1$ region, see Fig. 72. If these maxima were larger than one, inflation could in principle stop by violation of the slow-roll conditions. In the vacuum dominated approximation, however, we see from Eq. (6.8), that $\epsilon_{1} \simeq$ $\left(c^{2} / 2\right)\left(\phi_{0}^{2} / M_{\mathrm{PI}}^{2}\right) x^{2} \ln ^{2} x$. This means that the vev $x_{\text {end }}$ satisfies $x_{\text {end }} \ln x_{\text {end }}= \pm(\sqrt{2} / c)\left(M_{\mathrm{Pl}} / \phi_{0}\right)$. But we have established previously that the vacuum dominated condition precisely implies that $c M_{\mathrm{Pl}} / \phi_{0} \gg 1$ and one would have $\ln x_{\text {end }} \gg 1$. But for the model to be valid, we have already mentioned that the condition $|\ln x| \ll 1$ should be enforced. We conclude that the value of $x_{\text {end }}$ obtained above lies outside the regime of validity of the potential. The end of inflation either occurs by violation of slow-roll but in a regime where additional unknown corrections arise and modify the shape of $V(\phi)$, or by tachyonic instability. In this last case, inflation stops in a regime where our calculations are valid. This also means that we must consider an additional parameter in the


Fig. 149. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 4 inflation $\left[\alpha>0, \beta<0, x^{2}>-\alpha /(2 \beta)\right]$ models for $\beta=-10^{-5}$, in the plane ( $n_{S}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
model, namely $x_{\text {end }}$. In this article, this is the assumption made which implies that RMI is indeed a three parameters model.

We now turn to the calculation of the observable predictions. The first step is to obtain the slow-roll trajectory. One obtains

$$
\begin{align*}
N-N_{\mathrm{end}}= & \frac{1}{c}\left(\ln |\ln x|-\ln \left|\ln x_{\mathrm{end}}\right|\right)-\frac{1}{4} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}}\left(x^{2}-x_{\mathrm{end}}^{2}\right) \\
& +\frac{1}{4}\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}^{2}}\right)^{2}\left[\mathrm{Ei}(2 \ln x)-\mathrm{Ei}\left(2 \ln x_{\mathrm{end}}\right)\right], \tag{6.11}
\end{align*}
$$

where the exponential integral function Ei is defined by $\operatorname{Ei}(x) \equiv$ $-\int_{-x}^{+\infty} \mathrm{d} t e^{-t} / t$ [216,217]. This expression cannot be inverted analytically. However, in the limit $\left(c \phi_{0} / M_{\mathrm{PI}}\right) x \ll 1$ (the vacuum dominated regime), the above expression can be approximated by
$N-N_{\text {end }} \simeq \frac{1}{c}\left(\ln |\ln x|-\ln \left|\ln x_{\text {end }}\right|\right)$,
from which it follows that
$x(N)=\exp \left[e^{c\left(N-N_{\text {end }}\right)} \ln x_{\text {end }}\right]$.

The slow-roll predictions of the four models, RMI1, RMI2, RMI3 and RMI4 are presented in Figs. 167-170 for $|c|=10^{-2}$, $\phi_{0} / M_{\mathrm{PI}}<1 / \sqrt{|c|}$, and $1 / e<x_{\text {end }}<e$, respectively. In order to interpret them, it is interesting to use some approximations. From the trajectory (6.13), it is straightforward to calculate $x_{*}$. Recalling that inflation is supposed to stop at $x_{\text {end }}$, one obtains $x_{*}=$


Fig. 150. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 4 inflation $\left[\alpha>0, \beta<0, x^{2}>-\alpha /(2 \beta)\right]$ models for $\beta=-10^{-4}$, in the plane $\left(n_{\mathrm{S}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
$\exp \left(e^{-c \Delta N_{*}} \ln x_{\text {end }}\right)$. Then, using Eqs. (6.8)-(6.10) in the vacuum dominated limit, we find that
$\epsilon_{1 *} \simeq \frac{c^{2}}{2}\left(\frac{\phi_{0}}{M_{\mathrm{PI}}}\right)^{2} \exp \left(2 \mathrm{e}^{-c \Delta N_{*}} \ln x_{\mathrm{end}}\right) e^{-2 c \Delta N_{*}} \ln ^{2} x_{\mathrm{end}}$,
$\epsilon_{2 *} \simeq 2 c\left(1+e^{-c \Delta N_{*}} \ln x_{\text {end }}\right)$.
In fact, in order to compare with the existing literature, it turns out to be convenient to define the following quantity
$s \equiv c \ln x_{*}=-c e^{-c \Delta N_{*}} \ln x_{\text {end }}$.
For RMI1 and RMI4, $s>0$ while for RMI2 and RMI3 one has $s<0$. In terms of $s$ Eqs. (6.14) and (6.15) can be re-written as
$\epsilon_{1 *} \simeq \frac{s^{2}}{2}\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2} e^{-2 s / c}, \quad \epsilon_{2 *} \simeq 2 c\left(1-\frac{s}{c}\right)$.
These equations imply that the locus of the model predictions in the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ are given by $\epsilon_{2} \simeq 2(c-s)+4 \epsilon_{1} M_{\mathrm{Pl}}^{2} / \phi_{\rho}^{2}$. If we neglect $\epsilon_{1 *}$ (with respect to $\epsilon_{2 *}$ ) one recovers the formula derived in Refs. [578,581,583], namely $n_{S}-1 \simeq 2(s-c)$. The same route for the third slow-roll parameter gives $\epsilon_{2} \epsilon_{3} \simeq-2 c s$ and neglecting again $\epsilon_{1}$ gives the scalar running $\alpha_{\mathrm{S}} \simeq 2 s c$. The above analytic estimates agree well with the complete slow-roll predictions represented in Figs. 167-170.

From the CMB normalization, we obtain the following expression for the mass scale
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=720 \pi^{2} c^{2} \frac{Q_{\mathrm{rms}-\mathrm{Ps}}^{2}}{T^{2}} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}} \frac{x_{*}^{2} \ln ^{2}\left(x_{*}\right)}{\left\{1-\frac{\mathrm{c}}{2} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}}\left[-\frac{1}{2}+\ln \left(x_{*}\right)\right] x_{*}^{2}\right\}^{3}}$.


Fig. 151. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 4 inflation $\left[\alpha>0, \beta<0, x^{2}>-\alpha /(2 \beta)\right]$ models for $\beta=-10^{-3}$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

In the vacuum dominated regime, this expression can be approximated by
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}} \simeq 720 \pi^{2} s^{2} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}} \frac{\phi_{0}^{2}}{M_{\mathrm{Pl}}^{2}} e^{s / c}$.
One can then easily deduce the mass scale $M$ for a given value of $c$, $\phi_{0}$ and $x_{\text {end }}$, the three parameters of the model.

### 6.2. Valley hybrid inflation (VHI)

### 6.2.1. Theoretical justifications

Hybrid inflation is a two-fields model with the potential given by the following expression [168,253,352,584-587]
$V(\phi, \psi)=\frac{1}{2} m^{2} \phi^{2}+\frac{\lambda^{\prime}}{4}\left(\psi^{2}-\Delta^{2}\right)^{2}+\frac{\lambda}{2} \phi^{2} \psi^{2}$,
where $\phi$ is the inflaton, $\psi$ the waterfall field, $\lambda$ and $\lambda^{\prime}$ are two coupling constants and $\Delta$ a constant of dimension one. A priori, given the above potential, inflation can occur in different regimes. However, the standard lore is that inflation can proceed along the valley given by $\psi=0$ and, in this case, the potential reduces to an effective single field potential that can be written as
$V(\phi)=M^{4}\left[1+\left(\frac{\phi}{\mu}\right)^{p}\right]$,


Fig. 152. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 5 inflation $\left[\alpha<0, \beta>0, x^{2}<-\alpha /(2 \beta)\right]$ models for $\beta=10^{-6}$, in the plane $\left(n_{\mathrm{S}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The parameter $\alpha$ is varied between $\left|\alpha_{\min }(\beta)\right|<|\alpha|<10\left|\alpha_{\text {min }}(\beta)\right|$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
with $p=2$ and where one has used the following parameter redefinition
$M=\frac{\lambda^{\prime 1 / 4} \Delta}{\sqrt{2}}, \quad \mu=\sqrt{\frac{\lambda^{\prime}}{2}} \frac{\Delta^{2}}{m}$.
Inflation along the valley has been shown to be a dynamical attractor of the two-field dynamics in Refs. [588,589]. However, as recently shown in Ref. [590], the hybrid potential can also support an inflationary phase along a mixed valley-waterfall trajectory, which is genuinely a two-fields dynamics. As we use a single field description here, those effects cannot be described by the potential of Eq. (6.21). For this reason, we will refer to the single field approximation as the "valley hybrid regime". Let us stress that, if the waterfall inflationary regime occurs, then it will erase any observable effects coming the valley hybrid regime. As a result, Eq. (6.21) is a good description of hybrid inflation only if the model parameters are such that the waterfall regime remains subdominant. According to Ref. [590,591], this is the case provided
$\sqrt{\lambda^{\prime}} \frac{\Delta^{3}}{m} \ll M_{\mathrm{Pl}}^{2}$,
a condition that will be assumed in the following. The effective potential (6.21) was also obtained in Ref. [592] in the context of supergravity brane inflation, and in Ref. [537] in the context


Fig. 153. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 5 inflation $\left[\alpha\langle 0, \beta\rangle 0, x^{2}<-\alpha /(2 \beta)\right]$ models for $\beta=10^{-5}$, in the plane $\left(n_{\mathrm{S}}, r\right)$ (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$.The parameter $\alpha$ is varied between $\left|\alpha_{\text {min }}(\beta)\right|<$ $|\alpha|<10\left|\alpha_{\min }(\beta)\right|$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
of hilltop supernatural inflation. It depends on three parameters, namely $M, \mu$ and $p$. In fact, as mentioned before, $p=2$ for the two-field model given in Eq. (6.20) but we will consider the most general situation with $p>0$ unspecified. Let us stress again that all multifield effects such as the generation of isocurvature modes or cosmic strings cannot be accounted within the single field dynamics [156,593-595].

It is also worth mentioning that the potential (6.21) with $p=2$ can also be obtained in the supergravity context [596-599]. The main idea is to consider a supergravity model which is not Rsymmetry invariant and described by the following Kähler and super-potentials:
$K=X X^{\dagger}+\frac{b}{6 M^{2}}\left(X X^{\dagger}\right)^{2}-\frac{c}{9 M^{2}} X X^{\dagger}\left[X^{2}+\left(X^{\dagger}\right)^{2}\right]$,
$W=f X$,
Here $X$ is a superfield, $M<M_{\mathrm{PI}}$ a mass scale and $b, c$ two dimensionless constants, a priori of order one. The quantity $f$ is a constant of dimension two that can be viewed as the supersymmetry breaking scale. From these expressions, the scalar potential reads
$V=f^{2}\left[1-\frac{2 b}{3 M^{2}} X X^{\dagger}+\frac{c}{3 M^{2}}\left(X^{2}+X^{\dagger 2}\right)+\mathcal{O}\left(\frac{1}{M^{4}}\right)\right]$,


Fig. 154. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 5 inflation $\left[\alpha\langle 0, \beta\rangle 0, x^{2}<-\alpha /(2 \beta)\right]$ models for $\beta=10^{-4}$, in the plane ( $n_{\mathrm{s}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$.The parameter $\alpha$ is varied between $\left|\alpha_{\min }(\beta)\right|<$ $|\alpha|<10\left|\alpha_{\min }(\beta)\right|$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
or, re-writing $X=\alpha+i \beta$, it reads
$V \simeq f^{2}\left[1+\frac{2}{3 M^{2}}(b-c) \alpha^{2}-\frac{2}{3 M^{2}}(b+c) \beta^{2}\right]$.
For a field evolution along the $\alpha$ direction, we recover a potential of the VHI type with $p=2$ ( $b-c$ must be positive). In this setup, $\alpha / M \ll 1$ is required in order for the field $\alpha$ to be approximately canonically normalized, the Kähler potential being not minimal. It is also interesting to comment on the $\eta$-problem in this model since this is a generic issue in supergravity. If one calculates the slow-roll parameter $\eta \equiv M_{\mathrm{Pl}}^{2} V_{\alpha \alpha} / V$, one finds that
$\eta=\frac{4 M_{\mathrm{Pl}}^{2}}{3 M^{2}}(b-c)$.
Therefore, one must take $M \lesssim M_{\mathrm{Pl}}$ and fine-tune the difference $b-c$ to a small number.

### 6.2.2. Slow-roll analysis

We now turn to the slow-roll analysis of the VHI scenario. Recall that we consider the following potential
$V(\phi)=M^{4}\left[1+\left(\frac{\phi}{\mu}\right)^{p}\right]$,


Fig. 155. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 6 inflation $\left[\alpha\langle 0, \beta\rangle 0, x^{2}>-\alpha /(2 \beta)\right]$ models for $\beta=10^{-5}$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The parameter $\alpha$ is varied between $\left|\alpha_{\min }(\beta)\right|<$ $|\alpha|<10^{4}\left|\alpha_{\min }(\beta)\right|$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
where the parameter $M$ and $\mu$ have been expressed in terms of the parameters of the two-field model in Eq. (6.22). The first three Hubble flow functions in the slow-roll approximation can be derived from Eq. (6.29) in a straightforward fashion. Defining the quantity $x$ by the following expression
$x \equiv \frac{\phi}{\mu}$,
they read
$\epsilon_{1}=\frac{p^{2}}{2}\left(\frac{M_{P 1}}{\mu}\right)^{2} \frac{x^{2 p-2}}{\left(1+x^{p}\right)^{2}}$,
$\epsilon_{2}=2 p\left(\frac{M_{\mathrm{PI}}}{\mu}\right)^{2} x^{p-2} \frac{x^{p}-p+1}{\left(1+x^{p}\right)^{2}}$,
and
$\epsilon_{3}=p\left(\frac{M_{\mathrm{PI}}}{\mu}\right)^{2} x^{p-2} \frac{2 x^{2 p}-(p-1)(p+4) x^{p}+(p-1)(p-2)}{\left(1+x^{p}\right)^{2}\left(x^{p}-p+1\right)}$.

A specific feature of hybrid inflation in comparison to large and small field models is that $\epsilon_{2}$ and $\epsilon_{3}$ can be negative (see Fig. 73).


Fig. 156. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 6 inflation $\left[\alpha\langle 0, \beta\rangle 0, x^{2}>-\alpha /(2 \beta)\right]$ models for $\beta=10^{-1}$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The parameter $\alpha$ is varied between $\left|\alpha_{\min }(\beta)\right|<$ $|\alpha|<10^{4}\left|\alpha_{\min }(\beta)\right|$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

In particular
$\epsilon_{2} \underset{x \rightarrow 0}{\simeq}-2 p(p-1)\left(\frac{M_{P 1}}{\mu}\right)^{2} x^{p-2}$,
and $\epsilon_{3}$ blows up in the limit $x^{p} \rightarrow p-1$. Together with the potential, the three Hubble flow functions have been represented in Fig. 73.

The slow-roll trajectory is obtained by integrating Eq. (2.11) with the valley hybrid potential and reads
$N-N_{\text {end }}=\frac{1}{2 p} \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\left[-x^{2}+x_{\mathrm{end}}^{2}+\frac{2}{2-p}\left(x_{\mathrm{end}}^{2-p}-x^{2-p}\right)\right]$,
which is, up to a sign, the same as for the SFI models [see Eq. (5.5)]. The case $p=2$ requires special attention, but as for SFI, is recovered as the limit $p \rightarrow 2$ in the previous equation. One obtains
$N-N_{\text {end }}=\frac{1}{4} \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}\left[-x^{2}+x_{\text {end }}^{2}-2 \ln \left(\frac{x}{x_{\text {end }}}\right)\right]$,
which is again very similar to SFI, up to a sign. The trajectory (6.34) cannot be inverted analytically in the general case. It is however possible to perform this inversion for many integer values of $p$, but those expressions will be omitted for the sake of clarity. We simply


Fig. 157. Reheating consistent slow-roll predictions for the spontaneous symmetry breaking 6 inflation $\left[\alpha\langle 0, \beta\rangle 0, x^{2}>-\alpha /(2 \beta)\right]$ models for $\beta=1$, in the plane $\left(n_{\mathrm{S}}, r\right)$ (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The parameter $\alpha$ is varied between $\left|\alpha_{\min }(\beta)\right|<$ $|\alpha|<10^{4}\left|\alpha_{\text {min }}(\beta)\right|$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
give an approximate solution valid only in the limit $x \ll 1$ and $p>2$
$x \simeq\left[x_{\mathrm{end}}^{2-p}+p(p-2) \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}}\left(N-N_{\mathrm{end}}\right)\right]^{1 /(2-p)}$.
If the waterfall inflation does not take place, i.e. under the condition (6.23), valley hybrid inflation ends by a tachyonic instability in the small field regime $x<1$, also referred to as "the vacuum dominated regime". From the two-fields potential (6.20), one sees that the transverse direction becomes tachyonic at the inflaton value
$\phi_{\text {end }}=\sqrt{\frac{\lambda^{\prime}}{\lambda}} \Delta$.
In the single field approach, $x_{\text {end }}$ is therefore an extra-parameter and VHI is a three parameters model according to our classification. However, as can be seen in Fig. 73, one should pay attention to the various domains in which inflation can take place. They are given by the behavior of $\epsilon_{1}(x)$.

If $p>1$, the slow-roll parameter $\epsilon_{1}$ vanishes when the field goes to zero and at infinity while it reaches a maximum for

$$
\begin{equation*}
x_{\epsilon_{1}^{\max }}=(p-1)^{1 / p}, \tag{6.38}
\end{equation*}
$$



Fig. 158. Reheating consistent slow-roll predictions for the IMI models in the plane $\left(n_{\mathrm{S}}, r\right)$ (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The parameter $x_{\text {end }}$ varies above $x_{\text {end }}^{\min }$ ( $\Delta N=65 e$-folds). It is not labeled since it is fully degenerate with the reheating temperature. The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid lines represent the locus of different IMI- $p$ models [for which $(1-2 / p) r=8\left(1-n_{S}\right)$, i.e. $\left.\epsilon_{1}=-(p / 4) \epsilon_{2}\right]$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
equals to
$\epsilon_{1}^{\max }=\frac{1}{2}\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{2}(p-1)^{\frac{2 p-2}{p}}$.
Defining
$\mu_{\epsilon} \equiv \frac{M_{\mathrm{Pl}}}{\sqrt{2}}(p-1)^{1-1 / p}$,
for all $\mu>\mu_{\epsilon}$, one has $\epsilon_{1}(x)<1$ and inflation can proceed all over the domain $x>0$. On the contrary, if $\mu<\mu_{\epsilon}$, then inflation can, a priori, proceed in two disconnected domains. Either $0<x<x_{\epsilon_{1}=1}^{-}$ or $x>x_{\epsilon_{1}=1}^{+}$where $x_{\epsilon_{1}=1}^{ \pm}$are the two roots of $\epsilon_{1}=1$, i.e. the solutions of
$x^{2 p}+2 x^{p}-\frac{p^{2}}{2}\left(\frac{M_{P 1}}{\mu}\right)^{2} x^{2 p-2}+1=0$.
This equation cannot be solved explicitly in the general case but, as for the trajectory, there are explicit analytic expressions for many


Fig. 159. Reheating consistent slow-roll predictions for the brane inflation models with $p=2$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid line represent the locus of the points such that $r=(8 / 3)\left(1-n_{S}\right)$, i.e. $\epsilon_{2}=4 \epsilon_{1}$, on which this model must lie for $\mu \gg M_{\mathrm{Pl}}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
integer values of $p$. For instance, for $p=2$, one gets
$x_{\epsilon_{1}=1}^{ \pm(p=2)}=\frac{1}{\sqrt{2}} \frac{M_{\mathrm{Pl}}}{\mu}\left(1 \pm \sqrt{1-2 \frac{\mu^{2}}{M_{\mathrm{Pl}}^{2}}}\right)$.
The positive sign corresponds to the largest root while the minus one to the smallest (see Fig. 73). In the limit $\mu \ll M_{\mathrm{Pl}}$, one has $x_{\epsilon_{1}=1}^{+} \simeq p M_{\mathrm{PI}} /(\sqrt{2} \mu)$ which is also the expression of $x_{\text {end }}$ for the large field model LFI (see Section 4.2). This does not come as a surprise since in that situation Eq. (6.29) is indeed dominated by the monomial term. In fact, the two above-mentioned domains precisely corresponds to a large field one for $x>x_{\epsilon_{1}=1}^{+}$and a vacuum dominated one for $x<x_{\epsilon_{1}=1}^{-}$. It is a common mistake to assume that the large field domain remains unobservable due to the existence of the vacuum dominated one. In fact, as shown in Ref. [588], the large field regime becomes observable provided $\mu \ll \mu_{\epsilon}$. In that situation, after having crossed $x_{\epsilon_{1}=1}^{+}$, the field fastrolls in the region $\epsilon_{1}(x)>1$. Then, it enters the domain $x<x_{\epsilon_{1}=1}^{-}$ with a strong initial velocity and, as a consequence, crosses the whole vacuum dominated region, still in fast-roll, to reach $x_{\text {end }}$. All observable predictions in such a situation are therefore similar to that obtained in the LFI models. Let us notice that, if there exists a mechanism that can gently put the field without a strong initial


Fig. 160. Reheating consistent slow-roll predictions for the brane inflation models with $p=3$ in the plane ( $n_{S}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid line represent the locus of the points such that $r=(8 / 3)\left(1-n_{\mathrm{S}}\right)$, i.e. $\epsilon_{2}=4 \epsilon_{1}$, on which this model must lie for $\mu \gg M_{\mathrm{Pl}}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
velocity inside the $x<x_{\epsilon_{1}=1}^{-}$domain, then inflation can still occur in the vacuum dominated region, even though $\mu<\mu_{\epsilon}$. But if the field is coming from the region $x>x_{\epsilon_{1}=1}^{+}$, then this regime does not exist anymore.

For $p=1, \epsilon_{1}(x)$ is a decreasing function of the field and takes a finite value $M_{\mathrm{Pl}}^{2} /\left(2 \mu^{2}\right)$ for $x \rightarrow 0$. The behavior is similar to the case $p>1$ and if $\mu>M_{\mathrm{Pl}} / \sqrt{2}$ inflation can take place all over $x>x_{\text {end }}$. However, if $\mu<M_{\mathrm{PI}} / \sqrt{2}$ then the vacuum dominated region does not exist anymore and $x_{\epsilon_{1}=1}=x_{\epsilon_{1}=1}^{+}=M_{\mathrm{PI}} /(\sqrt{2} \mu)-1$. One should also notice that if $p=1$ the relation $\epsilon_{2}=4 \epsilon_{1}$ applies.

Finally, for $p<1, \epsilon_{1}(x)$ is a decreasing function of the field but it blows up when $x \rightarrow 0$. In that situation, inflation stops at $x=\max \left(x_{\epsilon_{1}=1}^{-}, x_{\text {end }}\right)$ but the field will still fast-roll till the tachyonic instability develops at $x_{\text {end }}$. As a result, even if for some cases $x_{\epsilon_{1}=1}^{-}>x_{\text {end }}$, the observable predictions remain mostly the same.

According to the previous discussion, for $p>1$, the VHI effective potential is therefore adequate to describe the vacuum dominated regime only, i.e. for $x_{\text {end }}<x<x_{\epsilon_{1}=1}^{-}$where $x_{\text {end }}$ is the instability point given by Eq. (6.37). In that situation, solving Eq. (2.47) together with the trajectory (6.34) gives the observable field value $x_{*}$ at which the pivot mode crossed the Hubble radius during inflation. The potential parameter $M$ is fixed from the amplitude of


Fig. 161. Reheating consistent slow-roll predictions for the brane inflation models with $p=4$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid line represent the locus of the points such that $r=(8 / 3)\left(1-n_{\mathrm{S}}\right)$, i.e. $\epsilon_{2}=4 \epsilon_{1}$, on which this model must lie for $\mu \gg M_{\mathrm{Pl}}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
the CMB anisotropies
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=720 \pi^{2} p^{2} \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \frac{x_{*}^{2 p-2}}{\left(1+x_{*}^{p}\right)^{3}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
The reheating consistent slow-roll predictions are displayed in Figs. 171-175 for $p=0.5, p=1, p=1.5, p=2$ and $p=3$, respectively. For $p>1$ and $x_{\epsilon_{1}^{\max }}>1, x_{\text {end }}$ is varied between 0 and an upper bound such that $x_{\text {in }}<x_{\epsilon_{1}=1}^{-}$. One the other hand, if $x_{\epsilon_{1}}^{\max }<1$, then one simply takes $x_{\text {end }}<10$. For $p \leq 1, x_{\text {end }}$ is varied on a wider range, with no particular constraints. For $p=1$, the predictions lie on the line $\epsilon_{2}=4 \epsilon_{1}$ as expected whereas for $p>1$ one recovers a blue spectral index when $x_{\epsilon_{1}}^{\max }>1$, while a red spectral index can be obtained when $x_{\epsilon_{1}^{\max }}<1$ and $x_{*}>x_{\epsilon_{1}^{\max }}$, with $x_{*}<1$ (that is to say, the large field regime).

### 6.3. Dynamical supersymmetric inflation (DSI)

### 6.3.1. Theoretical justifications

This model has been studied in Refs. [439,600]. As for the IMI scenario, see Section 5.18, the model is based on Ref. [553] which has shown that inverse power law potentials naturally arise in supersymmetric theories. The fact that we have an inverse power


Fig. 162. Reheating consistent slow-roll predictions for the brane inflation models in the string framework ( $p=4, \mu \ll M_{\mathrm{Pl}}$, for the fundamental parameters displayed in the figures), in the plane ( $\left.n_{\mathrm{S}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The green points delimitate the prediction points such that inflation end by slow roll violation (for $\mu / M_{P l}>0.02$, above the green points) from the ones where inflation end by tachyonic instability (below the green points). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
law behavior, rather than the usual positive power law behavior, can be traced back to the presence of non-perturbative effects, such as for instance gaugino condensation, see Section 5.18. Based on the previous considerations, one can write that
$V=V_{0}+\frac{\Lambda_{3}^{p+4}}{\phi^{p}}+\frac{\phi^{q+4}}{M_{\mathrm{Pl}}^{q}}$,
where the last term encodes a correction to $V(\phi)$ due to a nonrenormalizable operator. It is Planck suppressed since $M_{\mathrm{Pl}}$ is the only explicit scale present in the theory. This term implies that there is a minimum located at
$\phi_{V \text { min }}=\left(\frac{p}{q+4} \Lambda_{3}^{p+4} M_{\mathrm{Pl}}^{q}\right)^{\frac{1}{p+q+4}}$.
This means that the extra term can be neglected in the region $\phi \ll \phi_{V^{\text {min }}}$ and, in the following, we assume that this is the case. The difference with the IMI scenario is the presence of the constant term $V_{0}$ which will be assumed to be dominant.


Fig. 163. Reheating consistent slow-roll predictions for the KKLT inflation models with $p=2$ in the plane ( $n_{\mathrm{s}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid line represent the locus of the points such that $r=(8 / 3)\left(1-n_{\mathrm{S}}\right)$, i.e. $\epsilon_{2}=4 \epsilon_{1}$, on which BI lies for $\mu \gg M_{\mathrm{PI}}$ and deviates from KKLTI. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

### 6.3.2. Slow-roll analysis

In this sub-section, we now turn to the slow-roll analysis of the DSI scenario. For this purpose, we rewrite the potential as
$V(\phi)=M^{4}\left[1+\left(\frac{\phi}{\mu}\right)^{-p}\right]$,
where $p$ is a free index parameter and where we defined
$V_{0}=M^{4}, \quad \mu^{p}=\frac{\Lambda_{3}^{p+4}}{M^{4}}$.
As already mentioned, in order for inflation to take place in the vacuum dominated regime, we must assume that $\phi \gg \mu$. In Refs. [439,600], it was argued that natural values for $\Lambda_{3}$ and $M$ are $10^{6} \mathrm{GeV}$ and $10^{10} \mathrm{GeV}$, respectively. This means that a scale of order $\mu \simeq 10^{6+14 / p} \mathrm{GeV}$ is a reasonable prior for $\mu$.

The potential (6.46), as well as its logarithm, is displayed in Fig. 74. It is a decreasing function of the field, hence inflation proceeds from the left to the right. Defining the quantity
$x \equiv \frac{\phi}{\mu}$,


Fig. 164. Reheating consistent slow-roll predictions for the KKLT inflation models with $p=3$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid line represent the locus of the points such that $r=(8 / 3)\left(1-n_{S}\right)$, i.e. $\epsilon_{2}=4 \epsilon_{1}$, on which BI lies for $\mu \gg M_{\mathrm{PI}}$ and deviates from KKLTI. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
the first three Hubble flow functions in the slow-roll approximation read
$\epsilon_{1}=\frac{p^{2}}{2}\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{2} \frac{x^{-2 p-2}}{\left(1+x^{-p}\right)^{2}}$,
$\epsilon_{2}=-2 p\left(\frac{M_{\mathrm{PI}}}{\mu}\right)^{2} x^{-p-2} \frac{x^{-p}+p+1}{\left(1+x^{-p}\right)^{2}}$,
and

$$
\begin{align*}
\epsilon_{3} & =-p\left(\frac{M_{\mathrm{PI}}}{\mu}\right)^{2} x^{-p-2} \\
& \times \frac{\left[2 x^{-2 p}+(p+1)(p-4) x^{-p}+(p+1)(p+2)\right]}{\left(1+x^{-p}\right)^{2}\left(x^{-p}+p+1\right)} . \tag{6.50}
\end{align*}
$$

Let us already notice that, from these expressions, one has

$$
-2 \epsilon_{1}-\epsilon_{2}
$$

$$
\begin{equation*}
=\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{2} \frac{p x^{-p-2}}{\left(1+x^{-p}\right)^{2}}\left[p x^{-p}+2 p(p+1) x^{-p-2}\right]>0 \tag{6.51}
\end{equation*}
$$

which implies a blue spectral index for the scalar power spectrum since, at first order, $n_{\mathrm{S}}-1=-2 \epsilon_{1 *}-\epsilon_{2 *}$. The three slowroll parameters become very small at large fields $x \gg 1$. There


Fig. 165. Reheating consistent slow-roll predictions for the KKLT inflation models with $p=4$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid line represent the locus of the points such that $r=(8 / 3)\left(1-n_{\mathrm{S}}\right)$, i.e. $\epsilon_{2}=4 \epsilon_{1}$, on which BI lies for $\mu \gg M_{\mathrm{PI}}$ and deviates from KKLTI. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
is a value $x_{\epsilon_{1}=1}$ such that $\epsilon_{1}=1$. For $x$ such that $x<x_{\epsilon_{1}=1}$, $\epsilon_{1}>1$ and inflation cannot take place. This value has to be determined numerically, but since the natural values for $\mu$ are such that $\mu / M_{\mathrm{Pl}} \ll 1$, an approximate expression can be derived
$x_{\epsilon_{1}=1} \simeq\left(\frac{p}{\sqrt{2}} \frac{M_{\mathrm{PI}}}{\mu}\right)^{1 /(p+1)}$.
Because the potential is decreasing with $x$, inflation can only take place in the domain $x>x_{\epsilon_{1}=1} \gg 1$ if $\mu \ll M_{\text {Pl }}$. It cannot stop by slow-roll violation and another mechanism such as, e.g. a tachyonic instability, has to be introduced. We will denote by $x_{\text {end }}$ the field value at which this occurs. It represents an extra parameter of the model. Obviously, it must be such that $x_{\epsilon_{1}=1}<x_{\text {end }} \ll x_{V \text { min }}$.

Let us now turn to the slow-roll trajectory. It can be integrated explicitly from Eq. (2.11) and one obtains
$N_{\text {end }}-N=\frac{\mu^{2}}{2 p M_{\mathrm{Pl}}^{2}}\left(x_{\text {end }}^{2}+\frac{2}{p+2} x_{\text {end }}^{p+2}-x^{2}-\frac{2}{p+2} x^{p+2}\right)$.
In the $\mu / M_{\mathrm{PI}} \ll 1$ limit, one has $x>x_{\epsilon_{1}=1} \gg 1$, and the previous trajectory can be approximated by
$N_{\text {end }}-N \simeq \frac{\mu^{2}}{p(p+2) M_{\mathrm{Pl}}^{2}}\left(x_{\text {end }}^{p+2}-x^{p+2}\right)$.


Fig. 166. Reheating consistent slow-roll predictions for the KKLT inflation models in the string framework ( $p=4, \mu \ll M_{\mathrm{Pl}}$, for the fundamental parameters displayed in the figures), in the plane ( $\left.n_{\mathrm{S}}, r\right)$ (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The green points delimit the prediction points such that inflation end by slow roll violation (for $\mu / M_{\mathrm{Pl}}>0.02$, above the green points) from the ones where inflation end by tachyonic instability (below the green points). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

This expression can be analytically inverted to get the observable field value $x_{*}$ in terms of $\Delta N_{*}=N_{\text {end }}-N_{*}$ as
$x_{*} \simeq\left[x_{\mathrm{end}}^{p+2}-\frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} p(p+2) \Delta N_{*}\right]^{\frac{1}{p+2}}$.
One can notice that the total amount of $e$-folds is bounded because $x_{\text {end }} \ll x_{V \text { min }}$ and cannot take infinitely large values. In order to get a number of $e$-folds, $\Delta N>\Delta N_{\min }, x_{\text {end }}$ should be sufficiently large with $x_{\text {end }}>x_{\text {end }}^{\min }$. More precisely, setting $x_{\text {ini }}=x_{\epsilon_{1}=1}$, one has

$$
\begin{align*}
x_{\mathrm{end}}^{\min } & \simeq\left[p(p+2) \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \Delta N_{\min }+\left(\frac{p}{\sqrt{2}} \frac{M_{\mathrm{Pl}}}{\mu}\right)^{\frac{p+2}{p+1}}\right]^{\frac{1}{p+2}} \\
& \simeq\left[p(p+2) \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} \Delta N_{\min }\right]^{\frac{1}{p+2}} . \tag{6.56}
\end{align*}
$$

In practice one wants $\Delta N_{\text {min }}>50$ to solve the problems of the standard Big-Bang scenario. Whether this value is compatible, or not, with the condition $x_{\text {end }} \ll x_{V \text { min }}$ depends on the value of $M^{4}$ appearing in Eq. (6.45), which is itself determined by the amplitude


Fig. 167. Reheating consistent slow-roll predictions for the running mass inflation 1 models ( $c>0, x<1$ ) with $c=0.01, \phi_{0} / M_{\mathrm{Pl}}<1 / \sqrt{c}, 1 / e<x_{\text {end }}<1$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The energy scale at which reheating ends and the field vev when inflation stops $x_{\text {end }}=\phi_{\text {end }} / \phi_{0}$ are degenerated, which is the reason why they are not displayed. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
of the CMB anisotropies. This one reads

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=720 \pi^{2} p^{2}\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{2} x_{*}^{-2 p-2}\left(1+x_{*}^{-p}\right)^{-3} \frac{\mathrm{Q}_{\mathrm{mls}-\mathrm{PS}}^{2}}{T^{2}} \tag{6.57}
\end{equation*}
$$

In the limit $\mu / M_{\mathrm{Pl}} \ll 1$, one has $x_{*} \gg 1$ and this expression can be approximated by
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}} \simeq 720 \pi^{2} p^{2} \frac{M_{\mathrm{Pl}}^{2}}{\mu^{2}} x_{*}^{-2 p-2} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
Therefore, from Eq. (6.45), one has
$x_{V^{\min }} \simeq\left[720 \pi^{2} \frac{p^{3}}{q+4}\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{6+q} x_{*}^{-2 p-2} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}\right]^{\frac{1}{p+q+4}}$,
with $x_{*}$ depending on $x_{\text {end }}$ through Eq. (6.55). One can see that the previous expression decreases with $x_{*}$ and the condition $x_{\text {end }} \ll$ $x_{V \text { min }}$ imposes an upper bound on $x_{\text {end }}<x_{\text {end }}^{\max }$ with
$x_{\mathrm{end}}^{\max } \simeq\left[720 \pi^{2} \frac{p^{3}}{q+4} \frac{Q_{\mathrm{mms}-\mathrm{PS}}^{2}}{T^{2}}\left(\frac{M_{\mathrm{Pl}}}{\mu}\right)^{q+6}\right]^{1 /(3 p+q+6)}$.
The prior condition on $x_{\text {end }}$ is therefore of the type $x_{\text {end }}^{\min }<x_{\text {end }} \ll$ $x_{\text {end }}^{\max }$, with $x_{\text {end }}^{\min }$ defined by Eq. (6.56) and $x_{\text {end }}^{\max }$ defined by Eq. (6.60).


Fig. 168. Reheating consistent slow-roll predictions for the running mass inflation 2 models ( $c>0, x>1$ ) with $c=0.01, \phi_{0} / M_{\mathrm{Pl}}<1 / \sqrt{c}, 1<x_{\text {end }}<e$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The energy scale at which reheating ends and the field vev when inflation stops $x_{\text {end }}$ are degenerated and not represented. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

For any $q>0$, these two equations show that there exists an upper bound $\mu<\mu_{\text {max }}$ under which the condition $x_{\text {end }}^{\min } \ll x_{\text {end }}^{\max }$ is satisfied. It reads
$\frac{\mu_{\max }}{M_{\mathrm{Pl}}} \simeq \frac{\left(720 \pi^{2} \frac{p^{3}}{q+4} \frac{\frac{Q}{\mathrm{rms}}_{\mathrm{T}^{2}}^{2}}{\mathrm{P}^{2}}\right)^{(p+2) /(p q)}}{\left[p(p+2) \Delta N_{\mathrm{min}}\right]^{(3 p+q+6) /(p q)}}$,
and has been represented in Fig. 75. One can see that a typical value $\mu / M_{\mathrm{PI}} \simeq 10^{10} \mathrm{GeV}$ (see Ref. [439]) is not allowed for realistic values of $p$ and $q$. As such, the prior space for $p, \mu$, and $x_{\text {end }}$ is constrained and should be handled carefully.

The reheating consistent slow-roll predictions of the dynamical supersymmetric models are displayed in Figs. 176-178 for $p=$ $2, p=3$ and $p=4$, respectively, and with $10^{-10} M_{\mathrm{Pl}}<$ $\mu<\mu_{\max }$ (where $\mu_{\max }$ has been calculated taking $q=8$ and $\Delta N_{\min }=60$ to cover a large prior space). The reheating equation of state parameter $\bar{w}_{\text {reh }}$ has been taken to 0 but since there is no potential minimum around which the inflaton field can oscillate at the end of inflation, this parameter is a priori unspecified and can take different values. In any case the reheating temperature is strongly degenerated with the parameter $x_{\text {end }}^{\min }<x_{\text {end }}<x_{\text {end }}^{\max }$ preventing their inference. One can check that the spectral index is blue, as announced earlier, making these models disfavored by


Fig. 169. Reheating consistent slow-roll predictions for the running mass inflation 3 models ( $c<0, x<1$ ) with $c=-0.01, \phi_{0} / M_{\mathrm{PI}}<1 / \sqrt{-c}, 1 / e<x_{\text {end }}<1$, in the plane ( $\left.n_{\mathrm{S}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The energy scale at which reheating ends and the field $v e v$ when inflation stops $x_{\text {end }}$ are degenerated and have not been represented. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
the observations. The typical amount of gravitational waves is very small, in agreement with the results of Ref. [439].

### 6.4. Generalized mixed inflation (GMLFI)

This model is a generalization of MLFI (see Section 4.3) and is, by definition, the sum of two monomial functions with arbitrary power indices. The corresponding potential can be written as
$V=M^{4}\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{p}\left[1+\alpha\left(\frac{\phi}{M_{\mathrm{Pl}}}\right)^{q}\right]$,
where $\alpha, p$ and $q$ are three dimensionless positive parameters. It can be seen as a generalization of the large field inflation potential (LFI, see Section 4.2), which is recovered when $\alpha \rightarrow 0$ or $\alpha \rightarrow \infty$. The parameter $\alpha$ therefore controls the relative weight of the two terms. Since the potential is an increasing function of the inflaton $v e v$, inflation proceeds from the right to the left and occurs in the large field regime $\phi / M_{\mathrm{PI}} \gg 1$. Defining the quantity $x$ by
$x \equiv \frac{\phi}{M_{\mathrm{PI}}}$,


Fig. 170. Reheating consistent slow-roll predictions for the running mass inflation 4 models ( $c<0, x>1$ ) with $c=-0.01, \phi_{0} / M_{\mathrm{Pl}}<1 / \sqrt{-c}, 1<x_{\text {end }}<e$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The energy scale at which reheating ends and the field $v e v x_{\text {end }}$ are degenerated and not displayed. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
the first three Hubble flow functions in the slow-roll approximation can be expressed as

$$
\begin{align*}
& \epsilon_{1}=\frac{1}{2 x^{2}}\left[\frac{p+\alpha(p+q) x^{q}}{1+\alpha x^{q}}\right]^{2},  \tag{6.64}\\
& \epsilon_{2}=\frac{2}{x^{2}} \frac{p+\alpha^{2}(p+q) x^{2 q}+\alpha\left(2 p+q-q^{2}\right) x^{q}}{\left(1+\alpha x^{q}\right)^{2}}, \tag{6.65}
\end{align*}
$$

and

$$
\begin{align*}
\epsilon_{3}= & \frac{1}{x^{2}\left(1+\alpha x^{q}\right)^{2}}\left[p q^{2}+\alpha^{2} q^{2}(p+q) x^{2 q}\right. \\
& \left.+\alpha q^{2}\left(2 p+q-q^{2}\right) x^{q}\right]^{-1}\left\{2 q^{2}\left[p^{2}+\alpha^{4}(p+q)^{2}\right] x^{4 q}\right. \\
& +\alpha^{2} q^{2}\left[12 p^{2}+6 p q(2-q)+(q-2)(q-1) q^{2}\right] x^{2 q} \\
& +\alpha^{3} q^{3}(p+q)\left[8 \frac{p}{q}+(1-q)(4+q)\right] x^{3 q} \\
& \left.+\alpha p q^{2}\left[8 p+q\left(4+q^{2}-3 q\right)\right] x^{q}\right\} . \tag{6.66}
\end{align*}
$$



Fig. 171. Reheating consistent slow-roll predictions for the valley hybrid inflation models with $p=0.5$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The color of the data points encodes the value of $\mu$, while different data blocks correspond to different values of $x_{\text {end }}$. Inside a given bock, the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

They are decreasing functions of the field, vanishing when $x \rightarrow \infty$ and diverging when $x \rightarrow 0$. Together with the potential and its logarithm, the Hubble flow functions are represented in Fig. 76.

In Fig. 76, one sees that inflation ends by slow-roll violation at $x=x_{\text {end }}$, the solution of the equation $\epsilon_{1}\left(x_{\text {end }}\right)=1$. From Eq. (6.64), one obtains
$\sqrt{2} \alpha x_{\text {end }}^{q+1}+\sqrt{2} x_{\text {end }}= \pm\left[p+\alpha(p+q) x_{\text {end }}^{q}\right]$.
One can check that, for $\alpha=0$, one recovers the LFI- $p$ result $x_{\text {end }}=$ $p / \sqrt{2}$ (see Section 4.2) and that, for $\alpha \rightarrow \infty$, one gets $x_{\text {end }}=$ $(p+q) / \sqrt{2}$, which correspond again to the LFI- $p+q$ solution. The above equation cannot be solved analytically for arbitrary values of $p, q$. This is possible only in some particular cases, namely $q=$ $0, q=1$ or $q=2$. For $q=0$, this is LFI whereas $q=2$ corresponds to MLFI, both solutions being given in Sections 4.2 and 4.3 , respectively. For $q=1$, one obtains

$$
\begin{align*}
x_{\text {end }}= & \frac{\sqrt{2}}{4}(p+1)-\frac{1}{2 \alpha} \\
& +\frac{\sqrt{4+4 \sqrt{2} \alpha(p-1)+2 \alpha^{2}(p+1)^{2}}}{4 \alpha} \tag{6.68}
\end{align*}
$$

but, in general, $x_{\text {end }}$ has to be determined numerically.


Fig. 172. Reheating consistent slow-roll predictions for the valley hybrid inflation models with $p=1$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The color of the data points encodes the value of $\mu$, while different data blocks correspond to different values of $x_{\text {end }}$. Inside a given bock, the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid line represent the locus of the points such that $\epsilon_{2}=4 \epsilon_{1}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The slow-roll trajectory can be integrated explicitly using Eq. (2.11) and this leads to

$$
\begin{align*}
& N_{\text {end }}-N=\frac{1}{2(p+q)} x^{2}\left\{1+\frac{q}{p_{2}} F_{1}\left[1, \frac{2}{q}, 1+\frac{2}{q},\right.\right. \\
& \left.\left.\quad-\alpha q\left(\frac{1}{p}+\frac{1}{q}\right) x^{q}\right]\right\}-\frac{1}{2(p+q)} x_{\mathrm{end}}^{2} \\
& \quad \times\left\{1+\frac{q}{p_{2}} F_{1}\left[1, \frac{2}{q}, 1+\frac{2}{q},-\alpha q\left(\frac{1}{p}+\frac{1}{q}\right) x_{\mathrm{end}}^{q}\right]\right\} . \tag{6.69}
\end{align*}
$$

Here, ${ }_{2} F_{1}$ stands for the Gauss hypergeometric function [216,217]. Since it is equal to unity when its last argument vanishes, one can check that, in the limit $\alpha \rightarrow 0$, one recovers the slow-roll trajectory for the LFI- $p$ models while the limit $\alpha \rightarrow \infty$ leads to the trajectory of the LFI- $(p+q)$ models. Finally, since ${ }_{2} F_{1}(1,1,2, x)=$ $-\ln (1-x) / x$, one can also check that the MLFI case corresponds to $p=q=2$. The previous expression can only be inverted for $q=0$ (LFI) and $q=2$ (MLFI), and they have been already discussed in Sections 4.2 and 4.3, respectively. The case $q=1$ can also be simplified using ${ }_{2} F_{1}(1,2,3, x)=-2 / x-2 \ln (1-x) / x^{2}$. In general, one has to inverse this slow-roll trajectory numerically.


Fig. 173. Reheating consistent slow-roll predictions for the valley hybrid inflation models with $p=1.5$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The color of the data points encodes the value of $\mu$, while different data blocks correspond to different values of $x_{\text {end }}$. Inside a given bock, the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The parameter $M$ can be determined from the amplitude of the CMB anisotropies and the Hubble crossing vev $x_{*}$. One obtains
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=720 \pi^{2} \frac{\left[p+\alpha(p+q) x_{*}^{q}\right]^{2}}{x_{*}^{p+2}\left(1+\alpha x_{*}^{q}\right)^{3}} \frac{Q_{\text {rms-PS }}^{2}}{T^{2}}$.
The reheating consistent slow-roll predictions for the generalized mixed large field models are displayed in Figs. 179-181 for $(p=2$ and $q=1),(p=2$ and $q=3)$ and $(p=3$ and $q=2)$, respectively. As for MLFI, the predictions lie between the LFI-p and LFI- $(p+q)$ models, but can actually exit this region for large enough values of $\alpha$. This means that, if one starts from a pure $V \propto \phi^{p+q}$ potential and adds a small $\propto \phi^{p}$ term, then this extra term has the effect of increasing the "effective value" of the power index of the potential. Moreover, since for large field inflation models, the $p$-model fits the data better than the $(p+q)$-one, it follows that small values for the parameter $\alpha$ are favored, together with high reheating temperatures.

### 6.5. Logarithmic potential inflation (LPI)

### 6.5.1. Theoretical justifications

This class of model assumes that inflation is driven by a composite state in a strongly interacting theory, see Refs. [518,601,


Fig. 174. Reheating consistent slow-roll predictions for the valley hybrid inflation models with $p=2$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The color of the data points encodes the value of $\mu$, while different data blocks correspond to different values of $x_{\text {end }}$. Inside a given bock, the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

602]. Let us consider the following model, see Section 5.14 for more details
$\mathcal{L}_{\mathrm{GI}}=-\varphi^{-3 / 2} \partial_{\mu} \varphi \partial^{\mu} \varphi-\frac{\varphi}{2} \ln \left(\frac{\varphi}{\Lambda^{4}}\right)$,
where $\Lambda$ is a mass scale. Moreover, let us consider the situation where the model has a general non-minimal coupling to gravity of the form
$S=\int \mathrm{d}^{4} \boldsymbol{x} \sqrt{-g}\left[-\frac{1}{2}\left(M^{2}+\xi \varphi^{1 / 2}\right) R+\mathscr{L}_{\mathrm{GI}}\right]$.
The coupling to gravity is characterized by the parameter $\xi$. Then, the action in the Einstein frame reads [518,601,602]

$$
\begin{align*}
S= & \int \mathrm{d}^{4} \boldsymbol{x} \sqrt{-g}\left[-\frac{1}{2} M_{\mathrm{Pl}}^{2} R-\Omega^{-2}\left(1+\frac{3 \xi^{2} \varphi^{1 / 2}}{4 M_{\mathrm{Pl}}^{2}} \Omega^{-2}\right)\right. \\
& \left.\times \varphi^{-3 / 2} \partial_{\mu} \varphi \partial^{\mu} \varphi-\Omega^{-4} V_{\mathrm{GI}}\right], \tag{6.73}
\end{align*}
$$

where $V_{\mathrm{GI}}$ refers to the potential in Eq. (6.71) and $\Omega^{2}=$ $\left(M^{2}+\xi \varphi^{1 / 2}\right) / M_{\mathrm{Pl}}^{2}$. If $\xi \neq 0$ and if we are in the large field limit, then $\Omega^{2} \simeq \xi \varphi^{1 / 2} / M_{\mathrm{Pl}}^{2}$ and the canonically normalized field $\phi$ is such that $\phi \propto \ln \varphi$. In that case the potential reduces to $\Omega^{-4} V_{\mathrm{GI}} \propto$ $\ln \varphi \propto \phi$. Therefore, we have obtained a LFI model with $p=1$,


Fig. 175. Reheating consistent slow-roll predictions for the valley hybrid inflation models with $p=3$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane ( $\epsilon_{1}, \epsilon_{2}$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The color of the data points encodes the value of $\mu$, while different data blocks correspond to different values of $x_{\text {end }}$. Inside a given bock, the annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right.$ ). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
see Section 4.2. On the other hand, if one assumes that $\xi=0$, then $\varphi=\phi^{4} /(4 \sqrt{2})^{4}$ and
$V=2 \Lambda^{4}\left(\frac{\phi}{\phi_{0}}\right)^{4} \ln \left(\frac{\phi}{\phi_{0}}\right)$,
with $\phi_{0} \equiv 4 \sqrt{2} \Lambda$. This resembles the potential found in Section 5.14 which, for $\beta=0$ (see the precise definition in that section), was such that $V \propto \phi^{4} \ln ^{2}\left(\phi / \phi_{0}\right)$. These considerations motivate the next section devoted to the slow-roll analysis of this class of scenarios.

### 6.5.2. Slow-roll analysis

Based on the previous discussion, we now turn to the slow-roll analysis of the models described by the following potential
$V(\phi)=M^{4}\left(\frac{\phi}{\phi_{0}}\right)^{p}\left(\ln \frac{\phi}{\phi_{0}}\right)^{q}$.
We have just seen that, for $p=4$ and $q=2$, the model discussed in Ref. [518] is recovered, see Section 5.14, while for $p=4$ and $q=1$, this model matches with the so-called Glueball Inflation of Ref. [601]. This class of models has also been studied on general grounds in Ref. [603]. In the following, we keep $p$ and $q$ unspecified.


Fig. 176. Reheating consistent slow-roll predictions for the dynamical supersymmetric inflation models with $p=2,10^{-10}<\mu / M_{\mathrm{Pl}}<\mu_{\max } / M_{\mathrm{Pl}}$, and $x_{\text {end }}^{\min }<$ $x_{\text {end }}<x_{\text {end }}^{\max }$ in the plane ( $\left.n_{s}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The parameter $x_{\text {end }}$ increases along the direction specified by the arrows, and is degenerate with the energy scale at which reheating ends. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Defining the quantity $x$ by the following relation
$x \equiv \frac{\phi}{\phi_{0}}$,
the potential has a local maximum at $x=x_{V^{\max }}$ and a local minimum (at which the potential vanishes) at $x=x_{V=0}$ with
$x_{V \max }=e^{-q / p}, \quad x_{V=0}=1$.
For $x>x_{V=0}, V(x)$ increases and finally diverge when $x$ goes to infinity. The potential is always definite positive in the $x>1$ branch, whereas it is definite positive in the $x<1$ branch only if $q$ is an even integer. The first three Hubble flow functions in the slow-roll approximation are given by
$\epsilon_{1}=\frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}} \frac{(q+p \ln x)^{2}}{2 x^{2} \ln ^{2} x}, \quad \epsilon_{2}=2 \frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}} \frac{q+q \ln x+p \ln ^{2} x}{x^{2} \ln ^{2} x}$,
and
$\epsilon_{3}=\frac{M_{\mathrm{Pl}}^{2}}{\phi_{0}^{2}}(q+p \ln x) \frac{2 q+3 q \ln x+2 q \ln ^{2} x+2 p \ln ^{3} x}{x^{2} \ln ^{2} x\left(q+q \ln x+p \ln ^{2} x\right)}$.
Together with the potential, they are displayed in Fig. 77.
As can be checked on this figure, and assuming $q$ is even, the behavior of $\epsilon_{1}(x)$ exhibits three domains in which inflation can


Fig. 177. Reheating consistent slow-roll predictions for the dynamical supersymmetric inflation models with $p=3,10^{-10}<\mu / M_{\mathrm{Pl}}<\mu_{\max } / M_{\mathrm{Pl}}$, and $x_{\mathrm{end}}^{\min }<$ $x_{\text {end }}<x_{\text {end }}^{\max }$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The parameter $x_{\text {end }}$ increases along the direction specified by the arrows, and is degenerated with the energy scale at which reheating ends. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
occur and can naturally end. Either $x>1$ and inflation proceeds from the right to the left (LPI1), or $x_{V}{ }^{\max }<x<1$ and inflation proceeds from the left to the right (LPI2), or $0<x<x_{V^{\max }}$ and inflation proceeds from the right to the left (LPI3), see the three arrows in Fig. 77. For these three cases, the slow-roll trajectory can be integrated analytically and one has

$$
\begin{align*}
N & -N_{\mathrm{end}}=\left(\frac{\phi_{0}}{M_{\mathrm{Pl}}}\right)^{2}\left\{-\frac{x^{2}-x_{\mathrm{end}}^{2}}{2 p}+\frac{q}{p^{2}} e^{-2 q / p}\right. \\
& \left.\times\left[\operatorname{Ei}\left(\frac{2 q}{p}+2 \ln x\right)-\operatorname{Ei}\left(\frac{2 q}{p}+2 \ln x_{\mathrm{end}}\right)\right]\right\} . \tag{6.80}
\end{align*}
$$

Let us remark that for $x \rightarrow+\infty$ (LPI1), one recovers the large field inflation (LFI) trajectory of Section 4.2 with $p$ becoming the same parameter of LFI.

In the three above described regimes, inflation ends at the field value $x_{\text {end }}$ solution of $\epsilon_{1}\left(x_{\text {end }}\right)=1$, i.e. verifying
$p \ln \left(x_{\text {end }}\right)+q \mp \sqrt{2} \frac{\phi_{0}}{M_{\mathrm{Pl}}} x_{\text {end }} \ln x_{\text {end }}=0$.
This is a transcendental equation that cannot be solved analytically for any values of $p$ and $q$. It can nevertheless be solved numerically in each of the three above-mentioned situations. Together with Eq. (2.47), Eq. (6.80) uniquely determines the observable field value


Fig. 178. Reheating consistent slow-roll predictions for the dynamical supersymmetric inflation models with $p=4,10^{-10}<\mu / M_{\mathrm{PI}}<\mu_{\max } / M_{\mathrm{PI}}$, and the prior $x_{\text {end }}^{\min }<x_{\text {end }}<x_{\text {end }}^{\max }$ in the plane $\left(n_{\mathrm{S}}, r\right)$ (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The parameter $x_{\text {end }}$ increases along the direction specified by the arrows and is degenerated with the energy scale at which reheating ends. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
$x_{*}$ at which the pivot scale crossed out the Hubble radius during inflation. Therefore, according to our classification, LPI is a three parameters model with $p, q$ and $\phi_{0}$.

Finally, the parameter $M$ is fixed by the amplitude of the CMB anisotropies to
$\frac{M^{4}}{M_{\mathrm{Pl}}^{4}}=720 \pi^{2}\left(\frac{M_{\mathrm{Pl}}}{\phi_{0}}\right)^{2} \frac{\left(q+p \ln x_{*}\right)^{2}}{x_{*}^{2+p} \ln ^{2+q} x_{*}} \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}$.
The reheating consistent slow-roll predictions for the LPI1 models with $p=4$ are represented in Figs. $182-184$ for $q=2, q=1$ and $q=3$, respectively. The predictions for LPI2 are displayed in Figs. 185-187 for $(p=1, q=2),(p=2, q=2)$ and $(p=3$, $q=4$ ), respectively. For the LPI3 scenario, the predictions have been plotted in Figs. 188-190 for ( $p=1, q=2$ ), $(p=2, q=2)$ and ( $p=3, q=4$ ), respectively. One can see that the current CMB data generically require LPI inflation to take place with superPlanckian values for $\phi_{0}$ while some combinations of $p$ and $q$ are already disfavored at more than two-sigma.

### 6.6. Constant $n_{S} D$ inflation (CNDI)

This model has been studied in Ref. [516]. Its potential is designed to produce a power law power spectrum $\propto k^{n}$ (where $n$ is a constant). In this sense, the approach followed here is similar to


Fig. 179. Reheating consistent slow-roll predictions for the generalized mixed inflation models with $p=2$ and $q=1$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid lines represent the locus of the LFI- $p$ and LFI$(p+q)$ models (for which $\epsilon_{2}=(4 / p) \epsilon_{1}$ and $\epsilon_{2}=4 \epsilon_{1} /(p+q)$ respectively). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
the one investigated in Sections 4.20, 4.21 and 5.15. The potential studied in this section is given by
$V(\phi)=\frac{M^{4}}{\left\{1+\beta \cos \left[\alpha\left(\frac{\phi-\phi_{0}}{M_{\mathrm{Pl}}}\right)\right]\right\}^{2}}$,
where $\alpha$ and $\beta$ are two dimensionless parameters. Since the potential is an even function of $x \equiv\left(\phi-\phi_{0}\right) / M_{\mathrm{PI}}$ and is $2 \pi$ periodic, it can be studied without loss of generality in the range $x \in[0, \pi / \alpha]$ only (with $\alpha>0, \beta>0$ ). The potential and its logarithm are displayed in Fig. 78 (top panels) for two different representative values of $\beta$. If $\beta<1$ (blue curve), it is an increasing function of the field, hence inflation proceeds from the right to the left. On the contrary, if $\beta \geq 1$ (pink curve), it diverges at $x_{V \rightarrow \infty}=$ $\arccos (-1 / \beta) / \alpha$. Then, for $x<x_{V \rightarrow \infty}$ it is an increasing function of $x$ and inflation proceeds from the right to the left, whereas for $x>x_{V \rightarrow \infty}$ it is an decreasing function of $x$ and inflation proceeds from the left to the right.

The three first slow-roll parameters are given by the following expressions
$\epsilon_{1}=\frac{2 \alpha^{2} \beta^{2} \sin ^{2}(\alpha x)}{[1+\beta \cos (\alpha x)]^{2}}, \quad \epsilon_{2}=\frac{-4 \alpha^{2} \beta[\beta+\cos (\alpha x)]}{[1+\beta \cos (\alpha x)]^{2}}$,


Fig. 180. Reheating consistent slow-roll predictions for the generalized mixed inflation models with $p=2$ and $q=3$, in the plane ( $n_{\mathrm{s}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid lines represent the locus of the LFI- $p$ and LFI$(p+q)$ models (for which $\epsilon_{2}=(4 / p) \epsilon_{1}$ and $\epsilon_{2}=4 \epsilon_{1} /(p+q)$ respectively). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
and
$\epsilon_{3}=\frac{-2 \alpha^{2} \beta\left[2 \beta^{2}-1+\beta \cos (\alpha x)\right] \sin ^{2}(\alpha x)}{[\beta+\cos (\alpha x)][1+\beta \cos (\alpha x)]^{2}}$.
They are displayed in Fig. 78 (bottom panels). Let us now study in more detail the behavior of $\epsilon_{1}$ and $\epsilon_{2}$. It depends on whether $\beta$ is larger or smaller than 1 . If $\beta<1$, the first slow-roll parameter $\epsilon_{1}$ vanishes at $x=0$ and $x=\pi / \alpha$, and reaches a maximum in between at $x_{\epsilon_{2}=0}$. This maximum is larger than one provided $\alpha>\alpha_{\text {min }}(\beta)$, where
$\alpha_{\text {min }}(\beta)=\sqrt{\frac{1-\beta^{2}}{2 \beta^{2}}}$.
In that case, inflation can stop by slow-roll violation, at the position $x_{\text {end }}$ given by
$x_{\text {end }}=x_{\epsilon_{1}=1}^{+}=\frac{1}{\alpha} \arccos \left[\frac{\alpha \sqrt{2 \beta^{2}\left(1+2 \alpha^{2}\right)-2}-1}{\beta+2 \alpha^{2} \beta}\right]$,
and proceeds in the range $\left[x_{\text {end }}, \pi / \alpha\right]$ (from the right to the left). On the other hand, the second slow-roll parameter $\epsilon_{2}$ is a monotonous


Fig. 181. Reheating consistent slow-roll predictions for the generalized mixed inflation models with $p=3$ and $q=2$, in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. The black solid lines represent the locus of the LFI- $p$ and LFI$(p+q)$ models (for which $\epsilon_{2}=(4 / p) \epsilon_{1}$ and $\epsilon_{2}=4 \epsilon_{1} /(p+q)$ respectively). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
increasing function of $x$, which vanishes at $x_{\epsilon_{2}=0}=\arccos (-\beta) / \alpha$. If $\beta \geq 1$, as can be seen in Fig. 78, the first slow-roll parameter $\epsilon_{1}$ diverges at $x_{V \rightarrow \infty}=\arccos (-1 / \beta) / \alpha$, so that inflation cannot stop by slow-roll violation in that case. This means that inflation must end by another mechanism and, therefore, that the model depends on an additional parameter. The second slow-roll parameter $\epsilon_{2}$ is always negative and also diverges at $x_{V \rightarrow \infty}$. Let us notice that, for $\beta<1$ and $\alpha>\alpha_{\text {min }}(\beta)$, and for $\beta>1$ (for any $\alpha$ ), we will need below the other solution of $\epsilon_{1}=1$, namely
$x_{\epsilon_{1}=1}^{-}=\frac{1}{\alpha} \arccos \left[-\frac{\alpha \sqrt{2 \beta^{2}\left(1+2 \alpha^{2}\right)-2}+1}{\beta+2 \alpha^{2} \beta}\right]$.
We are now in a position where the slow-roll trajectory can be determined. It turns out that this one can be integrated analytically and reads

$$
\begin{align*}
N-N_{\text {end }}= & \frac{1}{2 \alpha^{2}}\left\{-\ln [\sin (\alpha x)]-\frac{1}{\beta} \ln \left[\tan \left(\alpha \frac{x}{2}\right)\right]\right. \\
& \left.+\ln \left[\sin \left(\alpha x_{\text {end }}\right)\right]+\frac{1}{\beta} \ln \left[\tan \left(\alpha \frac{x_{\text {end }}}{2}\right)\right]\right\} \tag{6.89}
\end{align*}
$$

Because of the logarithmic functions, a sufficient number of $e$-folds can be realized only if the initial conditions are fine-tuned and $x_{\text {ini }}$ is chosen to be extremely close to $\pi / \alpha$.


Fig. 182. Reheating consistent slow-roll predictions for the logarithmic potential inflation 1 models for $p=4$ and $q=2$ in the plane ( $n_{\mathrm{s}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Indeed, inserting Eq. (6.87) into Eq. (6.89), the total number of $e$-folds during inflation becomes a function of $x_{\text {ini }}$ and of the two parameters $\alpha$ and $\beta$. For given values of those parameters, one can check that $\left(N_{\text {end }}-N_{\text {ini }}\right)\left(x_{\text {ini }}\right)$ remains always small compared to unity, unless $x_{\text {ini }} \rightarrow \pi / \alpha$ where it blows up. Let us write $x_{\text {ini }}$ as $\pi / \alpha+\delta x_{\text {ini }}$ with $\delta x_{\text {ini }} \ll 1$ and defining $A \equiv \ln \left[\sin \left(\alpha x_{\text {end }}\right)\right]+$ $\ln \left[\tan \left(\alpha x_{\text {end }} / 2\right)\right] / \beta$, one arrives at
$\delta x_{\mathrm{ini}} \simeq\left[\alpha\left(\frac{\alpha}{2}\right)^{-1 / \beta} e^{-A}\right]^{\beta /(1-\beta)} e^{-2 \alpha^{2} \beta\left(N_{\mathrm{end}}-N_{\mathrm{ini}} /(1-\beta)\right.}$.
The coefficient between the squared brackets only depends on $\alpha$ and $\beta$ which are, a priori, coefficients of order one. On the other hand, the argument of the exponential is $2\left(N_{\text {end }}-N_{\text {ini }}\right)>120$, times a negative term of order one. This means that $\delta x_{\text {ini }}$ must be exponentially small to obtain a significant number of $e$-folds and one can question the physical relevance of such a fine-tuning. The typical predictions of the model (taking $x_{*} \simeq \pi / \alpha$ ) actually are $\epsilon_{1} \simeq 0, \epsilon_{2} \simeq 4 \alpha^{2} \beta /(1-\beta)$, and $\epsilon_{3} \simeq 0$. It follows that the condition $\alpha>\alpha_{\min }(\beta)$ implies $\epsilon_{2}>2(1+\beta) / \beta>4$, which is is completely ruled out by the observations. Therefore, we conclude that the case $\beta<1$ is not of cosmological interest.

The only remaining possibility is $\beta>1$. Inflation cannot end by slow-roll violation and $x_{\text {end }}$ is an additional parameter, making the model a three parameters one. In the range $\alpha x_{\text {end }} \ll 1$, one has $\epsilon_{1} \ll 1$ and $\epsilon_{2} \simeq-4 \alpha^{2} \beta /(1+\beta)$ such that the spectral index is given by $n_{\mathrm{S}} \simeq 1+4 \alpha^{2} \beta /(\beta+1)$. Therefore, it is indeed a constant.


Fig. 183. Reheating consistent slow-roll predictions for the logarithmic potential inflation 1 models for $p=4$ and $q=1$ in the plane ( $n_{\mathrm{s}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The CMB normalization gives the mass scale $M$ as

$$
\begin{equation*}
\left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4}=2880 \alpha^{2} \beta^{2} \pi^{2} \sin ^{2}\left(\alpha x_{*}\right) \frac{Q_{\mathrm{rms}-\mathrm{PS}}^{2}}{T^{2}}, \tag{6.91}
\end{equation*}
$$

which has to be numerically evaluated when if $\alpha x_{*}$ is not small. The predictions of CNDI inflation are displayed in Figs. 191 and 192. We see that, in the regime $\alpha x_{\text {end }} \ll 1$, the spectral index is constant, as expected. However, this occurs in a regime where the predictions are not consistent with the observations (the spectrum is too blue). On the other hand, when $\alpha x_{\text {end }}$ is no longer small, we observe strong deviations from $n_{\mathrm{S}} \simeq 1+4 \alpha^{2} \beta /(\beta+1)$ but, for intermediate values of $\alpha \simeq 0.3$, this renders the predictions compatible with the data. Obviously, these considerations bear some resemblance with the findings of Sections 4.20, 4.21 and 5.15.

## 7. Conclusions

Let us very briefly recap our main findings and present some directions for future works.

In this article, we have discussed the question of how the inflationary theory can be constrained given that we now have at our disposal high accuracy cosmological data. We have argued that this can be done by means of the slow-roll approximation which has the advantage of being relatively model independent. Although this approximation cannot be used if one has to deal with more complicated models, it produces interesting but limited information on inflation. Concretely, it leads to the Hubble flow posterior distributions $P\left(\epsilon_{n} \mid C_{\ell}^{\text {meas }}\right)$. This is interesting since it gives


Fig. 184. Reheating consistent slow-roll predictions for the logarithmic potential inflation 1 models for $p=4$ and $q=3$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
a general constraint on the derivatives of the inflaton potential. But, at the same time, this does not answer some legitimate fundamental questions one might have about the plethora of inflationary scenarios studied so far. For instance, it does not tell us rigorously which constraints exist on the parameters of a given model. Indeed, suppose that we are interested in LFI, $V(\phi) \propto \phi^{p}$. It is obvious that we would like to know for which values of $p$ this class of models is compatible with the data and for which values it is not.

In order to complement the slow-roll approximation and to address the above mentioned issues, we have argued that it is interesting to scan the inflationary landscape model by model and have provided the public code ASPIC to do so. Such a strategy has to be done for all the inflationary scenarios since it would be arbitrary to consider only a restricted class while ignoring the others. In fact, this question deserves to be discussed in more detail. One could indeed imagine that it is not necessary to consider all the models one by one and that considering a representative for each class is sufficient. Indeed, to simplify the discussion, it is common to distinguish three broad types of scenarios: large field models (LFI), small field models (SFI) and Hybrid models (VHI). Such a classification is not very precise and biased because it pushes to the front line these three models. It could be reasonably argued that a better classification is the one of Schwarz and Terrero-Escalante introduced in Ref. [444]. For a scalar field, the ratio of the kinetic energy to the total energy density is given by $\epsilon_{1} / 3=\dot{\phi}^{2} /(2 \rho)$. Because $\epsilon_{2}$ is, by definition, the logarithmic derivative of $\epsilon_{1}$ with respect to the $e$-fold number, the kinetic contribution to the total energy density increases if $\epsilon_{2}>0$ and decreases if $\epsilon_{2}<0$. On the


Fig. 185. Reheating consistent slow-roll predictions for the logarithmic potential inflation 2 models for $p=4$ and $q=2$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
other hand, we also have
$\frac{\mathrm{d}\left(\dot{\phi}^{2} / 2\right)}{\mathrm{d} t}=H \frac{\dot{\phi}^{2}}{2}\left(\epsilon_{2}-2 \epsilon_{1}\right)$,
and, therefore, the absolute value of the kinetic energy increases if $\epsilon_{2}>2 \epsilon_{1}$ whereas it decreases if $\epsilon_{2}<2 \epsilon_{1}$. This allows us to identify three different regions: $\epsilon_{2}>0$ and $2 \epsilon_{1}<\epsilon_{2}$ (region 1 ), $\epsilon_{2}<2 \epsilon_{1}$ (region 2 ), $\epsilon_{2}<0<2 \epsilon_{1}$ (region 3).

These three regions are identified in Fig. 79 together with Planck and WMAP9 bounds ${ }^{9}$. If we use the first order slow-roll expressions, the condition $\epsilon_{2}>0$ is equivalent to $r<8\left(1-n_{\mathrm{S}}\right)$ while $\epsilon_{2}>2 \epsilon_{1}$ amounts to $r<4\left(1-n_{\mathrm{S}}\right)$. These two lines are also represented in Fig. 79 (solid black lines). We have also superimposed the predictions of LFI, SFI and VHI (upper panel). We see that the three regions defined above roughly correspond to the cases large field, small field and hybrid. However, the correspondence is not perfect and we notice, for instance, that the predictions of VHI can penetrate region 2.

Having identified three broad classes of scenarios, the question is whether testing only a representative model for each class could be sufficient. In Fig. 80, we have considered the predictions of six different models that all belong to region 1 . This plot clearly shows that inside this region, these six models span different domains

[^8]

Fig. 186. Reheating consistent slow-roll predictions for the logarithmic potential inflation 2 models for $p=4$ and $q=1$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
that are separated enough to be distinguishable within current and future data. Given the quality of the current data, working only with broad classes of models seems to be no longer justified. Therefore, if one really wants to scan the inflationary landscape, the approach advocated in this paper is well-suited.

With ASPIC, we have provided a new tool to treat any model of inflation and this has led us to derive observational predictions for 74 models. ASPIC is an evolutive project and therefore the next steps will be to complete and upgrade it with new models. Finally, the ultimate goal is to identify which ASPIC model is performing the best for explaining cosmological data. In order to carry out this task, an appropriate method is to use Bayesian evidence and model comparison. Then, we should be able to identify, in a statistically well-defined manner, what might be called "the best model of inflation" [178-180].

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## Appendix. Reheating consistent slow-roll predictions

## A.1. Higgs Inflation (HI)

See Fig. 81.


Fig. 187. Reheating consistent slow-roll predictions for the logarithmic potential inflation 2 models for $p=4$ and $q=3$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

## A.2. Radiatively corrected higgs inflation (RCHI)

See Fig. 82.

## A.3. Large field inflation (LFI)

See Fig. 83.

## A.4. Mixed large field inflation (MLFI)

See Fig. 84.

## A.5. Radiatively corrected massive inflation (RCMI)

See Fig. 85.

## A.6. Radiatively corrected quartic inflation (RCQI)

See Figs. 86 and 87.

## A.7. Natural inflation (NI)

See Fig. 88.

## A.8. Exponential SUSY inflation (ESI)

See Figs. 89 and 90.


Fig. 188. Reheating consistent slow-roll predictions for the logarithmic potential inflation 3 models for $p=4$ and $q=2$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

## A.9. Power law inflation (PLI)

See Fig. 91.

## A.10. Kähler moduli inflation I (KMII)

See Fig. 92.

## A.11. Horizon Flow Inflation at first order (HF1I)

See Fig. 92.

## A.12. Colemann-Weinberg inflation (CWI)

See Figs. 94 and 95.

## A.13. Loop inflation (LI)

See Figs. 96 and 96.
A.14. $R+R^{2 p}$ inflation (RpI)

See Figs. 98-100.
A.15. Double well inflation (DWI)

See Fig. 101.


Fig. 189. Reheating consistent slow-roll predictions for the logarithmic potential inflation 3 models for $p=4$ and $q=1$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

## A.16. Mutated hilltop inflation (MHI)

See Fig. 102.

## A.17. Radion gauge inflation (RGI)

See Fig. 103.

## A.18. MSSM inflation (MSSMI)

See Fig. 104.

## A.19. Renormalizable inflection point inflation (RIPI)

See Fig. 105.

## A.20. Arctan inflation (AI)

See Fig. 106.

## A.21. Constant $n_{\mathrm{S}} A$ inflation (CNAI)

See Fig. 107.

## A.22. Constant $n_{S} B$ inflation (CNBI)

See Fig. 108.


Fig. 190. Reheating consistent slow-roll predictions for the logarithmic potential inflation 3 models for $p=4$ and $q=3$ in the plane ( $n_{\mathrm{s}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The annotations trace the energy scale at which reheating ends and correspond to $\log \left(g_{*}^{1 / 4} T_{\text {reh }} / \mathrm{GeV}\right)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

## A.23. Open string tachyonic inflation (OSTI)

See Fig. 109.

## A.24. Witten-O'Raifeartaigh inflation (WRI)

See Fig. 109.

## A.25. Small field inflation (SFI)

See Figs. 111-113.

## A.26. Intermediate inflation (II)

See Fig. 114.

## A.27. Kähler moduli inflation II (KMIII)

See Fig. 115.

## A.28. Logamediate inflation (LMI)

See Figs. 116-121 and 121.

## A.29. Twisted inflation (TWI)

See Fig. 122


Fig. 191. Reheating consistent slow-roll predictions for the constant $n_{S} D$ inflation models for $\beta=0.1$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right)$ (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The energy scale at which reheating ends is not annotated since it is degenerated with the parameter $x_{\text {end }}$. The black solid lines stand for the points such that $n_{S}=1+4 \alpha^{2} \beta /(\beta+1)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

## A.30. GMSSM Inflation (GMSSMI)

See Figs. 123 and 124.

## A.31. Generalized renormalizable inflection point inflation (GRIPI)

See Figs. 125 and 126.

## A.32. Brane SUSY breaking Inflation (BSUSYBI)

See Fig. 127.

## A.33. Tip inflation (TI)

See Figs. 128-130.

## A.34. $\beta$ exponential inflation (BEI)

## See Fig. 131.

## A.35. Pseudo natural inflation (PSNI)

See Fig. 132.

## A.36. Non canonical Kähler inflation (NCKI)



Fig. 192. Reheating consistent slow-roll predictions for the constant $n_{S} D$ inflation models for $\beta=5$ in the plane ( $n_{\mathrm{S}}, r$ ) (top panel) and the plane $\left(\epsilon_{1}, \epsilon_{2}\right.$ ) (bottom panel). The two pink solid contours are the one and two-sigma Planck confidence intervals (marginalized over second order slow-roll). The energy scale at which reheating ends is not annotated since it is degenerated with the parameter $x_{\text {end }}$. The black solid lines stand for the points such that $n_{S}=1+4 \alpha^{2} \beta /(\beta+1)$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

## A.37. Constant spectrum inflation (CSI)

See Figs. 135 and 136.

## A.38. Orientifold inflation (OI)

See Fig. 137.

## A.39. Constant $n_{S}$ C inflation (CNCI)

See Fig. 138.

## A.40. Supergravity brane inflation (SBI)

See Figs. 139-141.

## A.41. Spontaneous symmetry breaking inflation 1 (SSBII)

See Figs. 142-144
A.42. Spontaneous symmetry breaking inflation 2 (SSBI2)

See Fig. 145.
A.43. Spontaneous symmetry breaking inflation 3 (SSBI3)

See Figs. 146-148.

## A.44. Spontaneous symmetry breaking inflation 4 (SSBI4)

See Figs. 149-151.
A.45. Spontaneous symmetry breaking inflation 5 (SSBI5)

See Figs. 152-154.
A.46. Spontaneous symmetry breaking inflation 6 (SSBI6)

See Figs. 155-157.

## A.47. Inverse monomial inflation (IMI)

See Fig. 158.

## A.48. Brane inflation (BI)

See Figs. 159-162.

## A.49. KKLT inflation (KKLTI)

See Figs. 163-166.

## A.50. Running mass inflation 1 (RMI1)

See Fig. 167.

## A.51. Running mass inflation 2 (RMI2)

See Fig. 168.

## A.52. Running mass inflation 3 (RMI3)

See Fig. 169.

## A.53. Running mass inflation 4 (RMI4)

See Fig. 170.

## A.54. Valley hybrid inflation (VHI)

See Figs. 171-175.

## A.55. Dynamical supersymmetric inflation (DSI)

See Figs. 176-178.

## A.56. Generalized mixed inflation (GMLFI)

See Figs. 179-181.

## A.57. Logarithmic potential inflation 1 (LPI1)

See Figs. 182-184.

## A.58. Logarithmic potential inflation 2 (LPI2)

See Figs. 185-187.

## A.59. Logarithmic potential inflation 3 (LPI3)

See Figs. 188-190.

## A.60. Constant $n_{S} D$ inflation (CNDI)

See Figs. 191 and 192.

## References

[1] A.H. Guth, The inflationary universe: A possible solution to the horizon and flatness problems, Phys. Rev. D23 (1981) 347-356.
[2] A.D. Linde, A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems, Phys. Lett. B108 (1982) 389-393.
[3] A. Albrecht, P.J. Steinhardt, Cosmology for grand unified theories with radiatively induced symmetry breaking, Phys. Rev. Lett. 48 (1982) 1220-1223.
[4] A.D. Linde, Chaotic inflation, Phys. Lett. B129 (1983) 177-181.
[5] A.D. Linde, Inflationary cosmology, Lect. Notes Phys. 738 (2008) 1-54. arXiv: 0705.0164.
[6] J. Martin, Inflation and precision cosmology, Braz. J. Phys. 34 (2004) 1307-1321. astro-ph/0312492.
[7] J. Martin, Inflationary cosmological perturbations of quantum-mechanical origin, Lect. Notes Phys. 669 (2005) 199-244. hep-th/0406011.
[8] J. Martin, Inflationary perturbations: The cosmological Schwinger effect, Lect. Notes Phys. 738 (2008) 193-241. arXiv:0704.3540.
[9] A.A. Starobinsky, Relict gravitation radiation spectrum and initial state of the universe, JETP Lett. 30 (1979) 682-685 (in Russian).
[10] V.F. Mukhanov, G. Chibisov, Quantum fluctuation and nonsingular universe, JETP Lett. 33 (1981) 532-535 (in Russian).
[11] S. Hawking, The development of irregularities in a single bubble inflationary universe, Phys. Lett. B115 (1982) 295. Revised version.
[12] A.A. Starobinsky, Dynamics of phase transition in the new inflationary universe scenario and generation of perturbations, Phys. Lett. B117 (1982) 175-178.
[13] A.H. Guth, S.Y. Pi., Fluctuations in the new inflationary universe, Phys. Rev. Lett. 49 (1982) 1110-1113.
[14] J.M. Bardeen, P.J. Steinhardt, M.S. Turner, Spontaneous creation of almost scale - free density perturbations in an inflationary universe, Phys. Rev. D 28 (1983) 679.
[15] E.D. Stewart, D.H. Lyth, A More accurate analytic calculation of the spectrum of cosmological perturbations produced during inflation, Phys. Lett. B302 (1993) 171-175. gr-qc/9302019.
[16] V.F. Mukhanov, H.A. Feldman, R.H. Brandenberger, Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions, Phys. Rep. 215 (1992) 203-333.
[17] A.R. Liddle, P. Parsons, J.D. Barrow, Formalizing the slow roll approximation in inflation, Phys. Rev. D50 (1994) 7222-7232. http://xxx.lanl.gov/abs/astroph/9408015.
[18] C. Kiefer, D. Polarski, A.A. Starobinsky, Quantum to classical transition for fluctuations in the early universe, Internat. J. Modern Phys. D7 (1998) 455-462. http://xxx.lanl.gov/abs/gr-qc/9802003.
[19] L. Grishchuk, Y. Sidorov, Squeezed quantum states of relic gravitons and primordial density fluctuations, Phys. Rev. D42 (1990) 3413-3421.
[20] D. Polarski, A.A. Starobinsky, Semiclassicality and decoherence of cosmological perturbations, Classical Quantum Gravity 13 (1996) 377-392. http://xxx.lanl.gov/abs/gr-qc/9504030.
[21] C. Kiefer, D. Polarski, Why do cosmological perturbations look classical to us? Adv. Sci. Lett. 2 (2009) 164-173. arXiv:0810.0087.
[22] D. Sudarsky, Shortcomings in the understanding of why cosmological perturbations look classical, Int. J. Mod. Phys. D 20 (2011) 509-552. arXiv:0906.0315.
[23] J. Martin, V. Vennin, P. Peter, Cosmological inflation and the quantum measurement problem, Phys. Rev. D86 (2012) 103524. arXiv:1207.2086.
[24] J. Martin, The quantum state of inflationary perturbations, J. Phys. Conf. Ser. 405 (2012) 012004. arXiv:1209.3092.
[25] S. Alexander, R.H. Brandenberger, D. Easson, Brane gases in the early universe, Phys. Rev. D62 (2000) 103509. http://xxx.lanl.gov/abs/hep-th/0005212.
[26] P.J. Steinhardt, N. Turok, Cosmic evolution in a cyclic universe, Phys. Rev. D65 (2002) 126003. http://xxx.lanl.gov/abs/hep-th/0111098.
[27] J. Khoury, B.A. Ovrut, N. Seiberg, P.J. Steinhardt, N. Turok, From big crunch to big bang, Phys. Rev. D65 (2002) 086007. http://xxx.lanl.gov/abs/hepth/0108187.
[28] J. Khoury, B.A. Ovrut, P.J. Steinhardt, N. Turok, The Ekpyrotic universe: Colliding branes and the origin of the hot big bang, Phys. Rev. D64 (2001) 123522. http://xxx.lanl.gov/abs/hep-th/0103239.
[29] J. Martin, P. Peter, N. Pinto Neto, D.J. Schwarz, Passing through the bounce in the ekpyrotic models, Phys. Rev. D65 (2002) 123513. http://xxx.lanl.gov/abs/hep-th/0112128.
[30] P. Steinhardt, N. Turok, A cyclic model of the universe, Science 296 (2002) 1436-1439.
[31] F. Finelli, R. Brandenberger, On the generation of a scale invariant spectrum of adiabatic fluctuations in cosmological models with a contracting phase, Phys. Rev. D65 (2002) 103522. http://xxx.lanl.gov/abs/hep-th/0112249.
[32] R. Brandenberger, D.A. Easson, D. Kimberly, Loitering phase in brane gas cosmology, Nuclear Phys. B623 (2002) 421-436. http://xxx.lanl.gov/abs/hepth/0109165.
[33] R. Kallosh, L. Kofman, A.D. Linde, Pyrotechnic universe, Phys. Rev. D 64 (2001) 123523. http://xxx.lanl.gov/abs/hep-th/0104073.
[34] J. Martin, P. Peter, N. Pinto-Neto, D.J. Schwarz, Comment on density perturbations in the ekpyrotic scenario, Phys. Rev. D67 (2003) 028301. http://xxx.lanl.gov/abs/hep-th/0204222.
[35] P. Peter, N. Pinto-Neto, Primordial perturbations in a non singular bouncing universe model, Phys. Rev. D66 (2002) 063509. http://xxx.lanl.gov/abs/hepth/0203013.
[36] S. Tsujikawa, R. Brandenberger, F. Finelli, On the construction of nonsingular pre-big bang and ekpyrotic cosmologies and the resulting density perturbations, Phys. Rev. D66 (2002) 083513. http://xxx.lanl.gov/abs/hepth/0207228.
[37] L. Kofman, A.D. Linde, V.F. Mukhanov, Inflationary theory and alternative cosmology, J. High Energy Phys. 0210 (2002) 057. http://xxx.lanl.gov/abs/hepth/0206088.
[38] J. Khoury, P.J. Steinhardt, N. Turok, Designing cyclic universe models, Phys. Rev. Lett. 92 (2004) 031302. http://xxx.lanl.gov/abs/hep-th/0307132.
[39] J. Martin, P. Peter, On the causality argument in bouncing cosmologies, Phys. Rev. Lett. 92 (2004) 061301. http://xxx.lanl.gov/abs/astro-ph/0312488.
[40] J. Martin, P. Peter, Parametric amplification of metric fluctuations through a bouncing phase, Phys. Rev. D68 (2003) 103517. http://xxx.lanl.gov/abs/hepth/0307077.
[41] J. Martin, P. Peter, On the properties of the transition matrix in bouncing cosmologies, Phys. Rev. D69 (2004) 107301. http://xxx.lanl.gov/abs/hepth/0403173.
[42] A. Nayeri, R.H. Brandenberger, C. Vafa, Producing a scale-invariant spectrum of perturbations in a Hagedorn phase of string cosmology, Phys. Rev. Lett. 97 (2006) 021302. http://xxx.lanl.gov/abs/hep-th/0511140.
[43] P. Peter, E.J. Pinho, N. Pinto-Neto, A Non inflationary model with scale invariant cosmological perturbations, Phys. Rev. D75 (2007) 023516. http://xxx.lanl.gov/abs/hep-th/0610205.
[44] F. Finelli, P. Peter, N. Pinto-Neto, Spectra of primordial fluctuations in two-perfect-fluid regular bounces, Phys. Rev. D77 (2008) 103508. arXiv:0709.3074.
[45] L.R. Abramo, P. Peter, K-Bounce, JCAP 0709 (2007) 001. arXiv:0705.2893.
[46] F.T. Falciano, M. Lilley, P. Peter, A Classical bounce: Constraints and consequences, Phys. Rev. D77 (2008) 083513. arXiv:0802.1196.
[47] A. Linde, V. Mukhanov, A. Vikman, On adiabatic perturbations in the ekpyrotic scenario, JCAP 1002 (2010) 006. arXiv:0912.0944.
[48] L.R. Abramo, I. Yasuda, P. Peter, Non singular bounce in modified gravity, Phys. Rev. D81 (2010) 023511. arXiv:0910.3422.
[49] R. Brandenberger, Matter bounce in Horava-Lifshitz cosmology, Phys. Rev. D80 (2009) 043516. arXiv:0904.2835.
[50] R.H. Brandenberger, String gas cosmology: progress and problems, Classical Quantum Gravity 28 (2011) 204005. arXiv:1105.3247.
[51] R.H. Brandenberger, The Matter Bounce Alternative to Inflationary Cosmology. arXiv:1206.4196.
[52] Y.-F. Cai, D.A. Easson, R. Brandenberger, Towards a nonsingular bouncing cosmology, JCAP 1208 (2012) 020. arXiv: 1206.2382.
[53] Y.-F. Cai, R. Brandenberger, P. Peter, Anisotropy in a Nonsingular Bounce. arXiv:1301.4703.
[54] M.S. Turner, Coherent scalar field oscillations in an expanding universe, Phys. Rev. D28 (1983) 1243.
[55] L. Kofman, A.D. Linde, A.A. Starobinsky, Towards the theory of reheating after inflation, Phys. Rev. D56 (1997) 3258-3295. http://xxx.lanl.gov/abs/hepph/9704452.
[56] B.A. Bassett, S. Tsujikawa, D. Wands, Inflation dynamics and reheating, Rev. Modern Phys. 78 (2006) 537-589. http://xxx.lanl.gov/abs/astro-ph/0507632.
[57] A. Mazumdar, J. Rocher, Particle physics models of inflation and curvaton scenarios, Phys. Rep. 497 (2011) 85-215. arXiv: 1001.0993
[58] F. Finelli, R.H. Brandenberger, Parametric amplification of gravitational fluctuations during reheating, Phys. Rev. Lett. 82 (1999) 1362-1365. http://xxx.lanl.gov/abs/hep-ph/9809490.
[59] B.A. Bassett, D.I. Kaiser, R. Maartens, General relativistic preheating after inflation, Phys. Lett. B455 (1999) 84-89. http://xxx.lanl.gov/abs/hepph/9808404.
[60] F. Finelli, R.H. Brandenberger, Parametric amplification of metric fluctuations during reheating in two field models, Phys. Rev. D62 (2000) 083502. http://xxx.lanl.gov/abs/hep-ph/0003172.
[61] K. Jedamzik, M. Lemoine, J. Martin, Collapse of small-scale density perturbations during preheating in single field inflation, JCAP 1009 (2010) 034. arXiv:1002.3039.
[62] K. Jedamzik, M. Lemoine, J. Martin, Generation of gravitational waves during early structure formation between cosmic inflation and reheating, JCAP 1004 (2010) 021. arXiv: 1002.3278.
[63] R. Easther, R. Flauger, J.B. Gilmore, Delayed reheating and the breakdown of coherent oscillations, JCAP 1104 (2011) 027. arXiv:1003.3011.
[64] J. Martin, C. Ringeval, First CMB constraints on the inflationary reheating temperature, Phys. Rev. D82 (2010) 023511. arXiv: 1004.5525.
[65] Planck Collaboration, P. Ade et al., Planck 2013 results. I. Overview of products and scientific results. arXiv:1303.5062.
[66] Planck Collaboration, P. Ade et al., Planck 2013 results. XV. CMB power spectra and likelihood. arXiv:1303.5075.
[67] Planck Collaboration, P. Ade et al., Planck 2013 Results. XXIV. Constraints on primordial non-Gaussianity. arXiv:1303.5084.
[68] Planck Collaboration, P. Ade et al., Planck 2013 results. XXV. Searches for cosmic strings and other topological defects. arXiv:1303.5085.
[69] Planck Collaboration, P. Ade et al., Planck 2013 results. XXII. Constraints on inflation. arXiv:1303.5082.
[70] Planck Collaboration, P. Ade et al., Planck 2013 results. XVI. Cosmological parameters. arXiv:1303.5076.
[71] J.-M. Lamarre, J.-L. Puget, P.A.R. Ade, F. Bouchet, G. Guyot, A.E. Lange, F. Pajot, A. Arondel, K. Benabed, J.-L. Beney, A. Benoît, J.-P. Bernard, R. Bhatia, Y. Blanc, J.J. Bock, E. Bréelle, T.W. Bradshaw, P. Camus, A. Catalano, J. Charra, M. Charra, S.E. Church, F. Couchot, A. Coulais, B.P. Crill, M.R. Crook, K. Dassas, P. De bernardis, J. Delabrouille, P. De marcillac, J.-M. Delouis, F.-X. Désert, C. Dumesnil, X. Dupac, G. Efstathiou, P. Eng, C. Evesque, J.-J. Fourmond, K. Ganga, M. Giard, R. Gispert, L. Guglielmi, J. Haissinski, S. Henrot-Versillé, E. Hivon, W.A. Holmes, W.C. Jones, T.C. Koch, H. Lagardère, P. Lami, J. Landé, B. Leriche, C. Leroy, Y. Longval, J.F. Macías-Pérez, T. Maciaszek, B. Maffei, B. Mansoux, C. Marty, S. Masi, C. Mercier, M.-A. Miville-Deschênes, A. Moneti, L. Montier, J.A. Murphy, J. Narbonne, M. Nexon, C.G. Paine, J. Pahn, O. Perdereau, F. Piacentini, M. Piat, S. Plaszczynski, E. Pointecouteau, R. Pons, N. Ponthieu, S. Prunet, D. Rambaud, G. Recouvreur, C. Renault, I. Ristorcelli, C. Rosset, D. Santos, G. Savini, G. Serra, P. Stassi, R.V. Sudiwala, J.-F. Sygnet, J.A. Tauber, J.-P. Torre, M. Tristram, L. Vibert, A. Woodcraft, V. Yurchenko, D. Yvon, Planck prelaunch status: The HFI instrument, from specification to actual performance, Astron. Astrophys. 520 (2010) A9.
[72] C. Bennett, D. Larson, J. Weiland, N. Jarosik, G. Hinshaw, et al., Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Final Maps and Results. arXiv:1212.5225.
[73] G. Hinshaw, D. Larson, E. Komatsu, D. Spergel, C. Bennett, et al., Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results. arXiv:1212.5226.
[74] Supernova Search Team Collaboration, A.G. Riess, et al., Cosmological results from high-z supernovae, Astrophys. J. 594 (2003) 1-24. http://xxx.lanl.gov/abs/astro-ph/0305008.
[75] Supernova Search Team Collaboration, A.G. Riess, et al., Type Ia supernova discoveries at $z>1$ from the hubble space telescope: Evidence for past deceleration and constraints on dark energy evolution, Astrophys. J. 607 (2004) 665-687. http://xxx.lanl.gov/abs/astro-ph/0402512.
[76] A.G. Riess, L.-G. Strolger, S. Casertano, H.C. Ferguson, B. Mobasher, et al., New Hubble space telescope discoveries of type ia supernovae at $z>1$ : Narrowing constraints on the early behavior of dark energy, Astrophys. J. 659 (2007) 98-121. http://xxx.lanl.gov/abs/astro-ph/0611572.
[77] A.G. Riess, L. Macri, S. Casertano, H. Lampeitl, H.C. Ferguson, et al., A 3 telescope and wide field camera 3, Astrophys. J. 730 (2011) 119. arXiv:1103.2976.
[78] SDSS Collaboration, J.K. Adelman-McCarthy, et al., The sixth data release of the sloan digital sky survey, Astrophys. J. Suppl. 175 (2008) 297-313. arXiv:0707.3413.
[79] SDSS Collaboration, K.N. Abazajian, et al., The seventh data release of the sloan digital sky survey, Astrophys. J. Suppl. 182 (2009) 543-558. arXiv:0812. 0649.
[80] Euclid Collaboration, J. Amiaux et al., Euclid mission: building of a reference survey. arXiv:1209.2228
[81] M.S. Turner, M.J. White, J.E. Lidsey, Tensor perturbations in inflationary models as a probe of cosmology, Phys. Rev. D48 (1993) 4613-4622. http://xxx.lanl.gov/abs/astro-ph/9306029.
[82] M. Maggiore, Gravitational wave experiments and early universe cosmology, Phys. Rept. 331 (2000) 283-367. http://xxx.lanl.gov/abs/gr-qc/9909001.
[83] H. Kudoh, A. Taruya, T. Hiramatsu, Y. Himemoto, Detecting a gravitationalwave background with next-generation space interferometers, Phys. Rev. D73 (2006) 064006. http://xxx.lanl.gov/abs/gr-qc/0511145.
[84] S. Kuroyanagi, C. Gordon, J. Silk, N. Sugiyama, Forecast constraints on inflation from combined CMB and gravitational wave direct detection experiments, Phys. Rev. D81 (2010) 083524. arXiv:0912.3683.
[85] S. Kawamura, M. Ando, N. Seto, S. Sato, T. Nakamura, et al., The Japanese space gravitational wave antenna: DECIGO, Classical Quantum Gravity 28 (2011) 094011.
[86] P. Amaro-Seoane, S. Aoudia, S. Babak, P. Binetruy, E. Berti, et al. eLISA: Astrophysics and cosmology in the millihertz regime. arXiv:1201.3621.
[87] S. Kuroyanagi, C. Ringeval, T.Takahashi, Early universe tomography with CMB and gravitational waves, Phys. Rev. D 87 (2013) 083502. arXiv:1301.1778.
[88] D. Gorbunov, A. Tokareva, $R^{2}$-inflation with conformal SM Higgs field, JCAP 1312 (2013) 021. arXiv:1212.4466.
[89] J. Dunkley, E. Calabrese, J. Sievers, G. Addison, N. Battaglia, et al. The Atacama Cosmology Telescope: likelihood for small-scale CMB data. arXiv:1301.0776.
[90] J.L. Sievers, R.A. Hlozek, M.R. Nolta, V. Acquaviva, G.E. Addison, et al. The Atacama Cosmology Telescope: Cosmological parameters from three seasons of data. arXiv:1301.0824.
[91] Z. Hou, C. Reichardt, K. Story, B. Follin, R. Keisler, et al. Constraints on Cosmology from the Cosmic Microwave Background Power Spectrum of the 2500-square degree SPT-SZ Survey. arXiv:1212.6267.
[92] K. Story, C. Reichardt, Z. Hou, R. Keisler, K. Aird, et al. A Measurement of the Cosmic Microwave Background Damping Tail from the 2500 -square-degree SPT-SZ survey. arXiv:1210.7231.
[93] CMBPol Study Team Collaboration, D. Baumann, et al., CMBPol mission concept study: Probing inflation with CMB polarization, AIP Conf. Proc. 1141 (2009) 10-120. arXiv:0811.3919.
[94] B. Crill, P. Ade, E. Battistelli, S. Benton, R. Bihary, et al. SPIDER: A Balloon-borne Large-scale CMB Polarimeter. arXiv:0807.1548.
[95] M. Zaldarriaga, S.R. Furlanetto, L. Hernquist, 21 Centimeter fluctuations from cosmic gas at high redshifts, Astrophys. J. 608 (2004) 622-635. http://xxx.lanl.gov/abs/astro-ph/0311514.
[96] A. Lewis, A. Challinor, The 21 cm angular-power spectrum from the dark ages, Phys. Rev. D76 (2007) 083005. http://xxx.lanl.gov/abs/astro-ph/0702600.
[97] M. Tegmark, M. Zaldarriaga, The fast Fourier transform telescope, Phys. Rev. D79 (2009) 083530. arXiv:0805.4414.
[98] V. Barger, Y. Gao, Y. Mao, D. Marfatia, Inflationary potential from 21 cm tomography and Planck, Phys. Lett. B673 (2009) 173-178. arXiv:0810.3337.
[99] Y. Mao, M. Tegmark, M. Mcquinn, M. Zaldarriaga, O. Zahn, How accurately can 21 cm tomography constrain cosmology? Phys. Rev. D78 (2008) 023529. arXiv:0802.1710.
[100] P. Adshead, R. Easther, J. Pritchard, A. Loeb, Inflation and the scale dependent spectral index: Prospects and strategies, JCAP 1102 (2011) 021. arXiv:1007.3748.
[101] S. Clesse, L. Lopez-Honorez, C. Ringeval, H. Tashiro, M.H. Tytgat, Background reionization history from omniscopes, Phys. Rev. D86 (2012) 123506. arXiv:1208.4277.
[102] A. Golovnev, V. Mukhanov, V. Vanchurin, Vector inflation, JCAP 0806 (2008) 009. arXiv:0802.2068.
[103] P. Adshead, M. Wyman, Chromo-Natural Inflation: Natural inflation on a steep potential with classical non-Abelian gauge fields, Phys. Rev. Lett. 108 (2012) 261302. arXiv:1202.2366.
[104] A. Maleknejad, M. Sheikh-Jabbari, Gauge-flation: Inflation From Non-Abelian Gauge Fields. arXiv:1102.1513.
[105] A. Maleknejad, M. Sheikh-Jabbari, Non-abelian gauge field inflation, Phys. Rev. D84 (2011) 043515. arXiv:1102.1932.
[106] A. Maleknejad, M. Sheikh-Jabbari, J. Soda, Gauge Fields and Inflation. arXiv:1212.2921.
[107] S. Avila, J. Martin, D. Steer, Superimposed Oscillations in Brane Inflation. arXiv:1304.3262.
[108] A. Berera, Warm inflation, Phys. Rev. Lett. 75 (1995) 3218-3221. http://xxx. lanl.gov/abs/astro-ph/9509049.
[109] J. Yokoyama, A.D. Linde, Is warm inflation possible? Phys. Rev. D60 (1999) 083509. http://xxx.lanl.gov/abs/hep-ph/9809409.
[110] M. Bastero-Gil, A. Berera, R.O. Ramos, Dissipation coefficients from scalar and fermion quantum field interactions, JCAP 1109 (2011) 033. arXiv:1008.1929.
[111] S. Bartrum, A. Berera, J.G. Rosa, Warming up for Planck. arXiv:1303.3508.
[112] M. Alishahiha, E. Silverstein, D. Tong, DBI in the sky, Phys. Rev. D70 (2004) 123505. http://xxx.lanl.gov/abs/hep-th/0404084.
[113] D. Langlois, S. Renaux-Petel, D.A. Steer, T. Tanaka, Primordial perturbations and non-Gaussianities in DBI and general multi-field inflation, Phys. Rev. D78 (2008) 063523. arXiv:0806.0336.
[114] D. Langlois, S. Renaux-Petel, D.A. Steer, Multi-field DBI inflation: Introducing bulk forms and revisiting the gravitational wave constraints, JCAP 0904 (2009) 021. arXiv:0902.2941.
[115] A. Gangui, F. Lucchin, S. Matarrese, S. Mollerach, The three point correlation function of the cosmic microwave background in inflationary models, Astrophys. J. 430 (1994) 447-457. http://xxx.lanl.gov/abs/astro-ph/9312033.
[116] A. Gangui, NonGaussian effects in the cosmic microwave background from inflation, Phys. Rev. D50 (1994) 3684-3691. http://xxx.lanl.gov/abs/astroph/9406014.
[117] A. Gangui, J. Martin, Cosmic microwave background bispectrum and slow roll inflation, Mon. Not. Roy. Astron. Soc. (1999) http://xxx.lanl.gov/abs/astroph/9908009.
[118] L.-M. Wang, M. Kamionkowski, The cosmic microwave background bispectrum and inflation, Phys. Rev. D61 (2000) 063504. http://xxx.lanl.gov/abs/ astro-ph/9907431.
[119] J.M. Maldacena, Non-Gaussian features of primordial fluctuations in single field inflationary models, J. High Energy Phys. 0305 (2003) 013. http://xxx.lanl.gov/abs/astro-ph/0210603.
[120] P. Creminelli, M. Zaldarriaga, Single field consistency relation for the 3-point function, JCAP 0410 (2004) 006. http://xxx.lanl.gov/abs/astro-ph/0407059.
[121] C. Cheung, A.L. Fitzpatrick, J. Kaplan, L. Senatore, On the consistency relation of the 3-point function in single field inflation, JCAP 0802 (2008) 021. arXiv:0709.0295.
[122] J. Ganc, E. Komatsu, A new method for calculating the primordial bispectrum in the squeezed limit, JCAP 1012 (2010) 009. arXiv: 1006.5457.
[123] A. De Felice, S. Tsujikawa, Shapes of primordial non-Gaussianities in the Horndeski's most general scalar-tensor theories, JCAP 1303 (2013) 030. arXiv:1301.5721.
[124] D. Seery, J.E. Lidsey, Primordial non-Gaussianities in single field inflation, JCAP 0506 (2005) 003. http://xxx.lanl.gov/abs/astro-ph/0503692.
[125] X. Chen, Running non-Gaussianities in DBI inflation, Phys. Rev. D72 (2005) 123518. http://xxx.lanl.gov/abs/astro-ph/0507053.
[126] X. Chen, M.-x. Huang, S. Kachru, G. Shiu, Observational signatures and non-Gaussianities of general single field inflation, JCAP 0701 (2007) 002. http://xxx.lanl.gov/abs/hep-th/0605045.
[127] X. Chen, Primordial non-Gaussianities from inflation models, Adv. Astron. 2010 (2010) 638979. arXiv:1002.1416.
[128] X. Chen, R. Easther, E.A. Lim, Large non-Gaussianities in single field inflation, JCAP 0706 (2007) 023. http://xxx.lanl.gov/abs/astro-ph/0611645.
[129] X. Chen, R. Easther, E.A. Lim, Generation and characterization of large nonGaussianities in single field inflation, JCAP 0804 (2008) 010. arXiv:0801.3295.
[130] S. Hannestad, T. Haugbolle, P.R. Jarnhus, M.S. Sloth, Non-Gaussianity from axion monodromy inflation, JCAP 1006 (2010) 001. arXiv:0912.3527.
[131] R. Flauger, E. Pajer, Resonant non-Gaussianity, JCAP 1101 (2011) 017. arXiv: 1002.0833.

132] P. Adshead, C. Dvorkin, W. Hu., E.A. Lim, Non-Gaussianity from step features in the inflationary potential, Phys. Rev. D85 (2012) 023531. Typos fixed, supersedes journal version arXiv:1110.3050.
[133] J. Martin, L. Sriramkumar, The scalar bi-spectrum in the Starobinsky model: The equilateral case, JCAP 1201 (2012) 008. arXiv:1109.5838.
[134] X. Chen, Folded resonant non-Gaussianity in general single field inflation, JCAP 1012 (2010) 003. arXiv: 1008.2485.
[135] A. Gangui, J. Martin, M. Sakellariadou, Single field inflation and nonGaussianity, Phys. Rev. D66 (2002) 083502. http://xxx.lanl.gov/abs/astroph/0205202.
[136] R. Holman, A.J. Tolley, Enhanced non-gaussianity from excited initial states, JCAP 0805 (2008) 001. arXiv:0710.1302.
[137] W. Xue, B. Chen, alpha-vacuum and inflationary bispectrum, Phys. Rev. D79 (2009) 043518. arXiv:0806.4109.
[138] P.D. Meerburg, J.P. van der Schaar, P.S. Corasaniti, Signatures of initial state modifications on bispectrum statistics, JCAP 0905 (2009) 018. arXiv:0901.4044.
[139] A. Ashoorioon, G. Shiu, A note on calm excited states of inflation, JCAP 1103 (2011) 025. arXiv:1012.3392.
[140] J.-L. Lehners, S. Renaux-Petel, Multifield cosmological perturbations at third order and the ekpyrotic trispectrum, Phys. Rev. D80 (2009) 063503. arXiv:0906.0530.
[141] S. Renaux-Petel, S. Mizuno, K. Koyama, Primordial fluctuations and nonGaussianities from multifield DBI Galileon inflation, JCAP 1111 (2011) 042. arXiv:1108.0305.
[142] R. Trotta, Bayes in the sky: Bayesian inference and model selection in cosmology, Contemp. Phys. 49 (2008) 71-104. arXiv:0803.4089.
[143] J.E. Lidsey, A.R. Liddle, E.W. Kolb, E.J. Copeland, T. Barreiro, et al., Reconstructing the inflation potential: An overview, Rev. Mod. Phys. 69 (1997) 373-410. http://xxx.lanl.gov/abs/astro-ph/9508078.
[144] H. de Oliveira, C.A. Terrero-Escalante, Troubles for observing the inflaton potential, JCAP 0601 (2006) 024. http://xxx.lanl.gov/abs/astro-ph/0511660.
[145] J. Martin, C. Ringeval, V. Vennin, K-inflationary power spectra at second order. arXiv:1303.2120.
[146] J.B. Jimenez, M. Musso, C. Ringeval, Exact Mapping between Tensor and Most General Scalar Power Spectra. arXiv:1303.2788.
[147] D. Boyanovsky, H.J. de Vega, N.G. Sanchez, Clarifying inflation models: Slow-roll as an expansion in 1/N efolds, Phys. Rev. D73 (2006) 023008. http://xxx.lanl.gov/abs/astro-ph/0507595.
[148] C. Destri, H.J. de Vega, N. Sanchez, MCMC analysis of WMAP3 and SDSS data points to broken symmetry inflaton potentials and provides a lower bound on the tensor to scalar ratio, Phys. Rev. D77 (2008) 043509. http://xxx.lanl.gov/abs/astro-ph/0703417.
[149] C. Burigana, C. Destri, H. de Vega, A. Gruppuso, N. Mandolesi, et al., Forecast for the Planck precision on the tensor to scalar ratio and other cosmological parameters, Astrophys. J. 724 (2010) 588-607. arXiv: 1003.6108.
[150] D. Boyanovsky, C. Destri, H. de Vega, N. Sanchez, The effective theory of inflation in the standard model of the universe and the CMB + LSS data analysis, Int. J. Mod. Phys. A24 (2009) 3669-3864. arXiv:0901.0549.
[151] S.M. Leach, A.R. Liddle, Constraining slow-roll inflation with WMAP and 2dF, Phys. Rev. D68 (2003) 123508. http://xxx.lanl.gov/abs/astro-ph/0306305.
[152] J. Martin, C. Ringeval, Inflation after WMAP3: Confronting the slowroll and exact power spectra to CMB data, JCAP 0608 (2006) 009. http://xxx.lanl.gov/abs/astro-ph/0605367.
[153] L. Lorenz, J. Martin, C. Ringeval, Constraints on kinetically modified inflation from WMAP5, Phys. Rev. D78 (2008) 063543. arXiv:0807.2414.
[154] F. Finelli, J. Hamann, S.M. Leach, J. Lesgourgues, Single-field inflation constraints from CMB and SDSS data, JCAP 1004 (2010) 011. arXiv:0912.0522.
[155] D.K. Hazra, L. Sriramkumar, J. Martin, BINGO: A code for the efficient computation of the scalar bi-spectrum. arXiv:1201.0926.
[156] C. Ringeval, P. Brax, C. van de Bruck, A.-C. Davis, Boundary inflation and the WMAP dta, Phys. Rev. D73 (2006) 064035. http://xxx.lanl.gov/abs/astroph/0509727.
[157] C. Ringeval, The exact numerical treatment of inflationary models, Lect. Notes Phys. 738 (2008) 243-273. http://xxx.lanl.gov/abs/astro-ph/0703486.
[158] L. Lorenz, J. Martin, C. Ringeval, Brane inflation and the WMAP data: a Bayesian analysis, JCAP 0804 (2008) 001. arXiv:0709.3758.
[159] M.J. Mortonson, H.V. Peiris, R. Easther, Bayesian analysis of inflation: Parameter estimation for single field models, Phys. Rev. D83 (2011) 043505. arXiv: 1007.4205.
[160] A.R. Liddle, S.M. Leach, How long before the end of inflation were observable perturbations produced? Phys. Rev. D68 (2003) 103503. http://xxx.lanl.gov/ abs/astro-ph/0305263.
[161] R. Easther, H.V. Peiris, Bayesian analysis of inflation II: Model selection and constraints on reheating, Phys. Rev. D85 (2012) 103533. arXiv:1112.0326.
[162] A.A. Starobinsky, Spectrum of adiabatic perturbations in the universe when there are singularities in the inflation potential, JETP Lett. 55 (1992) 489-494.
[163] J. Silk, M.S. Turner, Double inflation, Phys. Rev. D35 (1987) 419.
[164] P. Peter, D. Polarski, A.A. Starobinsky, Confrontation of double inflationary models with observations, Phys. Rev. D50 (1994) 4827-4834. http://xxx.lanl.gov/abs/astro-ph/9403037.
[165] D. Polarski, A.A. Starobinsky, Structure of primordial gravitational waves spectrum in a double inflationary model, Phys. Lett. B356 (1995) 196-204. http://xxx.lanl.gov/abs/astro-ph/9505125.
[166] D. Parkinson, S. Tsujikawa, B.A. Bassett, L. Amendola, Testing for double inflation with WMAP, Phys. Rev. D71 (2005) 063524. http://xxx.lanl.gov/abs/ astro-ph/0409071.
[167] S. Tsujikawa, D. Parkinson, B.A. Bassett, Correlation - consistency cartography of the double inflation landscape, Phys. Rev. D67 (2003) 083516. http://xxx.lanl.gov/abs/astro-ph/0210322.
[168] A.D. Linde, Hybrid inflation, Phys. Rev. D49 (1994) 748-754. http://xxx.lanl. gov/abs/astro-ph/9307002.
[169] D.H. Lyth, E.D. Stewart, More varieties of hybrid inflation, Phys. Rev. D54 (1996) 7186-7190. http://xxx.lanl.gov/abs/hep-ph/9606412.
[170] A.R. Liddle, A. Mazumdar, F.E. Schunck, Assisted inflation, Phys. Rev. D58 (1998) 061301. http://xxx.lanl.gov/abs/astro-ph/9804177.
[171] A. Ashoorioon, H. Firouzjahi, M. Sheikh-Jabbari, M-flation: Inflation from matrix valued scalar fields, JCAP 0906 (2009) 018. arXiv:0903.1481.
[172] A. Ashoorioon, H. Firouzjahi, M.M. Sheikh-Jabbari, Matrix inflation and the landscape of its potential, JCAP 1005 (2010) 002. arXiv:0911.4284.
[173] A. Ashoorioon, M. Sheikh-Jabbari, Gauged M-flation its UV sensitivity and spectator species, JCAP 1106 (2011) 014. arXiv:1101.0048.
[174] S. Tsujikawa, J. Ohashi, S. Kuroyanagi, A. De Felice, Planck constraints on single-field inflation. arXiv:1305.3044.
[175] S. Unnikrishnan, V. Sahni, Resurrecting power law inflation in the light of Planck results. arXiv:1305.5260.
[176] S. Choudhury, A. Mazumdar, S. Pal, Low and High scale MSSM inflation, gravitational waves and constraints from Planck. arXiv:1305.6398.
[177] J. Martin, C. Ringeval, R. Trotta, Hunting down the best model of inflation with bayesian evidence, Phys. Rev. D83 (2011) 063524. arXiv:1009.4157.
[178] J. Martin, C. Ringeval, R. Trotta, V. Vennin, The Best Inflationary Models After Planck. arXiv:1312.3529.
[179] J. Martin, Inflation after Planck: and the winners are. arXiv:1312.3720.
[180] C. Ringeval, Fast Bayesian inference for slow-roll inflation. arXiv:1312.2347.
[181] M.B. Hoffman, M.S. Turner, Kinematic constraints to the key inflationary observables, Phys. Rev. D64 (2001) 023506. http://xxx.lanl.gov/abs/astroph/0006321.
[182] D.J. Schwarz, C.A. Terrero-Escalante, A.A. Garcia, Higher order corrections to primordial spectra from cosmological inflation, Phys. Lett. B517 (2001) 243-249. http://xxx.lanl.gov/abs/astro-ph/0106020.
[183] J. Martin, D.J. Schwarz, WKB approximation for inflationary cosmological perturbations, Phys. Rev. D67 (2003) 083512. http://xxx.lanl.gov/abs/astroph/0210090.
[184] R. Casadio, F. Finelli, M. Luzzi, G. Venturi, Improved WKB analysis of cosmological perturbations, Phys. Rev. D71 (2005) 043517. http://xxx.lanl. gov/abs/gr-qc/0410092.
[185] R. Casadio, F. Finelli, M. Luzzi, G. Venturi, Higher order slow-roll predictions for inflation, Phys. Lett. B625 (2005) 1-6. http://xxx.lanl.gov/abs/grqc/0506043.
[186] R. Casadio, F. Finelli, M. Luzzi, G. Venturi, Improved WKB analysis of slowroll inflation, Phys. Rev. D72 (2005) 103516. http://xxx.lanl.gov/abs/grqc/0510103.
[187] J.-O. Gong, E.D. Stewart, The density perturbation power spectrum to second order corrections in the slow roll expansion, Phys. Lett. B510 (2001) 1-9. http://xxx.lanl.gov/abs/astro-ph/0101225.
[188] J. Choe, J.-O. Gong, E.D. Stewart, Second order general slow-roll power spectrum, JCAP 0407 (2004) 012. http://xxx.lanl.gov/abs/hep-ph/0405155.
[189] S.M. Leach, A.R. Liddle, J. Martin, D.J. Schwarz, Cosmological parameter estimation and the inflationary cosmology, Phys. Rev. D66 (2002) 023515. http://xxx.lanl.gov/abs/astro-ph/0202094.
[190] C. Ringeval, T. Suyama, J. Yokoyama, Magneto-reheating constraints from curvature perturbations. arXiv:1302.6013.
[191] D.H. Lyth, E.D. Stewart, Thermal inflation and the moduli problem, Phys. Rev. D53 (1996) 1784-1798. http://xxx.lanl.gov/abs/hep-ph/9510204.
[192] T. Biswas, A. Notari, Can inflation solve the hierarchy problem? Phys. Rev. D74 (2006) 043508. http://xxx.lanl.gov/abs/hep-ph/0511207.
[193] ATLAS Collaboration, G. Aad, et al., Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC, Phys. Lett. B716 (2012) 1-29. arXiv:1207.7214.
[194] CMS Collaboration, S. Chatrchyan, et al., Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC, Phys. Lett. B716 (2012) 30-61. arXiv:1207.7235.
[195] F. Bezrukov, M. Shaposhnikov, The standard model Higgs Boson as the inflaton, Phys. Lett. B659 (2008) 703-706. arXiv:0710.3755.
[196] F.L. Bezrukov, A. Magnin, M. Shaposhnikov, Standard Model Higgs boson mass from inflation, Phys. Lett. B675 (2009) 88-92. arXiv:0812.4950.
[197] F. Bezrukov, M. Shaposhnikov, Standard Model Higgs Boson mass from inflation: Two loop analysis, J. High Energy Phys. 0907 (2009) 089. arXiv:0904.1537.
[198] J. Garcia-Bellido, J. Rubio, M. Shaposhnikov, D. Zenhausern, Higgs-dilaton cosmology: From the early to the late universe, Phys. Rev. D84(2011) 123504. arXiv:1107.2163.
[199] N.D. Birrell, P.C.W. Davies, Quantum Fields In Curved Space, Cambridge Univ. Pr., 1982.
[200] G. Esposito-Farese, D. Polarski, Scalar tensor gravity in an accelerating universe, Phys. Rev. D63 (2001) 063504. http://xxx.lanl.gov/abs/gr-qc/0009034.
[201] J. Garcia-Bellido, D.G. Figueroa, J. Rubio, Preheating in the standard model with the Higgs-inflaton coupled to gravity, Phys. Rev. D79 (2009) 063531. arXiv:0812.4624.
[202] O. Bertolami, P. Frazao, J. Paramos, Reheating via a generalized nonminimal coupling of curvature to matter, Phys. Rev. D83 (2011) 044010. arXiv:1010.2698.
[203] H. Motohashi, A. Nishizawa, Reheating after $f(R)$ inflation, Phys. Rev. D86 (2012) 083514. arXiv: 1204.1472.

204] J. Ellis, D.V. Nanopoulos, K.A. Olive, A No-Scale Supergravity Realization of the Starobinsky Model. arXiv:1305.1247.
[205] W. Buchmuller, V. Domcke, K. Kamada, The Starobinsky Model from Superconformal D-Term Inflation. arXiv:1306.3471.
[206] R.N. Lerner, J. Mcdonald, Gauge singlet scalar as inflaton and thermal relic dark matter, Phys. Rev. D80 (2009) 123507. arXiv:0909.0520.
[207] J. Elias-Miro, J.R. Espinosa, G.F. Giudice, H.M. Lee, A. Strumia, Stabilization of the electroweak vacuum by a scalar threshold effect, J. High Energy Phys. 1206 (2012) 031. arXiv:1203.0237.
[208] C. Arina, J.-O. Gong, N. Sahu, Unifying darko-lepto-genesis with scalar triplet inflation, Nuclear Phys. B865 (2012) 430-460. arXiv:1206.0009.
[209] J. Barbon, J. Espinosa, On the naturalness of Higgs inflation, Phys. Rev. D79 (2009) 081302. arXiv:0903.0355.
[210] A. Barvinsky, A.Y. Kamenshchik, A. Starobinsky, Inflation scenario via the Standard Model Higgs boson and LHC, JCAP 0811 (2008) 021. arXiv:0809.2104.
[211] A. De simone, M.P. Hertzberg, F. Wilczek, Running inflation in the standard model, Phys. Lett. B678 (2009) 1-8. arXiv:0812.4946.
[212] A. Barvinsky, A.Y. Kamenshchik, C. Kiefer, A. Starobinsky, C. Steinwachs, Higgs boson renormalization group and naturalness in cosmology, Eur. Phys. J. C72 (2012) 22 19. arXiv:0910.1041.
[213] F. Bezrukov, A. Magnin, M. Shaposhnikov, S. Sibiryakov, Higgs inflation: consistency and generalisations, J. High Energy Phys. 1101 (2011) 016. arXiv:1008.5157.
[214] C.F. Steinwachs, A.Y. Kamenshchik, Non-minimal Higgs Inflation and Frame Dependence in Cosmology. arXiv: 1301.5543.
[215] F. Bezrukov, G.K. Karananas, J. Rubio, M. Shaposhnikov, Higgs-dilaton cosmology: an effective field theory approach, Phys. Rev. D87 (2013) 096001. arXiv:1212.4148.
[216] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, nineth ed., National Bureau of Standards, Washington, US, 1970.
[217] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals Series Products, Academic Press, New York, London, 1965.
[218] A. Vilenkin, Eternal inflation and chaotic terminology. gr-qc/0409055.
[219] A.D. Linde, Chaotic inflating universe, JETP Lett. 38 (1983) 176-179.
[220] M. Madsen, P. Coles, Chaotic inflation, Nuclear Phys. B298 (1988) 701-725.
[221] G. Lazarides, Q. Shafi, A predictive inflationary scenario without the gauge singlet, Phys. Lett. B308 (1993) 17-22. http://xxx.lanl.gov/abs/hepph/9304247.
[222] L. Kofman, A.D. Linde, A.A. Starobinsky, Reheating after inflation, Phys. Rev. Lett. 73 (1994) 3195-3198. http://xxx.lanl.gov/abs/hep-th/9405187.
[223] G. Lazarides, Q. Shafi, Topological defects and inflation, Phys. Lett. B372 (1996) 20-24. http://xxx.lanl.gov/abs/hep-ph/9510275.
[224] M. Kawasaki, M. Yamaguchi, T. Yanagida, Natural chaotic inflation in supergravity, Phys. Rev. Lett. 85 (2000) 3572-3575. http://xxx.lanl.gov/abs/hepph/0004243.
[225] D. Baumann, A. Dymarsky, I.R. Klebanov, L. Mcallister, Towards an explicit model of D-brane inflation, JCAP 0801 (2008) 024. arXiv:0706.0360.
[226] E. Silverstein, A. Westphal, Monodromy in the CMB: Gravity waves and string inflation, Phys. Rev. D78 (2008) 106003. arXiv:0803.3085.
[227] R.H. Brandenberger, A. Knauf, L.C. Lorenz, Reheating in a brane monodromy inflation model, J. High Energy Phys. 0810 (2008) 110. arXiv:0808.3936.
[228] K. Nakayama, F. Takahashi, Higgs chaotic inflation in standard model and NMSSM, JCAP 1102 (2011) 010. arXiv:1008.4457.
[229] F. Takahashi, Linear inflation from running kinetic term in supergravity, Phys. Lett. B693 (2010) 140-143. arXiv:1006.2801.
[230] K. Nakayama, F. Takahashi, Running kinetic inflation, JCAP 1011 (2010) 009. arXiv:1008.2956.
[231] A. Vilenkin, Quantum fluctuations in the new inflationary universe, Nuclear Phys. B226 (1983) 527.
[232] A. Vilenkin, The birth of inflationary universes, Phys. Rev. D27 (1983) 2848.
[233] A. Goncharov, A.D. Linde, V.F. Mukhanov, The global structure of the inflationary universe, Int. J. Mod. Phys. A2 (1987) 561-591.
[234] A.D. Linde, D.A. Linde, A. Mezhlumian, From the big bang theory to the theory of a stationary universe, Phys. Rev. D49 (1994) 1783-1826 http://xxx.lanl.gov/abs/gr-qc/9306035.
[235] A.A. Starobinsky, Stochastic De Sitter (Inflationay) Stage in the Early Universe.
[236] J. Martin, M. Musso, Solving stochastic inflation for arbitrary potentials, Phys. Rev. D73 (2006) 043516. http://xxx.lanl.gov/abs/hep-th/0511214.
[237] J. Martin, M. Musso, On the reliability of the Langevin perturbative solution in stochastic inflation, Phys. Rev. D73 (2006) 043517. http://xxx.lanl.gov/abs/hep-th/0511292.
[238] R. Mohapatra, A. Perez-Lorenzana, C.A. de Sousa Pires, Inflation in models with large extra dimensions driven by a bulk scalar field, Phys. Rev. D62 (2000) 105030. http://xxx.lanl.gov/abs/hep-ph/0003089.
[239] F. Cao, Generalized chaotic inflation. astro-ph/0205207.
[240] M. Bellini, Fresh inflation: A warm inflationary model from a zero temperature initial state, Phys. Rev. D63 (2001) 123510. http://xxx.lanl.gov/abs/gr$\mathrm{qc} / 0101062$.
[241] M. Bellini, Fresh inflation with nonminimally coupled inflaton field, Gen. Rel. Grav. 34 (2002) 1953-1961. http://xxx.lanl.gov/abs/hep-ph/0205171.
[242] M. Bellini, Fresh inflation with increasing cosmological parameter, Phys. Rev. D67 (2003) 027303. http://xxx.lanl.gov/abs/gr-qc/0211044.
[243] C.-S. Chen, C.-M. Lin, Type II seesaw Higgs triplet as the inflaton for chaotic inflation and leptogenesis, Phys. Lett. B695 (2011) 9-12. arXiv:1009.5727.
[244] A. Bouaouda, R. Zarrouki, H. Chakir, M. Bennai, F-term braneworld inflation in light of five-year WMAP observations, Internat. J. Modern Phys. A25 (2010) 3445-3451. arXiv: 1010.4884.
[245] V.N. Senoguz, Q. Shafi, Chaotic inflation radiative corrections and precision cosmology, Phys. Lett. B668 (2008) 6-10. arXiv:0806.2798.
[246] K. Freese, J.A. Frieman, A.V. Olinto, Natural inflation with pseudo-NambuGoldstone bosons, Phys. Rev. Lett. 65 (1990) 3233-3236.
[247] F.C. Adams, J.R. Bond, K. Freese, J.A. Frieman, A.V. Olinto, Natural inflation: Particle physics models, power law spectra for large scale structure, and constraints from COBE, Phys. Rev. D47 (1993) 426-455. http://xxx.lanl.gov/abs/hep-ph/9207245.
[248] R. Peccei, H.R. Quinn, Constraints imposed by CP conservation in the presence of instantons, Phys. Rev. D16 (1977) 1791-1797.
[249] R. Peccei, H.R. Quinn, CP conservation in the presence of instantons, Phys. Rev. Lett. 38 (1977) 1440-1443.
[250] D. Lyth, Axions and inflation: Sitting in the vacuum, Phys. Rev. D45 (1992) 3394-3404.
[251] L. Knox, A. Olinto, Initial conditions for natural inflation, Phys. Rev. D48 (1993) 946-949.
[252] J. Garcia-Bellido, A.D. Linde, D. Wands, Density perturbations and black hole formation in hybrid inflation, Phys. Rev. D54 (1996) 6040-6058. http://xxx.lanl.gov/abs/astro-ph/9605094.
[253] D.H. Lyth, A. Riotto, Particle physics models of inflation and the cosmological density perturbation, Phys. Rep. 314 (1999) 1-146. http://xxx.lanl.gov/abs/hep-ph/9807278.
[254] S. Tsujikawa, T. Torii, Spinodal effect in the natural inflation model, Phys. Rev. D62 (2000) 043505. http://xxx.lanl.gov/abs/hep-ph/9912499.
[255] X. Wang, B. Feng, M. Li., X.-L. Chen, X. Zhang, Natural inflation Planck scale physics and oscillating primordial spectrum, Internat. J. Modern Phys. D14 (2005) 1347. http://xxx.lanl.gov/abs/astro-ph/0209242.
[256] K. Freese, W.H. Kinney, On: Natural inflation, Phys. Rev. D70 (2004) 083512. http://xxx.lanl.gov/abs/hep-ph/0404012.
[257] C. Savage, K. Freese, W.H. Kinney, Natural inflation: Status after WMAP 3-year data, Phys. Rev. D74 (2006) 123511. http://xxx.lanl.gov/abs/hepph/0609144.
[258] G. Panotopoulos, Cosmic strings and natural inflation, J. High Energy Phys. 0706 (2007) 080. arXiv:0706.2747.
[259] T.W. Grimm, Axion inflation in type II string theory, Phys. Rev. D77 (2008) 126007. arXiv:0710.3883.
[260] K. Freese, C. Savage, W.H. Kinney, Natural inflation: The status after WMAP 3-year data, Int. J. Mod. Phys. D16 (2008) 2573-2585. arXiv:0802.0227.
[261] S. Mohanty, A. Nautiyal, Natural inflation at the GUT scale, Phys. Rev. D78 (2008) 123515. arXiv:0807.0317.
[262] A. Ashoorioon, K. Freese, J.T. Liu, Slow nucleation rates in chain inflation with QCD axions or monodromy, Phys. Rev. D79 (2009) 067302. arXiv:0810.0228.
[263] M.E. Olsson, Inflation assisted by heterotic axions, JCAP 0704 (2007) 019. http://xxx.lanl.gov/abs/hep-th/0702109.
[264] D. Maity, Kinetic Gravity Braiding and axion inflation. arXiv:1209.6554.
[265] K. Freese, A coupling of pseudoNambu-Goldstone bosons to other scalars and role in double field inflation, Phys. Rev. D50 (1994) 7731-7734. http://xxx.lanl.gov/abs/astro-ph/9405045.
[266] W.H. Kinney, K. Mahanthappa, Natural inflation from Fermion loops, Phys. Rev. D52 (1995) 5529-5537. http://xxx.lanl.gov/abs/hep-ph/9503331.
[267] W.H. Kinney, K.T. Mahanthappa, Inflation at low scales: General analysis and a detailed model, Phys. Rev. D53 (1996) 5455-5467. http://xxx.lanl.gov/abs/hep-ph/9512241.
[268] G.G. Ross, G. German, Hybrid natural inflation from non Abelian discrete symmetry, Phys. Lett. B684 (2010) 199-204. arXiv:0902.4676.
[269] G. German, A. Mazumdar, A. Perez-Lorenzana, Angular inflation from supergravity, Modern Phys. Lett. A17 (2002) 1627-1634. http://xxx.lanl.gov/abs/hep-ph/0111371.
[270] D. Bailin, A. Love, Supersymmetric Gauge Field Theory and String Theory, IOP (Graduate student series in physics), 1994.
[271] N. Arkani-Hamed, H.-C. Cheng, P. Creminelli, L. Randall, Extra natural inflation, Phys. Rev. Lett. 90 (2003) 221302. http://xxx.lanl.gov/abs/hepth/0301218.
[272] N. Arkani-Hamed, H.-C. Cheng, P. Creminelli, L. Randall, Pseudonatural inflation, JCAP 0307 (2003) 003. http://xxx.lanl.gov/abs/hep-th/0302034.
[273] D.E. Kaplan, N.J. Weiner, Little inflatons and gauge inflation, JCAP 0402 (2004) 005. http://xxx.lanl.gov/abs/hep-ph/0302014.
[274] H. Firouzjahi, S.H. Tye, Closer towards inflation in string theory, Phys. Lett. B584 (2004) 147-154. http://xxx.lanl.gov/abs/hep-th/0312020.
[275] J.P. Hsu, R. Kallosh, Volume stabilization and the origin of the inflaton shift symmetry in string theory, J. High Energy Phys. 0404 (2004) 042. http://xxx.lanl.gov/abs/hep-th/0402047.
[276] R. Gonzalez felipe, N. Santos, Natural inflation in 5-D warped backgrounds, Phys. Rev. D78 (2008) 023519. arXiv:0711.0022.
[277] B.A. Ovrut, S. Thomas, Instanton induced periodic potentials in nonlinear sigma models, Phys. Lett. B267 (1991) 227-232.
[278] J.E. Kim, Axion and almost massless quark as ingredients of quintessence, J. High Energy Phys. 9905 (1999) 022. http://xxx.lanl.gov/abs/hepph/9811509.
[279] S.C. Park, Orbifold GUT inflation, JCAP 0711 (2007) 001. arXiv:0704.3920.
[280] J. Preskill, M.B. Wise, F. Wilczek, Cosmology of the invisible axion, Phys. Lett. B120 (1983) 127-132.
[281] L. Abbott, P. Sikivie, A cosmological bound on the invisible axion, Phys. Lett. B120 (1983) 133-136.
[282] M. Dine, W. Fischler, The not so harmless axion, Phys. Lett. B120 (1983) 137-141.
[283] A.D. Linde, Inflation and axion cosmology, Phys. Lett. B201 (1988) 437.
[284] J.E. Kim, H.P. Nilles, M. Peloso, Completing natural inflation, JCAP 0501 (2005) 005. http://xxx.lanl.gov/abs/hep-ph/0409138.
[285] S. Dimopoulos, S. Kachru, J. Mcgreevy, J.G. Wacker, N-flation, JCAP 0808 (2008) 003. http://xxx.lanl.gov/abs/hep-th/0507205.
[286] Y.N. Obukhov, Spin driven inflation, Phys. Lett. A182 (1993) 214-216. http://xxx.lanl.gov/abs/gr-qc/0008015.
[287] E.D. Stewart, Inflation supergravity and superstrings, Phys. Rev. D51 (1995) 6847-6853. http://xxx.lanl.gov/abs/hep-ph/9405389.
[288] G. Dvali, S.H. Tye, Brane inflation, Phys. Lett. B450 (1999) 72-82. http://xxx.lanl.gov/abs/hep-ph/9812483.
[289] M. Cicoli, C. Burgess, F. Quevedo, Fibre inflation: Observable gravity waves from IIB string compactifications, JCAP 0903 (2009) 013. arXiv:0808.0691.
[290] G.F. Giudice, H.M. Lee, Unitarizing Higgs inflation, Phys. Lett. B694 (2011) 294-300. arXiv:1010.1417.
[291] L. Abbott, M.B. Wise, Constraints on generalized inflationary cosmologies, Nuclear Phys. B244 (1984) 541-548.
[292] V. Sahni, Scalar field fluctuations and infrared divergent states in cosmological models with power law expansion, Class. Quant. Grav. 5 (1988) L113.
[293] V. Sahni, The energy density of relic gravity waves from inflation, Phys. Rev. D42 (1990) 453-463.
[294] B. Ratra, P. Peebles, Cosmological consequences of a rolling homogeneous scalar field, Phys. Rev. D37 (1988) 3406.
[295] P.G. Ferreira, M. Joyce, Cosmology with a primordial scaling field, Phys. Rev. D58 (1998) 023503. http://xxx.lanl.gov/abs/astro-ph/9711102.
[296] D. La., P.J. Steinhardt, Extended inflationary cosmology, Phys. Rev. Lett. 62 (1989) 376.
[297] E.W. Kolb, First order inflation, Phys. Scripta T36 (1991) 199-217.
[298] Y. Kitada, K.-i. Maeda, Cosmic no hair theorem in power law inflation, Phys. Rev. D45 (1992) 1416-1419.
[299] L.E. Mendes, A.B. Henriques, Inflation in a simple Kantowski-Sachs model, Phys. Lett. B254 (1991) 44-48.
[300] N. Banerjee, S. Sen, Power law inflation and scalar field cosmology with a causal viscous fluid, Phys. Rev. D57 (1998) 4614-4619.
[301] M. Fairbairn, M.H. Tytgat, Inflation from a tachyon fluid? Phys. Lett. B546 (2002) 1-7. http://xxx.lanl.gov/abs/hep-th/0204070.
[302] M. Sami, P. Chingangbam, T. Qureshi, Aspects of tachyonic inflation with exponential potential, Phys. Rev. D66 (2002) 043530. http://xxx.lanl.gov/abs/hep-th/0205179.
[303] V.H. Cardenas, Tachyonic quintessential inflation, Phys. Rev. D73 (2006) 103512. http://xxx.lanl.gov/abs/gr-qc/0603013.
[304] J.M. Aguirregabiria, L.P. Chimento, A.S. Jakubi, R. Lazkoz, Symmetries leading to inflation, Phys. Rev. D67 (2003) 083518. http://xxx.lanl.gov/abs/grqc/0303010.
[305] K. Becker, M. Becker, A. Krause, M-theory inflation from multi M5-brane dynamics, Nuclear Phys. B715 (2005) 349-371. http://xxx.lanl.gov/abs/hepth/0501130.
[306] A. Ashoorioon, A. Krause, Power Spectrum and Signatures for Cascade Inflation. hep-th/0607001.
[307] M. Bennai, H. Chakir, Z. Sakhi, On inflation potentials in randall-sundrum braneworld model, Eur. J. Phys. 9 (2006) 84-93. arXiv:0806.1137.
[308] F. Lucchin, S. Matarrese, Power law inflation, Phys. Rev. D32 (1985) 1316.
[309] J. Yokoyama, K.-i. Maeda, On the dynamics of the power law inflation due to an exponential potential, Phys. Lett. B207 (1988) 31.
[310] A.R. Liddle, Power law inflation with exponential potentials, Phys. Lett. B220 (1989) 502.
[311] B. Ratra, Inflation in an exponential potential scalar field model, Phys. Rev. D45 (1992) 1913-1952.
[312] B. Ratra, Quantum mechanics of exponential potential inflation, Phys. Rev. D40 (1989) 3939.
[313] H.-J. Schmidt, New exact solutions for power law inflation Friedmann models, Astron. Nachr. 311 (1990) 165. http://xxx.lanl.gov/abs/gr-qc/0109004.
[314] R. Maartens, D. Taylor, N. Roussos, Exact inflationary cosmologies with exit, Phys. Rev. D52 (1995) 3358-3364.
[315] E.J. Copeland, A.R. Liddle, D. Wands, Exponential potentials and cosmological scaling solutions, Phys. Rev. D57 (1998) 4686-4690. http://xxx.lanl.gov/abs/ gr-qc/9711068.
[316] S. Hirai, T. Takami, Length of inflation and WMAP data in the case of powerlaw inflation. astro-ph/0506479.
[317] J.M. Heinzle, A.D. Rendall, Power-law inflation in spacetimes without symmetry, Commun. Math. Phys. 269 (2007) 1-15. http://xxx.lanl.gov/abs/grqc/0506134.
[318] J.P. Conlon, F. Quevedo, Kahler moduli inflation, J. High Energy Phys. 0601 (2006) 146. http://xxx.lanl.gov/abs/hep-th/0509012.
[319] J.R. Bond, L. Kofman, S. Prokushkin, P.M. Vaudrevange, Roulette inflation with Kahler moduli and their axions, Phys. Rev. D75 (2007) 123511. http://xxx.lanl.gov/abs/hep-th/0612197.
[320] H.-X. Yang, H.-L. Ma., Two-field Kahler moduli inflation on large volume moduli stabilization, JCAP 0808 (2008) 024. arXiv:0804.3653.
[321] S. Krippendorf, F. Quevedo, Metastable SUSY breaking, de Sitter moduli stabilisation and Kahler moduli inflation, J. High Energy Phys. 0911 (2009) 039. arXiv:0901.0683.
[322] J.J. Blanco-Pillado, D. Buck, E.J. Copeland, M. Gomez-Reino, N.J. Nunes, Kahler moduli inflation revisited, J. High Energy Phys. 1001 (2010) 081. arXiv:0906.3711.
[323] M. Kawasaki, K. Miyamoto, Kahler moduli double inflation, JCAP 1102 (2011) 004. arXiv:1010.3095.
[324] S. Lee, S. Nam, Káhler moduli inflation and WMAP7, Internat. J. Modern Phys. A26 (2011) 1073-1096. arXiv:1006.2876.
[325] A.R. Liddle, On the inflationary flow equations, Phys. Rev. D68 (2003) 103504. http://xxx.lanl.gov/abs/astro-ph/0307286.
[326] E.J. Copeland, I.J. Grivell, E.W. Kolb, A.R. Liddle, On the reliability of inflaton potential reconstruction, Phys. Rev. D58 (1998) 043002. http://xxx.lanl.gov/abs/astro-ph/9802209.
[327] E. Ramirez, A.R. Liddle, Stochastic approaches to inflation model building, Phys. Rev. D71 (2005) 123510. http://xxx.lanl.gov/abs/astro-ph/0502361.
[328] S.R. Coleman, E.J. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking, Phys. Rev. D7 (1973) 1888-1910.
[329] P.M. Stevenson, The Gaussian effective potential. 1. Quantum mechanics, Phys. Rev. D30 (1984) 1712.
[330] P.M. Stevenson, The Gaussian effective potential. 2. Lambda phi**4 field theory, Phys. Rev. D32 (1985) 1389-1408.
[331] P.M. Stevenson, I. Roditi, The Gaussian effective potential. III. phi**6 theory and bound states, Phys. Rev. D33 (1986) 2305-2315.
[332] P.M. Stevenson, Dimensional continuation and the two lambda phi**4 in four-dimensions theories, Z. Phys. C35 (1987) 467.
[333] P.M. Stevenson, R. Tarrach, The return of lambda phi**4, Phys. Lett. B176 (1986) 436.
[334] P.M. Stevenson, B. Alles, R. Tarrach, O(n) symmetric lambda phi**4 theory: the Gaussian effective potential approach, Phys. Rev. D35 (1987) 2407.
[335] P.M. Stevenson, G. Hajj, J. Reed, Fermions and the Gaussian effective potential, Phys. Rev. D34 (1986) 3117.
[336] G. Hajj, P.M. Stevenson, Finite temperature effects on the Gaussian effective potential, Phys. Rev. D37 (1988) 413.
[337] R. Ibanez-Meier, I. Stancu, P.M. Stevenson, Gaussian effective potential for the U(1) Higgs model, Z. Phys. C70 (1996) 307-320. http://xxx.lanl.gov/abs/hepph/9207276.
[338] L. Abbott, Gravitational effects on the $\operatorname{SU}(5)$ breaking phase transition for a Coleman-Weinberg potential, Nuclear Phys. B185 (1981) 233.
[339] J.R. Ellis, D.V. Nanopoulos, K.A. Olive, K. Tamvakis, Primordial supersymmetric inflation, Nuclear Phys. B221 (1983) 524.
[340] A. Albrecht, L.G. Jensen, P.J. Steinhardt, Inflation in SU(5) gut models coupled to gravity, Nucl. Phys. B239 (1984) 290.
[341] Q. Shafi, A. Vilenkin, Inflation with SU(5), Phys. Rev. Lett. 52 (1984) 691-694.
[342] A. Albrecht, R.H. Brandenberger, On the realization of new inflation, Phys. Rev. D31 (1985) 1225.
[343] M.U. Rehman, Q. Shafi, J.R. Wickman, GUT inflation and proton decay after WMAP5, Phys. Rev. D78 (2008) 123516. arXiv:0810.3625.
[344] R. Langbein, K. Langfeld, H. Reinhardt, L. von Smekal, Natural slow roll inflation, Mod. Phys. Lett. A11 (1996) 631-646. http:/|xxx.lanl.gov/abs/hepph/9310335.
[345] P. Gonzalez-Diaz, Primordial Kaluza-Klein inflation, Phys. Lett. B176 (1986) 29-32.
[346] J. Yokoyama, Chaotic new inflation and primordial spectrum of adiabatic fluctuations, Phys. Rev. D59 (1999) 107303.
[347] Y.-g. Gong, Constraints on inflation in Einstein-Brans-Dicke frame, Phys. Rev. D59 (1999) 083507. http://xxx.lanl.gov/abs/gr-qc/9808057.
[348] P. Binetruy, G. Dvali, D term inflation, Phys. Lett. B388 (1996) 241-246. http://xxx.lanl.gov/abs/hep-ph/9606342.
[349] E. Halyo, Hybrid inflation from supergravity D terms, Phys. Lett. B387 (1996) 43-47. http://xxx.lanl.gov/abs/hep-ph/9606423.
[350] G. Dvali, Natural inflation in SUSY and gauge mediated curvature of the flat directions, Phys. Lett. B387 (1996) 471-477. http://xxx.lanl.gov/abs/hepph/9605445.
[351] G. Dvali, Q. Shafi, S. Solganik, D-brane inflation. hep-th/0105203.
[352] L. Covi, Models of inflation supersymmetry breaking and observational constraints. hep-ph/0012245.
[353] A. Safsafi, A. Bouaouda, R. Zarrouki, H. Chakir, M. Bennai, Supersymmetric braneworld inflation in light of WMAP7 observations, Int. J. Theor. Phys. 51 (2012) 1774-1782.
[354] T. Matsuda, Successful D term inflation with moduli, Phys. Lett. B423 (1998) 35-39. http://xxx.lanl.gov/abs/hep-ph/9705448.
[355] J. Espinosa, A. Riotto, G.G. Ross, D-term inflation in superstring theories, Nuclear Phys. B531 (1998) 461-477. http://xxx.lanl.gov/abs/hep-ph/9804214.
[356] C.F. Kolda, D.H. Lyth, D term inflation and M theory. hep-ph/9812234.
[357] E. Halyo, D term inflation in type I string theory, Phys. Lett. B454 (1999) 223-227. http://xxx.lanl.gov/abs/hep-ph/9901302.
[358] D. Suematsu, D term inflation and neutrino mass, J. High Energy Phys. 0210 (2002) 014. http://xxx.lanl.gov/abs/hep-ph/0207041.
[359] A.-C. Davis, M. Majumdar, Inflation in supersymmetric cosmic string theories, Phys. Lett. B460 (1999) 257-262. http://xxx.lanl.gov/abs/hep-ph/9904392.
[360] J. Urrestilla, A. Achucarro, A. Davis, D term inflation without cosmic strings, Phys. Rev. Lett. 92 (2004) 251302. http://xxx.lanl.gov/abs/hep-th/0402032.
[361] C.-M. Lin, J. Mcdonald, Supergravity modification of D-term hybrid inflation: Solving the cosmic string and spectral index problems via a right-handed sneutrino, Phys. Rev. D74 (2006) 063510. http://xxx.lanl.gov/abs/hepph/0604245.
[362] C.-M. Lin, J. Mcdonald, Supergravity and two-field inflation effects in righthanded sneutrino modified D-term inflation, Phys. Rev. D77 (2008) 063529. arXiv:0710.4273.
[363] M. Kawasaki, F. Takahashi, Inflation model with lower multipoles of the CMB suppressed, Phys. Lett. B570 (2003) 151-153. http://xxx.lanl.gov/abs/hepph/0305319.
[364] M. Gomez-Reino, I. Zavala, Recombination of intersecting D-branes and cosmological inflation, J. High Energy Phys. 0209 (2002) 020. http://xxx.lanl.gov/abs/hep-th/0207278.
[365] E. Halyo, P-term inflation on D-branes. hep-th/0405269.
[366] A. Hebecker, S.C. Kraus, D. Lust, S. Steinfurt, T. Weigand, Fluxbrane inflation, Nuclear Phys. B854 (2012) 509-551. arXiv:1104.5016.
[367] N.T. Jones, H. Stoica, S.H. Tye, Brane interaction as the origin of inflation, J. High Energy Phys. 0207 (2002) 051. http://xxx.lanl.gov/abs/hep-th/0203163.
[368] E. Halyo, Inflation on fractional branes: D-brane inflation as D term inflation, J. High Energy Phys. 0407 (2004) 080. http://xxx.lanl.gov/abs/hepth/0312042.
[369] K. Dasgupta, J.P. Hsu, R. Kallosh, A.D. Linde, M. Zagermann, D3/D7 brane inflation and semilocal strings, J. High Energy Phys. 0408 (2004) 030. http://xxx.lanl.gov/abs/hep-th/0405247.
[370] J. Mcdonald, F term hybrid inflation, the eta problem and extra dimensions, J. High Energy Phys. 0212 (2002) 029. http://xxx.lanl.gov/abs/hepph/0201016.
[371] G. Panotopoulos, D-term inflation in D-brane cosmology, Phys. Lett. B623 (2005) 185-191. http://xxx.lanl.gov/abs/hep-ph/0503071.
[372] E. Halyo, Inflation in Wess-Zumino Models. arXiv:1001.4812.
[373] C. Vayonakis, Natural values of coupling constants and cosmological inflation in a supersymmetric model, Phys. Lett. B123 (1983) 396.
[374] A.A. Starobinsky, A new type of isotropic cosmological models without singularity, Phys. Lett. B91 (1980) 99-102.
[375] K. Stelle, Classical gravity with higher derivatives, Gen. Rel. Grav. 9 (1978) 353-371.
[376] P. Teyssandier, P. Tourrenc, The Cauchy problem for the $\mathrm{R}+\mathrm{R}^{* *} 2$ theories of gravity without torsion, J. Math. Phys. 24 (1983) 2793.
[377] K.-i. Maeda, Towards the Einstein-Hilbert action via conformal transformation, Phys. Rev. D39 (1989) 3159.
[378] D. Wands, Extended gravity theories and the Einstein-Hilbert action, Classical Quantum Gravity 11 (1994) 269-280. http://xxx.lanl.gov/abs/grqc/9307034.
[379] A. De Felice, S. Tsujikawa, J. Elliston, R. Tavakol, Chaotic inflation in modified gravitational theories, JCAP 1108 (2011) 021. arXiv:1105.4685.
[380] A. De Felice, S. Tsujikawa, $f(\mathrm{R})$ theories, Living Rev. Rel. 13 (2010) 3. arXiv: 1002.4928.
[381] L. Kofman, A.D. Linde, A.A. Starobinsky, Inflationary universe generated by the combined action of a scalar field and gravitational vacuum polarization, Phys. Lett. B157 (1985) 361-367.
[382] S. Kaneda, S.V. Ketov, N. Watanabe, Slow-roll inflation in ( $\mathrm{R}+\mathrm{R}^{*} 4$ ) gravity, Classical Quantum Gravity 27 (2010) 145016. arXiv:1002.3659.
[383] S.V. Ketov, A.A. Starobinsky, Embedding $\left(R+R^{2}\right)$-inflation into supergravity, Phys. Rev. D83 (2011) 063512. arXiv:1011.0240.
[384] J. Goldstone, Field theories with superconductor solutions, Nuovo Cim. 19 (1961) 154-164.
[385] E. Witten, Superconducting strings, Nuclear Phys. B249 (1985) 557-592.
[386] P. Peter, Spontaneous current generation in cosmic strings, Phys. Rev. D49 (1994) 5052-5062. http://xxx.lanl.gov/abs/hep-ph/9312280.
[387] B. Carter, P. Peter, Supersonic string models for Witten vortices, Phys. Rev. D52 (1995) 1744-1748. http://xxx.lanl.gov/abs/hep-ph/9411425.
[388] P. Peter, Surface current carrying domain walls, J. Phys. A29 (1996) 5125-5136. http://xxx.lanl.gov/abs/hep-ph/9503408.
[389] P. Peter, C. Ringeval, Fermionic current carrying cosmic strings: Zero temperature limit and equation of state. hep-ph/0011308.
[390] C. Ringeval, Equation of state of cosmic strings with fermionic current carriers, Phys. Rev. D63 (2001) 063508. http://xxx.lanl.gov/abs/hep-ph/0007015.
[391] C. Ringeval, Fermionic massive modes along cosmic strings, Phys. Rev. D64 (2001) 123505. http://xxx.lanl.gov/abs/hep-ph/0106179.
[392] A.D. Linde, D.A. Linde, Topological defects as seeds for eternal inflation, Phys. Rev. D50 (1994) 2456-2468. http://xxx.lanl.gov/abs/hep-th/9402115.
[393] A. Vilenkin, Topological inflation, Phys. Rev. Lett. 72 (1994) 3137-3140. http://xxx.lanl.gov/abs/hep-th/9402085.
[394] A.M. Green, A.R. Liddle, Open inflationary universes in the induced gravity theory, Phys. Rev. D55 (1997) 609-615. http://xxx.lanl.gov/abs/astroph/9607166.
[395] J. Garcia-Bellido, A.R. Liddle, Complete power spectrum for an induced gravity open inflation model, Phys. Rev. D55 (1997) 4603-4613. http://xxx. lanl.gov/abs/astro-ph/9610183.
[396] A.D. Linde, Supergravity and inflationary universe, Pisma Zh. Eksp. Teor. Fiz. 37 (1983) 606-608 (in Russian).
[397] A.D. Linde, Primordial inflation without primordial monopoles, Phys. Lett. B132 (1983) 317-320.
[398] J. Casas, C. Munoz, Inflation from superstrings, Phys. Lett. B216 (1989) 37.
[399] J. Casas, J. Moreno, C. Munoz, M. Quiros, Cosmological implications of an anomalous $\mathrm{U}(1)$ : Inflation, cosmic strings and constraints on superstring parameters, Nuclear Phys. B328 (1989) 272.
[400] J. Cervantes-Cota, H. Dehnen, Induced gravity inflation in the standard model of particle physics, Nuclear Phys. B442 (1995) 391-412. http://xxx.lanl.gov/abs/astro-ph/9505069.
[401] S.H. Alexander, Inflation from D - anti-D-brane annihilation, Phys. Rev. D65 (2002) 023507. http://xxx.lanl.gov/abs/hep-th/0105032.

402] R. Easther, J. Khoury, K. Schalm, Tuning locked inflation: Supergravity versus phenomenology, JCAP 0406 (2004) 006. http://xxx.lanl.gov/abs/hepth/0402218.
[403] J.-O. Gong, Modular thermal inflation without slow-roll approximation, Phys. Lett. B637 (2006) 149-155. http://xxx.lanl.gov/abs/hep-ph/0602106.
[404] R. Kallosh, A.D. Linde, Testing string theory with CMB, JCAP 0704 (2007) 017. arXiv:0704.0647.
[405] G. Lazarides, A. Vamvasakis, Standard-smooth hybrid inflation, Phys. Rev. D76 (2007) 123514. arXiv:0709.3362.
[406] M.U. Rehman, Q. Shafi, Higgs inflation quantum smearing and the tensor to scalar ratio, Phys. Rev. D81 (2010) 123525. arXiv:1003.5915.
[407] F. Bauer, D.A. Demir, Higgs-palatini inflation and unitarity, Phys. Lett. B698 (2011) 425-429. arXiv: 1012.2900.
[408] A.O. Barvinsky, Standard model Higgs inflation: CMB, Higgs mass and quantum cosmology, Prog. Theor. Phys. Suppl. 190 (2011) 1-19. arXiv: 1012.4523.
[409] G. Barenboim, Inflation might be caused by the right: Handed neutrino, J. High Energy Phys. 0903 (2009) 102. arXiv:0811.2998.
[410] R. Kallosh, A. Linde, New models of chaotic inflation in supergravity, JCAP 1011 (2010) 011. arXiv: 1008.3375.
[411] L. Boubekeur, D. Lyth, Hilltop inflation, JCAP 0507 (2005) 010. hep$\mathrm{ph} / 0502047$. Latex, 20 pages, 5 figures. Minor changes, references added.
[412] K. Tzirakis, W.H. Kinney, Inflation over the hill, Phys. Rev. D75 (2007) 123510. http://xxx.lanl.gov/abs/astro-ph/0701432.
[413] B.K. Pal, S. Pal, B. Basu, Mutated hilltop inflation: A natural choice for early universe, JCAP 1001 (2010) 029. arXiv:0908.2302.
[414] B.K. Pal, S. Pal, B. Basu, A semi-analytical approach to perturbations in mutated hilltop inflation, Internat. J. Modern Phys. D21 (2012) 1250017. arXiv:1010.5924.
[415] M. Fairbairn, L. Lopez honorez, M. Tytgat, Radion assisted gauge inflation, Phys. Rev. D67 (2003) 101302. http://xxx.lanl.gov/abs/hep-ph/0302160.
[416] A. de la Macorra, S. Lola, Inflation in S dual superstring models, Phys. Lett. B373 (1996) 299-305. http://xxx.lanl.gov/abs/hep-ph/9511470.
[417] T. Gherghetta, C.F. Kolda, S.P. Martin, Flat directions in the scalar potential of the supersymmetric standard model, Nuclear Phys. B468 (1996) 37-58. http://xxx.lanl.gov/abs/hep-ph/9510370.
[418] K. Enqvist, A. Mazumdar, Cosmological consequences of MSSM flat directions, Phys. Rep. 380 (2003) 99-234. http://xxx.lanl.gov/abs/hep-ph/0209244.
[419] M. Dine, L. Randall, S.D. Thomas, Baryogenesis from flat directions of the supersymmetric standard model, Nuclear Phys. B458 (1996) 291-326. http://xxx.lanl.gov/abs/hep-ph/9507453.
[420] R. Allahverdi, K. Enqvist, J. Garcia-Bellido, A. Mazumdar, Gauge invariant MSSM inflaton, Phys. Rev. Lett. 97 (2006) 191304. http://xxx.lanl.gov/abs/ hep-ph/0605035.
[421] J. Garcia-Bellido, Flat direction MSSM (A-term) inflation, AIP Conf. Proc. 878 (2006) 277-283. http://xxx.lanl.gov/abs/hep-ph/0610152.
[422] R. Allahverdi, MSSM flat direction inflation, eConf C0605151 (2006) 0020. http://xxx.lanl.gov/abs/hep-ph/0610180.
[423] D.H. Lyth, MSSM inflation, JCAP 0704 (2007) 006. http://xxx.lanl.gov/abs/ hep-ph/0605283.
[424] R. Allahverdi, A. Mazumdar, Spectral tilt in A-term inflation. hepph/0610069.
[425] R. Allahverdi, B. Dutta, A. Mazumdar, Probing the parameter space for an MSSM inflation and the neutralino dark matter, Phys. Rev. D75 (2007) 075018. http://xxx.lanl.gov/abs/hep-ph/0702112.
[426] K. Enqvist, L. Mether, S. Nurmi, Supergravity origin of the MSSM inflation, JCAP 0711 (2007) 014. arXiv:0706.2355.
[427] R. Allahverdi, B. Dutta, A. Mazumdar, Attraction towards an inflection point inflation, Phys. Rev. D78 (2008) 063507. arXiv:0806.4557.
[428] K. Kamada, J. Yokoyama, On the realization of the MSSM inflation, Progr. Theoret. Phys. 122 (2010) 969-986. arXiv:0906.3402.
[429] R. Allahverdi, B. Dutta, Y. Santoso, MSSM inflation dark matter the LHC, Phys. Rev. D82 (2010) 035012. http://xxx.lanl.gov/abs/1004.2741.
[430] K. Enqvist, A. Mazumdar, P. Stephens, Inflection point inflation within supersymmetry, JCAP 1006 (2010) 020. arXiv: 1004.3724.
[431] K. Kohri, C.-M. Lin, Hilltop supernatural inflation and gravitino problem, JCAP 1011 (2010) 010. arXiv:1008.3200.
[432] A.D. Linde, A. Mezhlumian, Inflation with Omega not = 1, Phys. Rev. D52 (1995) 6789-6804. http://xxx.lanl.gov/abs/astro-ph/9506017.
[433] A.D. Linde, A Toy model for open inflation, Phys. Rev. D59 (1999) 023503. http://xxx.lanl.gov/abs/hep-ph/9807493.
[434] R.K. Jain, P. Chingangbam, J.-O. Gong, L. Sriramkumar, T. Souradeep, Punctuated inflation and the low CMB multipoles, JCAP 0901 (2009) 009. arXiv:0809.3915.
[435] R.K. Jain, P. Chingangbam, L. Sriramkumar, T. Souradeep, The tensor-to-scalar ratio in punctuated inflation, Phys. Rev.D82 (2010) 023509. arXiv:0904.2518.
[436] D.A. Lowe, S. Roy, Punctuated eternal inflation via AdS/CFT, Phys. Rev. D82 (2010) 063508. arXiv:1004.1402.
[437] R. Allahverdi, B. Dutta, A. Mazumdar, Unifying inflation and dark matter with neutrino masses, Phys. Rev. Lett. 99 (2007) 261301. arXiv:0708.3983.
[438] R. Allahverdi, A. Kusenko, A. Mazumdar, A-term inflation and the smallness of neutrino masses, JCAP 0707 (2007) 018. http://xxx.lanl.gov/abs/hepph/0608138.
[439] W.H. Kinney, A. Riotto, Dynamical supersymmetric inflation, Astropart. Phys. 10 (1999) 387-395. http://xxx.lanl.gov/abs/hep-ph/9704388.
[440] L.-M. Wang, V.F. Mukhanov, P.J. Steinhardt, On the problem of predicting inflationary perturbations, Phys. Lett. B414 (1997) 18-27. http://xxx.lanl.gov/abs/astro-ph/9709032.
[441] M. Drees, E. Erfani, Running spectral index and formation of primordial black hole in single field inflation models, JCAP 1201 (2012) 035. arXiv:1110.6052.
[442] M. Drees, E. Erfani, Dark matter primordial black holes and inflation models. arXiv:1205.4012.
[443] A. Vallinotto, E.J. Copeland, E.W. Kolb, A.R. Liddle, D.A. Steer, Inflationary potentials yielding constant scalar perturbation spectral indices, Phys. Rev. D69 (2004) 103519. http://xxx.lanl.gov/abs/astro-ph/0311005.
[444] D.J. Schwarz, C.A. Terrero-Escalante, Primordial fluctuations and cosmological inflation after WMAP 1.0, JCAP 0408 (2004) 003. http://xxx.lanl.gov/abs/ hep-ph/0403129.
[445] D. Veberic, Lambert w function for applications in physics, CoRR abs/1209.0735 2012.
[446] E. Witten, On background independent open string field theory, Phys. Rev. D46 (1992) 5467-5473. http://xxx.lanl.gov/abs/hep-th/9208027.
[447] E. Witten, Some computations in background independent off-shell string theory, Phys. Rev. D47 (1993) 3405-3410. http://xxx.lanl.gov/abs/hepth/9210065.
[448] A.A. Gerasimov, S.L. Shatashvili, On exact tachyon potential in open string field theory, J. High Energy Phys. 0010 (2000) 034. http://xxx.lanl.gov/abs/ hep-th/0009103.
[449] D. Kutasov, M. Marino, G.W. Moore, Some exact results on tachyon condensation in string field theory, J. High Energy Phys. 0010 (2000) 045. http://xxx.lanl.gov/abs/hep-th/0009148.
[450] L. Kofman, A.D. Linde, Problems with tachyon inflation, J. High Energy Phys. 0207 (2002) 004. http://xxx.lanl.gov/abs/hep-th/0205121.
[451] D. Choudhury, D. Ghoshal, D.P. Jatkar, S. Panda, On the cosmological relevance of the tachyon, Phys. Lett. B544 (2002) 231-238. http://xxx.lanl.gov/abs/hepth/0204204.
[452] J.E. Lidsey, D. Seery, Primordial non-gaussianity and gravitational waves: Observational tests of brane inflation in string theory, Phys. Rev. D75 (2007) 043505. http://xxx.lanl.gov/abs/astro-ph/0610398.
[453] J.A. Minahan, B. Zwiebach, Field theory models for tachyon and gauge field string dynamics, J. High Energy Phys. 0009 (2000) 029. http://xxx.lanl.gov/ abs/hep-th/0008231.
[454] E. Witten, Mass hierarchies in supersymmetric theories, Phys. Lett. B105 (1981) 267.
[455] L. O'Raifeartaigh, Spontaneous symmetry breaking for chiral scalar superfields, Nuclear Phys. B96 (1975) 331.
[456] E. Witten, Dynamical breaking of supersymmetry, Nuclear Phys. B188 (1981) 513.
[457] S. Dimopoulos, S. Raby, Geometric hierarchy, Nuclear Phys. B219 (1983) 479.
[458] A. Albrecht, S. Dimopoulos, W. Fischler, E.W. Kolb, S. Raby, et al., New inflation in supersymmetric theories, Nuclear Phys. B229 (1983) 528.
[459] E. Papantonopoulos, T. Uematsu, T. Yanagida, Natural chaotic inflation, Phys. Lett. B183 (1987) 282.
[460] M. Pollock, On the possibility of chaotic inflation from a softly broken superconformal invariance, Phys. Lett. B194 (1987) 518-522.
[461] K.-i. Kobayashi, T. Uematsu, Nonlinear realization of superconformal symmetry, Nuclear Phys. B263 (1986) 309.
[462] P. Binetruy, M. Gaillard, Candidates for the inflaton field in superstring models, Phys. Rev. D34 (1986) 3069-3083.
[463] W.H. Kinney, K. Mahanthappa, Inflation from symmetry breaking below the Planck scale, Phys. Lett. B383 (1996) 24-27. http://xxx.lanl.gov/abs/hepph/9511460.
[464] M. Kawasaki, M. Yamaguchi, A supersymmetric topological inflation model, Phys. Rev. D65 (2002) 103518. http://xxx.lanl.gov/abs/hep-ph/0112093.
[465] K. Kumekawa, T. Moroi, T. Yanagida, Flat potential for inflaton with a discrete R invariance in supergravity, Progr. Theoret. Phys. 92 (1994) 437-448. http://xxx.lanl.gov/abs/hep-ph/9405337.
[466] J.A. Adams, G.G. Ross, S. Sarkar, Natural supergravity inflation, Phys. Lett. B391 (1997) 271-280. hep-ph/9608336.
[467] K.-I. Izawa, T. Yanagida, Natural new inflation in broken supergravity, Phys. Lett. B393 (1997) 331-336. http://xxx.lanl.gov/abs/hep-ph/9608359.
[468] K. Izawa, M. Kawasaki, T. Yanagida, R invariant topological inflation, Progr. Theoret. Phys. 101 (1999) 1129-1133. http://xxx.lanl.gov/abs/hepph/9810537.
[469] W. Buchmuller, K. Hamaguchi, M. Ratz, T. Yanagida, Gravitino and goldstino at colliders. hep-ph/0403203.
[470] T. Banks, M. Berkooz, S. Shenker, G.W. Moore, P. Steinhardt, Modular cosmology, Phys. Rev. D52 (1995) 3548-3562. http://xxx.lanl.gov/abs/hepth/9503114.
[471] Y. Himemoto, M. Sasaki, Brane world inflation without inflaton on the brane, Phys. Rev. D63 (2001) 044015. http://xxx.lanl.gov/abs/gr-qc/0010035.
[472] N. Sago, Y. Himemoto, M. Sasaki, Quantum fluctuations in brane world inflation without inflaton on the brane, Phys. Rev. D65 (2002) 024014. http://xxx.lanl.gov/abs/gr-qc/0104033.
[473] X. Chen, Inflation from warped space, J. High Energy Phys. 0508 (2005) 045. http://xxx.lanl.gov/abs/hep-th/0501184.
[474] J.D. Barrow, Graduated inflationary universes, Phys. Lett. B235 (1990) 40-43.
[475] J.D. Barrow, P. Saich, The behavior of intermediate inflationary universes, Phys. Lett. B249 (1990) 406-410.
[476] J.D. Barrow, A.R. Liddle, Perturbation spectra from intermediate inflation, Phys. Rev.D47 (1993)5219-5223. http://xxx.lanl.gov/abs/astro-ph/9303011.
[477] J.D. Barrow, A.R. Liddle, C. Pahud, Intermediate inflation in light of the three-year WMAP observations, Phys. Rev. D74 (2006) 127305. http://xxx.lanl.gov/abs/astro-ph/0610807.
[478] J.D. Barrow, String-driven inflationary and deflationary cosmological models, Nuclear Phys. B310 (1988) 743-763.
[479] S. del Campo, R. Herrera, A. Toloza, Tachyon field in intermediate inflation, Phys. Rev. D79 (2009) 083507. arXiv:0904.1032.
[480] H. Farajollahi, A. Ravanpak, Tachyon field in intermediate inflation on the brane, Phys. Rev. D84 (2011) 084017. arXiv:1106.2211.
[481] S. del Campo, R. Herrera, Warm-intermediate inflationary universe model, JCAP 0904 (2009) 005. arXiv:0903.4214.
[482] S. del Campo, R. Herrera, J. Saavedra, Tachyon warm inflationary universe model in the weak dissipative regime, Eur. Phys. J. C59 (2009) 913-916. arXiv:0812.1081.
[483] R. Herrera, N. Videla, Intermediate inflation in Gauss-Bonnet braneworld, Eur. Phys. J. C67 (2010) 499-505. arXiv: 1003.5645.
[484] A. Cid, S. del Campo, Constraints from CMB in the intermediate Brans-Dicke inflation, JCAP 1101 (2011) 013. arXiv:1101.4588.
[485] A. Cid, S. del Campo, Intermediate inflation in the Jordan-Brans-Dicke theory, AIP Conf. Proc. 1471 (2012) 114-117. arXiv:1210.5273.
[486] J.D. Barrow, N.J. Nunes, Dynamics of logamediate inflation, Phys. Rev. D76 (2007) 043501. arXiv:0705.4426.
[487] P. Parsons, J.D. Barrow, Generalized scalar field potentials and inflation, Phys. Rev. D51 (1995) 6757-6763. http://xxx.lanl.gov/abs/astro-ph/9501086.
[488] J.L. Davis, T.S. Levi, M. Van Raamsdonk, K.R.L. Whyte, Twisted inflation, JCAP 1009 (2010) 032. arXiv:1004.5385.
[489] D.H. Lyth, A. Riotto, Generating the curvature perturbation at the end of inflation in string theory, Phys. Rev. Lett. 97 (2006) 121301. http://xxx.lanl.gov/abs/astro-ph/0607326.
[490] J. Bueno Sanchez, K. Dimopoulos, D.H. Lyth, A-term inflation and the MSSM, JCAP 0701 (2007) 015. http://xxx.lanl.gov/abs/hep-ph/0608299.
[491] R. Allahverdi, K. Enqvist, J. Garcia-Bellido, A. Jokinen, A. Mazumdar, MSSM flat direction inflation: Slow roll, stability, fine tunning and reheating, JCAP 0706 (2007) 019. http://xxx.lanl.gov/abs/hep-ph/0610134.
[492] A. Chatterjee, A. Mazumdar, Tuned MSSM Higgses as an inflaton, JCAP 1109 (2011) 009. arXiv:1103.5758.
[493] S. Hotchkiss, A. Mazumdar, S. Nadathur, Inflection point inflation: WMAP constraints and a solution to the fine-tuning problem, JCAP 1106 (2011) 002. arXiv:1101.6046.
[494] C.S. Aulakh, I. Garg, Supersymmetric seesaw inflation, Phys. Rev. D86 (2012) 065001. arXiv:1201.0519.
[495] C.S. Aulakh, Susy Seesaw Inflation and NMSO(10)GUT. arXiv: 1210.2042.
[496] E. Dudas, N. Kitazawa, S. Patil, A. Sagnotti, CMB Imprints of a Pre-Inflationary Climbing Phase. arXiv:1202.6630.
[497] J. Martin, C. Ringeval, Superimposed oscillations in the WMAP data? Phys. Rev. D69 (2004) 083515. http://xxx.lanl.gov/abs/astro-ph/0310382.
[498] J. Martin, C. Ringeval, Addendum to 'Superimposed oscillations in the WMAP data?', Phys. Rev. D69 (2004) 127303. http://xxx.lanl.gov/abs/astroph/0402609.
[499] J. Martin, C. Ringeval, Exploring the superimposed oscillations parameter space, JCAP 0501 (2005) 007. http://xxx.lanl.gov/abs/hep-ph/0405249.
[500] J. Trudeau, J.M. Cline, Warped radion inflation, J. High Energy Phys. 1202 (2012) 081. arXiv:1111.4257.
[501] C.P. Burgess, J.M. Cline, K. Dasgupta, H. Firouzjahi, Uplifting and inflation with D3 branes, J. High Energy Phys. 0703 (2007) 027. http://xxx.lanl.gov/abs/hepth/0610320.
[502] A. Krause, E. Pajer, Chasing brane inflation in string-theory, JCAP 0807 (2008) 023. arXiv:0705.4682.
[503] D. Baumann, A. Dymarsky, I.R. Klebanov, L. Mcallister, P.J. Steinhardt, A Delicate universe, Phys. Rev. Lett. 99 (2007) 141601. arXiv:0705.3837.
[504] O. DeWolfe, L. Mcallister, G. Shiu, B. Underwood, D3-brane vacua in stabilized compactifications, J. High Energy Phys. 0709 (2007) 121. http://xxx.lanl.gov/abs/hep-th/0703088.
[505] E. Pajer, Inflation at the tip, JCAP 0804 (2008) 031. arXiv:0802.2916.
[506] F. Chen, H. Firouzjahi, Dynamics of D3-D7 brane inflation in throats, J. High Energy Phys. 0811 (2008) 017. arXiv:0807.2817.
[507] I.R. Klebanov, M.J. Strassler, Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities, J. High Energy Phys. 0008 (2000) 052. http://xxx.lanl.gov/abs/hep-th/0007191.
[508] P. Candelas, X.C. de la Ossa, Comments on conifolds, Nuclear Phys. B342 (1990) 246-268.
[509] S. Kuperstein, Meson spectroscopy from holomorphic probes on the warped deformed conifold, J. High Energy Phys. 0503 (2005) 014. http://xxx.lanl.gov/abs/hep-th/0411097.
[510] S. Kachru, R. Kallosh, A.D. Linde, S.P. Trivedi, De Sitter vacua in string theory, Phys. Rev. D68 (2003) 046005. http://xxx.lanl.gov/abs/hep-th/0301240.
[511] J.S. Alcaniz, F. Carvalho, Beta-exponential inflation, Europhys. Lett. 79 (2007) 39001. http://xxx.lanl.gov/abs/astro-ph/0612279.
[512] C. Panagiotakopoulos, Hybrid inflation with quasicanonical supergravity, Phys. Lett. B402 (1997) 257-262. http://xxx.lanl.gov/abs/hep-ph/9703443.
[513] C. Panagiotakopoulos, Blue perturbation spectra from hybrid inflation with canonical supergravity, Phys. Rev. D55 (1997) 7335-7339. http://xxx.lanl.gov/abs/hep-ph/9702433.
[514] L.M. Hall, H.V. Peiris, Cosmological constraints on dissipative models of inflation, JCAP 0801 (2008) 027. arXiv:0709.2912.
[515] B. Kyae, Spectral index and non-Gaussianity in supersymmetric hybrid inflation, Eur. Phys. J. C72 (2012) 1857. arXiv:0910.4092.
[516] H. Hodges, G. Blumenthal, Arbitrariness of inflationary fluctuation spectra, Phys. Rev. D42 (1990) 3329-3333.
[517] G. Veneziano, S. Yankielowicz, An effective Lagrangian for the pure $N=1$ supersymmetric Yang-Mills theory, Phys. Lett. B113 (1982) 231.
[518] P. Channuie, J.J. Jorgensen, F. Sannino, Composite inflation from super Yang-Mills, orientifold and one-flavor QCD, Phys. Rev. D86 (2012) 125035. arXiv:1209.6362.
[519] Q. Shafi, V.N. Senoguz, Coleman-Weinberg potential in good agreement with wmap, Phys. Rev. D73 (2006) 127301. http://xxx.lanl.gov/abs/astroph/0603830.
[520] S. Choudhury, S. Pal, Brane inflation in background supergravity, Phys. Rev. D85 (2012) 043529. arXiv: 1102.4206
[521] S. Choudhury, S. Pal, Brane inflation: A field theory approach in background supergravity. arXiv:1209.5883.
[522] I. Moss, Primordial inflation with spontaneous symmetry breaking, Phys. Lett. B154 (1985) 120.
[523] B. Hu., D. O'Connor, Mixmaster inflation, Phys. Rev. D34 (1986) 2535.
[524] M. Dine, A. Riotto, An Inflaton candidate in gauge mediated supersymmetry breaking, Phys. Rev. Lett. 79 (1997) 2632-2635. http://xxx.lanl.gov/abs/hepph/9705386.
[525] A. Riotto, Inflation and the nature of supersymmetry breaking, Nuclear Phys. B515 (1998) 413-435. http://xxx.lanl.gov/abs/hep-ph/9707330.
[526] D. Cormier, R. Holman, Spinodal inflation, Phys. Rev. D60 (1999) 041301. http://xxx.lanl.gov/abs/hep-ph/9812476.
[527] D. Cormier, R. Holman, Spinodal decomposition and inflation: Dynamics and metric perturbations, Phys. Rev. D62 (2000) 023520. http://xxx.lanl.gov/abs/hep-ph/9912483.
[528] S. Bhattacharya, D. Choudhury, D.P. Jatkar, A.A. Sen, Brane dynamics in the Randall-Sundrum model, inflation and graceful exit, Classical Quantum Gravity 19 (2002) 5025-5038. http://xxx.lanl.gov/abs/hep-th/0103248.
[529] W.-F. Wang, Exact solution in the cosmological chaotic inflation model with induced gravity, Phys. Lett. A328 (2004) 255-260.
[530] T. Fukuyama, T. Kikuchi, W. Naylor, Electroweak inflation and reheating in the NMSSM. hep-ph/0511105.
[531] S. Antusch, Sneutrino hybrid inflation, AIP Conf. Proc. 878 (2006) 284-290. http://xxx.lanl.gov/abs/hep-ph/0608261.
[532] J. Blanco-Pillado, C. Burgess, J.M. Cline, C. Escoda, M. Gomez-Reino, et al., Racetrack inflation, J. High Energy Phys. 0411 (2004) 063. http://xxx.lanl.gov/ abs/hep-th/0406230.
[533] P. Brax, S.C. Davis, M. Postma, The Robustness of $n(s)<0.95$ in racetrack inflation, JCAP 0802 (2008) 020. arXiv:0712.0535.
[534] J.-O. Gong, N. Sahu, Inflation in minimal left-right symmetric model with spontaneous D-parity breaking, Phys. Rev. D77 (2008) 023517. arXiv:0705.0068.
[535] L.-Y. Lee, K. Cheung, C.-M. Lin, Comments on SUSY inflation models on the brane, Mod. Phys. Lett. A25 (2010) 2105-2110. arXiv:0912.5423.
[536] C.-M. Lin, K. Cheung, Reducing the spectral index in supernatural inflation, Phys. Rev. D79 (2009) 083509. arXiv:0901.3280.
[537] C.-M. Lin, Hilltop supernatural inflation, Prog. Theor. Phys. Suppl. 190 (2011) 20-25. arXiv:1012.2647.
[538] S. Khalil, A. Sil, Right-handed sneutrino inflation in SUSY B-L with inverse seesaw, Phys. Rev. D84 (2011) 103511. arXiv:1108.1973.
[539] S. Khalil, A. Sil, Sneutrino inflation in supersymmetric B-L with inverse seesaw, AIP Conf. Proc. 1467 (2012) 294-297.
[540] S. Antusch, D. Nolde, Káhler-driven Tribrid Inflation. arXiv:1207.6111.
[541] I. Masina, A. Notari, Standard model false vacuum inflation: Correlating the tensor-to-scalar ratio to the top quark and Higgs Boson masses, Phys. Rev. Lett. 108 (2012) 191302. arXiv:1112.5430.
[542] I. Masina, A. Notari, The Higgs mass range from standard model false vacuum inflation in scalar-tensor gravity, Phys. Rev. D85 (2012) 123506. arXiv:1112.2659.
[543] I. Masina, A. Notari, Inflation from the Higgs field false vacuum with hybrid potential, JCAP 1211 (2012) 031. arXiv: 1204.4155.
[544] P. Peebles, B. Ratra, Cosmology with a time variable cosmological constant, Astrophys. J. 325 (1988) L17.
[545] G. Huey, J.E. Lidsey, Inflation brane worlds and quintessence, Phys. Lett. B514 (2001) 217-225. http://xxx.lanl.gov/abs/astro-ph/0104006.
[546] A. Feinstein, Power law inflation from the rolling tachyon, Phys. Rev. D66 (2002) 063511. http://xxx.lanl.gov/abs/hep-th/0204140.
[547] M. Sami, Implementing power law inflation with rolling tachyon on the brane, Modern Phys. Lett. A18 (2003) 691. http://xxx.lanl.gov/abs/hepth/0205146.
[548] B. Wang, E. Abdalla, R.-K. Su., Dynamics and holographic discreteness of tachyonic inflation, Modern Phys. Lett. A18 (2003) 31-40. http://xxx.lanl.gov/abs/hep-th/0208023.
[549] L.R.W. Abramo, F. Finelli, Cosmological dynamics of the tachyon with an inverse power-law potential, Phys. Lett. B575 (2003) 165-171. http://xxx.lanl.gov/abs/astro-ph/0307208.
[550] P. Binetruy, Models of dynamical supersymmetry breaking and quintessence, Phys. Rev. D60 (1999) 063502. hep-ph/9810553.
[551] P. Brax, J. Martin, The robustness of quintessence, Phys. Rev. D61 (2000) 103502. http://xxx.lanl.gov/abs/astro-ph/9912046.
[552] T. Taylor, G. Veneziano, S. Yankielowicz, Supersymmetric QCD and its massless limit: An effective lagrangian analysis, Nuclear Phys. B218 (1983) 493.
[553] I. Affleck, M. Dine, N. Seiberg, Dynamical supersymmetry breaking in fourdimensions and its phenomenological implications, Nuclear Phys. B256 (1985) 557.
[554] C. Burgess, M. Majumdar, D. Nolte, F. Quevedo, G. Rajesh, et al., The inflationary brane anti-brane universe, J. High Energy Phys. 0107 (2001) 047. http://xxx.lanl.gov/abs/hep-th/0105204.
[555] G. Shiu, S.H. Tye, Some aspects of brane inflation, Phys. Lett. B516 (2001) 421-430. http://xxx.lanl.gov/abs/hep-th/0106274.
[556] J. Garcia-Bellido, Inflation from branes at angles. astro-ph/0306195.
[557] L. Pogosian, S.H. Tye, I. Wasserman, M. Wyman, Observational constraints on cosmic string production during brane inflation, Phys. Rev. D68 (2003) 023506. http://xxx.lanl.gov/abs/hep-th/0304188.
[558] T. Matsuda, F term D term and hybrid brane inflation, JCAP 0311 (2003) 003. http://xxx.lanl.gov/abs/hep-ph/0302078.
[559] T. Matsuda, Brane Q ball branonium and brane Q ball inflation, JCAP 0410 (2004) 014. http://xxx.lanl.gov/abs/hep-ph/0402223.
[560] H.-X. Yang, D3/D7 inflation in a Type-0B string background. hep-th/0504096.
[561] Q.-G. Huang, M. Li., J.-H. She, Brane inflation after wmap three year results, JCAP 0611 (2006) 010. http://xxx.lanl.gov/abs/hep-th/0604186.
[562] R. Bean, S.E. Shandera, S. Henry tye, J. Xu, Comparing brane inflation to WMAP, JCAP 0705 (2007) 004. http://xxx.lanl.gov/abs/hep-th/0702107.
[563] R.A. Battye, B. Garbrecht, A. Moss, H. Stoica, Constraints on brane inflation and cosmic strings, JCAP 0801 (2008) 020. arXiv:0710.1541.
[564] S.-H. Henry tye, Brane inflation: String theory viewed from the cosmos, Lect. Notes Phys. 737 (2008) 949-974. http://xxx.lanl.gov/abs/hep-th/0610221.
[565] R.H. Brandenberger, A.R. Frey, L.C. Lorenz, Entropy fluctuations in brane inflation models, Int. J. Mod. Phys. A24 (2009) 4327-4354. arXiv:0712.2178.
[566] L. Lorenz, Constraints on brane inflation from WMAP3. arXiv:0801.4891.
[567] Y.-Z. Ma., X. Zhang, Brane inflation revisited after WMAP five year results, JCAP 0903 (2009) 006. arXiv:0812.3421.
[568] D. Baumann, L. Mcallister, A microscopic limit on gravitational waves from Dbrane inflation, Phys. Rev. D75 (2007) 123508. http://xxx.lanl.gov/abs/hepth/0610285.
[569] L.C. Lorenz, Primordial fluctuations in string cosmology. arXiv:1002.2087.
[570] K.L. Panigrahi, H. Singh, Assisted inflation from geometric tachyon, J. High Energy Phys. 0711 (2007) 017. arXiv:0708.1679.
[571] P.S. Kwon, G.Y. Jun, K.L. Panigrahi, M. Sami, Inflation driven by single geometric tachyon with D-brane orbiting around NS5-branes, Phys. Lett. B712 (2012) 10-15. arXiv:1106.4118.
[572] P. Brax, C.A. Savoy, A. Sil, SQCD inflation \%26 SUSY breaking, J. High Energy Phys. 0904 (2009) 092. arXiv:0902.0972.
[573] R. Bean, X. Chen, H. Peiris, J. Xu., Comparing infrared dirac-born-infeld brane inflation to observations, Phys. Rev. D77 (2008) 023527. arXiv:0710.1812.
[574] S. Kachru, R. Kallosh, A.D. Linde, J.M. Maldacena, L.P. Mcallister, et al., Towards inflation in string theory, JCAP 0310 (2003) 013. http://xxx.lanl.gov/abs/hepth/0308055.
[575] E.D. Stewart, Flattening the inflaton's potential with quantum corrections, Phys. Lett. B391 (1997) 34-38. http://xxx.lanl.gov/abs/hep-ph/9606241.
[576] E.D. Stewart, Flattening the inflaton's potential with quantum corrections. 2 ., Phys. Rev. D56 (1997) 2019-2023. http://xxx.lanl.gov/abs/hep-ph/9703232.
[577] L. Covi, D.H. Lyth, L. Roszkowski, Observational constraints on an inflation model with a running mass, Phys. Rev. D60 (1999) 023509. http://xxx.lanl.gov/abs/hep-ph/9809310.
[578] L. Covi, D.H. Lyth, Running-mass models of inflation, and their observational constraints, Phys. Rev. D59 (1999) 063515. http://xxx.lanl.gov/abs/hepph/9809562.
[579] S.M. Leach, I.J. Grivell, A.R. Liddle, Black hole constraints on the running mass inflation model, Phys. Rev. D62 (2000) 043516. http://xxx.lanl.gov/abs/astroph/0004296.
[580] D.H. Lyth, Observational constraints on models of inflation from the density perturbation and gravitino production. hep-ph/0012065.
[581] L. Covi, D.H. Lyth, A. Melchiorri, New constraints on the running-mass inflation model, Phys. Rev. D67 (2003) 043507. http://xxx.lanl.gov/abs/hepph/0210395.
[582] K. Kadota, E.D. Stewart, Inflation on moduli space and cosmic perturbations, J. High Energy Phys. 0312 (2003) 008. http://xxx.lanl.gov/abs/hepph/0311240.
[583] L. Covi, D.H. Lyth, A. Melchiorri, C.J. Odman, The running-mass inflation model and WMAP, Phys. Rev. D70 (2004) 123521. http://xxx.lanl.gov/abs/astroph/0408129.
[584] A.D. Linde, Axions in inflationary cosmology, Phys. Lett. B259 (1991) 38-47.
[585] E.J. Copeland, A.R. Liddle, D.H. Lyth, E.D. Stewart, D. Wands, False vacuum inflation with Einstein gravity, Phys. Rev. D49 (1994) 6410-6433. http://xxx.lanl.gov/abs/astro-ph/9401011.
[586] C. Panagiotakopoulos, Hybrid inflation and supergravity. hep-ph/0011261.
[587] G. Lazarides, Supersymmetric hybrid inflation. hep-ph/0011130.
[588] S. Clesse, J. Rocher, Avoiding the blue spectrum and the fine-tuning of initial conditions in hybrid inflation, Phys. Rev. D79 (2009) 103507. arXiv:0809.4355.
[589] S. Clesse, C. Ringeval, J. Rocher, Fractal initial conditions and natural parameter values in hybrid inflation, Phys. Rev. D80 (2009) 123534. arXiv:0909.0402.
[590] S. Clesse, Hybrid inflation along waterfall trajectories, Phys. Rev. D83 (2011) 063518. arXiv:1006.4522.
[591] H. Kodama, K. Kohri, K. Nakayama, On the waterfall behavior in hybrid inflation, Progr. Theoret. Phys. 126 (2011) 331-350. arXiv:1102.5612.
[592] M. Bento, O. Bertolami, A. Sen, Supergravity inflation on the brane, Phys. Rev. D67 (2003) 023504. http://xxx.lanl.gov/abs/gr-qc/0204046.
[593] J. Rocher, M. Sakellariadou, Constraints on supersymmetric grand unified theories from cosmology, JCAP 0503 (2005) 004. http://xxx.lanl.gov/abs/hepph/0406120.
[594] M. Bastero-Gil, S.F. King, Q. Shafi, Supersymmetric hybrid inflation with nonminimal Kaehler potential, Phys. Lett. B651 (2007) 345-351. http://xxx.lanl. gov/abs/hep-ph/0604198.
[595] J. Martin, V. Vennin, Stochastic effects in hybrid inflation, Phys. Rev. D85 (2012) 043525. arXiv: 1110.2070.
[596] Z. Komargodski, N. Seiberg, From linear SUSY to constrained superfields, J. High Energy Phys. 0909 (2009) 066. arXiv:0907.2441.
[597] L. Alvarez-Gaume, C. Gomez, R. Jimenez, A minimal inflation scenario, JCAP 1103 (2011) 027. arXiv:1101.4948.
[598] L. Alvarez-Gaume, C. Gomez, R. Jimenez, Minimal inflation, Phys. Lett. B690 (2010) 68-72. arXiv:1001.0010.
[599] L. Alvarez-Gaume, C. Gomez, R. Jimenez, Phenomenology of the minimal inflation scenario: inflationary trajectories and particle production, JCAP 1203 (2012) 017. arXiv:1110.3984.
[600] W.H. Kinney, A. Riotto, A Signature of inflation from dynamical supersymmetry breaking, Phys. Lett. B435 (1998) 272-276. http://xxx.lanl.gov/abs/hepph/9802443.
[601] F. Bezrukov, P. Channuie, J.J. Joergensen, F. Sannino, Composite inflation setup and glueball inflation, Phys. Rev. D86 (2012) 063513. arXiv:1112.4054.
[602] P. Channuie, K. Karwan, Observational Constraints on Composite Inflationary Models. arXiv:1307.2880.
[603] J.D. Barrow, P. Parsons, Inflationary models with logarithmic potentials, Phys. Rev. D52 (1995) 5576-5587. http://xxx.lanl.gov/abs/astro-ph/9506049.


[^0]:    * Corresponding author. Tel.: +32 10472075.

    E-mail addresses: jmartin@iap.fr (J. Martin), christophe.ringeval@uclouvain.be (C. Ringeval), vennin@iap.fr (V. Vennin).

    1 http://cp3.irmp.ucl.ac.be/~ringeval/aspic.html.

[^1]:    2 See http://theory.physics.unige.ch/~ringeval/fieldinf.html.

[^2]:    ${ }^{3}$ In the figures, $\bar{w}_{\text {reh }}$ has been denoted by $w$ for simplicity.

[^3]:    4 One may also wonder about the influence of the cosmological constant on this result. In fact, one can show that it leads to a negligible correction. Indeed, it simply amounts to redefining $N_{0}$ by
    $N_{0} \rightarrow N_{0}+\frac{1}{3} \ln \left[1-\frac{\Omega_{\Lambda} \Omega_{\gamma}^{3}}{\Omega_{\mathrm{dm}}^{4}}\left(\frac{g_{\text {eq }}}{g_{0}}\right)^{3}\left(\frac{q_{0}}{q_{\mathrm{eq}}}\right)^{4}\right]$
    which is clearly a very tiny modification (the subscript "eq" denotes quantities at the equivalence time between radiation and matter).

[^4]:    5 A sign in these two equations differs from the ones typeset in Ref. [17], most probably due to a misprint.

[^5]:    ${ }^{6}$ For this purpose, it is convenient to write that $A_{d}^{c}=\left(\phi_{A}\right)_{a}^{b}\left(T_{b}^{a}\right)_{d}^{c}$ and $Z_{d}^{c}=$ $\left(\phi_{Z}\right)_{a}^{b}\left(T_{b}^{a}\right)_{d}^{c}$, where $T_{a}^{b}, a, b=1, \ldots, 5$ is a basis of $\operatorname{SU}(5)$ generators. Concretely, one has $\left(T_{b}^{a}\right)_{d}^{c}=\delta_{b}^{c} \delta_{d}^{a}-\delta_{b}^{a} \delta_{d}^{c} / 5$. As a consequence, the three F-term can be expressed as $F_{X}=\lambda_{2}\left[\operatorname{Tr}\left(\phi_{A}^{2}\right)-m^{2}\right], F_{Z}=\lambda_{1}\left[\phi_{A}^{2}-\operatorname{Tr}\left(\phi_{A}^{2}\right) \mathbb{1} / 5\right]$ and $F_{A}=$ $\lambda_{1}\left[\phi_{Z} \phi_{A}+\phi_{A} \phi_{Z}-2 \operatorname{Tr}\left(\phi_{Z} \phi_{A}\right) \mathbb{1} / 5\right]+2 \lambda_{2} \phi_{X} \phi_{A}$. These expressions are obtained by explicitly writing the superpotential in terms of the components $\left(\phi_{A}\right)_{b}^{a}$ and $\left(\phi_{Z}\right)_{b}^{a}$ and differentiating $W$ with respect to them. From $F_{X}=0$ it follows that $\operatorname{Tr}\left(\phi_{A}^{2}\right)=m^{2}$ and, therefore, $F_{Z}=0$ implies that $\phi_{A}^{2}=m^{2} \mathbb{1} / 5$. This last relation is compatible with $\operatorname{Tr}\left(\phi_{A}^{2}\right)=m^{2}$ but not with $\operatorname{Tr}\left(\phi_{A}\right)=0$ in five dimensions. The conditions $F_{X}=0$ and $F_{Z}=0$ are thus incompatible and supersymmetry is spontaneously broken in this model.

[^6]:    7 This exceeds the usual 64 bits precision on floating point numbers (FP64).

[^7]:    8 see Eqs. (1.1) and (2.9) in that reference.

[^8]:    9 The slight shift visible on the one- and two-sigma contours between the two plots come from the different priors used, either flat on $\epsilon_{1}$ or flat on $\log \epsilon_{1}$ (Jeffreys' prior).

